Partial Differential Equations

High frequency periodic solutions of semilinear equations

Geneviève Allain\textsuperscript{a}, Anne Beaulieu\textsuperscript{b}

\textsuperscript{a} Laboratoire d’analyse et de mathématiques appliquées, Université Paris-Est, UMR CNRS 8050, Faculté de sciences et technologie, 61, avenue du Général-de-Gaulle, 94010 Créteil cedex, France
\textsuperscript{b} Laboratoire d’analyse et de mathématiques appliquées, Université Paris-Est, UMR CNRS 8050, 5, boulevard Descartes, 77454 Marne-la-Vallée cedex 2, France

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Abstract

We are interested with positive solutions of 
\[-\varepsilon^2 \Delta u + f(u) = 0\] in 
\(S^1 \times \mathbb{R}\), i.e. periodic solutions in the first coordinate \(x_1\). The model function for \(f\) is \(f(u) = u - u^p\), \(p > 1\). We prove that for \(\varepsilon\) large enough, any positive solution is a function of the second coordinate only.

Résumé

Solutions périodiques de haute fréquence d’équations semi-linéaires. On s’intéresse aux solutions positives de 
\[-\varepsilon^2 \Delta u + f(u) = 0\] dans 
\(S^1 \times \mathbb{R}\), c’est-à-dire aux solutions périodiques en \(x_1\), la première coordonnée. Le cas modèle est \(f(u) = u - u^p\), \(p > 1\). Nous prouvons que, pour \(\varepsilon\) suffisamment grand, toute solution positive est une fonction de \(x_2\) seulement.

I. Introduction

Let \(N \geq 2\). Under some conditions on \(f\), following Kwong and Zhang, [9], there exists a ground-state solution \(w_0\), that is a radial positive solution, of

\[-\Delta u + f(u) = 0\] in \(\mathbb{R}^{N-1}\).

(1)

Dancer, [5], studied the bifurcation of solutions, which are periodic in one variable, of

\[-\varepsilon^2 \Delta u + f(u) = 0\] in 
\(S^1 \times \mathbb{R}^{N-1}\)

(2)

around \(w_x(x_1, x’) = w_0(\frac{x’}{\varepsilon})\), which is seen as a bounded solution in \(\mathbb{R}^N\), depending only on \(N - 1\) variables. There exists a sequence \((\varepsilon_j)\) of positive parameters, with \(\varepsilon_j = \varepsilon_0/(j + 1)\) for \(j \in \mathbb{N}\), such that there is a curve of positive solutions of (2) in \(L^\infty(\mathbb{R}^N)\) which are \(2\pi\)-periodic in \(x_1\), and decay to zero, uniformly in \(x_1\), as \(|x’| \to \infty\) and which...
bifurcate from \( w_{\varepsilon} \). We could ask whether \( w_{\varepsilon} \) is the only positive bounded periodic solution of (2) for \( \varepsilon > \varepsilon_0 \). In all what follows we suppose that \( N = 2 \) and we give a partial answer to this question in this case.

The model function for \( f \) is \( f(u) = u - u^p \), \( p > 1 \), but we give more general assumptions for a continuous function \( f \):

There exists \( s_0 > 0 \) such that \( f \) is non-decreasing in \([0, s_0]\). \( f(0) = 0 \) and \( f'(0) \) exists.

There exists \( p > 1 \) and \( K > 0 \) such that for any \( u > 0 \), \( -Ku^p \leq f(u) - f'(0)u < 0 \).

**Theorem 1.1.** Let \( f \) be a \( C^1 \) function in \( \mathbb{R}^+ \), that satisfies the hypotheses (3), (4) and (5), such that \( f' \) is decreasing in \( \mathbb{R}^+ \), \( f \) has a maximum for some \( c > 0 \) and \( f'' \) exists and is continuous, except in isolated points of \( \mathbb{R}^+ \). Then there exists \( \varepsilon > 0 \) such that for \( \varepsilon > \varepsilon \) any positive solution of (2) that tends to 0 as \( |x_2| \) tends to infinity, uniformly in \( x_1 \in S^1 \), can only be a function of the variable \( x_2 \).

Therefore, when \( f(u) = u - u^p \), \( p > 1 \), for \( \varepsilon > \varepsilon \), the solutions are the null solution and the functions \( w_0(\frac{\varepsilon^2}{\varepsilon}) \), and the functions obtained by translation from these. Since the conjecture of De Giorgi, (see [1]), several authors ([6,1,3], ...) have proved that the solution of some other semilinear elliptic equations on \( \mathbb{R}^N \) depends only on one variable.

2. Some properties of solutions

**Theorem 2.1.** Let \( f \) be a function that verifies (3) and (4). Let \( (x_1, x_2) \mapsto u(x_1, x_2) \) be a positive solution of (2) that tends to 0 as \( x_2 \) tends to infinity, uniformly in \( x_1 \in S^1 \). Then there exists \( t_0 \in \mathbb{R} \) such that \( u(x_1, t_0 - x_2) = u(x_1, t_0 + x_2) \) for all \( (x_1, x_2) \in S^1 \times \mathbb{R} \) and \( u \) decreases with respect to \( x_2 \) for \( x_2 \geq t_0 \).

The proof of this theorem is similar to [2]. It uses the moving plane method like [7,4].

**Theorem 2.2.** Let \( f \) be a function that verifies (3), (4) and (5). Then for all \( \varepsilon > 0 \), there exists \( C > 0 \), depending only on \( \varepsilon \), \( f'(0) \) and \( p \), and decreasing with respect \( \varepsilon \), such that if \( u \) is any positive solution of (2) that satisfies the hypotheses of Theorem 1.1 and that is even in \( x_2 \), we have

\[
\sup_{S^1 \times \mathbb{R}^+} u \leq C \left( \inf_{S^1 \times \{0\}} u + \frac{K}{\varepsilon^2} \inf_{S^1 \times \{0\}} u^p \right),
\]

**Proof.** The claim follows from the Harnack inequalities. First, we apply Theorem 8.17 of [8] with \( Lu = \Delta u \) and the equation \( Lu = \frac{1}{\varepsilon^2} f(u) \) and \( R = \pi \). We get for all \( n > 1 \) and all \( q > 2 \) a constant \( C \) that depends on \( n \) and \( q \), such that for all positive solution \( u \) and all \( \varepsilon > 0 \) we have

\[
\sup_{B_{2\varepsilon}(0)} u \leq C \left( R^{-\frac{2}{n}} \|u\|_{L^p(B_{2\varepsilon}(0))} + \frac{1}{\varepsilon^2} R^{-\frac{4}{q}} \left( \frac{1}{B_{2\varepsilon}(0)} \int (f'(0)u + Ku^p)^{\frac{q}{2}} \right)^{\frac{2}{q}} \right).
\]

that gives

\[
\sup_{B_{\varepsilon}(0)} u \leq C \left( R^{-\frac{2}{n}} \|u\|_{L^p(B_{\varepsilon}(0))} + \frac{1}{\varepsilon^2} R^{-\frac{4}{q}} \left( \|u\|_{L^q(B_{\varepsilon}(0))}^q + K \|u\|_{L^p(B_{\varepsilon}(0))}^p \right) \right)
\]

where \( C \) depends only on \( q \) and \( n \). Now we apply Theorem 8.18 of [8] for \( Lu = \varepsilon^2 \Delta u - f'(0)u \), the equation \( Lu \leq 0 \) and \( R = \pi \). We get a constant \( C > 0 \), that depends on \( n \) and on \( \frac{K}{\varepsilon^2} \) such that for all non-negative \( u \) satisfying \( Lu \leq 0 \) we have

\[
R^{-\frac{2}{n}} \|u\|_{L^p(B_{\varepsilon}(0))} \leq C \inf_{B_{\varepsilon}(0)} u.
\]
But the constant $C$ is a decreasing function of $\varepsilon$. Indeed, if $\varepsilon_1 < \varepsilon_2$ and if $\varepsilon_2^2 \Delta u - f'(0) u \leq 0$, then $\varepsilon_1^2 \Delta u - f'(0) u \leq 0$. So, if $C(\varepsilon_1)$ and $C(\varepsilon_2)$ are the best constants in (9), respectively for $\varepsilon_1$ and $\varepsilon_2$, we have $C(\varepsilon_2) \leq C(\varepsilon_1)$. On the other hand we have $\sup_{B_R(0)} u = \sup_{S^1 \times \mathbb{R}^+} u$ and $\inf_{B_R(0)} u \leq \inf_{S^1 \times \mathbb{R}^+} u$. Combining (8) and (9), we get (6).

3. Proof of Theorem 1.1

We may suppose that $u$ is even in $x_2$ and consequently that $\frac{\partial u}{\partial x_2}(x_1, 0) = 0$ for all $x_1 \in S^1$. Let us define $\Psi(x_2) = \frac{1}{2\pi} \int_0^{2\pi} u(x_1, x_2) \, dx_1$. Integrating (2) on $[0, 2\pi]$ we obtain

$$-\varepsilon^2 \Psi''(x_2) + \frac{1}{2\pi} \int_0^{2\pi} f(u) \, dx_1 = 0.$$ 

The hypotheses on $f$ give $-\varepsilon^2 \Psi''(x_2) \geq -f(\Psi(x_2))$.

By the decaying property of $u$ in $x_2$, we have that $\Psi'(x_2) < 0$. Multiplying by $\Psi'$, integrating on $[0, +\infty[$ and using the Neumann condition on $u$ we get

$$F(\Psi(0)) \geq 0,$$

where $F(u) = \int_0^u f(t) \, dt$. It follows from the assumptions on $f$ that $F$ tends to $-\infty$ when $u$ tends to $+\infty$. Let $C_*$ be such that $F(u)$ is non-positive for $u > C_*$. We have

$$\Psi(0) \leq C_*,$$

that leads to $\inf_{x_1 \in S^1} u(x_1, 0) \leq C_*$ and then, thanks to (6), for $\varepsilon \geq \varepsilon_1$, where $\varepsilon_1 > 0$ is given, we have

$$\sup_{S^1 \times \mathbb{R}^+} u \leq C,$$

where $C$ depends on $\varepsilon_1$ and is valid for any solution $u$ of (2). Now we multiply (2) by $\frac{\partial u}{\partial x_2}$ and we integrate on $S^1 \times \mathbb{R}^+$. We obtain

$$\frac{\varepsilon^2}{2} \int_0^{2\pi} \left( \frac{\partial u}{\partial x_1}(x_1, 0) \right)^2 \, dx_1 + \int_0^{2\pi} F(u(x_1, 0)) \, dx_1 = 0.$$ (13)

Using (10) we get

$$\frac{\varepsilon^2}{2} \int_0^{2\pi} \left( \frac{\partial u}{\partial x_1}(x_1, 0) \right)^2 \, dx_1 \leq \int_0^{2\pi} \left( -F(u(x_1, 0)) + F(\Psi(0)) \right) \, dx_1,$$

that leads to

$$\frac{\varepsilon^2}{2} \int_0^{2\pi} \left( \frac{\partial u}{\partial x_1}(x_1, 0) \right)^2 \, dx_1 \leq \int_0^{2\pi} \left( F(u(x_1, 0)) - F(\Psi(0)) - (u(x_1, 0) - \Psi(0)) f(\Psi(0)) \right) \, dx_1.$$ (14)

However, by (11) and (12), given $\varepsilon_1 > 0$, there exists $M > 0$ such that $|f'(v)| \leq M$ for all $v$ between $\Psi(0)$ and $u(x_1, 0)$, $x_1 \in S^1$. Thus we have, for all $x_1 \in S^1$ and for all $\varepsilon > \varepsilon_1$,

$$\left| F(u(x_1, 0)) - F(\Psi(0)) - (u(x_1, 0) - \Psi(0)) f(\Psi(0)) \right| \leq M \left| u(x_1, 0) - \Psi(0) \right|^2.$$ (15)

On the other hand the Poincaré inequality gives

$$\int_0^{2\pi} \left( u(x_1, 0) - \Psi(0) \right)^2 \, dx_1 \leq 4\pi^2 \int_0^{2\pi} \left( \frac{\partial u}{\partial x_1} \right)^2 \, dx_1.$$ (16)
We deduce from (14)–(16) that there exists $C > 0$ such that for all $\varepsilon > \varepsilon_1$,
\[
\varepsilon^2 \frac{2}{2} \pi \int_0^{2\pi} (u(x_1, 0) - \Psi(0))^2 \, dx_1 \leq C \int_0^{2\pi} (u(x_1, 0) - \Psi(0))^2 \, dx_1.
\]
This inequality gives that there exists $\bar{\varepsilon} > 0$ such that for $\varepsilon > \bar{\varepsilon}$ any solution of (2) verifies $\frac{\partial u}{\partial x_1}(x_1, 0) = 0$, for all $x_1 \in [0, 2\pi]$. Let us prove that such a solution verifies in fact $\frac{\partial u}{\partial x_1}(x_1, 0) = 0$, for all $x_1 \in [0, 2\pi]$ and for all $x_2 \in [0, +\infty[$. As $f$ is twice differentiable in $[0, +\infty[$, except at isolated points, we may argue as follows. By derivation of (2) we get
\[
-\varepsilon^2 \Delta \frac{\partial u}{\partial x_1} + f'(u) \frac{\partial u}{\partial x_1} = 0. \tag{17}
\]
Then we multiply this equation by $\frac{\partial^2 u}{\partial x_1 \partial x_2}$ and we integrate on $S^1 \times \mathbb{R}^+$. We obtain
\[
\int_0^{+\infty} \int_0^{2\pi} f''(u) \left( \frac{\partial u}{\partial x_2} \right)^2 \, dx_1 \, dx_2 = 0.
\]
However, $f$ is concave and $u$ decreases with respect to $x_2$, so we have $\frac{\partial u}{\partial x_1} = 0$ in $S^1 \times \mathbb{R}^+$.

**Remark 1.** For $N > 2$, the positive solutions of (2) are radially symmetric and decreasing in $r = |(x_2, \ldots, x_N)|$. But our above proof does not work for $x_2$ replaced by $r$. In this case we are unable to prove (10) because the equation for $\psi$ is not the same one.

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**References**