Stability of relaxation models
for conservation laws

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These notes intend to give an introduction to the recent development of relaxation models, the associated stability conditions, and discrete approximations.

1 Relaxation models

A system of conservation laws is a system of partial differential equations of the form

\[ \partial_t u + \partial_x F(u) = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}, \]

where \( u(t, x) \in \mathbb{R}^p \), and \( F(u) \in \mathbb{R}^p \). The classical features for such systems are that

- The Cauchy problem has bounded, but discontinuous, solutions \( u \)
- The nonlinearity \( F \) induces nonuniqueness

The idea of approximation by relaxation to the system (1) is as follows:

- Build solutions of (1) as limits \( u = \lim u_\epsilon, \ u_\epsilon(t, x) = Lf_\epsilon(t, x) \), obtained from solutions \( f_\epsilon \) to another (simpler) system of conservation laws
• This solution \( f_\epsilon \) is forced in the limit, by a relaxation process, to lie in a manifold of equilibrium, \( f_\epsilon(t, x) \in \mathcal{M} \).

• This manifold \( \mathcal{M} \) can be parametrized by \( u \equiv Lf \), i.e. we have

\[
  f \in \mathcal{M} \iff f = M(u), \quad \text{and} \quad LM(u) = u.
\]  

(2)

Example: the Jin Xin model.

The most simple example is given by \([13]\) (\( \epsilon \) is dropped):

\[
  \begin{align*}
    \partial_t f_1 - c \partial_x f_1 &= \frac{M_1(u) - f_1}{\epsilon}, \\
    \partial_t f_2 + c \partial_x f_2 &= \frac{M_2(u) - f_2}{\epsilon},
  \end{align*}
\]

(3)

where \( f(t, x) = (f_1(t, x), f_2(t, x)) \in \mathbb{R}^p \times \mathbb{R}^p, u(t, x) = Lf(t, x), c > 0, \)

\[
  Lf = f_1 + f_2,
\]

\[
  M(u) = \left( \frac{u - F(u)/c}{2}, \frac{u + F(u)/c}{2} \right).
\]

(4)

• One has \( \partial_t u + c \partial_x (f_2 - f_1) = 0 \)

• The right-hand side forces \( f - M(u) \to 0 \), i.e. \( f \to \mathcal{M} \), thus \( c(f_2 - f_1) \simeq c(M_2(u) - M_1(u)) = F(u) \)

2 Hyperbolic relaxation: general framework

A general framework is given by \([9]\):

\[
  \partial_t f + \partial_x A(f) = \frac{Q(f)}{\epsilon},
\]

(5)

where \( f(t, x) \in \mathbb{R}^q, q > p, L : \mathbb{R}^q \to \mathbb{R}^p \) is linear, and the maxwellian equilibrium \( M(u) \) satisfies consistency relations

\[
  \begin{align*}
    LM(u) &= u, \\
    LA(M(u)) &= F(u).
  \end{align*}
\]

(6)

The relaxation term must satisfy

\[
  LQ(f) = 0, \quad Q(f) = 0 \quad \text{iff} \quad f = M(u) \quad \text{for some} \quad u.
\]

(7)

Example: BGK relaxation term \( Q(f) = M(Lf) - f \).
3 Kinetic relaxation models

The kinetic relaxation models occur as relaxation models when

- The space $\mathbb{R}^q = (\mathbb{R}^p)^\Xi$ is a space of functions, $f = f(\xi), \xi \in \Xi$, with $\Xi$ a measure space with positive measure $d\xi$,
- The nonlinearity is $A(f)(\xi) = a(\xi)f(\xi)$ for some function $a(\xi) \in \mathbb{R}$,
- The linear operator is $Lf = \int_\Xi f(\xi) d\xi$,
- The maxwellian becomes $M(u) = M(u, \xi)$, and the consistency relations become moment relations

$$\int M(u, \xi) d\xi = u, \quad \int a(\xi)M(u, \xi) d\xi = F(u).$$

Thus kinetic relaxation models identify with semilinear diagonal relaxation models, possibly in infinite dimension. Such models arise naturally in the kinetic theory of gases, like the Boltzmann equation. Such models are described in [17].

4 Parabolic relaxation

Parabolic relaxation comes from a different scaling in (5):

$$\partial_t f + \frac{1}{\epsilon} \partial_x A(f) = \frac{Q(f)}{\epsilon^2},$$

leading to different features than above:

- The limit $\epsilon \to 0$ is a parabolic equation, like in [11], [14], [12], [15], [7]
- Incompressible models can be obtained at the limit, see [1], [10], and the talk by F. Golse in this congress
5 Relaxation limit

How to justify the relaxation limit $\epsilon \to 0$?

Several methods are used in different situations:

1. Whenever the limit $u$ is smooth (in a Sobolev space).
   Method: relative entropy method to estimate the distance to the limit solution [16]

2. When the limit equation is mildly nonlinear (like incompressible Navier-Stokes, with viscosity).
   Method: control the compactness and the size of the solution [10]

3. When the limit $u$ is a discontinuous weak solution.
   Method: obtain $L^\infty$ bounds on the solution, and get compactness (BV estimates, or compensated compactness)

In these notes, I am more interested in the last situation.

6 Stability of the relaxation limit

Relaxation models were first discussed in [19]. In particular, a main idea was there to write structural necessary/sufficient conditions for the relaxation limit to hold.

Example: For Jin-Xin’s model, a stability condition is the subcharacteristic condition

$$\sigma(F'(u)) \subset [-c, c],$$

(10)

where $\sigma$ denotes the spectrum, and $F'(u)$ is the derivative of $F(u)$ with respect to $u$.

Several stability conditions exist. We shall discuss here the following ones:

- The entropy extension condition (EEC)
- The reduced stability condition (RSC)
- The interlacing subcharacteristic condition (ISC)
• The Chapman-Enskog dissipativity condition (CED)

Except (ISC), they involve entropy inequalities.

6.1 Entropy

The notion of entropy is used in hyperbolic conservation laws for:
• selecting admissible solutions (Lax)
• getting a priori bounds
• proving compactness (DiPerna)

For the conservation law
\[ \partial_t u + \partial_x F(u) = 0, \]  
(11)
an entropy is a scalar function \( \eta(u) \), such that there exist another scalar function \( G(u) \), called the entropy flux, satisfying
\[ G'(u) = \eta'(u)F'(u). \]  
(12)

Interest: smooth solutions to (11) satisfy
\[ \partial_t \eta(u) + \partial_x G(u) = 0. \]  
If \( \eta \) is a convex entropy, a weak solution \( u \) to (11) is said \( \eta \) entropy satisfying if
\[ \partial_t \eta(u) + \partial_x G(u) \leq 0. \]  
(13)

6.2 Relaxation system: Entropy Extension Condition (EEC)

This condition is due to [9].

Consider a conservation law \( \partial_t u + \partial_x F(u) = 0 \), and an associated relaxation system \( \partial_t f + \partial_x A(f) = Q(f)/\epsilon \), satisfying the previously stated conditions.

Definition. Given a convex entropy \( \eta \), we say that (EEC) holds if there exist a convex entropy \( \mathcal{H}(f) \) with entropy flux \( \mathcal{G}(f) \), such that
\[ \mathcal{H}(M(u)) = \eta(u) + \text{cst}, \]
\[ \mathcal{G}(M(u)) = G(u) + \text{cst}, \]  
(14)
and that the \textit{minimization principle} holds,
\[ H(M(u)) \leq H(f) \quad \text{whenever} \quad u = Lf. \quad (15) \]
The relaxation term must satisfy also \[ H'(f)Q(f) \leq 0. \]

**Interest of the Entropy Extension Condition (EEC):**

Starting from an \textit{entropy solution} \( f \) of the relaxation system, one automatically gets an \textit{entropy solution} \( u \) (if \( f \) converges weakly, and \( u \) converges strongly).

Thus it enables in the good cases to
- select admissible solutions (Lax)
- get a priori bounds
- prove compactness (DiPerna)

For the last point, compensated compactness works when there is a whole family of entropies with entropy extensions, i.e. in the scalar case \( p = 1 \), and for some good models [2].

In the case of only a single entropy extension, it can work also for special structures [18], and in the kinetic case with continuous variable \( \xi \) [3].

### 6.3 Relaxation system: Reduced Stability Condition (RSC)

This condition is introduced in [6].

Consider a conservation law \( \partial_t u + \partial_x F(u) = 0 \), and an associated relaxation system \( \partial_t f + \partial_x A(f) = Q(f)/\varepsilon \), satisfying again the minimal consistency conditions.

We assume hyperbolicity of both systems, i.e. that \( F'(u) \) and \( A'(f) \) are diagonalizable, and we denote by \( P_\lambda[\ldots] \) the projector onto the eigenspace, for any eigenvalue \( \lambda \).

**Definition.** Given a convex entropy \( \eta \), we say that (RSC) holds if for any \( u \) and any \( \lambda \)
\[ LP_\lambda[A'(M(u))]M'(u) \text{ is symmetric nonnegative for } \eta''(u). \quad (16) \]
• It implies that this operator must be diagonalizable with nonnegative eigenvalues
• It involves only maxwellian states

6.4 Relaxation system: Interlacing Subcharacteristic Condition (ISC)

Consider a conservation law \( \partial_t u + \partial_x F(u) = 0 \), and an associated relaxation system \( \partial_t f + \partial_x A(f) = Q(f)/\epsilon \), satisfying again the minimal consistency conditions.

We assume hyperbolicity of both systems, i.e. that \( F'(u) \) and \( A'(f) \) are diagonalizable. We denote by

\[
\begin{align*}
\lambda_1[F'(u)] & \leq \cdots \leq \lambda_p[F'(u)], \\
\lambda_1[A'(M(u))] & \leq \cdots \leq \lambda_q[A'(M(u))],
\end{align*}
\]

(17)

the eigenvalues repeated with multiplicities.

**Definition.** We say that (ISC) holds if for any \( u \)

\[
\lambda_k[A'(M(u))] \leq \lambda_k[F'(u)] \leq \lambda_{q-p+k}[A'(M(u))], \quad \text{for any} \quad 1 \leq k \leq p. \quad (18)
\]

6.5 Relaxation system: Chapman-Enskog Dissipativity (CED)

Consider a conservation law \( \partial_t u + \partial_x F(u) = 0 \), and an associated relaxation system \( \partial_t f + \partial_x A(f) = Q(f)/\epsilon \) with BGK relaxation \( Q(f) = M(Lf - f) \), satisfying again the minimal consistency conditions. Then formally, with \( u_\epsilon = Lf_\epsilon \),

\[
\partial_t u_\epsilon + \partial_x F(u_\epsilon) = \epsilon \partial_x (D(u_\epsilon) \partial_x u_\epsilon),
\]

(19)

up to terms in \( \epsilon^2 \), with

\[
D(u) = L A'(M(u))^2 M'(u) - F'(u)^2
\]

(20)
Definition. Let $\eta$ be a convex entropy. We say that (CED) holds if (19) is $\eta$-symmetrically entropy dissipative, i.e. if $D(u)$ is symmetric nonnegative for $\eta''(u)$. It implies that

$$
\partial_t \eta(u) + \partial_x G(u) - \epsilon \partial_x (\eta'(u) D(u) \partial_x u) = -\epsilon D(u)^T \eta''(u) \cdot \partial_x u \cdot \partial_x u \leq 0.
$$

(21)

6.6 Comparison of the stability conditions

Theorem 1 [9]

$$(EEC) \implies (ISC),$$

$$(EEC) \implies (CED).$$

(22)

None of $(ISC)$ or $(CED)$ imply the other.

Theorem 2 [6]

$$(EEC) \implies (RSC) \implies (ISC),$$

$$(EEC) \implies (RSC) \implies (CED).$$

(23)

Theorem 3 [4] In the kinetic case,

$$(EEC) \iff (RSC).$$

(24)

7 Discrete approximations

Relaxation approximations enable to build numerical schemes for conservation laws $\partial_t u + \partial_x F(u) = 0$, by the transport-projection approach [8]. It can be summarized as follows.

1. Start with $u^n(x)$ piecewise constant
2. Define $f^n(x) = M(u^n(x))$, which is piecewise constant
3. Solve the relaxation problem $\partial_t f + \partial_x A(f) = 0$ for $t^n < t < t^{n+1}$
4. define $u^{n+1}(x)$ by piecewise constant projection of $L f(t^{n+1}, x)$

- Builds an Approximate Riemann Solver that generates a conservative finite volume scheme
• In particular, kinetic relaxation models lead to Kinetic schemes
• Condition \((EEC)\) automatically gives entropy consistency
• Condition \((RSC)\) gives entropy consistency for data of small variation

For these numerical features, consult [5].

8 Concluding comments on relaxation models

• Relaxation approximations yield structurally well-behaved approxima-
tions:
  – Entropy conditions can be analyzed
  – Stability can be analyzed
  – Have the same hyperbolic structure as the limit (finite speed of
    propagation...), which is better than viscosity approximation

• Relaxation approximations can be used:
  – To prove existence to the Cauchy problem of the limit, even if
    until now they have not allowed to prove really new results
  – To build stable numerical methods

References

[1] C. Bardos, F. Golse, C.D Levermore, *The acoustic limit for the Boltz-

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