On the discrete conservation of the Gauss-Poisson equation of plasma Physics

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Abstract
We consider numerical methods which exactly preserve the Gauss-Poisson equation when solving the charge conservation and Maxwell-Ampère’s equations. Apart from the well-known leap-frog method, we present two situations where this property is verified, one with rectangular mesh and functions defined at the center of the cells, and one with a finite volume type of formulation on triangles.

Key-words. Gauss-Poisson equation, exact discrete conservation, discrete divergence.

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Sur la conservation discrète de l’équation de Gauss-Poisson de la physique des plasmas

Résumé
Nous considérons des méthodes numériques qui conservent exactement l’équation de Gauss-Poisson quand on résout les équations de conservation de la charge et de Maxwell-Ampère. Mise à part la méthode saute-mouton bien connue, nous présentons deux situations où cette propriété est vérifiée, une sur un maillage rectangulaire et avec des fonctions définies au centre des mailles, et une avec une formulation de type volumes finis sur des triangles.

Mots-clés. Equation de Gauss-Poisson, conservation discrète exacte, divergence discrète.
1 Introduction

We consider numerical methods for solving the charge conservation and Maxwell-Ampère’s equations of plasma Physics

\[ \partial_t \rho + \text{div} \, j = 0, \]  
\[ \partial_t E - \varepsilon_0 \varepsilon \, \text{curl} \, B = -\frac{1}{\varepsilon_0} j, \]

where \( \rho \) is the charge density, \( j \) is the current density and \( E, B \) are the electromagnetic fields. These equations have to be completed so as to determine the evolution of \( B \) and \( j \). For example, \( B \) can solve the Maxwell-Faraday equation

\[ \partial_t B + \text{curl} \, E = 0. \]

For \( j \), there are mainly two possibilities: either fluid equations or kinetic equations.

We do not assume here any peculiar form for these supplementary relations. We rather study the striking property of only (1.1)-(1.2) which lies in the Gauss-Poisson equation, which is obtained as follows. If we subtract (1.1) from the divergence of \( \varepsilon_0 \times (1.2) \), we get

\[ \partial_t (\varepsilon_0 \text{div} \, E - \rho) = 0. \]

Therefore, the law

\[ \varepsilon_0 \text{div} \, E = \rho \]

is preserved at any time if it is satisfied initially, and this holds independently of the choice of additional relations on \( B \) and \( j \).

When dealing with numerical methods, the accuracy at which (1.5) is true is very important to get physically relevant results. One usually uses a Poisson corrector [1], [7]. This procedure can be described as follows. From the density and electric field \( \rho^n \) and \( E^n \) at time \( n \Delta t \) (\( \Delta t \) denotes the timestep), we compute \( \rho^{n+1} \) and a prediction \( E_{pr}^{n+1} \) at time \( (n+1)\Delta t \) by a given method, then we set

\[ E^{n+1} = E_{pr}^{n+1} + \delta E^{n+1}, \]

where the correction \( \delta E^{n+1} \) has to solve

\[ \varepsilon_0 \text{div} \, \delta E^{n+1} = \rho^{n+1} - \varepsilon_0 \text{div} \, E_{pr}^{n+1}. \]

In practice, we choose \( \delta E^{n+1} \) to be a gradient field \( \delta E^{n+1} = -\nabla \phi_{cor}^{n+1} \), so we are led to the Poisson equation

\[ -\varepsilon_0 \Delta \phi_{cor}^{n+1} = \rho^{n+1} - \varepsilon_0 \text{div} \, E_{pr}^{n+1}. \]

The drawbacks of this method are well-known.

- It introduces an unnatural non-local dependence (and also a dependence on arbitrary boundary conditions).
It involves a high computational time, and it is difficult to use on vector or parallel computers.

An attempt to bypass this last (and most serious) difficulty is to use an approximate method for solving (1.8), see [9], [8].

Another method consists in draining the error in Gauss-Poisson equation by suitable dispersive equations so that it is absorbed in boundary conditions [10].

In an ideal method, there should exist an exact discrete conservation of the Gauss-Poisson equation (1.5). This situation can be roughly described as follows: we look for discrete versions of (1.1) and (1.2), and a discrete divergence $\nabla_d$ such that (1.4) is replaced by

$$\varepsilon_0 \nabla_d E^{n+1} - \rho^{n+1} = \varepsilon_0 \nabla_d E^n - \rho^n,$$

and this should be automatically satisfied without further correction. It was known for a long time [1] that such methods exist. However, at least in the context of the resolution of Vlasov’s equation by a PIC (particle in cell) method, these methods have been known to give very low resolution, compared to the Poisson correction method. The first method to work successfully with PIC methods was introduced by J. Villasenor and O. Buneman [11]. It consists in a precise current assignment by area weighting. Then, a general formulation by time-space finite element was introduced by J.W. Eastwood [4]. It has been found to be efficient in conjunction with PIC methods, and it can be used in complex geometries [5].

In this paper, we consider (1.1) and (1.2) in two dimensions, that is

$$\partial_t \rho + \nabla \cdot j = 0,$$

$$\partial_t E + c^2 (\nabla B)^\perp = \frac{1}{\varepsilon_0} j,$$

where $\rho$, $B$ are scalar and $j$, $E$ are vectors. Here, $z^\perp$ denotes the image of the two-dimensional vector $z$ by the rotation of angle $\pi/2$.

Our aim is not to build a method which satisfies the discrete Gauss-Poisson equation, this could only be done after specifying the additional relations on $B$ and $j$. Rather, we look for specific discrete forms of (1.10)-(1.11) that automatically satisfy an exact conservation of the type (1.9), disregarding the nature of the additional equations on $B$ and $j$, in the same way as (1.4) corresponds to (1.1)-(1.2). These specific discrete forms can be viewed as analogous of conservative forms used in the numerical computation of conservation laws.

Here we are mainly concerned with fluid equations, in which (1.1) can be easily written in conservative form. Therefore, we will not deal with the problem of realizing (1.1) for PIC methods, which is connected to what is usually called "current assignment".
2 The case of rectangles and staggered grid

The well-known situation where (1.9) happens to be verified can be described as follows. We assume that quantities are defined on a rectangular grid of lengths $\Delta x$ and $\Delta y$, according to Figure 1. The indices $i$ and $j$ refer to the space position ($x_i = i\Delta x$, $y_j = j\Delta y$), while $n$ refers to the time $n\Delta t$. If a numerical scheme for (1.10)-(1.11) can be put under the form

$$
\rho_{ij}^{n+1} - \rho_{ij}^n + \Delta t \left( j_{i+1/2,j}^{n+1/2} - j_{i-1/2,j}^{n+1/2} \right) + \frac{\Delta t}{\Delta y} \left( j_{i,j+1/2}^{n+1/2} - j_{i,j-1/2}^{n+1/2} \right) = 0,
$$

(2.1)

$$
(E_x)_{i,j+1/2}^{n+1} - (E_x)_{i,j-1/2}^n - \frac{c^2 \Delta t}{\Delta y} \left( B_{i+1/2,j+1/2}^{n+1/2} - B_{i+1/2,j-1/2}^{n+1/2} \right) = -\frac{\Delta t}{\varepsilon_0} j_{i+1/2,j}^{n+1/2},
$$

(2.2)

$$
(E_y)_{i,j+1/2}^{n+1} - (E_y)_{i,j-1/2}^n + \frac{c^2 \Delta t}{\Delta x} \left( B_{i+1/2,j+1/2}^{n+1/2} - B_{i-1/2,j+1/2}^{n+1/2} \right) = -\frac{\Delta t}{\varepsilon_0} j_{i,j+1/2}^{n+1/2},
$$

(2.3)

for some numerical fluxes $(j_{i+1/2,j}^{n+1/2})$, $(j_{i,j+1/2}^{n+1/2})$ and $(B_{i+1/2,j+1/2}^{n+1/2})$, then

$$
P_{ij}^n \equiv \frac{\varepsilon_0}{\Delta x} \left( (E_x)_{i+1/2,j}^n - (E_x)_{i-1/2,j}^n \right) + \frac{\varepsilon_0}{\Delta y} \left( (E_y)_{i,j+1/2}^n - (E_y)_{i,j-1/2}^n \right) - \rho_{ij}^n
$$

(2.4)

is in fact independent of $n$. It is easily obtained by plugging (2.1)-(2.3) into the expression giving $P_{ij}^{n+1} - P_{ij}^n$. Let us underline that, as stated in introduction, we have not defined what should be the numerical fluxes $(j_{i+1/2,j}^{n+1/2})$, $(j_{i,j+1/2}^{n+1/2})$ and $(B_{i+1/2,j+1/2}^{n+1/2})$. Their definition should of course depend on what equations we want to solve for $B$ and $j$. The above defined $P_{ij}^n$ is independent of $n$, independently of this choice. This property relies on the specific form of (2.1)-(2.3).
This formulation is adapted to the usual leap-frog method for Maxwell’s equations, because in that case (2.2)-(2.3) hold, while $B$ evolves according to Maxwell-Faraday’s equation

$$\partial_t B - \text{div} \, E^4 = 0,$$

which is discretized by

$$B_{i+1/2,j,1/2}^{n+1/2} - B_{i+1/2,j,1/2}^{n-1/2} + \frac{\Delta t}{\Delta x} \left( (E_y)_i^{n+1/2,j+1/2} - (E_y)_i^{n+1/2,j-1/2} \right)$$

$$- \frac{\Delta t}{\Delta y} \left( (E_x)_i^{n+1/2,j+1} - (E_x)_i^{n+1/2,j-1} \right) = 0.$$ 

(2.6)

Therefore, this situation is a special case of the more general one defined by (2.1)-(2.3) where $B$ can evolve according to a different law.

3 The case of rectangles and centered values

We now describe a situation where quantities are defined at the centers of the cells, according to Figure 2. If a numerical scheme for (1.10)-(1.11) can be put under the form

$$\rho_{ij}^{n+1} - \rho_{ij}^n + \frac{\Delta t}{\Delta x} (j_{i+1/2,j}^{n+1/2} - j_{i-1/2,j}^{n+1/2}) + \frac{\Delta t}{\Delta y} (j_{i,j+1/2}^{n+1/2} - j_{i,j-1/2}^{n+1/2}) = 0,$$

(3.1)

$$\left( E_x \right)_{ij}^{n+1} - \left( E_x \right)_{ij}^n - \frac{\epsilon^2 \Delta t}{\Delta y} \left( \frac{B_{i+1/2,j,1/2}^{n+1/2} + B_{i+1/2,j,1/2}^{n-1/2}}{2} - \frac{B_{i-1/2,j,1/2}^{n+1/2} + B_{i+1/2,j,1/2}^{n-1/2}}{2} \right) = - \frac{\Delta t}{\epsilon_0} \left( j_{i+1/2,j}^{n+1/2} + j_{i+1/2,j}^{n-1/2} \right),$$

(3.2)

$$\left( E_y \right)_{ij}^{n+1} - \left( E_y \right)_{ij}^n - \frac{\epsilon^2 \Delta t}{\Delta x} \left( \frac{B_{i+1/2,j,1/2}^{n+1/2} + B_{i+1/2,j,1/2}^{n-1/2}}{2} - \frac{B_{i-1/2,j,1/2}^{n+1/2} + B_{i+1/2,j,1/2}^{n-1/2}}{2} \right) = - \frac{\Delta t}{\epsilon_0} \left( j_{i+1/2,j}^{n+1/2} + j_{i+1/2,j}^{n-1/2} \right),$$

(3.3)

for some numerical fluxes $(j_{i+1/2,j}^{n+1/2})$, $(j_{i+1/2,j}^{n-1/2})$ and $(B_{i+1/2,j,1/2}^{n+1/2})$, then

$$P_{i+1/2,j,1/2}^n \equiv \frac{\epsilon_0}{2 \Delta x} \left( (E_x)_{i+1,j+1}^n - (E_x)_{i,j+1}^{n+1/2} + (E_x)_{i+1,j}^{n+1/2} - (E_x)_{i,j}^{n+1/2} \right)$$

$$+ \frac{\epsilon_0}{2 \Delta y} \left( (E_y)_{i+1,j+1}^n - (E_y)_{i+1,j}^{n+1/2} + (E_y)_{i,j+1}^{n+1/2} - (E_y)_{i,j}^{n+1/2} \right)$$

$$- \frac{1}{4} \left( \rho_{i+1,j}^n + \rho_{i,j+1}^n + \rho_{i,j}^n + \rho_{i+1,j+1}^n \right),$$

(3.4)

is in fact independent of $n$. As in Section 2, it is easily obtained by plugging (3.1)-(3.3) into $P_{i+1/2,j,1/2}^n - P_{i+1/2,j,1/2}^{n+1/2}$, and this result does not depend on the choice of evolution equations for $B$ and $j$. 


With \((\mathbf{B})\), we can reformulate \((\mathbf{E})\), \((\mathbf{H})\), and we also write a similar equation for the method, which is written as follows. We now consider centered values \(B_{ij}^{n+1/2}\) of \(B\), and the vertex values appearing in \((3.2), (3.3)\) are defined by
\[
B_{i+1/2,j+1/2}^{n+1/2} = \frac{1}{4} \left( B_{i,j}^{n+1/2} + B_{i,j+1}^{n+1/2} + B_{i+1,j}^{n+1/2} + B_{i+1,j+1}^{n+1/2} \right). \tag{3.5}
\]

With (3.5), we can reformulate (3.2), (3.3), and we also write a similar equation for the evolution of \(B\). We obtain
\[
\begin{align*}
(E_x)_{ij}^{n+1} - (E_x)_{ij}^n &= \frac{c^2 \Delta t}{8 \Delta y} \left( B_{i-1,j+1}^{n+1/2} + 2B_{i,j+1}^{n+1/2} + B_{i+1,j+1}^{n+1/2} \\
&\quad - B_{i-1,j-1}^{n+1/2} - 2B_{i,j-1}^{n+1/2} - B_{i+1,j-1}^{n+1/2} \right) - \frac{\Delta t j_{i-1/2,j+1/2}^{n+1/2} + j_{i+1/2,j+1/2}^{n+1/2}}{\varepsilon_0}, \tag{3.6}
\end{align*}
\]
\[
\begin{align*}
(E_y)_{ij}^{n+1} - (E_y)_{ij}^n &= \frac{c^2 \Delta t}{8 \Delta x} \left( B_{i+1,j-1}^{n+1/2} + 2B_{i,j-1}^{n+1/2} + B_{i+1,j-1}^{n+1/2} \\
&\quad - B_{i-1,j-1}^{n+1/2} - 2B_{i-1,j}^{n+1/2} - B_{i+1,j}^{n+1/2} \right) - \frac{\Delta t j_{i-1/2,j+1/2}^{n+1/2} + j_{i+1/2,j+1/2}^{n+1/2}}{\varepsilon_0}, \tag{3.7}
\end{align*}
\]
\[
\begin{align*}
B_{ij}^{n+1/2} - B_{ij}^{n-1/2} &= \frac{\Delta t}{8 \Delta y} \left( (E_x)_{i-1,j+1}^n + 2(E_x)_{i,j+1}^n + (E_x)_{i+1,j+1}^n \\
&\quad - (E_x)_{i-1,j-1}^n - 2(E_x)_{i,j-1}^n - (E_x)_{i+1,j-1}^n \right) + \frac{\Delta t}{8 \Delta x} \left( (E_y)_{i+1,j+1}^n + 2(E_y)_{i+1,j}^n + (E_y)_{i+1,j+1}^n \\
&\quad - (E_y)_{i-1,j+1}^n - 2(E_y)_{i-1,j}^n - (E_y)_{i-1,j+1}^n \right) = 0. \tag{3.8}
\end{align*}
\]
This leap-frog type method \((3.6)-(3.8)\) to solve Maxwell's equations (if \( j \equiv 0 \) in \((3.6)-(3.7)\)) with unknowns \( E_{ij}^n, B_{ij}^{n+1/2} \) seems to be new. We can compute its discrete dispersion relation, which is given by

\[
\sin^2 \left( \frac{\omega \Delta t}{2} \right) = \left[ \frac{c \Delta t}{2 \Delta x} \sin(k_x \Delta x) \left( 1 + \cos(k_y \Delta y) \right) \right]^2 + \left[ \frac{c \Delta t}{2 \Delta y} \sin(k_y \Delta y) \left( 1 + \cos(k_x \Delta x) \right) \right]^2,
\]

where \( \omega \) is the pulsation and \( k = (k_x, k_y) \) the wavevector. By expressing the right-hand side in terms of \( \cos^2(k_x \Delta x/2) \) and \( \cos^2(k_y \Delta y/2) \), we can check that its maximum over all real values of \( k_x, k_y \) is \( \max \left( (c \Delta t/2 \Delta x)^2, (c \Delta t/2 \Delta y)^2 \right) \). Since \( \omega \) has to be real for stability, this method is stable if and only if the CFL condition

\[
\max \left( \frac{c \Delta t}{\Delta x}, \frac{c \Delta t}{\Delta y} \right) \leq 2
\]

is satisfied.

Notice that if all functions are independent of \( y \), \((3.7), (3.8)\) reduce to \((2.3), (2.6)\), with double \( \Delta x \) and \( \Delta y \). The difference is that \((3.6)-(3.8)\) induce smaller phase velocities when \( k \) is not parallel to the coordinate axis, thus allowing larger Courant numbers. This method can be used to solve fluid problems, together with the splitting method, see [2].

4 The case of triangles and centered values

Let us now consider a mesh of triangles \((T_i)\). We are going to introduce a discrete formulation of \((1.10)-(1.11)\) that satisfies an exact conservation, as stated in \((1.9)\), independently of the evolution of \( B \) and \( j \). We assume that the charge density \( \rho \) and the electric field \( E \) are defined at the center of each triangle (see Figure 3), and we write a finite volume formulation for \((1.10)\)

\[
\rho_i^{n+1} - \rho_i^n + \frac{\Delta t}{|T_i|} \sum_{e \in T_i} |\epsilon| \phi_e^{n+1/2} \cdot n_{ie} = 0, \quad (4.1)
\]

where \( n \) refers to the time \( n \Delta t \), \( |T_i| \) is the area of \( T_i \), \( e \parallel T_i \) means that "\( e \) is an edge of \( T_i \)" , \( |\epsilon| \) is the length of \( e \), and \( n_{ie} \) is the unit normal to \( e \) which is exterior to \( T_i \). The numerical flux \( \phi_e^{n+1/2} \) is proportional to \( n_{ie} \) and is an approximation of the component of \( j \) in the \( n_{ie} \) direction,

\[
\phi_e^{n+1/2} \approx (j^{n+1/2} \cdot n_{ie}) n_{ie}. \quad (4.2)
\]

We write a discrete version of \((1.11)\) as follows,

\[
E_{i}^{n+1} - E_{i}^{n} + c^2 \frac{\Delta t}{|T_i|} \sum_{e \in T_i} |\epsilon| B_j^{n+1/2} n_{j}^{\perp} = -\frac{\Delta t}{\varepsilon_0} j_j^{n+1/2}, \quad (4.3)
\]
where \( G_i \) is the gravity center of \( T_i \), \( M_e \) is the middle of \( e \), and \( B_{n+i}^{n+1/2} \) is an approximation of the value of \( B \) on \( e \).

If a numerical scheme for (1.10)-(1.11) can be put under the form (4.1), (4.3) for some numerical fluxes \( \phi_{n+i}^{n+1/2} \) and \( B_{n+i}^{n+1/2} \) (that depend on the evolution equations for \( B \) and \( j \)), \( j_i^{n+1/2} \) being defined by (4.4); then there exists discrete quantities that approximate \( \varepsilon_0 \) \( \text{div} \ E - \rho \) and that are independent of time.

More precisely, consider a vertex \( V \), and let \( H \) be the union of the triangles \( T_i \) having \( V \) as a vertex (see Figure 4). Let \( \varphi \) be the continuous function, with compact support in \( H \), which is linear on each triangle of \( H \) and such that \( \varphi(V) = 1 \). Define \( E^n(x) \) to be constant by triangle equal to \( E_i^n \), and \( \rho^n(x) \) to be constant by triangle equal to \( \rho_i^n \). Then the quantity

\[
P^n_V \equiv \left( \varepsilon_0 \int_H \varphi \ \text{div} \ E^n - \int_H \varphi \ \rho^n \right) / \int_H \varphi \tag{4.5}
\]

is independent of \( n \).

In order to prove this property, let us define \( j^{n+1/2}(x) \) to be constant by cell equal to \( j_i^{n+1/2} \). Then, by (4.3) we get

\[
\int_H \varphi \ \text{div} \left( E^{n+1} - E^n + \frac{\Delta t}{\varepsilon_0} j^{n+1/2} \right) = - \int_H \nabla \varphi \cdot \left( E^{n+1} - E^n + \frac{\Delta t}{\varepsilon_0} j^{n+1/2} \right) \\
= - \sum_{T_i \subset H} |T_i| (\nabla \varphi)_i \cdot \left( E_i^{n+1} - E_i^n + \frac{\Delta t}{\varepsilon_0} j_i^{n+1/2} \right) \\
= \sum_{T_i \subset H} (\nabla \varphi)_i \cdot \left( c^2 \Delta t \sum_{e \subset T_i} |e| B_{n+i}^{n+1/2} n_{ie}^\perp \right) \\
= 0,
\]

because when \( e \) is an external edge of \( H \), \( (\nabla \varphi)_i \cdot n_{ie}^\perp = 0 \) since \( \varphi \) vanishes on the boundary of \( H \), and the interior contributions cancel two by two. Then, by (4.4),
Figure 4: set $H$ of triangles around a vertex $V$.
$e_{in}$: internal edge, $e_{ex}$: external edge.

\[ \int_{H} \varphi \text{div} \, j^{n+1/2} = \int_{H} j^{n+1/2} \cdot \nabla \varphi = -\sum_{T_{i} \in H} |T_{i}| j^{n+1/2} \cdot (\nabla \varphi); \]
\[ = -\sum_{T_{i} \in H} \sum_{\epsilon \notin T_{i}} \left[ \epsilon \phi_{\epsilon}^{n+1/2} \cdot n_{\epsilon} \quad \overrightarrow{G_{i}} \cdot (\nabla \varphi); \right] = -\sum_{T_{i} \in H} \sum_{\epsilon \notin T_{i}} \left[ \epsilon \phi_{\epsilon}^{n+1/2} \cdot n_{\epsilon} \quad (\varphi(M_{\epsilon}) - \varphi(G_{i})); \right] \]
\[ = \left( \int_{H} \varphi \right) \frac{1}{|H|} \sum_{\epsilon \in H} \epsilon \phi_{\epsilon}^{n+1/2} \cdot n_{H_{\epsilon}}. \]  

(4.7)

where $\epsilon \in H$ means that $\epsilon$ is an external edge of $H$, because again, the interior contributions cancel two by two, and $\varphi(G_{i}) = 1/3$. Therefore, (4.6) yields

\[ \varepsilon_{0} \int_{H} \varphi \text{div}(E^{n+1} - E^{n}) \int_{H} \varphi = -\frac{\Delta t}{|H|} \sum_{\epsilon \in H} \epsilon \phi_{\epsilon}^{n+1/2} \cdot n_{H_{\epsilon}}. \]  

(4.8)

Now, by averaging (4.1) over all triangles $T_{i} \subset H$, we get

\[ \rho_{H}^{n+1} - \rho_{H}^{n} + \frac{\Delta t}{|H|} \sum_{\epsilon \in H} \epsilon \phi_{\epsilon}^{n+1/2} \cdot n_{H_{\epsilon}} = 0, \]  

(4.9)

with

\[ \rho_{H}^{n} = \frac{1}{|H|} \sum_{T_{i} \in H} |T_{i}| \rho_{i}^{n} = \frac{\int_{H} \varphi \rho^{n}}{\int_{H} \varphi}. \]  

(4.10)

Finally, (4.8) and (4.9) lead to

\[ \varepsilon_{0} \int_{H} \varphi \text{div}(E^{n+1} - E^{n}) \int_{H} \varphi = \rho_{H}^{n+1} - \rho_{H}^{n}, \]  

(4.11)
which means that \( P_{V}^{n+1} - P_{V}^{n} = 0 \), and the proof is complete.

Notice that the above formulation makes use of both finite volume and finite element type of formulations. It does not seem to be possible to obtain similar results for the time-space finite element method of J.W. Eastwood [4]. However, there exists a finite volume method for Maxwell’s equations which satisfies (1.9) on Delaunay-Voronoi meshes, see F. Hermeline [6].

The formula (4.3) giving the approximation of curl \( B \), which has vanishing discrete divergence according to (4.6), can also be used in different situations where a curl is involved, as in the MHD equations, see J.-P. Croisille, R. Khanfir and G. Chanteur [3].

This formula can actually be interpreted as follows, in terms of non-conforming finite elements. Define the function \( B_{i}^{n+1/2}(x) \) for \( x \in T_{i} \) to be linear, with \( B_{i}^{n+1/2}(M_{e}) = B_{e}^{n+1/2} \) for any edge \( e \) of \( T_{i} \). Then, the function \( B^{n+1/2}(x) \) which coincides with \( B_{i}^{n+1/2} \) in each triangle satisfies

\[
-(\nabla B^{n+1/2})_{L} \equiv \text{curl } B^{n+1/2} = \left(\text{curl } B^{n+1/2}\right)_{r} + \left(\text{curl } B^{n+1/2}\right)_{s},
\]

where the regular part \( \left(\text{curl } B^{n+1/2}\right)_{r} \) is piecewise constant with value in \( T_{i} \) given by

\[
\left(\text{curl } B^{n+1/2}\right)_{r,i} = -\frac{1}{|T_{i}|} \sum_{e \in T_{i}} |e| B_{e}^{n+1/2} n_{e}^{i},
\]

and the singular part, which is small in the sense of measures, is of the form

\[
\left(\text{curl } B^{n+1/2}\right)_{s} = \sum_{e} \delta_{e}(x) b_{e}^{n+1/2}(x),
\]

with \( \delta_{e} \) the Dirac distribution supported by \( e \), and \( b_{e}^{n+1/2}(x) \) is defined for \( x \in e \) and is parallel to \( e \). Moreover, \( b_{e}^{n+1/2} \) is linear on \( e \) and \( b_{e}^{n+1/2}(M_{e}) = 0 \). Equation (4.3) can thus be written

\[
E^{n+1} - E^{n} - c^{2} \Delta t \left(\text{curl } B^{n+1/2}\right)_{r} = -\frac{\Delta t}{\varepsilon_{0}} B^{n+1/2}.
\]

Now, the above function \( \varphi \) which is used to evaluate the discrete divergence satisfies

\[
\int \nabla \varphi \cdot \left(\text{curl } B^{n+1/2}\right)_{s} = 0.
\]

Indeed, this quantity is well-defined because \( b_{e}^{n+1/2} \) is parallel to \( e \), and (4.16) holds because the average of \( b_{e}^{n+1/2} \) on \( e \) vanishes. Therefore,

\[
\int \nabla \varphi \cdot \left(\text{curl } B^{n+1/2}\right)_{r} = \int \nabla \varphi \cdot \text{curl } B^{n+1/2} = 0,
\]

and we recover the cancellation in (4.6).
Let us finally notice that the interpolation formula (4.4) is consistent. Actually, we have for any vector $v$

$$v = \frac{1}{|T_i|} \sum_{\epsilon \in \partial T_i} |\epsilon| (v \cdot n_\epsilon) \overrightarrow{G_i} M_\epsilon.$$  

(4.18)

In order to prove this identity, let us define the function

$$u(x) = \left( \overrightarrow{G_i} x \cdot w \right) v,$$

(4.19)

for a given vector $w$. Then $\text{div } u = v \cdot w$, and according to Stokes’ formula,

$$|T_i| v \cdot w = \int_{T_i} \text{div } u$$

$$= \int_{\partial T_i} u \cdot n$$

$$= \sum_{\epsilon \in \partial T_i} \int_{\epsilon} u \cdot n_\epsilon$$

$$= \sum_{\epsilon \in \partial T_i} |\epsilon| u(M_\epsilon) \cdot n_\epsilon$$

$$= \sum_{\epsilon \in \partial T_i} |\epsilon| \left( \overrightarrow{G_i} M_\epsilon \cdot w \right) (v \cdot n_\epsilon),$$

(4.20)

and (4.18) follows since (4.20) is true for any $w$.

5 Conclusion

Whenever possible, a method having exact discrete conservation of the Gauss-Poisson law for solving the continuity equation (1.1) and Maxwell-Ampère’s equation (1.2) has the great advantage of fast and easy to vectorize computations.

We have shown that in two dimensions and for different geometries, this property holds as soon as the scheme can be put under a coherent form involving numerical fluxes. Moreover, this is true independently of the way these fluxes are computed, and therefore independently of the evolution of $B$ and $j$.

However, such methods are very constraining. Therefore, in each situation, one should look carefully if there exists such a method which is adapted to the given set of equations and to the geometry.
References


