Inertial, barotropic and baroclinic instabilities of the Bickley jet in two - layer rotating shallow water model

François Bouchut
Laboratoire d’Analyse et de Mathématiques Appliquées,
Université Paris-Est/CNRS, Marne-la-Vallée, France

Bruno Ribstein
LMD, Univ. P.et M. Curie, and ENS,
24 rue Lhomond, 75005 Paris, France

Vladimir Zeitlin
∗
LMD, Univ. P.et M. Curie, and ENS,
24 rue Lhomond, 75005 Paris, France,
and Institut Universitaire de France
(Dated: October 25, 2011)

We undertake a detailed study of inertial instability of the barotropic Bickley jet and its nonlinear saturation in the 2 layer rotating shallow water (RSW) model on the $f$-plane, and compare it with the classical barotropic and baroclinic instabilities. We start with analytical and numerical investigation of the linear stability problem under hypothesis of strict homogeneity in the along-flow direction ("symmetric instability"). The unstable modes are identified and their parameters are determined. The dependence of the instability on Rossby and Burger numbers of the jet is investigated. The nonlinear development of the instability is then studied with the help of high-resolution well-balanced finite-volume numerical code recently developed for multi-layer RSW, which is initialized with the most unstable mode found from the linear stability analysis. It is shown that symmetric inertial instability is saturated by reorganization of the mean flow, without full homogenization of the anticyclonic region, where the unstable modes reside.

We then study along the same lines the fully two-dimensional problem, and compare the results with the one-dimensional analysis. The barotropic instability competes with inertial instability in this case. We show that for sufficiently strong anticyclonic shears the inertial instability still has the dominant growth rate in the long-wave sector. The "symmetric" one-dimensional inertial instability turns to be a limiting case of the ageostrophic baroclinic instability. Yet, the "asymmetric" inertial instability at small but non-zero wavenumbers has the higher growth rate. We study again the nonlinear development of the most unstable inertial mode, and show that homogenization of the region of strong anticyclonic shear of the flow takes place on average. The reorganization of the flow reveals a high degree of complexity, with coherent structure formation and inertia-gravity wave emission, both giving rise to the substantial enhancement of ageostrophic motions. We compare these processes for inertial baroclinic and barotropic instabilities.

I. INTRODUCTION

Inertial instability is a typical ageostrophic instability arising in rotating stratified flows at large enough Rossby numbers, such that the product of the Coriolis parameter $f$ and potential vorticity takes negative values in some regions of the flow. It is the geophysical analog of the centrifugal instability. Classical inertial instability is symmetric [1], [2] in the sense that it manifests itself in the perturbations invariant with respect to translations along the mean flow, although the name "symmetric" is often reserved for its moist counterpart [3].

Inertial/symmetric instability plays an important role in the redistribution of momentum in the ocean and the atmosphere. Obviously, it is easy to be realized, and hence ubiquitous, in the equatorial atmosphere and oceans, e.g. [4], [5]. Yet, it is relevant to the oceanic and atmospheric processes in mid-latitudes, too, e.g. [6], [7]. In what follows, we will be interested in this latter case, and use the $f$-plane approximation with a constant Coriolis parameter.

A general analysis of the development of the symmetric inertial instability was given in [8] and [9]. A number of numerical studies of nonlinear stages of the symmetric instability exist in the literature, revealing a rather complex structure of the resulting flow even in the simplest cases of destabilization of the barotropic jet on the $f$-plane in the homogeneous [10], [11], or linearly stratified fluid [12], [13].

In spite of these studies many aspects of the reorganization of the mean flow due to development of the inertial instability need further study. Among them a question of the mechanisms of homogenization of the barotropic geostrophic momentum and potential vorticity [12], [11] and of the structure of the baroclinic component of the reorganized flow, a question of the inertia-gravity wave...
emission, a question on the relation of symmetric and asymmetric inertial instabilities [14], [15] and a question about relation, and possible competition [16], [10], [17], [11], [15] of the inertial instability with the barotropic one. We should add that relation of the inertial instability to the classical baroclinic instability merits attention as well. Indeed, by increasing the Rossby number of e.g. a jet subject to the standard barotropic and baroclinic instabilities, one inevitably increases the amplitude of its anticyclonic shear, so the threshold of the inertial instability will be eventually reached. What is the interplay between the three instabilities then?

It should be emphasized that complexity resulting from the development of the inertial instability of a baroclinic flow is prohibitive, and renders high-resolution long-time simulations of such flows very expensive. That is why, being motivated by the questions above, we study the inertial instability in a simple two-layer rotating shallow water model which is a standard conceptual model in geophysical fluid dynamics [18], in particular for studying baroclinic [19] or ageostrophic [20] instabilities. The advantage of the model, beyond the relative simplicity of the linear stability analysis and of the straightforward physical interpretation of the results, is that it allows for high-resolution numerical simulations with new-generation finite-volume numerical schemes [21]. This gives a possibility to explore the late stages of the evolution of the instability at high resolution and relatively low computational cost. However, to have an efficient scheme, the rigid-lid upper boundary condition should be relaxed and replaced by the free-surface one [21].

It is important to emphasize that the step-like stratification of the two-layer model renders the situation different with respect to the case of continuous stratification, which is the standard framework for studying the inertial instability. Indeed, it is known that in the latter case the growth rate of inertially unstable modes increases with the vertical wavenumber, and viscosity and/or diffusivity is essential for the vertical scale selection of the instability, cf e.g. [22], [12], [23]. In the two-layer model the fluid motion is layerwise columnar, and there are only two vertical modes, the barotropic and the baroclinic one. So the question of the vertical scale selection does not arise. We thus do not need to introduce any explicit viscosity while studying nonlinear evolution of the instability, and will be able to achieve a very high resolution, see below. We should stress, however, that the nature of the instability, at least in the symmetric case, remains the same in both continuously stratified and layered models, cf. [13]: it is due to the trapped modes localized at the anticyclonic shear [35]. The squares of the eigenfrequencies of these modes are diminishing with increasing anticyclonic shear and eventually become negative, leading to instability. The vertical stratification is essential for the existence of the instability, the trapped modes of the localized jet not existing in the one-layer rotating shallow water model [24].

It is known that vertical overturning motions are essential in the process of saturation of the inertial instability in the continuously stratified case [22], [11], [13]. They are obviously not possible in the two-layer model. We have, however a proxy for such motions. Namely, as is known, the two-layer model loses hyperbolicity once the threshold of the shear Kelvin-Helmholtz (KH) type instability is reached [25]. Although KH billows are not resolved, the numerical model copes well with the hyperbolicity loss [21], and the non-hyperbolic regions of the computational domain are indicators of the developed KH instability.

Our strategy, which was already successfully tested for other ageostrophic instabilities, in particular in the framework of the two-layer model, e.g. [26], is the following: we first study the linear stability of a given flow (it will be a barotropic Bickley jet below) by the pseudospectral collocation method [27]. We identify and analyze the unstable modes, and then use them for initialization of fully nonlinear numerical simulations. We accomplish this program for both symmetric, one-dimensional, and full two-dimensional inertially unstable configurations, and compare the results. In the last case a comparative analysis of the evolution of unstable modes, corresponding to inertial and barotropic instabilities, is performed.

The paper is organized as follows: in section II we present the model and the background flow configuration. The linear stability problem under hypothesis of strict homogeneity in the along-flow direction is solved in section III A. The nonlinear development of the instability under the same hypothesis is studied in section III B. The results are then compared with fully two-dimensional problem in section IV, the linear stability analysis being presented in section IV A and the nonlinear evolution of symmetric and barotropic instabilities being presented, and compared, in section IV B. Finally, discussion and summary are given in section V.

II. THE MODEL AND THE BACKGROUND FLOW

We work with the two-layer rotating shallow water model on the $f$–plane with a free upper surface and flat bottom [36].

![Figure 1: Two-layer rotating shallow water model on the $f$–plane (Northern hemisphere)](image-url)
The equations of the model are:

\[
\begin{align*}
\begin{cases}
(\partial_t + u_1 \partial_x + v_1 \partial_y) u_1 - f v_1 + g \partial_z (h_1 + h_2) &= 0, \\
(\partial_t + u_1 \partial_x + v_1 \partial_y) v_1 + f u_1 + g \partial_y (h_1 + h_2) &= 0,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\partial_t h_1 + \partial_x (h_1 u_1) + \partial_y (h_1 v_1) &= 0, \\
(\partial_t + u_2 \partial_x + v_2 \partial_y) u_2 - f v_2 + g \partial_z (r h_1 + h_2) &= 0, \\
(\partial_t + u_2 \partial_x + v_2 \partial_y) v_2 + f u_2 + g \partial_y (r h_1 + h_2) &= 0, \\
\partial_t h_2 + \partial_x (h_2 u_2) + \partial_y (h_2 v_2) &= 0.
\end{cases}
\end{align*}
\]

(II.1)

Here, \( u_i \) and \( v_i \), \( i = 1, 2 \), are the \( x \) and \( y \) components of the velocity of the fluid layers (layer 1 on top of the layer 2), \( h_i \) are the thicknesses of the layers, \( r = \frac{\rho_2}{\rho_1} \leq 1 \) is the density ratio of the layers, \( f = \text{const} > 0 \) is the Coriolis parameter, and \( g \) is the gravity acceleration.

The potential vorticities (PV) of the layers

\[
q_i = \frac{\partial_y u_i - \partial_x v_i + f}{h_i}, \quad i = 1, 2 \quad \text{(II.2)}
\]

are Lagrangian invariants: \( \frac{\partial q_i}{\partial t} = 0 \).

The energy is a sum of potential and kinetic energies of the layers, \( E = E_1 + E_2 \), and is conserved in the absence of dissipation. The dimensionless parameters of this configuration, modulo a constant, as

\[
\begin{align*}
E_1 &= \int dx dy p_1 \left( \frac{h_1}{2} v_1^2 + g h_1 h_2 + g \frac{\delta^2}{4} \right), \\
E_2 &= \int dx dy p_2 \left( \frac{h_2}{2} v_2^2 + g \frac{\delta^2}{4} \right) \quad \text{(II.3)}
\end{align*}
\]

The geostrophic balance corresponds to the equilibrium between the Coriolis force and the pressure force in (II.1). A straight balanced jet, i.e., a geostrophically balanced configuration without dependence on one of the coordinates (say \( y \)) is an exact stationary solution of the model, as it is easy to see from (II.1). By the geostrophic balance, any localized balanced jet is at the same time a pressure front. In what follows we take a barotropic (in the sense of no vertical velocity shear), balanced localized Bickley jet (Fig. 2), an exact solution of (II.1), as the background flow:

\[
\begin{align*}
h_1 &= H_1(x) = H_{10}, \\
h_2 &= H_2(x) = H_{20} + \delta \tanh \left( \frac{x}{\ell} \right),
\end{align*}
\]

\[
\begin{align*}
U_1(x) &= U_2(x) = 0, \\
V_1(x) &= V_2(x) = V(x) = \frac{h_1}{h_2} \left( 1 - \tanh^2 \left( \frac{x}{\ell} \right) \right).
\end{align*}
\]

(II.4)

Here \( H_{10} \) and \( H_{20} \) are constant, the dimensionless parameters \( L \) and \( \delta \) measure the width and the intensity of the jet, respectively, with \( V_0 = \frac{v_0}{\ell} \) being the peak velocity. The dimensionless parameters of this configuration are the Burger number \( Bu = \frac{h_1}{\delta} \), the Rossby number \( Ro = \frac{\delta}{\ell} = \frac{\ell}{\delta h_1} \), the depth ratio \( d = \frac{h_2}{h_1} \), and \( H_0 = H_{10} + H_{20} \). It is easy to see that Ripa’s general stability conditions for multi-layer flows [28] are not satisfied for the jet, implying instability at any \( \delta \neq 0 \).

III. LINEAR AND NONLINEAR ANALYSES OF THE SYMMETRIC PROBLEM

In this section, we study the linear stability problem and nonlinear saturation of the symmetric inertial instability by assuming strict homogeneity in the along-flow direction (\( \partial_y (\ldots) = 0 \)).

A. General setting of the linear stability problem

We consider the small perturbations of the jet (II.4):

\[
h_i = H_i(x) + \eta_i(x, t), \quad u_i = u_i'(x, t), \quad v_i = V(x) + v_i'(x, t), \quad i = 1, 2.
\]

and apply a Fourier transform in time to the “1.5-dimensional” \( y \)-independent version of (II.1). The solution, thus, is sought in the form \( (u_i', v_i', \eta_i') = (u_{0i}(x), v_{0i}(x), \eta_{0i}(x)) e^{-i\omega t} + c.c. \), and the linearized equations are:

\[
\begin{align*}
-\omega \eta_{01} - f v_{01} + g \partial_z (\eta_{01} + \eta_{02}) &= 0, \\
\omega v_{01} + i(f + \partial_x V) u_{01} &= 0, \\
\omega \eta_{01} + i \partial_y (H_1 \eta_{01}) &= 0, \\
-\omega \eta_{02} - f v_{02} + g \partial_z (r \eta_{01} + \eta_{02}) &= 0, \\
\omega v_{02} + i(f + \partial_x V) u_{02} &= 0, \\
\omega \eta_{02} + i \partial_y (H_2 \eta_{02}) &= 0.
\end{align*}
\]

(III.2)

After some algebra the problem is reduced to a pair of coupled Schrödinger equations for the across-front velocities of the layers:

\[
\begin{pmatrix}
u_{10} \\ \nu_{20}
\end{pmatrix} + \frac{\partial^2}{\partial x^2} \begin{pmatrix}
u_{10} \\ \nu_{20}
\end{pmatrix} = g \partial_y \begin{pmatrix} H_1 \nu_{01} + H_2 \nu_{02} \\ r H_1 \nu_{01} + H_2 \nu_{02} \end{pmatrix}.
\]

(III.3)
The complete analysis of the spectrum of such system with nontrivial $\partial_s V$ is a rather hard mathematical problem, and can not be readily found in literature. This is a difference with the cases of barotropic jet in the two-layer model with rigid-lid upper boundary condition [25], or in the primitive equations with linear stratification after separation of spatial variables [13], where a single Schrödinger equation with well-known properties arises.

For the free-surface upper boundary condition, which we have to apply in order to be able to use an efficient numerical scheme later, the situation is thus more complex. We therefore will limit ourselves by an integral estimate to overcome it in the absence of the third term, which is inertially unstable when the r.h.s. of (III.5) is negative. Notice that the second term in the r.h.s. of (III.5) is positive-definite and $f(f + \partial_s V)$ should be sufficiently negative to overcome it in the absence of the third term, which is the only one containing the barotropic velocity. The third term is not sign-definite, but our numerical results show that it is generally small, as the unstable modes has only a small barotropic component, see below.

Equation (III.5) can be non-dimensionalized using the horizontal length scale $L$, the vertical length scale $H_0 = H_{10} + H_{20}$, and the velocity scale $U = \frac{f L}{g}$. In order to have $H_2 \geq 0$, we need $\frac{Bu}{1 + d} \geq Ro$, and the flow is inertially unstable when:

$$\omega^2 = 1 - Ro F + \frac{Bu}{1 + d}(1 - r) (G_1 + G_2) < 0 \, .$$  \hspace{1cm} (III.6)

Here

- $Ro F$ is the non-dimensional $\frac{\int ||\partial_s V|| H_b ||u_b||^2}{\int H_b ||u_b||^2}$,
- $\frac{Bu}{1 + d}(1 - r) G_1$ is the non-dimensional $\frac{\int ||\partial_s (H_b u_b)||^2}{\int H_b ||u_b||^2}$,

- $\frac{Bu}{1 + d}(1 - r) G_2$ is the non-dimensional $\frac{\int H_b u_b \partial_x^2 (H_b u_b)}{\int H_b ||u_b||^2}$.

Finally, as $\omega^2$ is $H_0$ - independent, we may set $H_0 = 1$. The formula (III.6) will be used for the control of the results of the numerical analysis of the linear stability below.

The eigenproblem (III.2) is solved numerically with the help of the collocation method [27]. As was already said, the inertial instability is related to the modes localized in the vicinity of the jet. We are, thus, not interested in the continuum spectrum corresponding to free inertia-gravity waves scattered by the jet, and impose decay boundary conditions far from the jet in the $x$ - direction. Practically, we require the eigenmodes to vanish at $x = \pm 10L$ (cf. Fig. 2).

To solve the eigenproblem, the system (III.2) is discretized on an irregular grid formed by the Chebyshev collocation points, which are unevenly spaced to avoid the Runge phenomenon \{$\pi j/2N = \cos(j\pi/N)$, $j = 0, 1, \ldots, N$ \} [27]. Numerical convergence is achieved starting from $N = 51$. The Chebyshev differentiation matrix, which will be denoted by $D$, is used for discrete differentiation. The discretized version of the system (III.2) thus is:

$$\omega(X) = [M](X) \, , \, \ X = \begin{pmatrix} \eta_{01} \\ \eta_{02} \\ \eta_{01} \\ \eta_{02} \end{pmatrix} \, , \, [M] = \begin{pmatrix} 0 & -f & gD & 0 & 0 \\ -f & -\partial_x V & 0 & 0 & 0 \\ -\partial_x H_1 - H_1 D & 0 & 0 & 0 & 0 \\ 0 & 0 & r gD & 0 & -f & gD \\ 0 & 0 & 0 & -\partial_x H_2 - H_2 D & 0 & 0 \end{pmatrix}$$  \hspace{1cm} (III.7)

The results of the stability analysis are as follows:

- Existence of the unstable modes: in a wide range of parameters at least one, sometimes two trapped unstable modes were found.
- Structure of the unstable modes: they are largely confined in the region of anticyclonic shear in the jet and are essentially baroclinic (compare upper- and lower-layer profiles in Fig. 3). Therefore, the instability is baroclinic by its nature.
- Consistency checks: we evaluate $\{F, G_1, G_2\}$ by using the calculated eigenmodes for different sets of parameters, and inject them into (III.6), in order to compare with the eigenvalue resulting from the direct collocation calculation. We get very good
matches all the time. For example, for the parameter set $Bu = 10, Ro = 5, d = 2, r = 0.5$ (this set will be used in direct numerical simulations below), the growth rate is approximately $0.676$ and $F \sim 0.458, G_1 \sim 0.480, G_2 \sim 0.021$. The growth rate calculated by plugging these values into (III.6) perfectly matches the eigenvalue. Moreover, the low value of $G_2$ with respect to the other terms in (III.6) is consistent with the observed baroclinic structure of the eigenmodes, see Fig. 3.

- Dependence of the growth rate on parameters $Bu, Ro, d, r$, guided by the previous estimates, negates the contribution of the barotropic velocity in (III.6), and assuming that the gradual change of parameters do not drastically change the form of the unstable modes, we see from this formula that the growth rate should increase with $Ro, d, r$ and decrease with $Bu$. This is fully consistent with the results of the direct computation presented in Fig. 4.

![Figure 3: The structure of the unstable mode for the parameter set $Bu = 10, Ro = 5, d = 2, r = 0.5$. The growth rate is $\text{Im}(\omega) = 0.676$. Dashed: layer 2; solid: layer 1. Left column: $v_0$. Middle column: $u_0$. Right column: $v_0$. Top: real, bottom: imaginary part.](image)

![Figure 4: Left: Stability diagram in the $Ro - Bu$ plane for $H_0 = 1, d = 2, r = 0.5$. Black lines: isolines of the growth rate. Contour interval: 0.1. Right: Evolution of the growth rate with $r$ for the parameter set $H_0 = 1, Bu = 10, Ro = 5, d = 2$.](image)

The gray lower curve on the left panel of Fig. 4 corresponds to the boundary in $\delta$, and hence in $Ro$ of (II.4), coming from the non-negativity of the thickness of the lower layer. The limit $r \rightarrow 1$ in the right panel of the Figure is singular: the numerical stability analysis always retrieves baroclinic unstable modes for $r \rightarrow 1$, i.e. for the fluid tending to barotropicity. It should be reminded that trapped modes, and hence inertial instability, are absent in the barotropic one-layer shallow water [24]. Higher modes of the instability with lower growth rates appear for sufficiently high $r$ (not shown).

### B. Nonlinear evolution of the symmetric instability

Recent progress in finite-volume numerical schemes for shallow-water equations allows for implementationally simple and reliable high-resolution modelling of fully nonlinear dynamics. We use this technique for studying nonlinear evolution of the instabilities of a barotropic jet.

We apply a recent two-layer extension [21] of the high-resolution finite-volume numerical method by Bouchut [29] which is:

1. weakly dissipative, with numerical dissipation concentrated in the high-gradient zones (shocks) and the energy necessarily decreasing across shocks (the "entropy" property of the shock-capturing scheme);
2. well balanced, i.e. preserves geostrophic equilibria and conserves well the potential vorticity in the absence of shocks;
3. allowing for variety of boundary conditions: periodic, free-slip, sponges, and treating easily the flows over topography.

In the context of multi-layer flows it copes well with the inherent difficulty of hyperbolicity loss in the case of strong velocity shear between the layers [21], [26]. The numerical simulations for symmetric configuration were performed with high spatial resolution $0.75 \times 10^{-2}L$ and lasted for a couple of hours on a standard personal computer. We systematically doubled resolution of the simulations to be sure of numerical convergence (not shown).

We present the fully nonlinear evolution of the symmetric one-dimensional inertial instability corresponding to the configuration with the parameters $H_0 = 1, Bu = 10, Ro = 5, d = 2, r = 0.5$. The boundary conditions are periodic in the along-flow direction and sponges are used at $x = \pm 15L$ in the cross-flow direction. The perturbation of the amplitude about 1% of the maximum height of the unperturbed configuration, corresponding to an unstable mode found by the linear stability analysis of the previous subsection, was superimposed on the background balanced jet.

In the one-dimensional configuration of this section, the nonlinear evolution may be sensible to the phase of the perturbation. Indeed one can see from Fig. 3 that the unstable mode is not symmetric with respect to the sign inversion and, at finite amplitude, will affect the mean flow differently depending on its sign (e.g. by pinching vs inflating the upper layer locally). This is why we performed simulations initialized with the perturbations of both signs. We present below only the one with pinching of the upper layer at the anticyclonic ($x > 0$) side of the jet, which is more interesting as it displays outcropping, i.e. local thinning of the upper-layer to zero, well resolved
by the code. The results for the inflating one are qualitatively and quantitatively similar, although outcropping is not produced. Note that PV is ill defined at the outcropping point, cf (II.2), and should be understood as a limit, which does exist.

To benchmark the simulation we calculated the growth rate of the perturbation:

$$\log \left( \int_{\text{domain}} |\eta'_1 + \eta'_2| \, dx \right)$$  \hspace{1cm} (III.8)

during several inertial periods and compared it with the results of the linear stability analysis. A scatter plot of (III.8) vs. $Im(\omega)t$ obtained by the linear analysis is presented in Fig. 5, showing perfect agreement at the initial stages of the development of the instability.

The development of the instability leads to the reorganization of the flow, the across-jet shear in $u$ disappears, and a new baroclinic mean flow emerges in $v$ and $h$, cf Figs. 6, and 7. It is worth emphasizing that the instability is saturated with the negative values of PV remaining in the anticyclonic region (see the right panels of the Figures). This behavior is due to the strict homogeneity with respect to translations, and was also observed in the continuously stratified case in [13]. We should remind at this point that existence of negative PV is a necessary, but not sufficient condition of the instability. At very long times the system tends to an almost steady state, but fluctuations around this state persist, most probably due to the interactions with the boundaries. The term “saturated state” used below means a time average of the late stages of the evolution presented in the bottom row of Figs. 6 and 7.

An important observation made in [11] in 2D simulations of nonlinear evolution of the barotropic inertial instability is the homogenization of the velocity shear over a region which may be predicted on the basis of the conservation of the geostrophic momentum $M = v_B + fx$ of the flow. We observe that this is only partially true in our 1.5 D simulations, although a tendency of the gradient of $M$ to become less negative is well observed in Fig. 8 where the evolution of the barotropic component
of the geostrophic momentum is presented. Note that in continuously stratified flow the homogenization is mainly achieved through overturning motions which are absent in shallow-water model.

Figure 8: Barotropic component of the geostrophic momentum of the unperturbed jet (bottom) with \( H_0 = 1, Bu = 10, Ro = 5, d = 2, r = 0.5 \), and of the saturated state resulting from the evolution of the most unstable mode with \( Im(\omega) = 0.676 \) added to the unperturbed configuration (top).

Figure 9: Evolution of the deviation of the total (solid), kinetic (black dashed) and potential (gray dashed) energy from its initial value over the whole calculation domain. The small-amplitude oscillations are due to the imperfect absorption at the sponges, implemented as discrete Neumann boundary condition.

Figure 10: Snapshot at \( t = 7.75 \) of the distribution of the dissipation during nonlinear evolution of the most unstable mode with \( Im(\omega) = 0.676 \) at \( H_0 = 1, Bu = 10, Ro = 5, d = 2, r = 0.5 \). The dissipation in the finite-volume scheme is calculated as a departure from the (discrete) balance of energy in each grid cell, and then summed up.

The energy of the flow is well preserved, the energy budget is presented in Fig. 9. The observed decrease in kinetic energy in favor of potential energy confirms that symmetric instability draws its energy from the kinetic energy of the unperturbed flow [1]. Strong dissipation events, cf. Fig. 10, are transient. They are related to the shock formation, visible in Figures 7 (second row, left). It should be reminded that our numerical scheme is shock-capturing entropy-satisfying, in a sense that strict energy decrease across shocks is guaranteed (energy is playing the role of entropy in the shallow-water system, which is equivalent to the two-dimensional gas dynamics). Obviously, shocks are easier to form in a thinner layer, which explains that the shock in \( u \) (Fig. 7, second row left) appears roughly at the outcropping region (Fig. 6, second row left).

In the two-layer numerical scheme we have a built-in diagnostics of hyperbolicity [21] which calculates the product of the squares of the differences of the eigenvalues (discriminant) of the characteristic velocities in each spatial direction. A hyperbolicity loss corresponds to negative values of the discriminant. The time evolution of the discriminant (not presented) shows that the problem always remains hyperbolic, despite the outcropping. It should be emphasized that the criterion for hyperbolicity loss coincides with the criterion of KH instability [25]. Our results, thus, are consistent with the analysis of [22], where strong shear generation and subsequent KH instabilities were described for developing equatorial inertial instability.

Gravity wave emission was also observed during the evolution of the instability (not shown).

C. Preliminary conclusions for the strictly symmetric 2-layer configuration

Thus, for the strictly symmetric configuration we have shown that

- The barotropic Bickley jet can be inertially unstable if Rossby number is high enough. The unstable mode is trapped in the jet core, and is essentially baroclinic.
- The flow becomes more stable with increasing Burger number.
- High-resolution DNS with a well-balanced finite volume numerical scheme reproduce the results of the linear stability theory at initial stages of the evolution. At later stages, a saturation of the instability and reorganization of the mean flow towards a new baroclinic state occurs, yet with negative PV values persisting in the anticyclonic region. The cross-stream gradient of the geostrophic momentum is partially homogenized.

IV. THE FULLY TWO-DIMENSIONAL PROBLEM

We now relax the strict one-dimensionality imposed in Section III and, thus, aim at asymmetric inertial instability [14], [15]. For the 2D problem we will follow the same strategy, by first studying the linear stability problem, and then using the unstable modes to initialize the DNS
of strongly ageostrophic jet with a high Rossby number, and study the instability life-cycle. The parameters of the jet are deliberately chosen to be the same as in the 1.5 dimensional problem above. To make a comparison of ageostrophic instabilities of such configuration with conventional geostrophic ones, we present in parallel results of the stability study of the same jet in geostrophic (order of magnitude smaller Rossby number), and intermediate cases.

A. The linear stability problem in two dimensions

We look for the perturbations of the background flow (II.4) in the form \((u', v', \eta'_j) = (u_0, v_0, \eta_0) e^{i(ky - \omega t)}\). The linearized equations of the system (II.1) are:

\[
\begin{align*}
-\omega(x)u_0 - f v_0 + g \partial_x (\eta_0 + \eta_0) &= 0, \\
-\omega(x)v_0 - i(f + \partial_x V(x))u_0 + kg (\eta_0 + \eta_0) &= 0, \\
-\omega(x)\eta_0 - i\partial_x (H_1(x)u_0) + k(H_1(x)v_0) &= 0, \\
-\omega(x)\eta_0 - f v_0 + g \partial_x (\eta_0 + \eta_0) &= 0, \\
-\omega(x)\eta_0 - i(f + \partial_x V(x))u_0 + kg (\eta_0 + \eta_0) &= 0, \\
-\omega(x)\eta_0 - i\partial_x (H_2(x)u_0) + k(H_2(x)v_0) &= 0,
\end{align*}
\]  

where \(\omega(x) = \omega - kV(x)\).

The analytic treatment of this problem being complicated already in 1.5 dimensions, we proceed directly with a direct numerical analysis with the help of the collocation method. Again, we impose zero boundary conditions on all fields at \(x = \pm 10L\) in order to get rid of the continuous spectrum we are not interested in. The discretized version of (IV.1) follows:

\[
\begin{align*}
\omega(x)(X) = [M](X), \\
(X) = \begin{pmatrix} iu_0 \\ v_0 \\ \eta_0 \\ iu_0 \\ v_0 \\ \eta_0 \end{pmatrix}, \\
[M] = \begin{pmatrix} 0 & -f & gD & 0 & 0 & gD \\ -f - \partial_x V & 0 & kg & 0 & 0 & kg \\ -\partial_x H_1 - H_1 D & kH_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & rgD & 0 & -f & gD \\ 0 & 0 & rkg & -f - \partial_x V & 0 & kg \\ 0 & 0 & 0 & -\partial_x H_2 - H_2 D & kH_1 & 0 \end{pmatrix}
\end{align*}
\]  

where \(D\) is the Chebyshev differentiation matrix. For \(k \neq 0\) convergence is achieved at \(N = 151\) collocation points.

The collocation method allows to treat any profile of the background flow and perturbations with arbitrary wavenumbers. The disadvantage of the method is that it is not especially designed for singular eigenproblems. The eigenproblem (IV.1) has a well-known critical-layer singularity occurring whenever the real part of the eigen phase velocity of the perturbation \(c = \frac{\sqrt{\omega}}{k}\) is equal to the local flow velocity: \(c = V(x)\). Such singularities give rise to the stable singular eigenmodes which form a continuous spectrum. These modes have Dirac-delta or stepfunction behaviour (depending on the variable) being, in fact, distributions, and not functions. Discrete counterparts of such singular eigenmodes are retrieved by a straightforward application of the collocation method. They may be nevertheless easily identified by their singular profiles and the fact that they accumulate with increasing resolution. These singular modes are either not differentiable or not localized, so a filtering procedure based on gradient limiters and localization of the mode can be applied to remove them.

In this paper we are mainly concentrated on the long-wave instabilities, by the obvious reason that symmetric inertial instability has an infinite wavelength along the jet. We therefore did not pay attention to short-wave \(KH\)-type instabilities which are not presented in the stability diagrams below. They correspond to the linear resonances of Poincaré modes propagating, respectively, at the interface and at the free surface along the jet.

Our main findings are as follows. The barotropic Bickley jet at high Rossby numbers has two leading long-wave instabilities: barotropic one, and baroclinic asymmetric inertial one. The second has the largest growth rate for sufficiently large Rossby number. Both unstable modes have dipolar across-jet structure and are far from the geostrophic balance. The growth rate of the baroclinic mode has a nonzero limit at along-jet wavenumber tending to zero for high enough Rossby numbers. The structure and the growth rate of the corresponding limiting unstable mode reproduce those of the symmetric inertial instability of the same configuration. The growth rate of
The symmetric inertial instability is smaller than that of the asymmetric one.

As compared to the classical instabilities of the geostrophically balanced jet at low Rossby numbers, where the first barotropic mode has the largest growth rate, followed by the first baroclinic and second barotropic modes, a swap of the instabilities is taking place with increasing Rossby number, with the baroclinic mode becoming more unstable than the barotropic one.

The spectrum and the structure of the unstable modes of the two-dimensional stability problem for the barotropic Bickley jet are presented below for three values of the Rossby number: $Ro = 0.5$ corresponding to a jet close to geostrophy, $Ro = 2$, and $Ro = 5$ corresponding to a strongly ageostrophic jet. Other parameters do not vary: $H_0 = 1$, $Bu = 10$, $d = 2$, $r = 0.5$.

We begin with a classical quasi-geostrophic configuration. Its stability diagram is presented in Fig. 11. The most unstable mode (circles) is essentially barotropic as follows from the Figs. 12, 13, where the cross-section at $y = 0$ and the two-dimensional structure of the mode are presented, respectively.

Note that the mode is almost balanced (velocity vectors aligned with isobars), which is consistent with the relatively small value of the Rossby number. It is monopolar in the across-jet direction, and of the same form as the most unstable mode of the Bickley jet in the barotropic one-layer RSW model, cf. [30], [31].

The second unstable branch (dots) of the dispersion diagram for the quasi-geostrophic flow corresponds to a baroclinic mode, as follows from Fig. 14 and Fig. 15. It
Figure 16: Unstable modes of the barotropic Bickley jet in the two-layer model. Intermediate case $Ro = 2$. Same conventions as in Fig. 11.

Figure 17: Unstable modes of the barotropic Bickley jet in the two-layer model. Strongly ageostrophic configuration $Ro = 5$. Same conventions as in Fig. 11. Big cross and big circle mark the modes used for initialization of the DNS below.

...is also monopolar and balanced.

The third, the less unstable mode (crosses), is dipolar and almost barotropic (not shown). We identify it with the second unstable barotropic mode which was also found in the one-layer model in [30], [31].

Thus, we confirm the standard wisdom on the stability of quasi-geostrophic jets: barotropic instability is most vigorous following by the baroclinic instability, both having monopolar structure in the across-jet direction, which allows to anticipate a classical vortex-street formation at the stage of nonlinear saturation. Obviously, the jet is stable with respect to symmetric perturbations with $k = 0$.

Figure 18: The cross-section at $y = 0$ of the most unstable mode of the strongly ageostrophic jet with $Ro = 5$ (big cross in Fig. 17). Dashed: layer 2; solid: layer 1. Same conventions as in Fig. 3.

Figure 19: 2D structure of the most unstable mode of the asymmetric inertial instability of the strongly ageostrophic jet with $Ro = 5$ (big cross in Fig. 17). Same conventions as in Fig. 13.

When the jet becomes essentially ageostrophic, $Ro = 2$, stability diagram changes, Fig. 16. The first barotropic and baroclinic instabilities swap, the latter
becoming more vigorous. What is also important, the unstable baroclinic mode acquires a dipolar in the across-jet direction component (not shown). Note that the growth rates of the two main instabilities are much stronger than in the geostrophic case, and there is the competition between the two: the baroclinic instability being stronger at the longwave end of the spectrum, while the barotropic one prevailing starting from $k_y \approx 3$. The jet is still stable with respect to symmetric perturbations.

Finally, the stability diagram for the strongly ageostrophic case is presented in Fig. 17. Two main instabilities are still present and competing, while the third (the second barotropic one) dies out having negligible, with respect to the first two ones, growth rates. However, the essential difference with Figs. 17 and 16 is that the jet becomes unstable with respect to symmetric perturbations (we checked that the threshold for this symmetric instability is $Ro \approx 3.94$). The phase velocity curves (upper panel) join each other at $k = 0$. Thus, we get a degeneracy for symmetric perturbation, which is, however, not complete and is lifted for the growth rates. One of them vanishes at this point [37] and leaves a single unstable mode. One would expect that it coincides with the symmetric unstable mode of the previous section. This is, indeed, the case, as may be seen in Fig. 18, where the structure of the most unstable mode corresponding to the maximal growth rate (dots) in Fig. 17 is displayed.

The resemblance of this figure with Fig. 3 is striking. We, thus, have a two-dimensional counterpart of the symmetric mode, which has the highest growth rate at the given values of parameters. Its two-dimensional structure is presented in Fig. 19. Note that this mode is clearly ageostrophic, as follows from the mutual orientation of the isobars and velocity vectors. It is also strongly asymmetric with respect to the jet axis, being localized in the zone of anticyclonic shear, and is essentially baroclinic. We, thus, have an asymmetric inertial instability, which may be also called baroclinic inertial to emphasize its character.

The most unstable mode of the second instability branch of Fig. 17 is presented in Fig. 20. This mode is almost completely barotropic (profiles of the upper and lower layer variables practically coincide). We, thus, identify the corresponding instability as barotropic. Its two-dimensional structure is presented in Fig. 21. The mode is also ageostrophic, which is not surprising as the value of Rossby number is high, but much more symmetric with respect to the jet axis than the inertial instability mode.

### B. Nonlinear evolution of the ageostrophic instabilities in the $x-y$ plane

In this section we present the results of the DNS of the nonlinear saturation of the two-dimensional instabilities. The simulations are performed with the finite-volume numerical scheme for the two-layer RSW recently developed in [21]. We will limit ourselves by displaying the simulations of the evolution of the strongly ageostrophic jet with parameters $H_0 = 1$, $Bu = 10$, $Ro = 5$, $d = 2$, $r = 0.5$, with a superimposed perturbation of the amplitude about 1% of the maximum height of the unperturbed configuration. The perturbation corresponds to the most unstable mode of either asymmetric inertial (the big cross), or barotropic (the big circle) instability in Fig. 17. The boundary conditions are sponges imposed at $x/L = \pm 30$.
in the cross-jet direction, and periodicity with the period of the perturbation in the along-jet direction. We doubled the width of the domain with respect to the 1.5D simulation because the saturation of the instabilities leads to lateral spreading of the jet. The result presented below were obtained with spatial resolution \( L/15 \). We made a control simulation with double resolution to be sure of numerical convergence (not shown). Although we perform the calculations at one wavelength of the perturbation, we checked that doubling the spatial period does not lead to modulational instabilities. During the nonlinear evolution of the instability, strong vertical shears between the layers may develop locally, leading to the hyperbolicity loss. As it was already shown [26], the numerical scheme copes well with this kind of situation, corresponding physically to the development of Kelvin-Helmholtz type small-scale shear instabilities. Indeed the threshold of hyperbolicity loss coincides in the model with the threshold of the KH instability [25]. Obviously, no KH roll-up is possible in the shallow-water model. What happens in the numerical scheme is that the related strong gradients trigger enhanced numerical dissipation. The scheme thus cures itself, the non-hyperbolicity zones remaining localized (see the examples below) and eventually disappearing. Note that the hyperbolicity loss is an endemic problem in the numerical modelling of multi-layer shallow water equations, our scheme being the first, to our knowledge, which copes with it satisfactory.

1. Asymmetric inertial instability

If the simulation is initialized with the most unstable inertial baroclinic instability mode (big cross in Fig. 17), the following scenario takes place. The instability reaches its nonlinear stage in several inertial periods, as may be seen in Figs. 22 - 25, where the evolution of the thicknesses and the PV of the layers is presented. A horizontal wave-breaking pattern develops with strong vertical shears accompanying this process and leading to hyperbolicity loss and enhanced dissipation appearing at \( t \approx 6 \) and persisting through the rest of the simulation. These processes are nevertheless localized and do not tend to fill the computational domain, which means that simulation remains reliable. At later stages secondary intense coherent vortices are formed and the flow is completely reorganized. The coherent structure formation is associated with the spreading of the flow in the cross-stream direction. We do not present the results for longer times because the secondary structures start to interact with the boundaries already at \( t \approx 22 \). The reliable simulations may be continued for longer times in a larger domain (at higher cost), but the tendency is clear already from the present cheap ones lasting several hours on the standard desktop computer. Again, we systematically doubled resolution to check numerical convergence (not shown).

The energy evolution during the simulation is presented in Fig. 26. It is worth noting that kinetic energy decreases faster than potential energy at the early stages of the evolution \( (t < 10) \), while at later stages it is the opposite. Such "inverse" (with respect to the classical baroclinic instability) behavior is consistent with what was observed in the symmetric case above.

The overall energy loss is non-negligible. This is due to persistent enhanced dissipation in the zones of strong vertical shear, cf. Fig. 25 showing that the two are well correlated. Strong shear generates shear (KH) instabil-

![Figure 22](image-url): Nonlinear evolution of the most unstable mode of the asymmetric inertial instability. Thickness of the layer 1 (top), and of the layer 2 (bottom). Contour interval 1/12.

![Figure 23](image-url): Same as in Fig. 22, but for later times.
Figure 24: Nonlinear evolution of the most unstable mode of the asymmetric inertial instability. Left: Isolines of the velocity shear $|u_2 - u_1|$ (thin lines), enhanced dissipation zones (black) and zones of hyperbolicity loss (gray). Contour interval 0.15 for $|u_2 - u_1|$. PV isolines for the layer 1 (middle) and for the layer 2 (right). Contour interval 2.5 (positive/negative PV: black/gray lines).

Figure 25: Same as Fig. 24, but for later times.

Figure 26: Nonlinear evolution of the most unstable mode of the asymmetric inertial instability. Normalized deviation of the total (solid), kinetic (black dashed), and potential (gray dashed) energy in the whole calculation domain from their initial values.

If one looks at the quantities averaged in the along-jet
Figure 27: Nonlinear evolution of the most unstable mode of the asymmetric inertial instability. Time evolution of the minimum (over the whole calculation domain) of the discriminant of the characteristic matrix.

Figure 28: Evolution of the $y$-averaged geostrophic momentum for the developing asymmetric inertial instability. Direction, one clearly observes a tendency to homogenize the cross-stream gradient of the averaged geostrophic momentum (Fig. 28) and PV (Fig. 29). The averaged jet intensity diminishes and it spreads. The average across-jet velocity becomes nonzero (Fig. 30) reflecting transverse motion due to secondary coherent structures.

As was mentioned in the Introduction, an important question is about inertial instability as a source of inertia-gravity waves. The inertia-gravity wave field is part of the ageostrophic motions. The initial jet being balanced, i.e. geostrophic, the initial ageostrophy is due uniquely to the superimposed unstable mode which has a very small amplitude (1% as compared to the unperturbed configuration). The intensity of the ageostrophic motions can be measured by the modulus of the divergence of velocity integrated over the computational domain and divided by the volume. The evolution of such norm both for baroclinic and barotropic components of velocity is presented in Fig. 31. It shows a net production of the ageostrophic component of the flow, especially in the baroclinic sector.

Yet, not all of the ageostrophic motions are propagating waves, as secondary fronts and vortices may have a significant ageostrophic component. In order to visualize the wave emission we present in Fig. 32 the Hovmöller diagram of the divergence of the baroclinic velocity. We clearly see the emission of baroclinic inertia-gravity waves starting with formation of secondary structures in Figs. 22 - 24. Similar diagram for the divergence of the barotropic velocity shows no pronounced emission (not shown). We should emphasize that much weaker

Figure 29: Evolution of the $y$-averaged thicknesses (left) and PV’s (right) of the two layers for the developing asymmetric inertial instability. Dashed: layer 2; solid: layer 1.

Figure 30: Evolution of the $y$-averaged components of velocity $u$ (left) and $v$ (right) for the developing asymmetric inertial instability. Dashed: layer 2; solid: layer 1.
Figure 31: Nonlinear evolution of the most unstable mode of the asymmetric inertial instability. Evolution of the norm of ageostrophic motion $s$. Baroclinic (dotted) and barotropic (solid) component.

Figure 32: Nonlinear evolution of the most unstable mode of the asymmetric inertial instability. Hovmöller diagram of the divergence of the baroclinic velocity at $y=0$. A characteristic value $\pm 0.1$ is plotted in black (positive) and gray (negative). Slopes corresponding to the phase velocities of linear barotropic (gray) and baroclinic (black) gravity waves at both sides of the jet are given for comparison.

(two orders of magnitude smaller than the value shown in Fig. 32) barotropic wave emission could be detected (not shown). It starts immediately and is a result of the adjustment of the jet due to discretization errors, as in other simulations of this type [31].

An estimate of the efficiency of the baroclinic inertia-gravity wave emission may be obtained from the frequency spectrum of the baroclinic divergence

$$\nabla \cdot \mathbf{v}_b(x, y, t) = \int \omega D(x, y, \omega) \exp(i \omega t) \, d\omega,$$

By taking an along-jet average $|D| = \langle |D| \rangle_y$ at some distance $x_*$ from the jet axis, we get a measure of ageostrophic motions of a given frequency at $x_*$. Propagating inertia-gravity waves are supra-inertial with $\omega > 1$ in non-dimensional terms.

Thus, the ratio $R = \int_{-\infty}^{\infty} d\omega |D|(x_*, \omega) / \int_0^{\infty} d\omega |D|(x_*, \omega)$ gives an indication of relative intensity of wave motions at distance $x_*$ from the jet (although, obviously, not all supra-inertial motions are waves). Sufficiently far from the jet axis (e.g. at $x_* = \pm 20L$) the calculated $R$ is about 60%. However, this estimate should be taken with caution, as our time series is not long.

2. Ageostrophic barotropic instability

Figure 33: Nonlinear evolution of the most unstable mode of the barotropic instability. Thickness of the layers 1 (left) and 2 (right). Same conventions as in Figs 22, 23.

We now briefly report a parallel simulation of the saturation of the barotropic instability of the same strongly ageostrophic jet. The simulation was initialized with the mode indicated by the big circle in Fig. 17 superimposed upon the jet [38]. The counterparts of Figs. 22 - 25 for this case are presented in Figs. 33 - 34. Several important differences with respect to the evolution of the inertial baroclinic instability are observed. The pattern at the earlier nonlinear stages is a typical breaking Rossby wave. The vertical shears develop slower and are less...
Figure 34: Nonlinear evolution of the most unstable mode of the ageostrophic barotropic instability. Velocity shear, dissipation, hyperbolicity and PV. Same conventions as in Figs. 24, 25.

intense, and the zones of enhanced dissipation are distributed differently. The hyperbolicity loss is transient, as follows from the evolution of the discriminant of the characteristic matrix presented in Figure 36.

An important difference is in the energy budget presented in Figure 35. From the very beginning and during the whole process of saturation the kinetic energy is fed by the potential energy, like in the classical baroclinic instability. The overall energy loss is less than in the case of inertial baroclinic instability, which is consistent with the transient character of the hyperbolicity loss and related enhanced dissipation.

Another difference is a much less efficient generation of the ageostrophic motions during the evolution of the instability, as follows from Fig. 37 representing the evolution of the norm of the ageostrophic motions. As compared to Fig. 31 one sees that in absolute terms the norm of the baroclinic divergence for the inertial instability reaches the values twice higher than for the barotropic one. The difference becomes even larger if the divergence norm is divided by the peak energy of the respective instabilities which could be inferred from Figs. 26, 35. As to the emission of inertia-gravity waves, the developing barotropic instability produces it as well, as follows from the Figure 38. The relative intensity of the wave emission calculated along the same lines as for inertial instability above gives similar results.

In spite of these differences, the characteristics of the flow averaged along the jet show qualitatively similar to Figs. 28 - 30 behaviour (not shown). Namely, the geostrophic momentum and PV are being homogenized, and the jet is spreading, although keeping all the time its barotropic character. As typical vertical shears are less...
V. DISCUSSION AND SUMMARY

We thus analyzed the inertial instability of a barotropic jet in the framework of the two-layer rotating shallow model on the $f$-plane, both in the symmetric one-dimensional and in the full two-dimensional settings. The model is a standard tool in GFD, and at the same time is a proxy for a step-like continuous stratification cf. [32] [39]. Unlike the case of smoothly stratified barotropic jet this configuration does not have accumulation of unstable modes at high vertical wavenumbers, which allows to treat it in the non-dissipative framework and to achieve a high resolution in direct numerical simulations of the saturation of the instability.

Intensification of any jet will lead, sooner or later, to the appearance of negative PV at the anticyclonic side of the jet. The conditions for inertial instability thus appear. We have shown that this instability resides in the baroclinic branch of the stability diagram of the barotropic Bickley jet, being stronger than the barotropic instability, also present in this range of the jet parameters, for large enough Rossby numbers and small enough wavenumbers. The "classical" symmetric inertial instability corresponds to the endpoint of the baroclinic branch at zero wavenumber in the along-jet direction. It is important to note that the growth rate of the asymmetric baroclinic inertial instability with small but non-zero along-jet wavenumber is higher than that of the purely symmetric one. When $Ro$ diminishes the stability diagram of the jet becomes conventional, displaying the standard geostrophic barotropic and baroclinic instabilities, although for high enough $Ro$ the baroclinic instability is stronger for small wavenumbers. The inertial instability, thus, acquires an interpretation as a long-wave essentially ageostrophic baroclinic instability.

It should be mentioned that the links between symmetric and asymmetric inertial instability have been discussed in the literature for continuously stratified flows. Thus weakly asymmetric inertial instability was studied in [15] in the general context of shear flows, and asymmetric equatorial inertial instability was found in [14]. As was already said in the Introduction, the competition between inertial and barotropic instabilities has been repeatedly discussed. We should stress, however, that the (relative) simplicity of our model allows to establish a clear link of the asymmetric inertial instability both with the symmetric one and with the ageostrophic baroclinic instability, and to have clear-cut conclusions about its competition with the barotropic instability.

In this context it should be emphasized that although the nature of the inertial instability on the $f$-plane and equatorial $\beta$-plane is the same, its interplay with other instabilities is, probably, not. (We remind that the classical instabilities in layered models are best understood in terms of phase-locking and resonances between linear waves e.g. [20], and that the spectrum of equatorial waves is special). That is why we restricted the present study to the $f$-plane case, the study of the equatorial case is in process.

We quantified and compared the linear unstable modes of the ageostrophic inertial baroclinic and barotropic instability, and then initialized direct high-resolution numerical simulations with these modes using a new-generation finite-volume scheme [21]. We demonstrated that the nonlinear saturation of the inertial baroclinic and barotropic instabilities follows scenarios displaying some common features, like homogenization of the cross-stream gradient of the geostrophic momentum in the unstable region, although the details of the evolution are rather different in two cases. In both cases the jet spread-
ing leads to formation of intense coherent vortices due to the breaking of unstable waves propagating on the jet background. Characteristic baroclinic inertia-gravity wave emission was found during the saturation process. The ageostrophic secondary motions are produced more efficiently by the inertial baroclinic instability. The energy budget is essentially different for the ageostrophic inertial baroclinic and barotropic instabilities. For the former, the kinetic energy feeds the potential energy at the early stages of the evolution, while the process is reversed at later stages. For the latter, the energy conventionally goes from the potential to kinetic one. During the simulations, hyperbolicity loss was observed due to formation of strong shears and subsequent development of the Kelvin-Helmholtz type shear instabilities. The inertial baroclinic and ageostrophic barotropic instabilities are essentially different with respect to these phenomena. For the latter the shear instabilities and enhanced dissipation are transient, while for the former they are persistent, leading to more intense production of small-scale ageostrophic motions and more pronounced dissipation of energy.

Acknowledgements

We are grateful to R. Plougonven for stimulating discussions and to J. Gula for helpful advice. We acknowledge constructive and helpful remarks of the two anonymous Referees, which led to substantial improvements. This work was supported by the French ANR grant “SVEMO”.


[35] The localized character of inertially unstable modes was repeatedly discussed in the literature, e.g. [33], [34], [15]

[36] An arbitrary topography may be easily included both in the linear stability analysis and in the nonlinear simulations

[37] while $\text{Re}(\omega) = 0$ when $k = 0$, we checked that

$$\lim_{k \to 0} \frac{\text{Re}(\omega)}{k} \neq 0.$$  

[38] Note that although we selected a pure barotropically unstable mode for this simulation, it is in the zone of competition between asymmetric inertial and barotropic instability. Arbitrary perturbation with this wavelength will be projected on both inertially and barotropically unstable modes.

[39] Yet, the conditions of realizability of the scenarios observed in our simulations in fully three-dimensional fluid with such stratification remain to be investigated - this was out of the scope of the present paper.