Introduction to quantum statistical models

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1 States and measurements

Let $\mathcal{H}$ be a separable Hilbert space, the space of wave functions. Typically, $\mathcal{H} = \mathbb{C}^d$ or $\mathcal{H} = \mathbb{L}_2(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} : \int |f|^2 < \infty\}$. Let $\mathcal{B}(\mathcal{H})$ denote the space of bounded linear operators on $\mathcal{H}$.

Definition 1 *A quantum state is an operator ρ on $\mathcal{H}$ which is*

- $\rho = \rho^*$, *self adjoint;*
- $\rho \geq 0$, *positive definite;*
- $tr(\rho) = 1$.

Example 1 *If $d = 2$ (quantum computing), the Bloch sphere*

$$\rho = \frac{1}{2}(I + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z),$$

where $r = (r_x, r_y, r_z)$ has euclidean norm less than or equal to 1 and

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are the Pauli matrices.

Example 2 *If $d = \infty$ (quantum optics), take, for example, the wave function $\psi_0(x) = \pi^{-1/4}\exp(-x^2/2)$ corresponding to the vacuum state $\rho_0 = |\psi_0 >< \psi_0|$. It is a pure state (rank 1).*

A faithful state is such that $\rho > 0$ (full rank).

Definition 2 *Let (Ω, Σ) be a measurable space. The application $\mathcal{M} : (\Omega, \Sigma) \to \mathcal{B}(\mathcal{H})$ is a POVM (positive operator valued measure) if it is such that*

- $\mathcal{M}(O) \geq 0$, *for all* $O \in \Sigma$*;*
- $\mathcal{M}(\bigcup_{n\in\mathbb{N}} O_n) = \sum_{n\in\mathbb{N}} \mathcal{M}(O_n)$ *for all sequence of disjoints events* $\{O_n\}_n$*;*
- $\mathcal{M}(\Omega) = \mathbb{I}$*, where* $\mathbb{I}$ *is the unity of* $\mathcal{H}$*.*

In particular, if $\mathbf{A}$ is an observable, i.e. a self adjoint operator on $\mathcal{H}$, the spectral theorem gives us a spectral measure which is, moreover, a PVM (projector valued measure) which is a particular case of POVM.

Example 3 *The position and momentum operators* $\mathbf{Q}$ *and* $\mathbf{P}$ *are observables (self-adjoint operators) defined by*

$$\mathbf{Q}f(x) = x \cdot f(x) \text{ and } \mathbf{P}f(x) = -i\frac{\partial f}{\partial x}(x).$$

They are not bounded.

Let us compute the law of the random variable resulting from measuring position operator on the vacuum state $\rho_0 = |\psi_0 >< \psi_0|$ *. The associated projector-valued measure is the spectral measure of the operator:*

$$\mathcal{M} : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to \mathcal{B}(\mathcal{H}),$$

given by $\mathcal{M}(O)f(x) = I_O(x) \cdot f(x)$*. The random variable* Q *is such that*

$$P_{\rho_0}(Q \in O) = Tr(\rho_0 \mathcal{M}(O)) = \int_O |\psi_0(x)|^2 dx = \int_O \frac{1}{\sqrt{\pi}} \exp(-x^2) dx, \quad \forall O \in \mathcal{B}(\mathbb{R}).$$

We recognize the Gaussian distribution $Q \sim N(0, 1/2)$*.*

Example 4 *Similar computations can be done for measuring the observable* $\mathbf{P}$ *in Fourier domain, where it becomes* $\mathbf{Q}$*. By measuring* $\mathbf{P}$ *on the vacuum state we also get a Gaussian random variable* $P \sim N(0, 1/2)$*.*

Example 5 *The operator number of photons* $\mathbf{N} = \frac{1}{2}(\mathbf{Q}^2 + \mathbf{P}^2 - 1)$ *has* $\mathbb{N}$ *as spectrum and is diagonalized by the Fock basis of* $\mathbb{L}_2(\mathbb{R})$*,* $\{|\psi_k >\}_{k\geq 0}$*, defined by:* $\mathbf{N}|\psi_k >= k|\psi_k >$*.*

Other examples include: the creation and annihilation operators A^+ and A^-, respectively,

$$A^+|\psi_k>=\sqrt{k+1}\cdot|\psi_{k+1}> \text{ and } A^-|\psi_k>=\sqrt{k}\cdot|\psi_{k-1}>.$$

We have the relations

$$A^+=\frac{1}{\sqrt{2}}(\mathbf{Q}-i\mathbf{P}) \text{ and } A^-=\frac{1}{\sqrt{2}}(\mathbf{Q}+i\mathbf{P}).$$

A **coherent state** *is a pure state $|\alpha>$ for some complex number α, defined by*

$$|\alpha>=e^{-|\alpha|^2/2}\sum_{k\geq 0}\frac{\alpha^k}{\sqrt{k!}}|\psi_k>.$$

By measuring the number of photons $\mathbf{N}=\sum_k k\cdot|\psi_k><\psi_k|$ on a coherent state

$$\rho=|\alpha><\alpha|=e^{-|\alpha|^2}\sum\sum_{n,m\geq 0}\frac{\alpha^n\bar{\alpha}^m}{\sqrt{n!m!}}|\psi_n><\psi_m|,$$

we obtain the random variable N on $\mathbb{N}$ such that

$$P_\rho(N=k)=Tr(\rho|\psi_k><\psi_k|)=e^{-|\alpha|^2}Tr(\sum_{n\geq 0}\frac{\alpha^n\bar{\alpha}^k}{\sqrt{n!k!}}|\psi_n><\psi_k|)=e^{-|\alpha|^2}\frac{|\alpha|^{2k}}{k!}.$$

We recognize the Poisson distribution $\mathcal{P}(|\alpha|^2)$.

The **Schrodinger cat state** *has wave function $|\psi>=c(|\alpha>+|-\alpha>)$, where c is a normalization constant.*

2 CCR and Weyl algebras

Let V be a vector space of finite dimension and S an antisymmetric form on $V\times V$ (i.e. bilinear and such that $S(x,y)=-S(y,x)$).

Definition 3 *The smallest unitary complex algebra $\mathcal{A}$ is the algebra of $CCR(V,S)$ if there exists a monomorphism $B:V\to\mathcal{A}$ satisfying*

$$[B(v),B(w)]=iS(v,w)\mathbb{I},$$

for all v, w in V.

With the involution $B(z)^*=B(z)$ we get a unitary *-algebra. This algebra is not normed.

We introduce the exponential form $W:V\to\mathcal{D}$

$$W(z)=e^{iB(z)}:=\sum_{n\geq 0}\frac{(iB(z))^n}{n!}.$$

Definition 4 *The smallest unitary complex algebra $\mathcal{D}$ generated by $\{W(z), z \in V\}$ is the algebra of Weyl $W(V,S)$ with the product*

$$W(v) \cdot W(w) = e^{-\frac{i}{2}S(v,w)} W(v+w),$$

and the involution $W(v)^* = W(-v)$, for all v, w in V and completed with respect to the norm $\| \sum_{i\in I} \alpha_i W(v_i)\| := \sum_{i \in I} |\alpha_i|$, for all finite sets I.

Moreover, we have $\|W(v) \cdot W(v)^*\| = \|W(v)\| \cdot \|W(v)^*\|$ and $W(V,S)$ is C^*-algebra.

Baker-Campbell-Hausdorff formula: $W(v+w) = W(v)W(w)e^{\frac{i}{2}S(v,w)}$. This formula implies that

$$W(v) \cdot W(w) = W(w) \cdot W(v) e^{iS(w,v)} = W(w) \cdot W(v) e^{-iS(v,w)}.$$

This implies for a state ρ: $Tr(\rho W(v+w)) = Tr(\rho W(v)W(w))e^{\frac{i}{2}S(v,w)}$. For more theory [8].

2.1 Quantum optics example, one mode Gaussian state

Several systems: the quantum harmonic oscilator, the quantum free particle, the monochromatic light in a cavity, are described by the same algebra of observables: position and momentum operators $\mathbf{Q}$ and $\mathbf{P}$. They verify the commutation relation

$$[\mathbf{Q}, \mathbf{P}] = i\mathbf{I}.$$

Thus, take $V = \mathbb{C}$, $B(1) = \mathbf{Q}$, $B(i) = \mathbf{P}$ and $S(v,w) = Im(\bar{v} \cdot w)$.

Due to the commutation relation a state ρ is such that $Tr(\rho\mathbf{QP}) = i + Tr(\rho\mathbf{PQ})$.

A state φ is Gaussian in this system if it has the moments of a Gaussian $Tr(\rho\mathbf{Q}^{2m+1}) = 0$ and $Tr(\rho\mathbf{Q}^{2m}) = (2m)!/(2^m m!)$ for all $m \in \mathbb{N}$, etc. It also has the "characteristic function" given by $Tr(\rho W(z)) = \exp(-\alpha(z,z)/2)$, where $\alpha : V \times V \to \mathbb{R}$ is bilinear, symmetric and positive form.

We have $W(1) = e^{i\mathbf{Q}}$ and $W(i) = e^{i\mathbf{P}}$, which gives for any complex number $z = x + iy$: $W(z) = e^{i(x\mathbf{Q}+y\mathbf{P})}$, acting as follows

$$\begin{aligned} W(z)f(\xi) &= e^{-\frac{i}{2}S(x,iy)} e^{ix\mathbf{Q}} e^{iy\mathbf{P}} f(\xi) = e^{-\frac{i}{2}xy} e^{ix\mathbf{Q}} f(\xi + y) \\ &= e^{-\frac{i}{2}xy + ix(\xi+y)} f(\xi+y) = e^{ix\xi + \frac{i}{2}xy} f(\xi+y). \end{aligned}$$

The Wigner function associated to a state ρ is $W_\rho(q,p)$ defined via its characteristic function:

$$\tilde{W}_\rho(x,y) = Tr(\rho W(x+iy)),\ \forall x,\ y \in \mathbb{R}.$$

A Gaussian state $\Phi_{\alpha,V}$ is described by

$$Tr(\Phi_{\alpha,V}W(z)) = \exp\left(-\frac{1}{2}(x - Re(\alpha), y - Im(\alpha))\, V \begin{pmatrix} x - Re(\alpha) \\ y - Im(\alpha) \end{pmatrix}\right),$$

where $V = \begin{pmatrix} Tr(\rho\mathbf{Q}^2) & Tr(\rho\mathbf{Q}\circ\mathbf{P}) \\ Tr(\rho\mathbf{Q}\circ\mathbf{P}) & Tr(\rho\mathbf{P}^2) \end{pmatrix}$, $A\circ B = \frac{1}{2}(AB + BA)$. The uncertainty principle states that $det(V) \geq 1/4$.

Most frequently used Gaussian states are summed up in the following table:

Gaussian state $\Phi_{\alpha,V}$	α	V
vacuum	0	I
coherent (laser)	α	I
thermal equilibrium	0	$\sigma\cdot I$
displaced thermal	α	$\sigma\cdot I$
squeezed	0	$\mathrm{diag}(e^{-\xi}, e^{\xi})$

2.2 Centered observables of a d-dimensional state

This construction is a key result in proving quantum local asymptotic normality for d-dimensional states, see [3], [4], [5] and was also used for limit theorems of quantum U-statistics in [2].

Let ρ be a quantum state on $\mathbb{C}^d$, with $d \geq 2$.

An observable $\mathbf{A}$ is said centered if $tr(\rho\mathbf{A}) = 0$. Let us denote by $\mathbb{L}_2^0(\rho)$ the set of centered observables. On this space we consider the scalar product $(\mathbf{A},\mathbf{B})_\rho = tr(\rho\cdot\mathbf{A}\circ\mathbf{B})$, where we recall that $A\circ B = (AB+BA)/2$. Note that $(\mathbf{A}, I)_\rho = tr(\rho\mathbf{A}) = 0$.

In the larger Hilbert space $\mathbb{L}^2(\rho)$ with inner product $(\cdot,\cdot)_\rho$, $\mathbb{L}_2^0(\rho)$ consists of all observables orthogonal to the identity I. On $\mathbb{L}_2(\rho)$ we define the symplectic form

$$\sigma(\mathbf{A},\mathbf{B}) = \frac{i}{2} tr(\rho\cdot[\mathbf{A},\mathbf{B}]),$$

where again $[\mathbf{A},\mathbf{B}] = \mathbf{AB} - \mathbf{BA}$ denotes the commutator.

Thus we can take $V = \mathbb{L}_2^0(\rho)$ and $S = \sigma$, to construct the $CCR(V,S)$ algebra $\mathcal{A}$. The monomorphism $B : \mathbb{L}_2^0(\rho) \to \mathcal{A}$ is such that

$$[B(\mathbf{A}), B(\mathbf{B})] = i\cdot tr(\rho\cdot[\mathbf{A},\mathbf{B}])\cdot I.$$

The corresponding Weyl operator is $\mathbf{W}(\mathbf{A}) = \exp(iB(\mathbf{A}))$. A Gaussian state in this setup is Φ such that $(\Phi, \mathbf{W}(\mathbf{A}))_\rho = \exp(-\|\mathbf{A}\|_\rho^2/2)$.

In order to make this algebra simpler to understand, let us consider a faithful, diagonal state $\rho = \mathrm{diag}(\mu_1, \mu_2, ..., \mu_d)$, with decreasing diagonal elements. The space $\mathbb{L}_2^0(\rho)$ can be written as the orthogonal sum of $\mathcal{H}_\rho$, the subspace of centered observables commuting with ρ, and its orthogonal $\mathcal{H}_\rho^\perp$.

The space $\mathcal{H}_\rho$ has dimension $d-1$ and has the basis $\{d_i = -\mu_i I + E_{i,i}\}$ for all i from 1 to $d-1$. Matrices $E_{i,j}$ have the entry (i,j) equal to 1 and all other 0. Note that $(d_i, d_j)_\rho = \mu_i \delta_{i,j} - \mu_i \mu_j$.

The orthogonal space $\mathcal{H}_\rho^\perp$ has dimension $d(d-1)$ and has the orthogonal basis

$$T_{j,k} = \begin{cases} \frac{i}{\sqrt{2|\mu_i - \mu_j|}}(E_{j,k} - E_{k,j}), & j < k \\ \frac{1}{\sqrt{2|\mu_i - \mu_j|}}(E_{j,k} + E_{k,j}), & j > k. \end{cases}$$

Note that $\sigma(T_{j,k}, T_{l,m}) = -1/2\delta_{jl}\delta_{k,m}$. This means that for each pair (j,k) we have a pair of canonical coordinates $\mathbf{Q}_{j,k}, \mathbf{P}_{j,k}$ of a harmonic oscillator.

In conclusion, we can map the $CCR(\mathbb{L}_2^0(\rho), \sigma)$ algebra into a joint space of $d-1$ classical Gaussian variables with variance matrix given by $(d_i, d_j)_\rho$, $1 \leq i, j \leq d-1$ and $d(d-1)/2$ harmonic oscillators with variances $\|T_{j,k}\|_\rho^2 = (\mu_j + \mu_k)/\sqrt{2|\mu_j - \mu_k|}$.

3 Fock space

Let us recall that $\mathcal{H}$ is a complex separable Hilbert space with norm $\|\cdot\|$ and $\mathcal{H}^{\otimes k}$ is the k-fold tensor product for any fixed $k \in \mathbb{N}$. Note that, for example, $\mathbb{L}_2(\mathbb{R})^{\otimes k}$ is isomorphic to $\mathbb{L}_2(\mathbb{R}^k)$. Thus a k-mode (multi mode) Gaussian state will be described by k pairs of observables $\mathbf{Q}_i, \mathbf{P}_i$ for i from 1 to k.

In order to introduce product state with arbitrary many modes, we define the Fock space $\Gamma(\mathcal{H}) = \bigoplus_{k\geq 0} \mathcal{H}^{\otimes k}$ where $\bigoplus$ denotes the orthogonal direct sum of Hilbert spaces, see [1], [7] . Let us define the boson Fock space $\Gamma_s(\mathcal{H}) = \bigoplus_{k\geq 0} \mathcal{H}^{\otimes_s k}$, where $\mathcal{H}^{\otimes_s k}$ is the boson (symmetric) tensor product, i.e. the projection of the k-fold tensor product $\mathcal{H}^{\otimes k}$ via the symmetrizing operator E:

$$Eu_1 \otimes \ldots \otimes u_k = \frac{1}{k!} \sum_{\sigma \in S_k} u_{\sigma(1)} \otimes \ldots \otimes u_{\sigma(k)},$$

for all $u_1, \ldots u_k$ in $\mathcal{H}$. The sum is taken over all permutations σ of k elements.

In a Fock space, the element $1 \oplus 0 \oplus 0 \oplus \ldots$ is called the vacuum and, for all $u \in \mathcal{H}$,

$$e(u) = \bigoplus_{k\geq 0} \frac{1}{\sqrt{k!}} u^{\otimes k}, \quad (\text{with } 0! = 1, u^{\otimes 0} = 1),$$

is called the exponential vector associated to u. The normalized exponential vector

$$\Phi(u) = \exp(-\|u\|^2/2) \cdot e(u)$$

is known as the coherent vector. Let us recall that

$$\langle e(u), e(v) \rangle = \lim_{k\to\infty} \sum_{l=0}^{k} \langle \frac{1}{\sqrt{l!}} u^{\otimes l}, \frac{1}{\sqrt{l!}} v^{\otimes l} \rangle = \sum_{l\geq 0} \frac{1}{l!} \langle u, v \rangle^l = e^{\langle u,v \rangle}.$$

Moreover, the linear span $\mathcal{D}$ of the set of all exponential (coherent) vectors is a dense set.

It is usual to define the **displacement operator**, for any u, v in $\mathcal{H}$:

$$D_v e(u) = e(u + v) \cdot \exp(-\|u + v\|^2/2 + \|u\|^2/2),$$

where the normalization is such that $\|D_v e(u)\| = \|e(u)\|$. Note that $D_v \Phi(u) = \Phi(u+v)$. This operator is not unitary.

The unitary **Weyl operator** is defined by

$$\mathbf{W}_v e(u) = e(u + v) \cdot \exp(-v^\star u - \|v\|^2/2) = D_v e(u) \cdot \exp(iIm(u^\star v)).$$

We can easily check that $\mathbf{W}_v \mathbf{W}_w = \mathbf{W}_{v+w} \cdot e^{-iIm(v^\star w)}$ and that $\mathbf{W}_v \mathbf{W}_w = \mathbf{W}_w \mathbf{W}_v \cdot e^{-2iIm(v^\star w)}$. It is known that, for any u, $\{\mathbf{W}_{tu}\}_{t\in\mathbb{R}}$ is a unitary group and continuous, hence there exists a self-adjoint operator $\mathbf{B}_u$ such that $\mathbf{W}_{tu} = e^{-it\mathbf{B}_u}$. Note that both $\mathbf{B}_u$ and $\mathbf{W}_u$ are defined on $\mathcal{D}$. Moreover, $\mathbf{W}_u$ is bounded and can be extended to the entire space, while $\mathbf{B}_u$ is not bounded and cannot be extended. We have

$$\mathbf{B}_v e(u) = \frac{1}{i} \frac{d}{dt} \mathbf{W}_{tv} e(u)|_{t=0} = i\langle v, u \rangle e(u) - i \frac{d}{dt} e(u + tv)|_{t=0}.$$

The operator $\mathbf{P} = \mathbf{B}_{-1}$ is known as the momentum operator and $\mathbf{Q} = \mathbf{B}_i$ is the position operator. The creation operator is defined by $\mathbf{A}^\star = (\mathbf{Q} - i\mathbf{P})/2$ and it is self-adjoint having as the adjoint the annihilation operator $\mathbf{A} = (\mathbf{Q} + i\mathbf{P})/2$.

3.1 Quantum Gaussian white noise model

In the continuous-time model, let us put $\mathcal{H} = \mathbb{L}_2(\mathbb{R}_+)$. An element ψ of the symmetric Fock space $\Gamma_s(\mathcal{H})$ can be seen as

$$\psi = \bigoplus_{k\geq 0} f_k(t_1, ..., t_k),$$

with f_k symmetric in all arguments, complex-valued and square integrable over $\mathbb{R}_+^k$. For any $t \geq 0$ and ψ in the symmetric Fock space of $\mathcal{H}$, we define the annihilation, creation and conservation process by, respectively,

$$\begin{aligned}
[\mathbf{A}_t\psi]_k\,(t_1,...,t_k) &= \int_0^t f_{k+1}(t_1,...,t_k,t_{k+1})dt_{k+1} \\
[\mathbf{A}_t^\star\psi]_k\,(t_1,...,t_k) &= \sum_{j=1}^k \chi_{[0,t]}(t_j)f_{k-1}(t_1,...,t_{j-1},t_{j+1},...,t_k) \\
[\mathbf{\Lambda}_t\psi]_k\,(t_1,...,t_k) &= \sum_{j=1}^k \chi_{[0,t]}(t_j)f_k(t_1,...,t_k),
\end{aligned}$$

where $\chi_{[0,t]}(\cdot)$ is the indicator function of the interval $[0,t]$.

The annihilation and creation operators are adjoint to one another, while the conservation operator is self-adjoint. They satisfy the following commutation relations:

$$\begin{aligned}
&[\mathbf{A}_t, \mathbf{A}_s] = [\mathbf{A}_t^\star, \mathbf{A}_s^\star] = 0, \\
&[\mathbf{A}_t, \mathbf{A}_s^\star] = (t \wedge s)\cdot I, \quad [\mathbf{\Lambda}_t, \mathbf{\Lambda}_s] = 0, \\
&[\mathbf{\Lambda}_t, \mathbf{A}_s] = -\mathbf{A}_{t\wedge s}, \quad [\mathbf{\Lambda}, \mathbf{A}_s^\star] = \mathbf{A}_{t\wedge s}^\star.
\end{aligned}$$

We define $\mathbf{Q}_t = \mathbf{A}_t + \mathbf{A}_t^\star$ and $\mathbf{P}_t = i(\mathbf{A}_t^\star - \mathbf{A}_t)$. Previous relations imply that

$$\begin{aligned}
&[\mathbf{Q}_t, \mathbf{Q}_s] = 0, \quad [\mathbf{P}_t, \mathbf{P}_s] = 0, \quad [\mathbf{Q}_t, \mathbf{P}_s] = 2i(t \wedge s)\cdot I, \\
&[\mathbf{\Lambda}_t, \mathbf{Q}_s] = -i\mathbf{P}_{t\wedge s}, \quad [\mathbf{\Lambda}_t, \mathbf{P}_s] = i\mathbf{Q}_{t\wedge s}.
\end{aligned}$$

It is proved in [6] that $\{\mathbf{Q}_t\}$, $\{\mathbf{P}_t\}$ and $\{\mathbf{\Lambda}_t\}$ are diagonalizable families of self-adjoint operators and that each of these families with distinguished vacuum vector is equivalent to a classical stochastic process.

The exponential vector writes

$$e(f) = \bigoplus_{k\geq 0} \frac{1}{\sqrt{k!}} f(t_1)\cdot \ldots \cdot f(t_k).$$

We get $\langle e(f), e(g)\rangle = \exp(\int_0^\infty \bar{f}(s)g(s)ds)$. The coherent vector is

$$\Phi(f) = \exp\left(-\frac{1}{2}\int_0^\infty |f(t)|^2 dt\right)\cdot e(f).$$

The previous operators act on exponential vectors as follows

$$\begin{aligned}
\langle e(f), \mathbf{A}_t e(g)\rangle &= \int_0^t g(s)ds \cdot \langle e(f), e(g)\rangle \\
\langle e(f), \mathbf{A}_t^\star e(g)\rangle &= \int_0^t \bar{f}(s)ds \cdot \langle e(f), e(g)\rangle \\
\langle e(f), \mathbf{\Lambda}_t e(g)\rangle &= \int_0^t \bar{f}(s)g(s)ds \cdot \langle e(f), e(g)\rangle
\end{aligned}$$

In consequence, these operators are diagonalized by exponential vectors.

Let $\{W_t, t \in \mathbb{R}_+\}$ be a Wiener process on a probability space $(\Omega, \mathcal{A}, P)$ and $\mathbb{L}_2(W)$ be the space of all complex-valued square-integrable functionals of the Wiener process. There exists a duality map J from $\Gamma_s(\mathbb{L}_2(\mathbb{R}_+))$ to $\mathbb{L}_2(W)$:

$$J(\psi) = \sum_{k\geq 0} \int_{0<t_1<\ldots<t_k} f_k(t_1, \ldots, t_k) dW_{t_1} \ldots dW_{t_k}.$$

In $\mathbb{L}_2(W)$, we can define the stochastic exponential functions:

$$\begin{aligned} J(\Phi(f)) &= e^{-\frac{1}{2}\int_0^\infty |f(t)|^2 dt} \cdot \sum_{k\geq 0} \int_{0<t_1<\ldots<t_k} f(t_1) \cdot \ldots \cdot f(t_k) dW_{t_1} \ldots dW_{t_k} \\ &= e^{-\frac{1}{2}\int_0^\infty |f(t)|^2 dt} \cdot \sum_{k\geq 0} \int_0^\infty \ldots \int_0^\infty \frac{1}{k!} f(t_1) \cdot \ldots \cdot f(t_k) dW_{t_1} \ldots dW_{t_k} \\ &= \exp\left(\int_0^\infty f(t) dW_t - \frac{1}{2}\int_0^\infty |f(t)|^2 dt\right). \end{aligned}$$

Then, $\langle e(f), \mathbf{Q}_t e(g)\rangle = E(J(\Phi(\bar{f}))W_t J(\Phi(g)))$. Indeed,

$$\begin{aligned} \langle e(f), \mathbf{Q}_t e(g)\rangle &= \langle e(f), \mathbf{A}_t e(g)\rangle + \langle e(f), \mathbf{A}_t^\star e(g)\rangle \\ &= \langle e(f), \int_0^t g(s) ds \cdot e(g)\rangle + \langle \int_0^t f(s) ds \cdot e(f), e(g)\rangle \\ &= \int_0^t (\bar{f} + g)(s) ds \cdot \exp\left(\int_0^\infty \bar{f} \cdot g\right), \end{aligned}$$

where $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$, and

$$\begin{aligned} E(J(\Phi(\bar{f}))W_t J(\Phi(g))) &= E\left(W_t \cdot \exp\left(\int_0^\infty (\bar{f} + g)(s) dW_s - \frac{1}{2}\int_0^\infty (|f|^2 + |g|^2)(s) ds\right)\right) \\ &= E(W_t \cdot J(\Phi(\bar{f} + g))) \cdot \exp\left(\int_0^\infty \bar{f} \cdot g\right) \\ &= \int_0^t (\bar{f} + g)(s) ds \cdot \exp\left(\int_0^\infty \bar{f} \cdot g\right). \end{aligned}$$

Let us recall that for $0 < s < t$ we have $\mathbb{L}_2(\mathbb{R}_+) = \mathbb{L}_2((0,s)) \oplus \mathbb{L}_2((s,t)) \oplus \mathbb{L}_2((t,\infty))$ and that Fock spaces factorize (see [7]) $\Gamma(\mathcal{H}) = \Gamma(\mathbb{L}_2((0,s))) \otimes \Gamma(\mathbb{L}_2((s,t))) \otimes \Gamma(\mathbb{L}_2((t,\infty)))$. All previous operators are adapted to this factorization, in the sense that, $\mathbf{Q}_t$ (and all others) acts only on $\Gamma(\mathbb{L}_2((0,t)))$. An exponential vector admits the factorization $e(f) = e(f \cdot \chi((0,t])) \otimes e(f \cdot \chi((t,\infty)))$.

We shall work from now on only on $\Gamma(\mathbb{L}_2((0,1)))$. The **quantum Gaussian white noise model** that we consider is the coherent vector with drift function $m \in \mathbb{L}_2((0,1))$

$$\Phi(\sqrt{n} m). \tag{1}$$

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