Existence of global strong solution and vanishing capillarity-viscosity limit in one dimension for the Korteweg system

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Résumé

In the first part of this paper, we prove the existence of a global strong solution for the Korteweg system in one dimension. In the second part, motivated by the processes of vanishing capillarity-viscosity limit in order to select the physically relevant solutions for a hyperbolic system, we show that the global strong solution of the Korteweg system converges in the case of a $\gamma$ law for the pressure ($P(\rho) = a \rho^\gamma$, $\gamma > 1$) to a weak entropy solution of the compressible Euler equations. In particular it justifies that the Korteweg system is suitable for selecting the physical solutions in the case where the Euler system is strictly hyperbolic. The problem remains open for a Van der Waals pressure, indeed in this case the system is not strictly hyperbolic and in particular the classical theory of Lax and Glimm (see [26, 13]) cannot be used.

1 Introduction

We are concerned with compressible fluids endowed with internal capillarity. The model we consider originates from the XIXth century works by Van der Waals and Korteweg [42, 25] and was actually derived in its modern form in the 1980s using the second gradient theory, see for instance [11, 22, 41]. The first investigations begin with the Young-Laplace theory which claims that the phases are separated by a hypersurface and that the jump in the pressure across the hypersurface is proportional to the curvature of the hypersurface. The main difficulty consists in describing the location and the movement of the interfaces.

Another major problem is to understand whether the interface behaves as a discontinuity in the state space (sharp interface, SI) or whether the phase boundary corresponds to a more regular transition (diffuse interface, DI). The diffuse interface models have the advantage to consider only one set of equations in a single spatial domain (the density takes into account the different phases) which considerably simplifies the mathematical and numerical study (indeed in the case of sharp interfaces, we have to treat a problem with free boundary).

Let us consider a fluid of density $\rho \geq 0$, velocity field $u \in \mathbb{R}$, we are now interested in the
following compressible capillary fluid model, which can be derived from a Cahn-Hilliard-like free energy (see the pioneering work by J.-E. Dunn and J. Serrin in [11] and also in [2, 5, 14, 19]). The conservation of mass and momentum writes:

\[
\begin{align*}
\rho \partial_t \rho^e + \partial_x (\rho^e u^e) &= 0, \\
\rho \partial_t (\rho^e u^e) + \partial_x (\rho^e (u^e)^2) - 2\epsilon \partial_x (\rho^e \partial_x u^e) + \partial_x (a(\rho^e)^\gamma) &= \epsilon^2 \partial_x K,
\end{align*}
\]

where the Korteweg tensor reads as following:

\[
\partial_x K = \partial_x (\rho^e \kappa(\rho^e) \partial_{xx} \rho^e + \frac{1}{2} (\kappa(\rho^e) + \rho^e \kappa'(\rho^e)) |\partial_x \rho^e|^2) - \partial_x (\kappa(\rho^e)(\partial_x \rho^e)^2).
\]

\[\kappa \text{ is the coefficient of capillarity and is a regular function of the form } \kappa(\rho) = k\rho^\alpha \text{ with } \alpha \in \mathbb{R}. \]

In the sequel we shall assume that \( \kappa(\rho) = \frac{k}{\rho} \) and take \( k = 1 \). The term \( \partial_x K \) allows to describe the variation of density at the interfaces between two phases, generally a liquid-vapour mixture. \( P(\rho) = a\rho^\gamma \) with \( \gamma \geq 1 \) is a general \( \gamma \) law pressure term. The small parameter \( \epsilon \) corresponds to the controlling parameter on the amplitude of the viscosity and of the capillarity.

**Remark 1** We would like to emphasize the fact that this choice of capillarity exhibits particular regime flows in the case of the compressible Euler system with quantic pressure (which corresponds here to the capillarity). Indeed, at least heuristically, the system is equivalent via the Madelung transform to the Gross-Pitaevskii equations which are globally well-posed for large initial data in dimension \( n = 1, 2, 3 \) (we refer to [4]). One of the main difficulty to pass from Gross-Pitaevskii to Quantic Euler consists in dealing with the vacuum. This is one of the reasons why the mathematical community is interested in building solitons for this type of problem (one of the main other reasons corresponds to give a negative answer to the problem of scattering and after to study the stability of the soliton). Finally we would also like to mention very interesting results of global weak solutions for the compressible quantic Euler equation with a regime \( \kappa(\rho) = \frac{1}{\rho} \) due to Antonelli and Marcati (see [1])

As in [15], the key idea in the present paper consists in introducing an effective velocity; indeed if we set \( v^e = u^e + \epsilon \partial_x (\ln \rho^e) \), we can write (1.1) into the following simply form (we refer to [15] for precise computations and more properties):

\[
\begin{align*}
\rho \partial_t \rho^e + \partial_x (\rho^e v^e) - \epsilon \partial_{xx} \rho^e &= 0, \\
\rho \partial_t (\rho^e v^e) + \partial_x (\rho^e (v^e)^2) - \epsilon \partial_x (\rho^e \partial_x v^e) + \partial_x (a(\rho^e)^\gamma) &= 0,
\end{align*}
\]

or, equivalently

\[
\begin{align*}
\rho \partial_t \rho^e + \partial_x (\rho^e v^e) - \epsilon \partial_{xx} \rho^e &= 0, \\
\rho \partial_t (\rho^e v^e) + \partial_x (\rho^e (v^e)^2) - \epsilon \partial_{xx}^2 (\rho^e v^e) + \partial_x (a(\rho^e)^\gamma) &= 0,
\end{align*}
\]

which is exactly the viscous approximation system \((1.40)\) introduced in [29].

We now consider the Cauchy problem of (1.3) when the fluid is away from vacuum. Namely, we shall study (1.3) with the following initial data:

\[
\rho^e(0, x) = \rho_0^e(x) > 0, \quad u^e(0, x) = u_0^e(x),
\]
such that:

$$\lim_{x \to \pm \infty} (\rho_0'(x), u_0'(x)) = (\rho^\pm, u^\pm), \text{ with } \rho^\pm > 0.$$ 

We would like to study in the following the limit process of system (1.3) when $\epsilon$ goes to 0 and to prove in particular that we converge to weak-entropy solution of the compressible Euler system:

$$\begin{cases}
\partial_t \rho + \partial_x (\rho v) = 0, \\
\partial_t (\rho v) + \partial_x (\rho v^2) + \partial_x (\rho a^{\gamma}) = 0,
\end{cases}$$

(1.6)

Let us now explain the interest of the capillary solutions for the hyperbolic systems of conservation laws.

In addition of modeling a liquid-vapour mixture, the Korteweg system also has purely theoretical interests consisting in the selection of the physically relevant solutions of the Euler model (in particular when the system is not strictly hyperbolic). The typical case corresponds to a Van der Waals pressure: indeed in this case the system is not strictly hyperbolic in the elliptic region (which corresponds to the region where the phase change occurs), in particular we can not use the classical results due to Glimm [13].

In the adiabatic pressure framework ($P(\rho) = \rho^\gamma$ with $\gamma > 1$), the system is strictly hyperbolic and the theory is classical. More precisely we are able to solve the Riemann problem when the initial Heaviside data is small in the BV space. Indeed we are in the context of the well known Lax result as the system is also genuinely nonlinear (we refer to [26]). It means we have existence of global $C^1$-piecewise solutions which are unique in the class of the weak-entropy solutions.

This result as been extent by Glimm in the context of small initial data in the BV-space by using a numerical scheme and approximating the initial BV data by a $C^1$-piecewise function (which implies to locally solve the Riemann problem via the Lax result). For the uniqueness of the solution we refer to the work of Bianchini and Bressan ([3]) who use a viscosity method.

In the setting of the Van der Waals pressure, as we said the existence of global solutions and the nature of physical relevant solutions remain completely open (essentially because the system is not strictly hyperbolic).

Heuristically, we also hope that the process of vanishing capillarity-viscosity limit selects the physical relevant solutions as it does for the stationary system (see [35]). This problem actually remains open.

1.1 Existence of global weak-entropy solutions for the Euler system

Before presenting the results of this paper let us recall the results on this topic in the last decades. We shall focus on the case of a $\gamma$ pressure law $P(\rho) = \rho^\gamma$ with $\gamma > 1$ and $a$ positive. Let us mention that these cases are the only ones well-known (essentially because the system is strictly hyperbolic in this case and that we can exhibit many entropy-flux pairs). Here the Lax-Glimm theory can be applied, however at the end of the 70’s, one was interested in relaxing the conditions on the initial data by only assuming $\rho_0$ and $u_0$ in $L^\infty$.

In the beginning of the 80’s Di Perna initiated this program, consisting in obtaining global weak-entropy solution for $L^\infty$ initial data.
Indeed in [9, 10], Di Perna proved the existence of global weak entropy solution of
(1.6) for \( \gamma = 1 + \frac{2}{d+1} \) and \( \gamma = 2k + \frac{3}{2k} + 1 \) (with \( k \geq 1 \), \( d \geq 2 \) by using the so-called “compensated compactness” introduced by Tartar in [39]. This result was extended by Chen in [6] in the case \( \gamma \in (1, \frac{5}{3}] \) and by Lions, Perthame and Tadmor in [30] in the case \( \gamma \in [3, \infty) \). In [29], Lions et al generalize this result to the general case \( \gamma \in (1, 3) \), and finally the case \( \gamma = 1 \) is treated by [20]. We would like to mention that these results are obtained through a vanishing artificial viscosity on both density and velocity.

The problem of vanishing physical viscosity limit of compressible Navier-Stokes equations to compressible Euler equations was until recently an open problem. However Chen and Perepelista in [7] proved that the solutions of the compressible Navier-Stokes system with constant viscosity coefficients converge to a weak-entropy solution of the Euler system with finite energy. This result was extended in [21] to the case of viscosity coefficients depending on the density.

Inspired by [7] and [21], we would like to show that the solution of the Korteweg system (1.3) converges to a weak-entropy solution of the Euler system with finite energy when the pressure is a \( \gamma \) law. To do this, we will prove for the first time up our knowledge the existence of global strong solution for the Korteweg system in one dimension in the case of Saint-Venant viscosity coefficients. By contrast, the problem of global strong solutions for compressible Navier-Stokes equations remains open in the case shallow-water viscosity coefficients (it means when \( \mu(\rho) = \rho \), indeed one of the main difficulties consists in controlling the vacuum, we refer to the works of Kanel [23] for the case of constant viscosity coefficients). This result justifies that the Korteweg system allows us to select the relevant physical solutions of the compressible Euler system at least when the pressure is adiabatic \( (P(\rho) = a\rho^\gamma \) with \( \gamma > 1 \)). The problem remains open in the case of a Van der Waals pressure.

1.2 Results

Let us now describe our main result. In the first theorem we prove the existence of global strong solution for the Korteweg system (1.3). Let us introduce \((\bar{\rho}, \bar{v})\) be a pair of smooth monotone positive functions of \( x \) satisfying \((\bar{\rho}(x), \bar{v}(x)) = (\rho^\pm, v^\pm)\) when \( \pm x \geq L_0 \) for some constant \( L_0 > 0 \).

**Theorem 1 (Di Perna [9])** Suppose that \((\rho_0 - 1, u_0 - 1) \in C^2 \cap H^2 \) and \( \rho_0 \geq c > 0 \).

There exists a global strong solution \((\rho, u)\) to the Cauchy problem with data \( \rho_0, u_0 \) such that :

\[(\rho(\cdot, t) - 1, u(\cdot, t) - 1) \in C^2 \cap H^2, \rho(t, \cdot) \geq c(t) > 0,\]

with \( c \) an appropriate function.

**Remark 2** Let us mention that it is possible to improve the initial data space (by choosing larger), moreover as the system is parabolic we have regularizing effects on the density and the regularity.

**Remark 3** Concerning the global existence of strong solution for the compressible Navier-Stokes equations, we would like to mention the results of Kanel in [23] and Kazhikhov and Shelukhin in [24] in the constant viscosity coefficient cases. Let us point out that
up our knowledge the only result of global strong solution for the compressible Navier-Stokes equations dealing with variable viscosity coefficients is due to Mellet and Vasseur in [31]. More precisely in this paper the viscosity coefficient is of the form \( \mu(\rho) \geq \rho^\alpha \) with \( 0 \leq \alpha < \frac{1}{2} \) close from the vacuum. In particular the problem of the existence of global strong solutions remains open in the case of the Saint-Venant system \( \mu(\rho) = \rho \), which corresponds to system (1.3) without capillarity. Indeed in this case it is tricky to control the vacuum.

In particular our result shows that provided an additional capillary term we are able to obtain the existence of global strong solution in one dimension for the Saint-Venant system.

Let us first recall some classical definitions on the entropy pairs and the energy estimates with which we shall work. Guided by [30], [7] and [21] we introduce:

**Definition 1.1** A pair of functions \((\eta(\rho,v),H(\rho,v))\) (or denoted \((\eta(\rho,m),H(\rho,m))\) if seen as functions of \( \rho \) and \( m = \rho v \)) is called an entropy-entropy flux pair of system (1.3), if for any smooth solution of (1.6) we have:

\[
[\eta(\rho,v)]_t + [H(\rho,v)]_x = 0,
\]

Furthermore \( \eta(\rho,v) \) is called a weak entropy if \( \eta(0,v) = 0 \) for any fixed \( v \).

**Definition 1.2** An entropy \( \eta(\rho,m) \) is convex if its Hessian \( \nabla^2 \eta(\rho,m) \) is nonnegative definite.

Such \( \eta \) satisfies the wave equation \( \theta = \frac{\gamma-1}{2} \):

\[
\partial_{\rho \rho} \eta = \frac{P'(\rho)}{\rho^2} = a \rho^{\gamma-3} \partial_{vv} \eta.
\]

From [30], we obtain an explicit representation of any weak entropy \((\eta,H)\). We refer to the sequel for more details. For instance, the mechanical energy and the associated flux:

\[
\eta^*(\rho,m) = \frac{m^2}{2\rho} + e(\rho), \quad H^*(\rho,m) = \frac{m^3}{2\rho^2} + me'(\rho), \quad (1.7)
\]

where \( e(\rho) = \frac{a}{\gamma-1} \rho^\gamma \) represents the gas internal energy in physics.

In the following we will work far away from the vacuum that it is why we shall introduce equilibrium states in order to avoid the vacuum. As introduced in the previous section, let \((\bar{\rho},\bar{v})\) be a pair of smooth monotone functions \((x)\) satisfying \((\bar{\rho}(x),\bar{v}(x)) = (\rho^\pm, v^\pm)\) when \( \pm x \geq L_0 \) for some \( L_0 > 0 \). The total mechanical energy for (1.6) in \( \mathbb{R} \) with respect to the pair of reference function \((\bar{\rho}(x),\bar{v}(x))\) is:

\[
E[\rho,v](t) = \int_{\mathbb{R}} (\eta^*(\rho,m) - \eta^*(\bar{\rho},m) - \nabla \eta^*(\bar{\rho},m).(\rho - \bar{\rho}, m - \bar{m})) dx
\]

\[
= \int_{\mathbb{R}} \left( \frac{1}{2} \rho(t,x)|v(t,x) - \bar{v}(x)|^2 + e^*(\rho(t,x), \bar{\rho}(x)) \right) dx \quad (1.8)
\]
where \( e^*(\rho, \bar{\rho}) = e(\rho) - e(\bar{\rho}) - e'(\bar{\rho})(\rho - \bar{\rho}) \geq 0 \). In the presence of capillarity, the total mechanical energy for system (1.1) with \( \kappa(\rho) = \frac{1}{\rho} \) is:

\[
E_1[\rho, u](t) = \int_{\mathbb{R}} \left( \frac{1}{2} \rho(t, x)|u(t, x) - \bar{u}(x)|^2 + e^*(\rho(t, x), \bar{\rho}(x)) + 2\epsilon^2(\partial_x \rho \frac{1}{2})^2 \right) dx \tag{1.9}
\]

and the total mechanical energy for system (1.3) is:

\[
E_2[\rho, v](t) = \int_{\mathbb{R}} \left( \frac{1}{2} \rho(t, x)|v(t, x) - \bar{v}(x)|^2 + e^*(\rho(t, x), \bar{\rho}(x)) \right) dx \tag{1.10}
\]

**Remark 4** Notice that due to the definition of the effective velocity, we have \( \bar{u} \equiv \bar{v} \).

**Definition 1.3** Let \((\rho_0, v_0)\) be given initial data with finite-energy with respect to the end states \((\rho^\pm, v^\pm)\) at infinity, that is, satisfying \( E[\rho_0, v_0] \leq E_0 < +\infty \). A pair of measurable functions \((\rho, v) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+ \times \mathbb{R}\) is called a finite-energy entropy solution of the Cauchy problem (1.6) if the following properties hold:

1. The total energy is locally bounded in time: there exists a function \( C(E, t) \), defined on \( \mathbb{R}_+ \times \mathbb{R}_+ \) and continuous in \( t \) for each \( E \in \mathbb{R}_+ \) such that for a.e \( t > 0 \):

\[
E[\rho, v](t) \leq C(E_0, t).
\]

2. The entropy inequality:

\[
\eta^\psi(\rho, v)_t + H^\psi(\rho, v)_x \leq 0,
\]

is satisfied in the sense of distributions for all test functions \( \psi(s) \in \{ \pm 1, \pm s, s^2 \} \).

3. The initial data \((\rho_0, v_0)\) are attained in the sense of distributions.

Let us now give the main conditions on the initial data (1.5), inspired from [7].

**Definition 1.4** We say that the initial data \((\rho_0^\epsilon, v_0^\epsilon)\) satisfy the condition \( \mathcal{H} \) if there exist positive constants \( C_0 \) and \( C_1 \) independent of \( \epsilon \) such that:

- \( \rho_0^\epsilon > 0 \) and \( \int_{\mathbb{R}} \rho_0^\epsilon(x)|u_0^\epsilon(x) - \bar{u}(x)| \leq C_0 < +\infty \),
- The energy is finite:

\[
E_1[\rho_0^\epsilon, u_0^\epsilon] + E_2[\rho_0^\epsilon, v_0^\epsilon] \leq C_1 < +\infty.
\]

- \( (\rho_0^\epsilon, v_0^\epsilon) \to (\rho_0, \rho_0v_0) \) in the sense of distributions as \( \epsilon \) goes to zero with \( \rho_0 \geq c > 0 \) a.e.

In the following theorem, we are interested in proving the convergence of the global solutions of system (1.3) to entropic solutions of the Euler system (1.6).

**Theorem 1.1** Let \( \gamma > 1 \) and \((\rho^\epsilon, v^\epsilon)\) with \( m^\epsilon = \rho^\epsilon v^\epsilon \) be the global strong solution of the Cauchy problem (1.3) with initial data \((\rho_0^\epsilon, v_0^\epsilon)\) which verify uniformly in \( \epsilon \) definition 1.4 and the assumptions of theorem 1. Then, when \( \epsilon \to 0 \), there exists a subsequence of \((\rho^\epsilon, m^\epsilon)\) that converges almost everywhere to a finite entropy solution \((\rho, pv)\) to the Cauchy problem (1.6) with initial data \((\rho_0, \rho_0v_0)\). More precisely we have:

- \( \rho^\epsilon \) strongly converges up to a subsequence to \( \rho \) in \( L^{\gamma+1-\alpha}_{loc}(\mathbb{R}_+ \times \mathbb{R}) \) for any \( 0 < \alpha < 1 \).
- \( (\rho^\epsilon)^{\frac{1}{3}} v^\epsilon \) strongly converges up to a subsequence to \( \rho^{\frac{1}{3}} v \) in \( L^{3-\alpha}_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}) \) for any \( \alpha > 0 \) small enough.

- \( (\rho^\epsilon)^{\frac{1}{2}} v^\epsilon \) strongly converges up to a subsequence to \( \rho^{\frac{1}{2}} v \) in \( L^{2+\frac{2\gamma}{\gamma+3} - \alpha}_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}) \) for any \( \alpha > 0 \) small enough.

**Remark 5** Here \( v \) is defined as follows:
- \( v(t,x) = \frac{m(t,x)}{\rho(t,x)} \) when \( \rho(t,x) \neq 0 \),
- \( v(t,x) = 0 \) almost everywhere in \( \{ \rho(t,x) = 0 \} \).

**Remark 6** Let us emphasize in particular that it is not necessary to bound uniformly \( (\rho^\epsilon_0, \rho_0, v_0^\epsilon) \) in \( L^\infty \) norm, however the fact that \( (\rho^\epsilon_0, \rho_0, v_0^\epsilon) \) belongs in \( L^\infty \) is important in order to obtain global strong solution via the use of the Riemann invariants.

**Remark 7** We would like to point out that Lions et al in [29] had obtained the existence of global weak entropy solution for \( \gamma > 1 \) by a viscosity vanishing process, and the considered regularizing system was exactly the Korteweg system modulo the introduction of the effective velocity.

**Remark 8** Let us point out that for the moment we only have the convergence of the effective velocity \( v^\epsilon \) to the weak-entropy solution \( v \) of the compressible Euler equations, the natural question is to know if the original velocity \( u^\epsilon \) converges also to \( v \). To do this it implies to prove the \( L^2_{\text{loc}} \) strong convergence of \( \epsilon \sqrt{\rho^\epsilon} \partial_x \ln \rho^\epsilon \) to 0. The inequality (4.46) is crucial but not sufficient, indeed we need of suitable estimates on \( \frac{1}{\rho^\epsilon} \) in Lebesgue spaces in terms of power of \( \frac{1}{\epsilon} \) in order to conclude. It seems natural to use estimates involving the maximum principle, unfortunately it seems not enough.

However via the inequality (4.46) we obtain the convergence of the physical quantity \( (\rho^\epsilon)^{\frac{\gamma-1}{2}} (\sqrt{\rho^\epsilon} u^\epsilon) \) (with \( \gamma > 1 \)) to the weak-entropy solution \( \rho^{\frac{1}{2}} v \) of the compressible Euler equations. As we can observe, the difficulty to prove the convergence of the physical quantity \( u^\epsilon \) is related with the existence or not of vacuum.

The most important tool of our problem for theorem 1.1 is the following compactness theorem established in [7].

**Theorem 1.2** *(Chen-Perepelitsa [7])* Let \( \psi \in C^2_0(\mathbb{R}) \) and let \( (\eta^\psi, H^\psi) \) be the weak entropy pair generated by \( \psi \) (we refer to the following for details). Assume that the sequence \( (\rho^\epsilon(t,x), v^\epsilon(t,x)) \) defined on \( \mathbb{R}_+ \times \mathbb{R} \) with \( m^\epsilon = \rho^\epsilon v^\epsilon \), satisfies the following conditions:

1. For any \(-\infty < a < b < +\infty \) and all \( t > 0 \), there exists \( C(t,a,b) > 0 \) independent of \( \epsilon \) such that:
   \[
   \int_0^t \int_a^b (\rho^\epsilon)^{\gamma+1} dxd\tau \leq C(t,a,b),
   \]  
   (1.11)

2. For any compact set \( K \subset \mathbb{R} \), there exists \( C(t,K) > 0 \) independent of \( \epsilon \) such that (with \( \theta = \frac{\gamma-1}{2} > 0 \)):
   \[
   \int_0^t \int_K ((\rho^\epsilon)^{\gamma+\theta} + \rho^\epsilon |v^\epsilon|^\beta) dxd\tau \leq C(t,K),
   \]  
   (1.12)
3. The sequence of entropy dissipation measures

\[ \eta^\psi(\rho^\epsilon, m^\epsilon)_t + H^\psi(\rho^\epsilon, m^\epsilon)_x \text{ is compact in } H^{-1}_{loc}(\mathbb{R}_+ \times \mathbb{R}). \]  

Then there exists a subsequence of \((\rho^\epsilon, m^\epsilon)\) (still denoted \((\rho^\epsilon, m^\epsilon)\)) and a pair of measurable functions \((\rho, m)\) such that:

\[ (\rho^\epsilon, m^\epsilon) \rightarrow (\rho, m), \text{ a.e as } \epsilon \rightarrow 0. \]  

Remark 9 Let us recall that the estimate \((1.12)\) was first derived by Lions et al in [30] by relying on the moment lemma introduced by Perthame in [34].

The paper is structured in the following way: in section 2 we recall some important results on the notion of entropy-flux pair for Euler system and on the kinetic formulation of Lions et al in [30]. In section 3, we give some elements of the proof of the theorem 1 by following DiPerna in [9] and in the last section 4.1 we prove theorem 1.1.

2 Mathematical tools

Let us now recall some estimates on the function \(e^*\) involved in the expressions of \(E_1\) and \(E_2\) (obtained by function study):

Lemma 1 There exist two constants \(C(\bar{\rho}) > 0\) (only depending on \(\bar{\rho}\)) and \(C > 0\) such that for all \(\rho \geq 0\) we have:

\[ \rho + \rho^\gamma \leq C(\bar{\rho})(1 + e^*(\rho, \bar{\rho})), \quad \rho(\rho^\theta - \bar{\rho}^\theta)^2 \leq C e^*(\rho, \bar{\rho}), \]

where \(\theta = \frac{\gamma - 1}{2} > 0\).

Next let us recall some properties on the entropy pair for system \((1.6)\) (we refer to [30] for more details). Smooth solutions of \((1.6)\) satisfy the conservation laws:

\[ \partial_t \eta(\rho, u) + \partial_x H(\rho, u) = 0, \]

if and only if:

\[ H_\rho = u \eta_\rho + \frac{P'(\rho)}{\rho} \eta_u, \quad H_u = \rho \eta_\rho + u \eta_u, \]

or equivalently

\[ \eta_{\rho\rho} = \frac{P'(\rho)}{\rho^2} \eta_{uu} = a \rho^{\gamma-3} \eta_{uu}. \]  

(2.15)

We supplement the equation 2.15 by giving initial conditions:

\[ \eta(0, u) = 0, \quad \eta_\rho(0, u) = \psi(u). \]  

(2.16)

Let us now state description result of \(\eta\) and \(H\), we refer to Lions-Perthame-Tadmor ([30]) for more details.

Proposition 2.1 For \(\rho \geq 0, u, \omega \in \mathbb{R},\)
- The fundamental solution of (2.15)-(2.16) (that is, corresponding to \( \eta_\rho(0,u) = \delta(u) \)) is given by:

\[
\chi(\rho, \omega) = r^{\frac{3-\gamma}{2(\gamma-1)}} > -\frac{1}{2}, \theta = \frac{\gamma-1}{2}
\]

and:

\[
t_\lambda^+ = \begin{cases} 
t_\lambda & \text{if } t > 0, \\
0 & \text{if } t \leq 0. 
\end{cases}
\]

- The solution of (2.15)-(2.16) is given by:

\[
\eta^\psi(\rho, \rho u) = \int_\mathbb{R} \psi(\xi) \chi(\rho, \xi - u) d\xi.
\]

- \( \eta \) is convex in \( (\rho, \rho u) \) for all \( \rho, u \) if and only if \( \psi \) is convex.

- The entropy flux \( H^\psi \) associated to \( \eta^\psi \) is given by:

\[
H^\psi(\rho, \rho u) = \int_\mathbb{R} \psi(\xi) \theta \xi + (1-\theta)u \chi(\rho, \xi - u) d\xi.
\]

More precisely, in our case (\( \gamma \)-law for the pressure) we have, as recalled in [7]:

\[
\eta^\psi(\rho, \rho u) = \rho \int_{-1}^{1} \psi(u + \rho^s s)(1-s^2)^\lambda ds,
\]

and

\[
H^\psi(\rho, \rho u) = \rho \int_{-1}^{1} (u + \theta \rho^s s) \psi(u + \rho^s s)(1-s^2)^\lambda ds.
\]

For instance, as recalled in the previous section, when \( \psi(s) = \frac{1}{2} s^2 \), the corresponding entropy pair is the mechanical energy and the associated flux:

\[
\eta^*(\rho, m) = \frac{m^2}{2\rho} + e(\rho), \quad H^*(\rho, m) = \frac{m^3}{2\rho^2} + me'(\rho),
\]

where \( e(\rho) = \frac{\kappa}{\gamma-1} \rho^\gamma \).

The following result gives important estimates on the entropy pair (we refer to [30], lemma 4, and [7] section 2):

**Proposition 2.2** For \( \psi(s) = \frac{1}{2} s|s| \), there exists a positive constant \( C > 0 \), only depending on \( \gamma > 1 \), such that the entropy pair \( (\eta^\psi, H^\psi) \) satisfies:

\[
|\eta^\psi(\rho, u)| \leq C(\rho|u|^2 + \rho^\gamma), \\
H^\psi(\rho, u) \geq C^{-1}(\rho|u|^3 + \rho^{\gamma+\theta}), \quad \text{for all } \rho \geq 0 \text{ and } u \in \mathbb{R}, \\
|\eta^\psi_m(\rho, u)| \leq C(|u| + \rho^\theta), \\
|\eta^\psi_{mm}(\rho, u)| \leq C\rho^{-1}.
\]

Let us now give recent results on the entropy pair \( (\eta^\psi, H^\psi) \) generated by \( \psi \in C^2_0(\mathbb{R}) \) (we refer to [7] for more details).
Proposition 2.3 For a $C^2$ function $\psi : \mathbb{R} \to \mathbb{R}$, compactly supported on the interval $[a, b]$, we have:

$$\text{supp}(\eta^\psi), \text{supp}(H^\psi) \subset \{(\rho, m) = (\rho, \rho u) : \rho + \rho^\theta \geq a, u - \rho^\theta \leq b\} : \quad (2.24)$$

Furthermore, there exists a constant $C_\psi$ such that, for any $\rho \geq 0$ and $u \in \mathbb{R}$, we have:

- For $\gamma \in (1, 3]$,
  $$|\eta^\psi(\rho, m)| + |H^\psi(\rho, m)| \leq C_\psi \rho. \quad (2.25)$$
- For $\gamma \in (3, +\infty)$,
  $$|\eta^\psi(\rho, m)| \leq C_\psi \rho, \quad |H^\psi(\rho, m)| \leq C_\psi (\rho + \rho^{\theta+1}). \quad (2.26)$$
- If $\eta^\psi$ is considered as a function of $(\rho, m)$, $m = \rho u$ then
  $$|\eta^\psi_m(\rho, m)| + |\rho \eta^\psi_{mm}(\rho, m)| \leq C_\psi, \quad (2.27)$$
  and, if $\eta^\psi_m$ is considered as a function of $(\rho, u)$, then
  $$|\eta^\psi_{mu}(\rho, \rho u)| + |1^{\theta-\gamma} \eta^\psi_{m\rho}(\rho, \rho u)| \leq C_\psi. \quad (2.28)$$

More precisely for the Korteweg system, we will need some additional estimates on the entropy-entropy flux pair:

Proposition 2.4 – Under the assumptions of proposition 2.2, there exists a constant $C$ such that we have:

$$\begin{cases} 
|\eta^\psi_\rho(\rho, m)| \leq C \rho^\theta (|u| + \rho^\theta) \\
|\eta^\psi_{\rho\rho}(\rho, m)| \leq C \rho^{\gamma-2}.
\end{cases}$$

- Under the assumptions of proposition 2.3, there exists a constant $C_\psi$ such that:

$$\begin{cases} 
|\eta^\psi_\rho(\rho, m)| \leq C_\psi (1 + \rho^\theta) \\
|\eta^\psi_{\rho\rho}(\rho, m)| \leq C_\psi \rho^{\gamma-2}.
\end{cases}$$

Proof: If we wish to use, as in [30], the classical method based on direct estimates of successive derivatives of the expression of $\eta^\psi_\rho$, we will obtain that for $\psi(s) = \frac{1}{2} |s| s$,

$$|\eta^\psi_\rho(\rho, m)| \leq C (|u|^2 + \rho^\theta + \rho^{\theta+1} + \rho |u|),$$

which, due to the term $|u|^2$ will cause problems in the proofs of lemmas 4.62 and 6 as we won’t be able to estimate terms such as $|u|^2 |\rho_x|$. The problem will persist when estimating $|\eta^\psi_{\rho\rho}|$.

In order to overcome this difficulty we simply use (2.15) which, combined with (2.23) and the fact that $\partial_v = \rho \partial_m$, ensures the estimates for $|\eta^\psi_{\rho\rho}|$ in both choices of $\psi$.

Concerning the order one derivative and for the case $\psi(s) = \frac{1}{2} |s| s$, writing the Taylor formula in $m$, and using that $|\eta^\psi_{mu}(\rho, u)| \leq C \rho^{\theta-1}$, we obtain that:

$$|\eta^\psi_\rho(\rho, m)| \leq |\eta^\psi_\rho(\rho, 0)| + C |m| \rho^{\theta-1}.$$

Computing the exact integral expression of $\eta^\psi_\rho$, we obtain that $|\eta^\psi_\rho(\rho, 0)| \leq \beta \rho^{2\theta}$ which gives the result.

In the case of a compactly supported $\psi$, we simply estimate the exact expression.

Let us now express the kinetic formulation of (1.6) introduced in ([30]).
Theorem 2.3 Let \((\rho, \rho v) \in L^\infty(\mathbb{R}^+, L^1(\mathbb{R}))\) have finite energy and \(\rho \geq 0\), then it is an entropy solution of (1.6) if and only if there exists a non-positive bounded measure \(m\) on \(\mathbb{R}^+ \times \mathbb{R}^2\) such that the function \(\chi(\rho, \xi - u)\) satisfies:

\[
\partial_t \chi + \partial_x \left[ \left( \theta \xi + (1 - \theta)u \right) \chi \right] = \partial_{\xi \xi} m(t, x, \xi).
\]

We would like to conclude this section by a useful lemma in order to obtain strong convergence.

**Lemma 2** Let \(K\) a compact subset of \(\mathbb{R}^N\) (with \(N \geq 1\)) and \(v^\epsilon\) a sequel such that:

- \(v^\epsilon\) is uniformly bounded in \(L^{p+\alpha}(K)\) with \(\alpha > 0\) and \(p \geq 1\),
- \(v^\epsilon\) converges almost everywhere to \(v\),

then \(v^\epsilon\) strongly converges to \(v\) in \(L^p(K)\) with \(v \in L^{p+\alpha}(K)\).

**Proof:** First by the Fatou lemma \(v\) is in \(L^{p+\alpha}(K)\). Next we have for any \(M > 0\):

\[
\int_K |v^\epsilon - v|^p dx \leq \int_{K \cap \{|v^\epsilon - v| \leq M\}} |v^\epsilon - v|^p dx + \int_{K \cap \{|v^\epsilon - v| \geq M\}} |v^\epsilon - v|^p dx.
\]

We are dealing with the second member of the right hand side, by Hölder inequality and Tchebychev lemma we have for a \(C > 0\):

\[
\int_{K \cap \{|v^\epsilon - v| \geq M\}} |v^\epsilon - v|^p dx \leq \left( \int_K |v^\epsilon - v|^p \right) \left( \frac{|v^\epsilon - v|^\alpha}{M^\alpha} \right) dx \leq \frac{C}{M^\alpha}.
\]

In particular we have shown the strong convergence of \(v^\epsilon\) to \(v\), indeed from the inequality (2.30) it suffices to use the Lebesgue theorem for the first term on the right hand side and the estimate (4) with \(M\) going to \(+\infty\).

### 3 Proof of theorem 1

We recall that the system (1.3) has the following form:

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho v) - \epsilon \partial_{xx} \rho &= 0, \\
\partial_t (\rho v) + \partial_x (\rho v^2) - \epsilon \partial_x (\partial_x \rho v) + \partial_x (a(\rho)^\gamma) &= 0,
\end{align*}
\]

and following DiPerna in [9] we are going to deal with the momentum \(m = \rho v\) in order to lead to a Korteweg system with constant viscosity coefficient (it allows us to simplify the system what enables us to get new energy estimates and to control the vacuum), we have then to consider the following system:

\[
\begin{align*}
\partial_t \rho + \partial_x (m) - \epsilon \partial_{xx} \rho &= 0, \\
\partial_t (m) + \partial_x \left( \frac{m^2}{\rho} \right) - \epsilon \partial_{xx} (m) + \partial_x (a(\rho)^\gamma) &= 0,
\end{align*}
\]

This system has been studied by Di Perna (see [9]) where he proves the existence of global strong solution (see theorem 4.1 p 29), we also refer to the works of Kanel in [23] and Kazhikhov and Shelukhin in [24] who show the global existence of strong solution for the compressible Navier-Stokes equations with constant viscosity. Let us first recall the theorem 4.1 p 29 of [9].
Theorem 2 ([9]) Suppose that \((\rho_0 - 1, u_0 - 1) \in C^2 \cap H^2\) and \(\rho_0 \geq c > 0\). There exists a global strong solution \((\rho, u)\) to the Cauchy problem with data \(\rho_0, u_0\) such that:

\[
(\rho(\cdot, t) - 1, u(\cdot, t) - 1) \in C^2 \cap H^2, \rho(t, \cdot) \geq c(t) > 0,
\]

with \(c\) an appropriate function.

In order to get such result, the first step consists in deriving pointwise control on higher derivatives of \((\rho, m)\). Indeed if we assume that:

\[
\left\| \left( \frac{1}{\rho}, \rho, m \right)(t, \cdot) \right\|_{L^\infty} \leq C(t, \left\| \frac{1}{\rho_0} \right\|_{L^\infty}, \left\| \rho_0 \right\|_{L^\infty}, \left\| m_0 \right\|_{L^\infty})
\]

with \(C\) is a finite valued function then we can prove that:

\[
\left\| (\rho, m)(t, \cdot) \right\|_{C^k} \leq a_k(t, \left\| \rho_0 \right\|_{C^k}, \left\| m_0 \right\|_{C^k}),
\]

\[
\left\| (\rho, m)(t, \cdot) \right\|_{H^k} \leq b_k(t, \left\| \rho_0 \right\|_{H^k}, \left\| m_0 \right\|_{C^k})
\]

with \(a_k\) and \(b_k\) denote finite valued functions depending on \(C(\cdot)\). This estimates are quite standard and are obtained by energy estimates and Gronwall lemma (see [9] for more details).

In view of justifying rigorously this last estimates we would need to work with regular solution on a finite time interval, it would be judicious to introduce a truncated viscosity coefficient \(\mu_n(\rho)\):

\[
\mu_n(\rho) = \max(\rho, \frac{1}{n}),
\]

then there exists approximated solutions \((\rho_n, v_n)\) defined for small time \((0, T_n)\) of the system (1.3). Finally it is sufficient to apply the estimates (3.35) and to conclude by a blow-up argument in order to prove the global existence of \((\rho_n, v_n)\). The last step consists in passing to the limit when \(n\) goes to \(\infty\) and to verify that the limit \((\rho, v)\) is solution of our problem.

In order to prove theorem 2, the main difficulty corresponds to obtain the estimate (3.34) which is the key point to obtain estimates (3.35). Following [9, 29] we have for any pair of entropy flux \((\eta(\rho_n, v_n), H(\rho_n, u_n))\) defined by (2.18) and (2.19) where \(\eta\) is a convex function of \((\rho_n, m_n)\) the following equality where we write \(\eta = \bar{\eta}(\rho_n, m_n)\):

\[
\partial_t \eta + \partial_x H = \epsilon \bar{\eta}_\rho \partial_{xx} \rho_n + \epsilon \bar{\eta}_m \partial_{xx} m_n,
\]

\[
= \epsilon \partial_{xx} \eta - \epsilon (\bar{\eta}_{\rho \rho}(\partial_x \rho_n)^2 + 2 \bar{\eta}_{\rho m}(\partial_x \rho_n)(\partial_x m_n) + \bar{\eta}_{mm}(\partial_x m_n)^2).
\]

Here we define \(\mu_n\) such that:

\[
\mu_n = \bar{\eta}_{\rho \rho}(\partial_x \rho_n)^2 + 2 \bar{\eta}_{\rho m}(\partial_x \rho_n)(\partial_x m_n) + \bar{\eta}_{mm}(\partial_x m_n)^2
\]

By proposition 2.1, we can check that \(\mu_n \geq 0\). We obtain then that:

\[
\partial_t \eta(\rho_n, v_n) + \partial_x H(\rho_n, v_n) - \epsilon \bar{\eta}_\rho \partial_{xx} \rho_n \leq 0 \text{ in } \mathbb{R} \times (0, +\infty).
\]

By applying the same method as for proving the theorem 2.3, we obtain the following kinetic formulation:

\[
\partial_t \chi + \partial_x [\theta \xi + (1 - \theta)v_n \chi] - \partial_{xx} \chi = \partial_t \bar{m}_n \text{ on } \mathbb{R}^2 \times (0, +\infty),
\]
where \( \bar{m}_n \) is a nonpositive bounded measure on \( \mathbb{R}^2 \times (0, +\infty) \). Finally we recover the classical maximum principle by multiplying (3.36) by the convex functions \( g(\xi) = (\xi - \xi_0)_+ \) and \( g(\xi) = (\xi - \xi_0)_- \) and integrating over \( \mathbb{R}^2 \times (0, +\infty) \). Indeed we have that:

\[
-C \leq \min_x (v_0 - \rho_0) \leq \max_x (v_0 + \rho_0) \leq C,
\]

and that:

\[
\text{supp}\xi = [v - \rho^\theta, v + \rho^\theta].
\]

for \( \xi_0 \) large enough, we can show that:

\[
\text{supp}\xi_0 \cap \text{supp}\chi = \emptyset.
\]

We have obtain then that:

\[
-C \leq \min_x (v_0 - \rho_0) \leq v_n - \rho_n \leq v_n + \rho_n \leq \max_x (v_0 + \rho_0) \leq C.
\]

This is nothing else than an application of the Riemann invariants. In particular we have shown that \( \rho_n \) and \( v_n \) are uniformly bounded in \( L^\infty(0, T_n, L^\infty(\mathbb{R})) \) or :

\[
\sup_{x \in \mathbb{R}, t \in (0, T_n)} (|\rho_n(t, x)| + |v_n(t, x)|) \leq C_0,
\]

with \( C_0 \) depending on the initial data \( (\rho_0, v_0) \). The last step in order to prove the estimate (3.34) is reduced to satisfy that \( \frac{1}{\rho_n} \) is bounded locally in time in \( L^\infty \) norm. To do this we just have to consider the first equation of (3.32) and to apply the maximum principle in taking into account the fact that \( v_n \) is uniformly bounded in \( L^\infty(\mathbb{T}) \) with \( T > 0 \). It achieves the proof of theorem 1.

4 Proof of theorem 1.1

4.1 Uniform estimates for the solutions of (1.3)

Let \( (\rho^\epsilon, v^\epsilon) \) be the global solutions of the Korteweg system (1.3) constructed by theorem 1 and satisfying:

\[
\rho^\epsilon(t, x) \geq c^\epsilon(t), \text{ for some } c^\epsilon(t) > 0,
\]

and

\[
\lim_{x \to \pm\infty} (\rho^\epsilon, v^\epsilon)(x, t) = (\rho^\pm, u^\pm).
\]

Here we are working around a non constant state \( (\bar{\rho}, \bar{v}) \) with :

\[
\lim_{x \to \pm\infty} (\bar{\rho}, \bar{v})(x, t) = (\rho^\pm, u^\pm).
\]

It is a simple extension of theorem 1. Our goal is now to check the properties (1.11), (1.12) and (1.13) in order to use the theorem 1.2 of Chen and Perepelitsa (see [7]) and prove theorem 1.1.

For more simplicity, we will drop the \( \epsilon \) and denote \( (\rho, v) = (\rho^\epsilon, v^\epsilon) \) and \( C > 0 \) will still denote a constant independent of \( \epsilon \).

As in [7] and [21] we start with the energy estimates for systems (1.1) and (1.4), indeed thanks to the capillarity and the introduction of the effective velocity we obtain better estimates. we refer to [15] for more details about the effective multidimensional velocity.
Lemma 3 Assume that $E_1[\rho_0, u_0] \leq E_0 < +\infty$ for some $E_0 > 0$ independent of $\epsilon$, then there exists $C(t) > 0$ depending on $E_0, t, \bar{\rho}$, and $\bar{u}$ but not on $\epsilon$ such that:

$$\sup_{0 \leq \tau \leq t} E_1[\rho, u](\tau) + \epsilon \int_0^t \int_{\mathbb{R}} \rho u_x^2 dx d\tau \leq C(t), \quad (4.40)$$

and:

$$\sup_{0 \leq \tau \leq t} E_2[\rho, v](\tau) + \epsilon \int_0^t \int_{\mathbb{R}} \rho v_x^2 dx d\tau + a \gamma \epsilon \int_0^t \int_{\mathbb{R}} \rho^{\gamma-2} \rho_x^2 dx d\tau \leq C(t). \quad (4.41)$$

Proof: The method is the same as in [7] and [21] and we will only point out what changes. Let us first deal with $E_1$ (the gradient $\nabla \eta^*$ is intended relatively to $(\rho, m)$):

$$\frac{d}{dt} E_1[\rho, u] = \int_\mathbb{R} (\partial_t \eta^*(\rho, pu) - \nabla \eta^*(\bar{\rho}, \bar{p}u)(\partial_t \rho, \partial_t (pu))) dx + \frac{\epsilon^2}{2} \int_\mathbb{R} \partial_t \left( \frac{(\rho x)^2}{\rho} \right) dx.$$

Then, thanks to the following entropy pair:

$$\eta^*(\rho, m) = \frac{m^2}{2\rho} + \frac{a}{\gamma - 1} \rho^{\gamma}, \quad H^*(\rho, m) = \frac{m^3}{2\rho^{\gamma}} + m \frac{a \gamma}{\gamma - 1} \rho^{\gamma-1},$$

we obtain:

$$\partial_t(\eta^*(\rho, m)) + \partial_x(H^*(\rho, m)) = 2\epsilon u \partial_x(\rho \partial_x u) + \epsilon^2 u \partial_x(\rho_{xx} - \frac{1}{\rho}(\rho_x)^2).$$

as the terms purely in $(\rho, m)$ and whose coefficient is $\epsilon^2$ neutralize each other, we have:

$$\frac{d}{dt} E_1[\rho, u] + 2\epsilon \int_\mathbb{R} \rho(\partial_x u)^2 dx = H^*(\rho^+, m^+) - H^*(\rho^-, m^-) - \int_\mathbb{R} \nabla \eta^*(\bar{\rho}, \bar{p}u)(\partial_t \rho, \partial_t (pu)) dx$$

We have to estimate the last integral:

$$- \int_\mathbb{R} \nabla \eta^*(\bar{\rho}, \bar{u})(\partial_t \rho, \partial_t (pu)) dx = C$$

$$+ \int_\mathbb{R} \left( (\bar{u} \partial_x \bar{u} - a \gamma \rho^{\gamma-2} \partial_x \bar{\rho}) \rho u + \partial_x \bar{u}(-\rho u^2 + 2\rho \partial_x u - a \rho \gamma) + \epsilon^2 (\partial_x x \rho - \frac{1}{\rho}(\rho_x)^2) \right) dx. \quad (4.42)$$

As it involves derivatives of $\bar{u}$ or $\bar{\rho}$, the summation is in fact for $x \in [-L_0, L_0]$, then every term can be estimated, using lemma 1 exactly as in [7] or [21], by $C(1 + E_1(t))$, except for the following two terms:

$$\left| \int_{-L_0}^{L_0} 2\epsilon \rho \partial_x u \partial_x \bar{u} dx \right| \leq \int_{-L_0}^{L_0} C \epsilon \rho |\partial_x u| dx \leq \int_{-L_0}^{L_0} C \epsilon (C \rho + \frac{\rho}{4C} |\partial_x u|^2) dx$$

$$\leq C(1 + E_1(t)) + \epsilon \int_{\mathbb{R}} \rho |\partial_x u|^2 dx, \quad (4.43)$$
Each term can be estimated by $C$ which implies that $H$

**Remark 12**
As in [21], the estimate

$$\sum_{i=1}^{N} \text{the last two terms who can be estimated by } C$$

which is estimated the same way and this concludes for the first estimate.

Concerning $E_2$, we do the same on system (1.4) with the entropy pair $\eta^*(\rho, \rho v)$ and $H^*(\rho, \rho v)$. We obtain that (denoting $m = \rho v$):

$$\partial_t(\eta^*(\rho, m)) + \partial_x(H^*(\rho, m)) = \epsilon \eta^*_m(\rho, m) \partial_{xx}(\rho v) + \epsilon \eta^*_m(\rho, m) \partial_{xx}\rho,$$

which implies that

$$\int_{\mathbb{R}} \partial_t \eta^*(\rho, m) dx + \epsilon \int_{\mathbb{R}} \rho(v_x)^2 dx + \epsilon \int_{\mathbb{R}} a^2 \rho \gamma^{-2} (\partial_x \rho)^2 dx = H^*(\rho^-, m^-) - H^*(\rho^+, m^+).$$

and then, estimating the integral $- \int_{\mathbb{R}} \nabla \eta^*$ in the case of the effective velocity, we obtain:

$$\frac{d}{dt} E_2[\rho, v] + \epsilon \int_{\mathbb{R}} \rho(v_x)^2 dx + \epsilon \int_{\mathbb{R}} a^2 \rho \gamma^{-2} (\rho_x)^2 dx = C + \int_{-L_0}^{L_0} (\rho \bar{v}_x v(v - \bar{v}) + \bar{v}_x a \rho \gamma^{-2} \rho_x \rho v - \epsilon \bar{v}_x \rho v - \epsilon \bar{v}_x \rho_x (v - \bar{v}) - \epsilon a^2 \rho \gamma^{-2} \rho_x \rho_x) dx.$$

Each term can be estimated by $C(1 + E_2(t))$ or $C(1 + E_2(t)) + \frac{\epsilon}{4} \int_{\mathbb{R}} \rho |v_x|^2 dx$, except for the last two terms who can be estimated by $C(1 + E_1(t) + E_2(t))$, as the majoration involves $\epsilon^2 \int_{\mathbb{R}} \rho^{-1} (\rho_x)^2 dx$. That concludes the proof of the lemma.

**Remark 10**
Thanks to the capillarity we obtain that:

$$\epsilon^2 \sup_{0 \leq \tau \leq t} \int_{\mathbb{R}} \rho^{-1} (\rho_x)^2 dx + \epsilon \int_{0}^{t} \int_{\mathbb{R}} \rho \gamma^{-2} (\rho_x)^2 dx d\tau \leq C(t),$$

(4.46)

These estimates have to be compared to those in lemma 3.2 from [7] and lemma 2.2 from [21], which are obtained thanks to additional assumptions on the initial density that are not needed for the Korteweg system. Moreover, our estimates also have to be compared to those obtained in [31] for the compressible Navier-Stokes system where the viscosity is more specific.

**Remark 11**
Let us mention that there is also a regularizing effect on the density, indeed combining the inequality (4.41) and (4.40) we observe that for any $T > 0$:

$$\epsilon^{3/2} \| \sqrt{\rho} \partial_{xx} \ln \rho \|_{L^2(L^2)} \leq C.$$

(4.47)

**Remark 12**
As in [21], the estimate

$$\epsilon \int_{0}^{t} \int_{\mathbb{R}} \rho u^2_x dx d\tau \leq C(t),$$

in (4.41) is much weaker than the corresponding one in [7]. That is why lemma 4 will be more tricky to obtain.
Let us now prove the following improved integrability estimate:

**Lemma 4** Under the same conditions as lemma 3, for any $-\infty < a < b < +\infty$ and all $t > 0$, we have:

$$\int_0^t \int_a^b \rho^{\gamma+1} dxd\tau \leq C(t, a, b), \quad (4.48)$$

where $C(t) > 0$ depends on $E_0$, $a$, $b$, $\gamma$, $t$, $\bar{\rho}$, $\bar{u}$ but not on $\epsilon$.

**Remark 13** The proof follows the same ideas than in the case of compressible Navier-Stokes equations when we wish to obtain a gain of integrability on the density. We refer to [28] for more details. The proof is also inspired from Huang et al in [21].

**Proof.** Choose $\omega \in C_0^\infty(\mathbb{R})$ such that:

$$0 \leq \omega(x) \leq 1, \quad \omega(x) = 1 \text{ for } x \in [a, b], \text{ and } \text{supp}\omega = (a - 1, b + 1).$$

By the momentum equation of (1.3) and by localizing, we have

$$(P(\rho)\omega)_x = -(\rho uv\omega)_x + (P(\rho) + \rho v)\omega_x - (\rho v_x \omega)_x + \epsilon(\rho v_x \omega)_x - \epsilon \rho v_x \omega_x. \quad (4.49)$$

Integrating (4.49) with respect to space variable over $(-\infty, x)$, we obtain:

$$P(\rho)\omega = -\rho uv \omega + (\int_{-\infty}^x \rho v \omega dy)_t + \int_{-\infty}^x [(\rho uv + P(\rho))\omega_x - \epsilon \rho v_x \omega_x] dy. \quad (4.50)$$

Multiplying (4.50) by $\rho \omega$, we have

$$\rho P(\rho)\omega^2 = -\rho^2 uv \omega^2 + \epsilon \rho^2 v_x \omega^2 - (\rho \omega \int_{-\infty}^x \rho v \omega dy)_t$$

$$- (p n x \omega \int_{-\infty}^x p u \omega dy) + \rho \omega \int_{-\infty}^x [(\rho uv + P(\rho))\omega_x - \epsilon \rho v_x \omega_x] dx,$$

$$= \epsilon \rho^2 v_x \omega^2 - (\rho \omega \int_{-\infty}^x \rho v \omega dy)_t - (\rho uv \int_{-\infty}^x \rho v \omega dy) x$$

$$+ \rho \omega \omega_x \int_{-\infty}^x \rho v \omega dy + \rho \omega \int_{-\infty}^x [(\rho uv + P(\rho))\omega_x - \epsilon \rho v_x \omega_x] dx, \quad (4.51)$$

We now integrate (4.51) over $(0, t) \times \mathbb{R}$ and we get:

$$\int_0^t \int_{\mathbb{R}} p v \omega^2 dxd\tau = \int_0^t \int_{\mathbb{R}} \rho^2 v_x \omega^2 - (\rho \omega \int_{-\infty}^x \rho v \omega dy) dx$$

$$+ \int_{\mathbb{R}} (p_0 \omega \int_{-\infty}^x \rho_0 v_0 \omega dy) dx + \int_0^t \int_{\mathbb{R}} (\rho v \omega_x \int_{-\infty}^x \rho v \omega dy) dxd\tau$$

$$+ \int_0^t \int_{\mathbb{R}} [(\rho v + P(\rho))\omega_x - \epsilon \rho v_x \omega_x] dx dxd\tau. \quad (4.52)$$

Let $A = \{ x : \rho(t, x) \geq 2\rho \}$, where $\bar{\rho} = \max(\rho^+, \rho^-)$, then we have the following estimates by (4.41):

$$|A| \leq \frac{C(t)}{e^*(2\bar{\rho}, \bar{\rho})} = d(t). \quad (4.53)$$
And for any \((t, x)\) with \(x \in A\) there exists a point \(x_0 = x_0(t, x)\) such that \(|x - x_0| \leq d(t)\) and \(\rho(t, x_0) = \bar{\rho}\). Here we choose \(\beta = \frac{n+1}{2} > 0\),

\[
\sup_{x \in \bar{A}} \rho^\beta(t, x) \leq \sup_{x \in A^c} \rho^\beta(t, x) + \sup_{x \in A} \rho^\beta(t, x) \leq \epsilon (2 \rho)^\beta + \sup_{x \in A} \rho^\beta(t, x),
\]

\[
\leq 2\epsilon (2 \rho)^\beta + \sup_{x \in A} \int_{\bar{A}} |\beta| |\epsilon \rho^\beta-1(t, x)\rho_x| dx,
\]

\[
\leq 2\epsilon (2 \rho)^\beta + \sup_{A^c} \left( \int_{\bar{A}} \int_{x_0(t, x) - d(t)} \rho^\beta dx + \int_\mathbb{R} \epsilon \rho^\beta \rho_x^2 dx \right),
\]

\[
\leq C(t) + |\beta| \sup_{x \in A} \left( \int_{x_0(t, x) - d(t)} C(\bar{\rho})(1 + e^\ast(\rho, \bar{\rho})) dx, \right.
\]

\[
\leq C(t).
\]

Using (4.54), Young inequalities and Hölder’s inequalities, the first term on the right hand side of (4.52) is treated as follows:

\[
\epsilon \int_0^t \int_\mathbb{R} \rho^2 v_x \omega^2 dx d\tau
\]

\[
\leq \frac{1}{2} \epsilon \int_0^t \int_\mathbb{R} \rho^3 \omega^4 dx d\tau + \frac{1}{2} \epsilon \int_0^t \int_\mathbb{R} \rho v_x^2 dx d\tau,
\]

\[
\leq C(t) + \epsilon \int_0^t \int_\mathbb{R} \rho^3 \omega^4 dx d\tau,
\]

In order to estimate the last integral, we have to distinguish two cases for \(\gamma > 1\):

- If \(\gamma \geq 3\), then there exists \(\theta_0 \in [0, 1]\) such that \(3 = (1 - \theta_0).0 + \theta_0.\gamma\), which implies by the Young inequalities that \(\rho^\beta \leq 1 + \rho^\gamma\) and then:

\[
\epsilon \int_0^t \int_\mathbb{R} \rho^3 \omega^4 dx d\tau \leq C \epsilon \int_0^t \int_{a-1}^{b+1} (1 + \rho^\gamma) dx d\tau \leq C(t, a, b).
\]

- If \(\gamma \in [1, 3]\) then (with \(\beta = \frac{n+1}{2}\)) we can write \(\epsilon \rho^\beta = \epsilon \rho^\beta \rho^{3-\beta}\) so that:

\[
\epsilon \int_0^t \int_\mathbb{R} \rho^3 \omega^4 dx d\tau \leq \int_0^t \int_\mathbb{R} C(t) \rho^{3-\beta} \omega^4 dx d\tau.
\]

As \(\beta = \frac{n+1}{2}\), we have \(1 < 3 - \beta < 2 < \gamma + 1\), so there exists some \(\theta_1 \in ]0, 1[\) such that \(3 - \beta = (1 - \theta_1).1 + \theta_1(\gamma + 1)\) and thanks again to the Young estimates (and as \(\omega\) is bounded by 1):

\[
C(t) \omega^4 \rho^{3-\beta} = \left( \frac{a \rho^{\gamma+1} \omega^2}{2} \right) \theta_1 \rho^{1-\theta_1} C(t) \omega^{4-2\theta_1} \left( \frac{2}{a} \right)^{\theta_1}
\]

\[
\leq \frac{a \rho^{\gamma+1} \omega^2}{2} + C(t)^{\frac{1}{1-\theta_1}} \left( \frac{2}{a} \right)^{\frac{\theta_1}{1-\theta_1}} \rho,
\]

\[
(4.56)
\]

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which gives that
\[ \epsilon \int_0^t \int_\mathbb{R} \rho^3 \omega^4 x dx d\tau \leq C_1(t) \int_0^t \int_{a-1}^{b+1} \epsilon p dx d\tau + \frac{a}{2} \int_0^t \int_\mathbb{R} \rho^{\gamma+1} \omega^2 x dx d\tau. \]

The last term is absorbed by the left-hand side and the other is estimated as usual (thanks to lemma 1) by \( C(t, \bar{\rho}, a, b) \).

So we obtain that in each case :
\[ \epsilon \int_0^t \int_\mathbb{R} \rho^2 \omega^2 x dx d\tau \leq C(t, \bar{\rho}, a, b, \bar{\rho}^\gamma \omega^2). \]  

By lemma 3 and the H"older inequality, we obtain
\[ \left| \int_0^x \rho \omega dy \right| \leq \left( \int_{\text{supp}(\omega)} |\rho| dy \right)^\frac{1}{2} \left( \int_{\text{supp}(\omega)} \rho^2 dy \right)^\frac{1}{2} \leq C(t). \]

Then :
\[ \left| \int_\mathbb{R} \left( \rho \int_{-\infty}^x \rho \omega dy \right) dx \right| + \left| \int_\mathbb{R} \left( \rho_0 \omega \int_{-\infty}^x \rho_0 \omega dy \right) dx \right| \]
\[ + \left| \int_0^t \int_\mathbb{R} \left( \rho \omega \int_{-\infty}^x \rho \omega dy \right) dx d\tau \right| \leq C(t). \]

Similarly, we have :
\[ \left| \int_0^t \int_\mathbb{R} \left( \rho \omega \int_{-\infty}^x (\rho v + P(\rho)) \omega dy \right) dx d\tau \right| \leq C(t), \]

and
\[ \epsilon | \int_0^t \int_\mathbb{R} \left( \rho \omega \int_{-\infty}^x \rho v_x \omega dy \right) dx d\tau | \leq \epsilon | \int_0^t \int_\mathbb{R} \left( \rho \omega \int_\mathbb{R} \rho |v| |\omega| dy \right) dx d\tau |, \]
\[ \leq \epsilon \left( \int_0^t \int_\mathbb{R} \rho \omega dx \right) \left( \int_\mathbb{R} \rho v_2^2 dy + \int_\mathbb{R} \rho v_2^2 dy \right) d\tau \leq C(t). \]

Substituting (4.55), (4.1), (4.59)-(4.61) into (4.52) achieves the proof of lemma 4.

Let us now turn to the last result of integrability gain on the density :

**Lemma 5** Assume that \((\rho_0, v_0)\) satisfy the conditions from lemma 3 and that there exists \(M_0 > 0\) independent of \(\epsilon\), such that
\[ \int_\mathbb{R} \rho_0(x) |v_0(x) - \bar{v}(x)| dx \leq M_0 < +\infty, \]

then for any compact set \(K \subset \mathbb{R}\), there exists \(C(t, K)\) independent of \(\epsilon\) such that :
\[ \int_0^t \int_K (\rho^{\gamma+\theta} + \rho |v|^3) dx d\tau \leq C(t, K), \]
Remark 14 As in [7] and [21], in order to prove the inequality (4.63), we will use the same ingredients as in [30] where this inequality was obtained for the first time.

Proof. As in [7] we work with the function \( \psi(s) = \frac{1}{2}|s|s \) of proposition 2.2. If we consider \( \eta_{\nu}(\rho) \) as a function depending of \((\rho, v)\), we have for all \( \rho \geq 0 \) and \( v \in \mathbb{R} \):

\[
\begin{aligned}
|\eta_{\nu m}(\rho, v)| &\leq C, \\
|\eta_{\nu m}(\rho, v)| &\leq C \rho^{\theta-1}.
\end{aligned}
\]  

(4.64)

For this weak entropy pair \((\eta^\nu, H^\nu)\), we observe that:

\[
\begin{aligned}
\eta^\nu(\rho, 0) &= \eta^\nu_\rho(\rho, 0) = 0, \\
H^\nu(\rho, 0) &= \frac{\theta}{2} \rho^{3\theta+1} \int_{\mathbb{R}} |s| [1 - s^2]^\lambda, \\
\end{aligned}
\]

and:

\[
\begin{aligned}
\eta_{\nu m}(\rho, 0) &= \beta \rho^\theta \text{ with } \beta = \int_{\mathbb{R}} |s| [1 - s^2]^\lambda ds.
\end{aligned}
\]

By Taylor formula, we have:

\[
\eta^\nu(\rho, m) = \beta \rho^\theta m + r(\rho, m),
\]

(4.65)

with, for some constant \( C > 0 \):

\[
r(\rho, m) \leq C \rho v^2,
\]

(4.66)

Now we introduce a new entropy pair \((\widehat{\eta}, \widehat{H})\) such that,

\[
\begin{aligned}
\widehat{\eta}(\rho, m) &= \eta^\nu(\rho, m - \rho v^-), \\
\widehat{H}(\rho, m) &= H^\nu(\rho, m - \rho v^-) + v^- \eta^\nu(\rho, m - \rho v^-),
\end{aligned}
\]

with \( m = \rho v \), which satisfies:

\[
\begin{aligned}
\eta(\rho, m) &= \beta \rho^{\theta+1}(v - v^-) + r(\rho, \rho(v - v^-)), \\
r(\rho, \rho(v - v^-)) &\leq C \rho(v - v^-)^2,
\end{aligned}
\]

(4.67)

and:

\[
\begin{aligned}
\widehat{H}_\rho &= v \widehat{\eta}_\rho + a \gamma \rho^{-2} \widehat{\eta}_v, \\
\widehat{H}_v &= \rho \widehat{\eta}_\rho + v \widehat{\eta}_v.
\end{aligned}
\]

Computing \((1.3)_1 \times \widehat{\eta}_\rho + (1.3)_2 \times \widehat{\eta}_v\) (recall that \( \widehat{\eta}_v = \rho \widehat{\eta}_m \)) we get:

\[
\begin{aligned}
\partial_t(\widehat{\eta}(\rho, v)) + \partial_x(\widehat{H}(\rho, v)) &= \epsilon \widehat{\eta}_\rho(\rho, v) \partial_{xx} \rho + \epsilon \widehat{\eta}_v(\rho, v) \partial_{xx} v + 2 \epsilon \widehat{\eta}_m(\rho, v) \rho_x v_x \\
&= \epsilon \widehat{\eta}_\rho(\rho, v) \partial_{xx} \rho - \epsilon \widehat{\eta}_v(\rho, v) \partial_{xx} v + 2 \epsilon \widehat{\eta}_m(\rho, v) \partial_x (\rho v_x). \quad (4.68)
\end{aligned}
\]

Integrating over \((0, t) \times (-\infty, x)_\), we have:

\[
\begin{aligned}
\int_{-\infty}^{x} (\widehat{\eta}(\rho, m) - \widehat{\eta}(\rho_0, m_0)) dy + \int_{0}^{t} \widehat{H}(\rho, v) d\tau &= t H^\nu(\rho^0, 0) + \epsilon \int_{0}^{t} (\widehat{\eta}_\rho \rho_x + \widehat{\eta}_v v_x) d\tau \\
&- \epsilon \int_{0}^{t} \int_{-\infty}^{x} (\widehat{\eta}_{\rho \rho}(\rho_x)^2 + \widehat{\eta}_{\mu \nu}(v_x)^2 + 2 \widehat{\eta}_{\mu \rho \nu} \rho_x v_x) dy d\tau.
\end{aligned}
\]

(4.69)
By using (4.64), we obtain as in [7] or [21]:

\[ |\epsilon \int_0^t \int_{-\infty}^x \widehat{\eta}_{\rho \rho |\rho |^2} \, dy \, d\tau| 
\leq C \epsilon \int_0^t \int_{\mathbb{R}} \rho v_x^2 \, dy \, d\tau \leq C(t), \quad (4.70) \]

\[ |\epsilon \int_0^t \int_{-\infty}^x \widehat{\eta}_{\rho \rho \rho \rho} \rho \rho v_x^2 \, dy \, d\tau| 
\leq C \epsilon \int_0^t \int_{\mathbb{R}} \rho^\theta \rho \rho v_x^2 \, dy \, d\tau, \quad (4.71) \]

The last term is estimated thanks to (2.4):

\[ |\epsilon \int_0^t \int_{-\infty}^x \widehat{\eta}_{\rho \rho \rho \rho} \rho \rho v_x^2 \, dy \, d\tau| \leq C \epsilon \int_0^t \int_{-\infty}^x \epsilon \rho^{\gamma - 2} \rho v_x^2 \, dy \, d\tau \leq C(t). \quad (4.72) \]

Substituting (4.70) and (4.71) into (4.69), then integrating over \( K \) and using (2.23), we obtain:

\[ \int_0^t \int_K \rho^\theta + \rho |v - v^-|^3 \, dx \, d\tau \leq C(t, K) + 2 \sup_{\tau \in [0, t]} \left| \int_{-\infty}^x \widehat{\eta}(\rho(y, \tau), (\rho v)(y, \tau)) \, dy \right| \]

\[ + C |v^-| \int_0^t \int_K |\eta^\psi(\rho, \rho(v - v^-))| \, dx \, d\tau + \int_0^t \int_K (\epsilon |\widehat{\eta}_{\rho \rho} | |\rho | v_x | + \epsilon |\widehat{\eta}_{\rho \rho} | |\rho | v_x |) \, dx \, d\tau. \]

Applying lemma 3, we have:

\[ \int_0^t \int_K |\eta^\psi(\rho, \rho(v - v^-))| \, dx \, d\tau \leq C(t, K). \quad (4.74) \]

Thanks to the estimates on the entropy pair provided by proposition 2.2, we have:

\[ \int_0^t \int_K \epsilon |\widehat{\eta}_{\rho \rho} | |\rho | v_x | \, dx \, d\tau \leq \int_0^t \int_K \epsilon \left( |v - v^-| + \rho^\theta \right) |\rho | v_x | \, dx \, d\tau \]

\[ \leq \int_0^t \int_K \epsilon \left( \rho^\theta |v - \bar{v}| + \rho |\bar{v} - v^-| \right) \rho^\theta |\rho | v_x | \, dx \, d\tau. \quad (4.75) \]

By Hölder’s inequality:

\[ \int_0^t \int_K \epsilon |\widehat{\eta}_{\rho \rho} | |\rho | v_x | \, dx \, d\tau \leq \frac{1}{2} \int_0^t \int_K \epsilon \left( |\rho | v - \bar{v}|^2 + |\rho |\bar{v} - v^-|^2 + 2|\rho | v_x |^2 + \rho^\gamma \right) \, dx \, d\tau. \]

Then, thanks to lemma 4.40, the estimates from lemma 1, and the fact that \( \bar{v} - v^- \) is bounded, we obtain:

\[ \int_0^t \int_K \epsilon |\widehat{\eta}_{\rho \rho} | |\rho | v_x | \, dx \, d\tau \leq C(t, K). \]

In order to estimate the last term, using proposition 2.4, we have (with \( \theta = (\gamma - 1)/2 \)):

\[ \int_0^t \int_K \epsilon |\widehat{\eta}_{\rho \rho} | |\rho | v_x | \, dx \, d\tau \leq C \int_0^t \int_K \epsilon (\rho^{\gamma - 1} + \rho^\theta |v - v^-|) |\rho | v_x | \, dx \, d\tau \]

\[ \leq C \int_0^t \int_K \epsilon \rho^\theta \left( \rho^\theta + \rho^\theta |v - \bar{v}| + |\bar{v} - v^-| \right) \epsilon \rho^{\frac{\gamma - 1}{2}} |\rho | v_x | \, dx \, d\tau, \quad (4.76) \]

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which is bounded the same way, thanks to Hölder’s inequality, by $C(t, K)$.

Now we are going to deal with the first term on the right hand side of (4.73). Thanks to (1.4) we have:

$$\begin{align*}
(\rho v - \rho v^-)_t + (\rho v^2 + P(\rho) - \rho w u^-)_x &= \epsilon (\rho v_x)_x.
\end{align*}$$

(4.77)

Integrating (4.77) over $[0, t] \times (-\infty, x)$ for $x \in K$, we get:

$$\begin{align*}
\int_{\infty}^{x} \rho(v - \rho v^-)dy &= \int_{-\infty}^{x} \rho_0(v_0 - \rho v^-)dy - \int_{0}^{t} (\rho v^2 + P(\rho) - \rho w u^- - P(\rho^-)) \\
&\quad + \epsilon \int_{0}^{t} \rho v_x d\tau.
\end{align*}$$

(4.78)

Furthermore:

$$\begin{align*}
| \int_{-\infty}^{x} \tilde{\eta}((\rho(y, \tau), (\rho v)(y, \tau))dy | \\
&\leq | \int_{-\infty}^{x} (\tilde{\eta}(\rho, \rho v) - \beta \rho^{\theta+1}(v - \bar{v}))dy | + | \int_{-\infty}^{x} \beta \rho^{\theta+1}(v - \bar{v})dy | \\
&\leq | \int_{-\infty}^{x} (r(\rho, \rho(v - \bar{v}))dy | + | \int_{-\infty}^{x} \beta(\rho - (\rho^-)\rho(v - \bar{v}))dy | \\
&\quad + \beta(\rho^-) | \int_{-\infty}^{x} \rho(v - \bar{v})dy |,
\end{align*}$$

(4.79)

By using (4.62), lemma 3 and 4, (4.78) and (4.79) we conclude the proof of the lemma.

4.2 $H_{loc}^{-1}(\mathbb{R}^2_+)$ Compactness

In this section we take advantage of the uniform estimates obtained in the previous section in order to prove the following lemma, which gives the $H_{loc}^{-1}(\mathbb{R}^2_+)$-compactness of the entropy dissipation measures of the sequence of solutions $(\rho^\epsilon, v^\epsilon)$ of the Korteweg system, for each entropy-entropy flux pair generated by a compactly supported function.

**Lemma 6** Let $\psi \in C_0^\infty(\mathbb{R})$ and $(\eta^\psi, H^\psi)$ be the weak entropy pair generated by $\psi$. Then for the solutions $(\rho^\epsilon, v^\epsilon)$ of system (1.3), the following sequence (with $m^\epsilon = \rho^\epsilon v^\epsilon$):

$$\eta^\psi(\rho^\epsilon, m^\epsilon)_t + H^\psi(\rho^\epsilon, m^\epsilon)_x$$

are compact in $H_{loc}^{-1}(\mathbb{R}^2_+)$

(4.80)

**Proof:** As in (4.68), a direct computation on $(1.3)_1 \times \eta^\psi_0(\rho^\epsilon, v^\epsilon) + (1.3)_2 \times \eta^\psi_0(\rho^\epsilon, v^\epsilon)$ gives (recall that $\eta^\psi_0(\rho^\epsilon, v^\epsilon) = \rho \eta^\psi_0(\rho^\epsilon, m^\epsilon)$):

$$\eta^\psi(\rho^\epsilon, m^\epsilon)_t + H^\psi(\rho^\epsilon, m^\epsilon)_x = \epsilon \eta^\psi_0(\rho^\epsilon, v^\epsilon)\rho^\epsilon_x - \epsilon \eta^\psi_0(\rho^\epsilon, v^\epsilon)v^\epsilon_x + 2\epsilon \eta^\psi_0(\rho^\epsilon, m^\epsilon)\partial_x(\rho^\epsilon v^\epsilon_x).$$

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Before estimating this, let us rewrite the right-hand side terms in a more suitable form:

$$2\epsilon \eta^\psi_m(\rho^t, m^t) \partial_x (\rho^t v^t_x) = 2\epsilon \left( (\eta^\psi_m(\rho^t, m^t) \rho^t v^t_x)_x - \rho^t \eta^\psi_m(\rho^t, m^t) \rho_x^t v^t_x - \rho^t \eta^\psi_m(\rho^t, m^t) (v^t_x)^2 \right),$$

and

$$\epsilon \left( \eta^\psi_\rho(\rho^t, v^t) \rho_x^t v^t_x - \eta^\psi_v(\rho^t, v^t) v^t_x \right) =$$

$$\epsilon \left( \eta^\psi_\rho(\rho^t, v^t) (\rho^t v^t_x)_x - \eta^\psi_v(\rho^t, v^t) (\rho^t v^t_x) - \eta^\psi_\rho(\rho^t, v^t) (\rho^t v^t_x)^2 + \eta^\psi_v(\rho^t, v^t) (v^t_x)^2 \right). \quad (4.81)$$

Let $K \subset \mathbb{R}$ be compact, using proposition 2.3 (2.28) and Hölder inequality, we get:

$$2\epsilon \int_0^t \int_K |\eta^\psi_m(\rho^t, m^t)| \rho^t (v^t_x)^2 + |\eta^\psi_\rho(\rho^t, m^t) \rho_x^t v^t_x \rho^t | \, dx \, dt \leq C \epsilon \int_0^t \int_K (\rho^t)^2 \, dx \, dt + C \epsilon \int_0^t \int_K (\rho^t)^{-2} (\rho^t_x)^2 \, dx \, dt \leq C(t, \psi, K). \quad (4.82)$$

Thanks to propositions 2.3 (2.27), and 2.4, we can estimate the same way:

$$\epsilon \int_0^t \int_K |\eta^\psi_\rho(\rho^t, v^t)| (\rho^t_x)^2 + |\eta^\psi_v(\rho^t, v^t) (v^t_x)^2 \, dx \, dt \leq C(t, \psi, K). \quad (4.83)$$

This shows that the quantity:

$$-2\epsilon \left( \rho^t \eta^\psi_\rho(\rho^t, m^t) \rho_x^t v^t_x + \rho^t \eta^\psi_m(\rho^t, m^t) (v^t_x)^2 \right) - \epsilon \eta^\psi_\rho(\rho^t, m^t) (\rho^t v^t_x)^2 + \epsilon \eta^\psi_v(\rho^t, m^t) (v^t_x)^2,$$

is bounded in $L^1([0, T] \times K)$ and thus is compact in $W^{-1, p_1}(\mathbb{R}^2_x)$, for any $1 < p_1 < 2$. Moreover as we have

$$|\eta^\psi_m(\rho^t, v^t_x)| \leq C\psi,$$

then we can write that for $p \in ]1, 2[$ (to be chosen later):

$$\int_0^t \int_K |2\epsilon \eta^\psi_m(\rho^t, m^t) \rho^t v^t_x|^p \, dx \, dt \leq C \epsilon \frac{2p}{p^p} \int_0^t \int_K \rho_x^t \left( \epsilon^{2p} \rho^t |v^t_x|^p \right) \, dx \, dt.$$

Thanks to the Hölder estimate with $r_1 = \frac{1}{1-p/2} > 1$ and $r_2 = \frac{2}{p} > 1$, we obtain

$$\|2\epsilon \eta^\psi_m(\rho^t, m^t) \rho^t v^t_x\|^p_{W^{-1, p}(K)} \leq C \epsilon \frac{2p}{1-p/2} \left( \int_0^t \int_K \rho^{\frac{p}{1-p/2}} \, dx \, dt \right)^{1-p/2} \left( \int_0^t \int_K \epsilon \rho |v^t_x|^2 \, dx \, dt \right)^{p/2}. \quad (4.85)$$

The second integral is bounded by $C(t)$, and for the first one to be bounded we need that:

$$0 \leq \frac{p/2}{1-p/2} \leq \gamma + 1,$$

which is equivalent to:

$$1 < p \leq \frac{2\gamma + 2}{\gamma + 2} < 2,$$

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and for such a $p$, we have:

$$\|2\epsilon (\eta_\rho^\psi (\rho^\epsilon, m^\epsilon) \rho^\epsilon v_x^\epsilon)_x\|_{W^{-1,p}(K)} \leq C(t, \psi) \epsilon^{\frac{1}{2}} \rightarrow 0. \quad (4.87)$$

As $\eta_\rho = \rho \eta_m$, the term $-\epsilon (\eta_\rho^\psi (\rho^\epsilon, v^\epsilon) v_x^\epsilon)_x$ is estimated the same way. Let us now turn to the last term. Thanks to proposition 2.4, we have:

$$|\eta_\rho^\psi (\rho, v) | \leq C \psi (1 + \rho^{\frac{\gamma-1}{2}}),$$

and then for $p \in ]1,2[$:

$$\int_0^t \int_K |\epsilon_\nu^\psi (\rho^\epsilon, v^\epsilon) \rho^\epsilon_x^\nu| dxd\tau \leq C \int_0^t \int_K \epsilon^p (1 + \rho^{\frac{\gamma-1}{2}}) |\rho^\epsilon_x^p| dxd\tau. \quad (4.88)$$

We first estimate the last term as previously:

$$\int_0^t \int_K \epsilon^p \rho^{\frac{\gamma-1}{2}} |\rho^\epsilon_x^p| dxd\tau = \epsilon^p \int_0^t \int_K \rho^\epsilon_x^p \left( \epsilon^2 \rho^\epsilon_x^p (\gamma-2) |\rho^\epsilon_x^p| \right) dxd\tau \leq C \epsilon^p \left( \int_0^t \int_K \rho^{\gamma-2} dxd\tau \right)^{1/2} \left( \int_0^t \int_K \epsilon^p \rho^{\gamma-2} dxd\tau \right)^{p/2}. \quad (4.89)$$

As previously the second integral is bounded by $C(t)$ and the first one is bounded if, as in (4.86)

$$1 < p \leq \frac{2\gamma + 2}{\gamma + 2} < 2,$$

and for such a $p$, we have:

$$\|\epsilon^p \rho^{\gamma-1} |\rho^\epsilon_x^p|\|_{W^{-1,p}(K)} \leq C(t, \psi) \epsilon^{\frac{1}{2}} \rightarrow 0. \quad (4.90)$$

The first term of (4.88) is less easy as we cannot use neither

$$\epsilon^2 \sup_{\tau \in [0,t]} \int_\mathbb{R} \rho^{-1} (\rho_x)^2 dx \leq C(t)$$

because the best we can have in doing so, is the following estimate

$$\int_0^t \int_K \epsilon^p |\rho^\epsilon_x^p| dxd\tau \leq \int_0^t \int_K \rho^{p/2} \left( \rho^{-p/2} \epsilon^p |\rho^\epsilon_x^p| \right) dxd\tau,$$

which uses all of the available power of $\epsilon$ (which prevents that it goes to zero as $\epsilon$ goes to zero), nor

$$\epsilon \int_0^t \int_\mathbb{R} \rho^{\gamma-2} (\rho_x)^2 dxd\tau \leq C(t)$$

because here we do not know the sign of the power of $\rho$. So basically using the energy estimates won’t give anything.

But we can use an interpolation of these two estimates which preserves the properties we need: involving a negative power of $\rho$ and a power of $\epsilon$ which is less than 2 (so that
in the end of the computation we have a positive power of $\epsilon$. Indeed, for all $\theta_0 \in [0,1]$ thanks to the energy estimates, we have:

$$
\int_0^t \int_K \epsilon^{2-\theta_0} \rho^{-1+\theta_0(\gamma-1)} (\rho_x)^2 dx d\tau \leq C(t).
$$

(4.91)

To satisfy both of the needed properties we have to choose $\theta_0 > 0$ such that $-1 + \theta_0(\gamma - 1) < 0$. This is satisfied when $\theta_0 \in [0, \frac{1}{\gamma-1}]$. For such a choice of $\theta_0$, we can go back to estimate (4.88):

$$
\int_0^t \int_K \epsilon^p |\rho_x|^2 dx d\tau = \epsilon^{\theta_0} \int_0^t \int_K \rho^{1/2} (1-\theta_0(\gamma-1)) \left( \epsilon^\frac{\theta_0}{p} (2-\theta_0) \rho^{\frac{\theta_0}{p} (1-\theta_0(\gamma-1)) |\rho_x|^p \right) dx d\tau.
$$

(4.92)

Thanks to the same Hölder estimate as before, we obtain that:

$$
\int_0^t \int_K \epsilon^p |\rho_x|^2 dx d\tau \leq \epsilon^{\theta_0} \left( \int_0^t \int_K \rho^{1/p/2} (1-\theta_0(\gamma-1)) dx d\tau \right)^{1-p/2} \times \left( \int_0^t \int_K \epsilon^{2-\theta_0} \rho^{-1+\theta_0(\gamma-1)} |\rho_x|^2 dx d\tau \right)^{p/2},
$$

(4.93)

which we can bound by $\epsilon^{\theta_0} C(t,K)$ provided that we have:

$$
0 \leq \frac{p\gamma/2}{1-p/2} (1-\theta_0(\gamma-1)) \leq \gamma + 1,
$$

(4.94)

**Remark 15** Notice that condition (4.86) implies the condition (4.94).

Finally, if $\theta_0 \in [0, \frac{1}{\gamma-1}]$ and for $p \in [1, \frac{2\gamma+2}{\gamma+1}] \subset [1,2]$, we have:

$$
\|e(\eta_\rho^p(\rho', m') \rho' \rho_x') x \|_{W^{-1,p}(K)} \leq C(t, \psi) \epsilon^{\frac{\theta_0}{2}} \epsilon \rightarrow 0.
$$

(4.95)

Which finally implies that

$$
\|2e(\eta_\rho^p(\rho', m') \rho' \rho_x') x + e(\eta_\rho^p(\rho', m') \rho' \rho_x') x \|_{W^{-1,p}(K)} \leq C(t, \psi) \epsilon^{\frac{\theta_0}{2}} \epsilon \rightarrow 0.
$$

(4.96)

so that it is confined in a compact subset of $W^{-1,p}_{\text{loc}}$. Using (4.85) and (4.96), we obtain

$$
\eta_\rho^p(\rho', m') x + H_\rho^p(\rho', m') x \text{ are compact in } W^{-1,p_1}_{\text{loc}}(\mathbb{R}^2_+) \text{ for some } 1 < p_1 < 2.
$$

(4.97)

Furthermore by (2.25)-(2.26), lemma 3-4 and (4.63), we have:

$$
\eta_\rho^p(\rho', m') x + H_\rho^p(\rho', m') x \text{ are uniformly bounded in } L^p_{\text{loc}}(\mathbb{R}^2_+) \text{ for } p_3 > 2.
$$

(4.98)

where $p_3 = \gamma + 1 > 2$ when $\gamma \in (1,3]$, and $p_3 = \frac{\gamma+\theta}{1+\theta} > 2$ when $\gamma > 3$. By interpolation we conclude the proof of the lemma 6.
5 Proof of theorem 1.1

From lemmas 3, we have verified the conditions (i)-(iii) of theorem 1.2 for the sequence of solutions \((\rho^\epsilon, m^\epsilon)\). Using theorem 1.2, there exists a subsequence \((\rho^\epsilon, m^\epsilon)\) and a pair of measurable functions \((\rho, m)\) such that

\[
(\rho^\epsilon, m^\epsilon) \to (\rho, m), \quad \text{a.e } \epsilon \to 0.
\]

(5.99)

It is easy to check that \((\rho, m)\) is a finite-energy entropy solution \((\rho, m)\) to the Cauchy problem (1.6) with initial data \((\rho_0, \rho_0 v_0)\) for the isentropic Euler equations with \(\gamma > 1\). Indeed by the lemma 2.30 and the estimate (1.11) we prove that \(\rho^\epsilon\) strongly converges up to a subsequence to \(\rho\) in \(L^{\gamma+1}_{{\text{loc}}} (\mathbb{R}^+ \times \mathbb{R})\) for any \(0 < \alpha < 1\). Similarly in setting \(v(t, x) = \frac{m(t,x)}{\rho(t,x)}\) when \(\rho(t, x) \neq 0\) and \(v(t, x) = 0\) almost everywhere in \(\{\rho(t, x) = 0\}\). By using the lemma 2.30 and the estimate (1.12) we obtain that \((\rho^\epsilon)^{\frac{1}{3}} v^\epsilon\) strongly converges up to a subsequence to \(\rho^\frac{1}{3} v\) in \(L^{3-\alpha}_{{\text{loc}}} (\mathbb{R}^+ \times \mathbb{R})\) for any \(0 < \alpha\) small enough. And finally \((\rho^\epsilon)^{\frac{1}{2}} v^\epsilon\) strongly converges up to a subsequence to \(\rho^\frac{1}{2} v\) in \(L^{\frac{3+\alpha}{2}}_{{\text{loc}}} (\mathbb{R}^+ \times \mathbb{R})\) for any \(0 < \alpha\) small enough, it suffices to use the two previous strong convergence and the H"older’s inequality. It achieves the proof of theorem 1.1.

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Références


