Geometrical methods for Equations of Hydrodynamical Type

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Outline

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   - Geodesics on diffeomorphisms groups
   - From the PDE to the geodesic equations

2. Local existence of the geodesics
   - Hilbert approximation manifolds
   - Smoothness of the metric and the spray
   - Sketch of proof

3. Geometric considerations
   - Local existence of geodesics in $\text{Diff}^\infty(\mathbb{S}^1)$
   - The minimization problem
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History

- It is known since Euler (1765), that the free motions of a rigid body correspond to the geodesics of a left invariant metric on the Euclidean group (the kinetic energy).

- In a famous paper (1966), Arnold has recast the evolution equations of an ideal fluid (with fixed boundary) as the geodesic equations for a right-invariant metric on the group of volume-preserving diffeomorphisms of the domain.

- This general approach is valid as soon as the configuration space of a Lagrangian system can be given the structure of a Lie group. It was applied successfully to many one-dimensional models in hydrodynamics.
A right-invariant metric on $\text{Diff}^{\infty}(\mathbb{S}^1)$ is defined by an inner product on $C^\infty(\mathbb{S}^1)$, the tangent space at $id$, and then translated on each tangent space.

In the sequel, this inner product is given by

$$< u, v > := \int_{\mathbb{S}^1} (Au)v \, dx, \quad u, v \in C^\infty(\mathbb{S}^1),$$

where $A$ (the inertia operator) is an invertible, symmetric, continuous, linear operator on $C^\infty(\mathbb{S}^1)$, which commutes with $D := d/dx$ (Fourier multiplier).
Euler equations on $\text{Diff}^\infty(S^1)$

The corresponding Euler equation on $C^\infty(S^1)$ is given by

$$u_t = -A^{-1} \{ (Au)_x u + 2(Au)u_x \}.$$

- For $A = I$ ($L^2$ metric), we get the **inviscid Burgers equation**

  $$u_t + 3uu_x = 0.$$

- For $A = I - D^2$ ($H^1$ metric), we get the **dispersionless Camassa-Holm equation**

  $$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0.$$
Euler equations on $\text{Diff}^\infty(S^1)/S^1$

- When $\ker A = \mathbb{R}$, the theory is still meaningful, provided we restrict to the subgroup, $\text{Diff}_1(S^1)$, of diffeomorphisms which fix one chosen point on the circle.
- For $A = D^2 (\dot{H}^1$ metric), we get the Hunter-Saxton equation
  \[ u_{xxt} + uu_{xxx} + 2u_xu_{xx} = 0. \]
- For $A = H D (\dot{H}^{1/2}$ metric), where $H$ is the Hilbert transform, we get the modified Constantin-Lax-Majda equation
  \[ H u_{xt} + u H u_{xx} + 2u_x H u_x = 0. \]
Euler equations on the Bott-Virasoro group

- The Bott-Virasoro group is an extension of $\text{Diff}^\infty(S^1)$ by $\mathbb{R}$. Euler equations for this group, are given by
  \[
  u_t = -A^{-1} [u(Au)_x + 2(Au)u_x - au_{xxx}], \quad a \in \mathbb{R}.
  \]

- For $A = I$ ($L^2$ metric), we get the Korteweg-de Vries equation
  \[
  u_t + 3uu_x - au_{xxx} = 0.
  \]

- For $A = I - D^2$ ($H^1$ metric), we get the general Camassa-Holm equation
  \[
  u_t - u_{txx} + 3uu_x - 2u_xu_{xx} - uu_{xxx} - au_{xxx} = 0.
  \]
Analytical consequence

- This geometric approach has been developed in the 70’ by Ebin-Marsden from an analytical point of view.
- Following their approach, if we can prove local existence and uniqueness of geodesics (ODE) on diffeomorphism groups then the PDE (Euler equation) is well-posed.
- The rich framework of classical Riemannian geometry can then be used to understand the geometric behaviour of solutions: for instance, sectional curvatures can be used to detect sensitivity to initial data, ...
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Retrieving the Lagrangian Formulation

- Let $\varphi(t)$ be the flow of the time dependent vector field $u(t)$, i.e. $\varphi$ solves $\varphi_t = u(t, \varphi)$, and let $\nu = \varphi_t$.

- Then $u(t)$ is a solution of the Euler equation

\[ u_t = -A^{-1} \{(Au)_x u + 2(Au)u_x\} \]

iff $(\varphi(t), \nu(t))$ is a solution of the second-order system, called the geodesic spray

\[ \begin{cases} 
\varphi_t = \nu, \\
\nu_t = R_{\varphi} \circ S \circ R_{\varphi^{-1}}(\nu),
\end{cases} \]

where $R_{\varphi}(u) := u \circ \varphi$, and

\[ S(u) := A^{-1} \{[A, u]u_x - 2(Au)u_x\}. \]
A fundamental observation

- The Euler equation

\[
u_t = -A^{-1} \{(Au)_x u + 2(Au)u_x\}
\]

is not an ODE on any reasonable Banach space of functions, due to the loss of order in the term

\[A^{-1} \{(Au)_x u\}.
\]

- However, if \(A\) is a differential operator of order \(r \geq 1\) then the quadratic operator

\[
S(u) := A^{-1} \{[A, u]u_x - 2(Au)u_x\}
\]

appearing in the spray is of order 0 because the commutator \([A, u]\) is of order \(\leq r - 1\).
Well-posedness for non-local inertia operators

- For this reason, the geometric theory on $\text{Diff}^\infty(S^1)$ has been applied successfully when $A$ is a differential operator: for instance the inertia operator for the $H^k$ metrics ($k \in \mathbb{N}$).

- One might expect, that for a larger class of inertia operators $A$, the quadratic operator $S$ to be of order 0 and the spray to induced a well defined ODE on suitable Hilbert approximation of the Fréchet manifold $\text{Diff}^\infty(S^1)$.

- We shall show that the theory extends well for a large class of (even singular) Fourier multipliers, including the important case of $H^s$ metrics ($s \in \mathbb{R}$), the Constantin-Lax-Majda equation and the Euler-Weil-Petersson equation.
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The group of diffeomorphisms of class $H^q$

- The Fréchet manifold $\text{Diff}^\infty(S^1)$ may be viewed as an inverse limit of Hilbert manifolds (ILH)

$$\text{Diff}^\infty(S^1) = \bigcap_q D^q(S^1),$$

where $D^q(S^1)$ is the set of $C^1$-diffeomorphisms which are of class $H^q$.

- For $q > 3/2$, $D^q(S^1)$ has the structure of a Hilbert manifold (modelled on $H^q(S^1)$) and a topological group.

- But $D^q(S^1)$ is not a Lie group: composition and inversion in $D^q(S^1)$ are continuous but not differentiable.

- However, for each $\varphi \in D^q(S^1)$, the right translation $R_\varphi$ is a smooth diffeomorphism of $D^q(S^1)$. 
Extending the metric to Hilbert approximations

If the inertia operator \( A \) is of finite order \( r \geq 0 \), the metric may be extended to define a continuous family of positive inner products on each tangent space.

\[
\langle v_1, v_2 \rangle_\varphi = \int_{S^1} v_1(A_\varphi v_2) \varphi_x \, dx.
\]

where \( A_\varphi := R_\varphi \circ A \circ R_{\varphi^{-1}} \).

**Difficulty 1**

**Is the metric smooth?** Even if \( A \) is a bounded operator of order \( r \), one cannot conclude directly that the mapping

\[
(\varphi, v) \mapsto A_\varphi(v) := R_\varphi \circ A \circ R_{\varphi^{-1}}(v)
\]

is smooth from \( \mathcal{D}^q(S^1) \times H^q(S^1) \) to \( H^{q-r}(S^1) \), because \( (\varphi, v) \mapsto R_\varphi(v) \) is not differentiable.
Weak versus strong metrics

**Definition**
A metric $g$ on a Banach manifold is **strong** if the Legendre map

$$P : T_qQ \rightarrow T^*_qQ, \quad v_q \mapsto g(v_q, \cdot)$$

is an isomorphism. It is **weak** if this map is only injective.

**Difficulty 2**
In the case we consider, the Legendre map

$$v \mapsto \varphi_x A_\varphi v, \quad T_\varphi \mathcal{D}^q(S^1) \cong H^q(S^1) \rightarrow H^{q-r}(S^1) \subsetneq T^*_\varphi \mathcal{D}^q(S^1)$$

is only injective (**weak metric**), and the geodesic spray may not exist on the Hilbert approximation manifolds! e.g. Burgers and (KdV).
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Smoothness of the spray

On \( T^D q(S^1) \), if the metric is smooth and the inertia operator is of order \( r \geq 1 \), existence and smoothness of the spray on \( T^D q(S^1) \) (Difficulty 2) result from the following lemma.

**Lemma**

Let \( A \) be a Fourier multiplier of order \( r \geq 1 \) and \( q \in \left( \frac{3}{2} + r, \infty \right) \). Suppose that \( A \) induces an isomorphism from \( H^q(S^1) \) onto \( H^{q-r}(S^1) \) and that the mapping

\[
\varphi \mapsto A \varphi = R \varphi \circ A \circ R \varphi^{-1}, \quad \mathcal{D}^q(S^1) \rightarrow \mathcal{L}(H^q(S^1), H^{q-r}(S^1))
\]

is smooth. Then the spray

\[
F(\varphi, v) : (\varphi, v) \mapsto S(\varphi)(v), \quad \mathcal{D}^q(S^1) \times H^q(S^1) \rightarrow H^q(S^1),
\]

where \( S(u) = A^{-1} \{ [A, u]u_x - 2(Au)u_x \} \) is smooth.
Smoothness of the metric

The smoothness of the metric (Difficulty 1) reduces to the following question:

Problem

Given a Fourier multiplier $A$ of order $r \geq 0$, under which conditions is the mapping

$$
\varphi \mapsto A_\varphi := R_\varphi \circ A \circ R_{\varphi^{-1}}, \quad D^q(S^1) \to \mathcal{L}(H^q(S^1), H^{q-r}(S^1))
$$

smooth?

Remark

If $A$ is a differentiable operator, then $\varphi \mapsto A_\varphi$ is real analytic for $q > r + 1/2$. Indeed, $A_\varphi$ is a linear differential operator whose coefficients are polynomial expressions of $1/\varphi_x$ and the derivatives of $\varphi$ up to order $r$ (e.g. $D_\varphi = (1/\varphi_x)D$).
A larger class of operators for which $A_\varphi$ is smooth

We have formulated a criteria on the symbol of $A$ which ensures that the map $\varphi \mapsto A_\varphi$ is smooth. It may not be optimal but it is satisfied:

- when $p(\xi) = (1 + |\xi|^2)^s$ ($s \geq 1/2$), which corresponds to the $H^s$ metric on $\text{Diff}^\infty(S^1)$,

- when $p(\xi) = |\xi|^{2s}$ ($s \geq 1/2$), which corresponds to the homogeneous metric $\dot{H}^s$ on $\text{Diff}^\infty(S^1)/S^1$ (for $s = 1/2$, we get modified Constantin-Lax-Majda equation),

- when $p(\xi) = |\xi| (\xi^2 - 1)$, we get the Euler-Weil-Petersson equation.

- and of course, when $P$ is a differentiable operator.
The criteria

Theorem (J. Escher & B. K., 2012)

Let \( A = \text{op}(p(k)) \) be a Fourier multiplier of order \( r \geq 1 \). Suppose that \( p \) extends to \( \mathbb{R} \) and satisfies the following conditions:

1. for each \( n \geq 1 \), \( f_n(\xi) := \xi^{n-1}p(\xi) \) is of class \( C^{n-1} \),
2. \( f_n^{(n-1)} \) is absolutely continuous,
3. there exists a constant \( C_n > 0 \) such that

\[
|f_n^{(n)}(\xi)| \leq C_n(1 + \xi^2)^{(r-1)/2}, \quad \text{a.e.}
\]

Then the map

\[
\varphi \mapsto A_\varphi := R_\varphi \circ A \circ R_{\varphi^{-1}}, \quad D^q(S^1) \to \mathcal{L}(H^q(S^1), H^{q-r}(S^1))
\]

is smooth for each \( q \in (\frac{3}{2} + r, \infty) \).
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The map \((\varphi, v) \mapsto A_\varphi(v)\) is smooth on the Fréchet manifold 
\(\text{Diff}^\infty(S^1) \times C^\infty(S^1)\). The \(n\)-th Gâteaux partial derivative in \(\varphi\) can be written as

\[
\partial^n A_\varphi(v, \delta \varphi_1, \cdots, \delta \varphi_n) = R_\varphi A_n R_{\varphi}^{-1}(v, \delta \varphi_1, \cdots, \delta \varphi_n)
\]

where \(A_n\) is a \((n+1)\)-linear operator.

In particular, for \(n = 1\), we get:

\[
A_1(u_0, u_1) = [u_1, A]u_{0x},
\]

and there is a recursive formula to express \(A_n\), which involves iterate commutators.
Proof 2/3: extension of $A_n$ to the Sobolev spaces

- Let $e_m(x) = \exp(2i\pi mx)$ for $m \in \mathbb{Z}$. Then
  \[ A_n(e_{m_0}, \ldots, e_{m_n}) = p_n(m_0, \ldots, m_n)e_{m_0+\ldots+m_n} \]
- For instance, $p_1(m_0, m_1) = (2i\pi)m_0[p(m_0) - p(m_0 + m_1)]$ and there is a recursive formula for $p_n$.
- The hypothesis on the symbol $p$ of $A$ leads to the inequality
  \[ |p_n(m_0, \ldots, m_n)| \leq C_n(1 + m_0^2)^{r/2} \cdots (1 + m_n^2)^{r/2}, \]
  from which we can deduce that $A_n$ extends to a bounded $(n + 1)$-linear operator from $H^q(S^1)$ to $H^{q-r}(S^1)$. 

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Proof 3/3: smoothness of $A_\varphi$

- The mappings $\varphi \mapsto R_\varphi$ and $\varphi \mapsto R_{\varphi^{-1}}$ from $D^q(S^1)$ to $L(H^\rho(S^1))$ are **locally bounded** for $3/2 < \rho \leq q$ (but not continuous).
- Hence, $\varphi \mapsto A_{n,\varphi} := R_\varphi A_n R_{\varphi^{-1}}$ is **locally bounded** from $D^q(S^1)$ to $L^{n+1}(H^q(S^1), H^{q-r}(S^1))$, for $n \geq 0$.
- Using the **mean value theorem**, we deduce inductively:
  
  $A_{n+1,\varphi}$ locally bounded $\Rightarrow$ $A_{n,\varphi}$ locally Lipschitz,

  and

  $A_{n+1,\varphi}$ locally Lipschitz $\Rightarrow$ $A_{n,\varphi}$ Fréchet differentiable.
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Invariance of the geodesic spray

The geodesic spray

\[ F(\varphi, v) := (v, S_{\varphi}(v)) \]

is invariant under right translations \( R_\eta \) (like the metric):

\[ F(R_\eta \varphi, R_\eta v) = R_\eta F(\varphi, v). \]

This property, is inherited by its flow \( \Psi_q \) on \( TD^q(S^1) \):

\[ \Psi_q (R_\eta \varphi, R_\eta v, t) = R_\eta \Psi_q (\varphi, v, t). \]
No loss, no gain in spacial regularity (Ebin-Marsden)

- In particular, specializing to $\eta = \tau_s$, the spatial translation and taking the derivative in $s$, we get

$$D_{(\varphi, v)} \psi_q (\varphi_0, v_0, t) \cdot (\varphi_0 x, v_0 x) = (\varphi_x(t), v_x(t)).$$

- Since $D_{(\varphi, v)} \psi_q (\varphi_0, v_0, t)$ is a bounded, linear operator on the Banach space $H^q(S^1) \times H^q(S^1)$, we obtain that if the initial data $(\varphi_0, v_0)$ is of class $H^{q+1}$ then $(\varphi(t), v(t))$ is of class $H^{q+1}$ (no loss).

- Going backward, we get that if $(\varphi(t), v(t))$ is of class $H^{q+1}$ for some $t > 0$ then $(\varphi_0, v_0)$ is of class $H^{q+1}$ (no gain).
Existence results in $\text{Diff}^\infty(S^1)$

**Theorem**

Suppose that $A$ is a Fourier multiplier, which symbol satisfies our hypothesis. Then, given any $(\varphi_0, v_0) \in T\text{Diff}^\infty(S^1)$, there exists a unique solution

$$(\varphi, v) \in C^\infty(J, T\text{Diff}^\infty(S^1))$$

of the geodesic equations defined on some interval $J = (-T, T)$ + smooth dependance on initial data.

**Corollary**

For each $u_0 \in C^\infty(S^1)$, there exists a unique solution $u \in C^\infty(J, C^\infty(S^1))$ of the corresponding Euler equation, defined on some interval $J = (-T, T)$ such that $u(0) = u_0$ + smooth dependance on initial data.
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Minimizing the arc-length

- On a **finite dimensional Riemannian manifold**, given two nearby points $x$ and $y$, there exists a unique geodesic, joining these two points which minimizes (globally) the **arc-length** (and the energy).

- This is no longer true for a **weak Riemannian metric** (pre-Hilbertian structure) on an infinite dimensional manifold. Indeed, it may happen that the lower bound of arc-lengths (and energy) between two points always vanishes, as for the $L^2$ metric on $\text{Diff}^\infty(\mathbb{S}^1)$ (Michor and Mumford, 2005).
The Riemannian exponential map

- In Riemannian geometry, the time one map of the geodesic flow \( \Psi(q, \cdot, 1) \) is called the **exponential map**. For a strong metric it is a local diffeomorphism at the origin.
- It permits to define local polar coordinates and is the main ingredient to show the existence of **minimal geodesics**.
- For a weak metric, it may not be a local diffeomorphism (for instance for the \( L^2 \) metric on \( \text{Diff}^\infty(S^1) \) (Burgers) and the Virasoro group (KdV) [A. Constantin & B. K. 2002]).

**Theorem (J.Escher & B.K 2012)**

The exponential mapping \( \exp \) at the unit element \( \text{id} \) for the \( H^s \)-metric on \( \text{Diff}^\infty(S^1) \) is a smooth local diffeomorphism from a neighbourhood of zero in \( \text{Vect}(S^1) \) to a neighbourhood of \( \text{id} \) on \( \text{Diff}^\infty(S^1) \) for each \( s \geq 1/2 \).
Geodesics are locally minimizing on $\text{Diff}^\infty(\mathbb{S}^1)$

**Corollary**

*Under the hypothesis of the main theorem, there exists a unique geodesic which minimizes locally the arc-length (weak version) of nearby states $\varphi_1, \varphi_2$ in $\text{Diff}^\infty(\mathbb{S}^1)$.***

**Remark**

This is the case for the $H^{1/2}$-metric on $\text{Diff}^\infty(\mathbb{S}^1)$. This is not in contradiction with the fact that the global infimum of arc-length between each pair of diffeomorphisms, in this case, vanishes identically (Bauer, Bruveris, Harms & Michor, 2012).
For Further Reading

J. Escher, B. Kolev and M. Wunsch

J. Escher and B. Kolev
Right-invariant Sobolev metrics $H^s$ on the diffeomorphisms group of the circle.