Optimal decay estimates on the framework of Besov spaces for hyperbolic systems with degenerate dissipation

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Consider the following hyperbolic system

\[
\begin{align*}
A^0 \partial_t w + \sum_{j=1}^{n} A^j \partial_{x_j} w + Lw &= 0, \\
w(0, x) &= w_0,
\end{align*}
\]

(1)

for \((t, x) \in [0, +\infty) \times \mathbb{R}^n\).

- \(w(t, x) : \mathbb{R}^N\)-valued function;
- \(A^j (j = 0, 1, \cdots, n)\) and \(L\) are constant matrices of order \(N\);
Assume that the equation of (1) is “symmetric hyperbolic" in the following sense:

(A1) Matrices $A_j (j = 0, \cdots, n)$ are real symmetric and, in addition, $A_0$ is positive definite. $L$ is real symmetric and nonnegative definite.

Also, assume (1) satisfies the “Shizuta-Kawashima" condition ([Shizuta & Kawashima, Hokkaido Math. J., 1985])

(A2) Let $\phi \in \mathbb{R}^N$ and $(\lambda, \omega) \in \mathbb{R} \times \mathbb{S}^{n-1}$. If $L\phi = 0$ and $\lambda A_0 \phi + A(\omega) \phi = 0$, then $\phi = 0$. 
The dissipative structure of (1) satisfies

\[ \Re \lambda(i\xi) \leq -c\eta_1(\xi) \quad \text{with} \quad \eta_1(\xi) = \frac{|\xi|^2}{1 + |\xi|^2} \]

for \( c > 0 \), which leads to the optimal decay estimate

\[ \| w \|_{L^2(\mathbb{R}^n)} \lesssim \| w_0 \|_{L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)}(1 + t)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2})} \quad (2) \]

for \( 1 \leq p < 2 \).
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In [X.-Kawashima (ARMA, 2015)], we give a new decay framework for (1):

\[ L^2(\mathbb{R}^n) \cap \dot{B}_{2,\infty}^{-s}(\mathbb{R}^n)(0 < s \leq n/2) \]

which can be regarded as the natural generalization, since

\[ L^p(\mathbb{R}^n) \hookrightarrow \dot{B}_{2,\infty}^{-s}(\mathbb{R}^n) \]

with \( s + n/2 = n/p \). In particular,

\[ L^1(\mathbb{R}^n) \hookrightarrow \dot{B}^0_{1,\infty}(\mathbb{R}^n) \hookrightarrow \dot{B}^{-n/2}_{2,\infty}(\mathbb{R}^n). \]
Theorem 1.1

Let the assumptions (A1)-(A2) hold. Suppose \( w_0 \in L^2(\mathbb{R}^n) \cap \dot{B}^{-s}_{2,\infty}(\mathbb{R}^n) \) for \( s > 0 \), then the solution of (1) has the decay estimate

\[
\| w \|_{L^2(\mathbb{R}^n)} \lesssim \| w_0 \|_{L^2(\mathbb{R}^n) \cap \dot{B}^{-s}_{2,\infty}(\mathbb{R}^n)} (1 + t)^{-s/2}. \tag{3}
\]

In particular, suppose \( w_0 \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) (1 \leq p < 2) \), one further has

\[
\| w \|_{L^2(\mathbb{R}^n)} \lesssim \| w_0 \|_{L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)} (1 + t)^{-n \frac{1}{2} \left( \frac{1}{p} - \frac{1}{2} \right)}. \tag{4}
\]
Sketch of Proof of Theorem 1.1

The proof is divided into several steps.

- **Step 1.** Pointwise energy estimate gives

\[
\frac{d}{dt} E[\hat{w}] + (c_0 - \kappa C)(I - P)\hat{w}^2 + \frac{c_1 \kappa |\xi|^2}{1 + |\xi|^2} |\hat{w}|^2 \leq 0 \quad (5)
\]

with

\[
E[\hat{w}] = \frac{1}{2} (A^0 \hat{w}, \hat{w}) + \frac{\kappa}{2} \text{Im} \left( \frac{|\xi|}{1 + |\xi|^2} K(\omega) A^0 \hat{w}, \hat{w} \right),
\]

where we chosen \(\kappa > 0\) so small that \(c_0 - \kappa C \geq 0\) and \(E[\hat{w}] \approx |\hat{w}|^2\), since \(A^0\) is positive definite.
Step 2. The low- and high-frequency decompositions (the unit decomposition):

\[ w = w_L + w_H \]

with \( w_L = \mathcal{F}^{-1} \left[ \phi(\xi) \hat{w}(\xi) \right] \) and \( w_H = \mathcal{F}^{-1} \left[ \varphi(\xi) \hat{w}(\xi) \right] \), where \( 1 \equiv \phi(\xi) + \varphi(\xi) \), and \( \phi, \varphi \in C_0^\infty(\mathbb{R}^n) \) \((0 \leq \phi(\xi), \varphi(\xi) \leq 1)\) satisfy

\[ \phi(\xi) \equiv 1, \text{ if } |\xi| \leq R; \quad \phi(\xi) \equiv 0, \text{ if } |\xi| \geq 2R \]

with \( R > 0 \).
Step 3. The high-frequency estimate

\[
\frac{d}{dt}(\varphi^2 E[\hat{w}]) + \frac{c_1 R^2}{1 + R^2} |\varphi \hat{w}|^2 \leq 0, \tag{6}
\]

which implies that

\[
\|w_H\|_{L^2} \leq Ce^{-c_2 t} \|w_0\|_{L^2}, \tag{7}
\]

for \(c_2 > 0\) depending on \(R\).
Step 4. The low-frequency estimate

\[ \frac{d}{dt} (\tilde{E}[\hat{w}]^2) + \frac{c_1}{1 + R^2} |\xi|^2 |\hat{w}_L|^2 \leq 0, \]  

(8)

where \( \tilde{E}[\hat{w}] := \{\phi^2 E[\hat{w}]\}^{1/2} \) and \( \tilde{E}[\hat{w}] \approx |\hat{w}_L| \).

Furthermore, Plancherel’s theorem gives

\[ \frac{d}{dt} \tilde{E}_L^2 + c_3 \| \nabla w_L \|^2_{L^2} \leq 0, \]  

(9)

where \( \tilde{E}_L \approx \| w_L \|_{L^2} \) and the constant \( c_3 > 0 \) depends on \( R \).
Step 5. Applying the interpolation inequality

\[ \| f \|_{L^2} \lesssim \| \nabla f \|_{L^2}^\theta \| f \|_{\dot{B}_{2,\infty}^{-s}}^{1-\theta} \left( \theta = \frac{s}{1+s} \right). \]  

(10)

to the low-frequency part \( w_L \), we obtain the differential equality from (9):

\[ \frac{d}{dt} \tilde{\mathcal{E}}_1^2 + C \| w_0 \|_{\dot{B}_{2,\infty}^{-s}}^{-2/s} \| w_L \|_{L^2}^{2(1+1/s)} \leq 0, \]  

(11)

which implies that

\[ \| w_L \|_{L^2} \lesssim \| w_0 \|_{\dot{B}_{2,\infty}^{-s}} (1 + t)^{-s/2}. \]  

(12)
Sketch of Proof of Theorem 1.1

- Step 6. Combining the low-frequency estimate and high-frequency estimate:

\[ \|w\|_{L^2} \leq \|w_L\|_{L^2} + \|w_H\|_{L^2} \leq \|w_0\|_{L^2 \cap \dot{B}^{-s}_{2,\infty}} (1 + t)^{-s/2}. \] (13)

Finally, note that the embedding
\[ L^p(\mathbb{R}^n) \hookrightarrow \dot{B}^{-s}_{2,\infty}(\mathbb{R}^n)(s = n(1/p - 1/2)), \]
we arrive at

\[ \|w\|_{L^2} \lesssim \|w_0\|_{L^2 \cap L^p}(1 + t)^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{2})}, \] (14)

which coincides with the corresponding result (2) in the classical framework. □
Consider the hyperbolic-parabolic composite system

\[
\begin{aligned}
    A^0 U_t + \sum_{j=1}^{n} A^j U_{x_j} &= \sum_{j,k} B^{j,k} U_{x_j x_k}, \\
    U(0, x) &= U_0.
\end{aligned}
\]  

(15)

Similarly, it can be shown that the solution of (15) admits the decay estimate

\[
\| U \|_{L^2(\mathbb{R}^n)} \lesssim \| U_0 \|_{L^2(\mathbb{R}^n) \cap \dot{B}^{-s}_{2,\infty}(\mathbb{R}^n)} (1 + t)^{-s/2},
\]

(16)

if \( U_0 \in L^2(\mathbb{R}^n) \cap \dot{B}^{-s}_{2,\infty}(\mathbb{R}^n) \) for \( s > 0 \).
Remark 1.1

In the proofs of Theorems 1.1, we see that the new framework allows to pay less attention on the traditional spectral analysis. Then,

**Littlewood-Paley decomposition \(\Rightarrow\) Unit decomposition**

which gives us the main **MOTIVATION** to study decay problems in spatially Besov spaces for nonlinear dissipative systems.
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Theorem 1.2 (Inhomogeneous Besov spaces)

If \( w_0 \in \dot{B}^\sigma_{2,1} \cap \dot{B}^{-s}_{2,\infty} \) for \( \sigma \geq 0 \) and \( s > 0 \), then the solutions \( w(t, x) \) of (1) has the decay estimate

\[
\| \Lambda^\ell w \|_{B^\sigma_{2,1} - \ell} \lesssim \| w_0 \|_{\dot{B}^\sigma_{2,1} \cap \dot{B}^{-s}_{2,\infty}} (1 + t)^{-\frac{\ell + s}{2}}
\]

(17)

for \( 0 \leq \ell \leq \sigma \). In particular, if \( w_0 \in \dot{B}^\sigma_{2,1} \cap L^p (1 \leq p < 2) \), one further has

\[
\| \Lambda^\ell w \|_{B^\sigma_{2,1} - \ell} \lesssim \| w_0 \|_{\dot{B}^\sigma_{2,1} \cap L^p} (1 + t)^{-\frac{n}{2p} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{\ell}{2}}
\]

(18)

for \( 0 \leq \ell \leq \sigma \).
Following the similar procedure of proof of Theorem 1.1, we can obtain Theorem 1.2 by deducing that

\[
\sum_{q \geq 0} 2^{q(\sigma - \ell)} \| \Delta_q \Lambda^{\ell} w \|_{L^2} \lesssim e^{-c_2 t} \sum_{q \geq 0} 2^{q(\sigma - \ell)} \| \Delta_q \Lambda^{\ell} w_0 \|_{L^2} \lesssim e^{-c_2 t} \| w_0 \|_{\dot{B}^{-s}_{2,1}}, \tag{19}
\]

and

\[
\| \Lambda^{\ell} \Delta^{-1} w \|_{L^2} \lesssim \| w_0 \|_{\dot{B}^{-s}_{2,1}} (1 + t)^{-\frac{\ell + s}{2}}. \tag{20}
\]
Theorem 1.3 (Homogeneous Besov spaces)

If $w_0 \in \dot{B}^\sigma_{2,1} \cap \dot{B}^{-s}_{2,\infty}$ for $\sigma \in \mathbb{R}, s \in \mathbb{R}$ satisfying $\sigma + s > 0$, then the solution $w(t, x)$ of (1) has the decay estimate

$$\|w\|_{\dot{B}^\sigma_{2,1}} \lesssim \|w_0\|_{\dot{B}^\sigma_{2,1} \cap \dot{B}^{-s}_{2,\infty}} (1 + t)^{-\frac{\sigma + s}{2}}. \quad (21)$$

In particular, if $w_0 \in \dot{B}^\sigma_{2,1} \cap L^p(1 \leq p < 2)$, one further has

$$\|w\|_{\dot{B}^\sigma_{2,1}} \lesssim \|w_0\|_{\dot{B}^\sigma_{2,1} \cap L^p} (1 + t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{\sigma}{2}}. \quad (22)$$
Due to the fact that the operator $\Delta_q$ consists with $\Delta_q$ for $q \geq 0$, it suffices to show the low-frequency estimate. In this case, we use a different idea inspired by Sohinger & Strain [Adv. Math., 2014]. Precisely,

$$2^{q\sigma} \| w_q \|_{L^2} \lesssim \| w_0 \|_{\dot{B}^{-s}_{2,\infty}} (1 + t)^{-\frac{\sigma + s}{2}} \left[ (2^q \sqrt{t})^{\sigma + s} e^{-\frac{1}{2} c_3 (2^q \sqrt{t})^2} \right], \quad (23)$$

which implies that

$$\sum_{q < 0} 2^{q\sigma} \| w_q \|_{L^2} \lesssim \| w_0 \|_{\dot{B}^{-s}_{2,\infty}} (1 + t)^{-\frac{\sigma + s}{2}}. \quad (24)$$
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Let us consider generally hyperbolic systems of balance laws

$$U_t + \sum_{j=1}^{n} F^j(U) x_j = G(U), \quad (25)$$

for \((t, x) \in [0, +\infty) \times \mathbb{R}^n\), where

- \(U(t, x) : \mathbb{R}^N\)-valued function taking values in an open set \(\mathcal{O}_U \subset \mathbb{R}^N\) (the state space);
- \(F^j, G : \mathbb{R}^N\)-valued smooth functions on \(\mathcal{O}_U\);

The system (25) is supplemented with the initial data

$$U_0 = U(0, x), \quad x \in \mathbb{R}^n. \quad (26)$$
Hyperbolic balance laws

- $G(U) \equiv 0$, conservation laws system (classical solutions blow-up);
- $G(U) \neq 0$, hyperbolic response, or relaxation schemes. Its form is given by

$$G(U) = \begin{pmatrix} 0 \\ g(U) \end{pmatrix},$$

with $0 \in \mathbb{R}^{N_1}$ and $g(U) \in \mathbb{R}^{N_2}$, where $N_1 + N_2 = N (N_1 \neq 0)$. 
An important example in gas dynamics is the following damped compressible Euler equations:

\[
\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\
&\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\rho) = -\rho \mathbf{u}
\end{aligned}
\]  

for \((t, x) \in [0, +\infty) \times \mathbb{R}^n\), where

- \(\rho(t, x)\) : the density of gas flow;
- \(\mathbf{u}(t, x)\) : the velocity field of gas flow.
In this position, decay results are stated as follows 
\((\sigma_c = 1 + n/2, \left[{X.-Kawashima \ (ARMA, \ 2015)\]}\right)\).

**Theorem 1.4**

Suppose that \(w_0 - \bar{w} \in B^{\sigma_c}_{2,1} \cap \dot{B}^{-s}_{2,\infty} (0 < s \leq n/2)\) and the norm \(E_0 := \|w_0 - \bar{w}\|_{B^{\sigma_c}_{2,1} \cap \dot{B}^{-s}_{2,\infty}}\) is sufficiently small. Then it holds that

\[
\|\Lambda^\ell w(t)\|_{X_1} \lesssim E_0 (1 + t)^{-\frac{s + \ell}{2}}
\]

for \(0 \leq \ell \leq \sigma_c - 1\), where \(X_1 := B^{\sigma_c - 1 - \ell}_{2,1}\) if \(0 \leq \ell < \sigma_c - 1\) and \(X_1 := \dot{B}^0_{2,1}\) if \(\ell = \sigma_c - 1\);


\[ \| \Lambda^\ell (I - P) w(t) \|_{X_2} \lesssim E_0 (1 + t)^{-\frac{s + \ell + 1}{2}} \]  

for \( 0 \leq \ell \leq \sigma_c - 2 \), where \( X_2 := B^{\sigma_c - 2 - \ell}_{2,1} \) if \( 0 \leq \ell < \sigma_c - 2 \) and \( X_2 := \dot{B}^0_{2,1} \) if \( \ell = \sigma_c - 2 \).
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Consider the Timoshenko system, which is a set of two coupled wave equations of the form

\[
\begin{align*}
\varphi_{tt} - (\varphi_x - \psi)_x &= 0, \\
\psi_{tt} - \sigma(\psi_x)_x - (\varphi_x - \psi) + \gamma \psi_t &= 0
\end{align*}
\]

for \((t, x) \in [0, +\infty) \times \mathbb{R}\). The system (30) is supplemented with the initial data

\[
(\varphi, \varphi_t, \psi, \psi_t)(x, 0) = (\varphi_0, \varphi_1, \psi_0, \psi_1)(x).
\]
By the change of variable introduced by [Ide, Haramoto & Kawashima (M3AS, 2008)]:

\[
\begin{align*}
\psi &= \varphi_x - \psi, \quad u = \varphi_t, \quad z = a\psi_x, \quad y = \psi_t,
\end{align*}
\]

with \( a > 0 \) being the sound speed defined by \( a^2 = \sigma'(0) \). System (30)-(31) can be rewritten as a Cauchy problem for the first-order hyperbolic system of \( U = (v, u, z, y)^\top \)

\[
\begin{align*}
U_t + A(U)U_x + LU &= 0, \\
U(x, 0) &= U_0(x),
\end{align*}
\]

(33)
Timoshenko system

where

\[
A(U) = -\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & a \\
0 & 0 & \frac{\sigma'(z/a)}{a} & 0 \\
\end{pmatrix}, \quad L = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & \gamma \\
\end{pmatrix}.
\]

Note that \(A(U)\) is a real symmetrizable matrix due to \(\sigma'(z/a) > 0\), the dissipative matrix \(L\) is nonnegative definite, however, \(L\) is not symmetric.
Consider the compressible isentropic Euler-Maxwell system in plasmas physics, which is given by

\[
\begin{align*}
\partial_t n + \nabla \cdot (nu) &= 0, \\
\partial_t (nu) + \nabla \cdot (nu \otimes u) + \nabla p(n) &= -n(E + u \times B) - nu, \\
\partial_t E - \nabla \times B &= nu, \quad \nabla \cdot E = n_\infty - n, \\
\partial_t B + \nabla \times E &= 0, \quad \nabla \cdot B = 0,
\end{align*}
\]

for \((t, x) \in [0, +\infty) \times \mathbb{R}^3\).
Euler-Maxwell system

Set \( w = (n, u, E, B)\) (\( \top \) transpose). Then (34) can be written in the vector form

\[
A^0(w)w_t + \sum_{j=1}^{3} A^j(w)w_{x_j} + L(w)w = 0, \tag{35}
\]

where the coefficient matrices are given explicitly as

\[
A^0(w) = \begin{pmatrix}
a(n) & 0 & 0 & 0 \\
0 & nl & 0 & 0 \\
0 & 0 & l & 0 \\
0 & 0 & 0 & l
\end{pmatrix}, \quad L(w) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & n(l - \Omega_B) & nl & 0 \\
0 & -nl & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]
Euler-Maxwell system

\[
\sum_{j=1}^{3} A_j^i(w) \xi_j = \begin{pmatrix}
    a(n)(u \cdot \xi) & p'(n)\xi & 0 & 0 \\
p'(n)\xi^\top & n(u \cdot \xi) I & 0 & 0 \\
0 & 0 & 0 & -\Omega_\xi \\
0 & 0 & \Omega_\xi & 0
\end{pmatrix}.
\]

Here, \( a(n) := p'(n)/n \) is the enthalpy function and \( \Omega_\xi E^\top = (\xi \times E)^\top \). Clearly, (35) is a symmetric hyperbolic system, since \( A_i^j(w) = (A^i_j)(w)^\top (j = 0, 1, 2, 3) \), \( A^0 > 0 \) and the dissipative matrix \( L(w) \geq 0 \), however, \( L(w) \) is not symmetric.
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Non-symmetric dissipation

It was shown that the non-symmetric dissipation affected the Timoshenko and Euler-Maxwell systems such that the weak dissipative mechanism of REGULARITY-LOSS was present. More precisely, their dissipative structure satisfies

$$\text{Re}\lambda(i\xi) \leq -c\eta_2(\xi) \quad \text{with} \quad \eta_2(\xi) = \frac{|\xi|^2}{(1 + |\xi|^2)^2}$$

for $c > 0$, which leads the decay property for the linearized solution $z_L$:

$$\|z_L\|_{L^2} \lesssim (1 + t)^{-3/4}\|z_0\|_{L^1} + (1 + t)^{-\ell/2}\|\partial_x^\ell z_0\|_{L^2}, \quad (36)$$

where $\ell$ is a non-negative integer.

$\Im$ [Ide & Kawashima (08’), Duan (11’), Ueda & Kawashima (11’)].
Non-symmetric dissipation

Remark 2.1

In this case, higher regularity than that for the global-in-time existence is usually assumed to obtain optimal decay rates. To overcome the main difficulty, we developed a new frequency-localization time-decay property to obtain the minimal decay regularity, which means the extra regularity is not necessary.
Theorem 2.1

Let \( \eta(\xi) \) be a positive, continuous and real-valued function in \( \mathbb{R}^n \) satisfying

\[
\eta(\xi) \sim \begin{cases} 
|\xi|^\sigma_1, & |\xi| \to 0; \\
|\xi|^{-\sigma_2}, & |\xi| \to \infty;
\end{cases}
\]

for \( \sigma_1, \sigma_2 > 0 \).

If \( f \in \dot{B}^{s+\ell}_{r,\alpha}(\mathbb{R}^n) \cap \dot{B}^{-\varrho}_{2,\infty}(\mathbb{R}^n) \) for \( s \in \mathbb{R}, \varrho \in \mathbb{R} \) and \( 1 \leq \alpha \leq \infty \) such that \( s + \varrho > 0 \), then it holds that
Theorem 2.1 (Contin.)

\[
\begin{align*}
\left\| 2^{qs} \sqrt{\Delta_q f e^{-\eta(\xi)t}} \right\|_{L^2} \lesssim & \quad (1 + t)^{-\frac{s+q}{\sigma_1}} \left\| f \right\|_{\dot{B}^{-\frac{q}{r}, \infty}} + (1 + t)^{-\frac{\ell}{\sigma_2} + \gamma_2(r, 2)} \left\| f \right\|_{\dot{B}^{s+\ell}_{r, \alpha}}, \quad (37)
\end{align*}
\]

for \( \ell > n\left(\frac{1}{r} - \frac{1}{2}\right) \) \(^a\) with \(1 \leq r \leq 2\), where \( \gamma_\sigma(r, p) := \frac{n}{\sigma}(\frac{1}{r} - \frac{1}{p}) \).

\(^a\)Let us remark that \( \ell \geq 0 \) in the case of \( r = 2 \).
In Theorem 2.1, two points need to be noticed:

**Remark 2.2**

- The low-frequency regularity is less restrictive than the usual $L^p$ space, due to the embedding

\[ L^p(\mathbb{R}^n) \hookrightarrow \dot{B}^{-\varrho}_{2,\infty}(\mathbb{R}^n) \quad (\varrho = n(1/p - 1/2), \quad 1 \leq p < 2); \]

- For the high-frequency part, it decays in time not only with algebraic rates of any order as long as the function is spatially regular enough, but also additional information related to the localized integrability is available.
Based on the decay inequality (37) in Theorem 2.1, we prove that ([X.-Mori-Kawashima (JDE, 2015)])

- The classical solution \( U(t, x) \) of Timoshenko system admits

\[
\| U \|_{L^2} \lesssim l_0 (1 + t)^{-\frac{1}{4}},
\]  

(38)

where \( l_0 := \| U_0 \|_{B^{3/2}_2 (\mathbb{R}) \cap \dot{B}^{-1/2}_2 (\mathbb{R})} \) is sufficiently small;
For the Euler-Maxwell system, we prove that (\[X.-Kawashima (2015)\])

*The classical solution* \(w(t, x)\) admits

\[
\| w - w_\infty \|_{L^2} \lesssim l_1 (1 + t)^{-\frac{3}{4}},
\]

where \(l_1 := \| w_0 - w_\infty \|_{B^{5/2}_{2,1} (\mathbb{R}^3) \cap \dot{B}^{-3/2}_{2,\infty} (\mathbb{R}^3)}\) is sufficiently small.
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Inspired by the recent efforts on Timoshenko and Euler-Maxwell systems, we consider the following general form

\[
\begin{aligned}
A^0 \partial_t w + \sum_{j=1}^{n} A^j \partial_{x_j} w + Lw &= 0, \\
w(0, x) &= w_0,
\end{aligned}
\]  

(40)

for \((t, x) \in [0, +\infty) \times \mathbb{R}^n\), where \(A^j = (A^j)^\top (j = 0, \ldots, n)\), \(A^0 > 0\) and \(L \geq 0\) but it is non-symmetric. Up to now, there is only two decay results for (40):

- **\(L^2\)-framework,**
  
  \[\text{[Ueda, Duan & Kawashima (ARMA, 2012)]}\;

- **\(L^p\)-framework** \((2 \leq p \leq \infty)\),
  
  \[\text{[X.-Mori-Kawashima (JMPA, 2015)]}\]  

However, the corresponding nonlinear application is left **OPEN!**
Inspired by the recent efforts on Timoshenko and Euler-Maxwell systems, we consider the following general form

\[
\begin{aligned}
& A^0 \partial_t w + \sum_{j=1}^{n} A^j \partial_x^j w + Lw = 0, \\
& w(0, x) = w_0,
\end{aligned}
\]  

(40)

for \((t, x) \in [0, +\infty) \times \mathbb{R}^n\), where \(A^j = (A^j)^\top (j = 0, \ldots, n)\), \(A^0 > 0\) and \(L \geq 0\) but it is non-symmetric. Up to now, there is only two decay results for (40):

- \(L^2\)-framework,
  - [Ueda, Duan & Kawashima (ARMA, 2012)];
- \(L^p\)-framework (\(2 \leq p \leq \infty\)),
  - [X.-Mori-Kawashima (JMPA, 2015)].

However, the corresponding nonlinear application is left OPEN!
References


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Thanks for Your Attention