Optimal bounds for the inviscid limit
of Navier–Stokes equations

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Abstract. We consider the inviscid limit of incompressible two-dimensional fluids with initial vorticity in $L^\infty$ and in some Besov space $B^{2,\infty}_\eta$ with low regularity index. We obtain a general result of strong convergence in $L^2$ which applies to the case of vortex patches with smooth boundaries. The rate of convergence we find is $\left(\frac{\nu t}{3}\right)^{3/4}$ (where $\nu$ stands for the viscosity and $t$, for the time). It improves the $\left(\frac{\nu t}{2}\right)^{1/2}$ rate given by P. Constantin and J. Wu in (Nonlinearity 8 (1995), 735–742). Besides, it is shown to be optimal in the case of circular vortex patches.

0. Introduction

Given a solenoidal vector field $\nu^0$ in some convenient functional space, we address the question of convergence of the solutions $v_\nu$ to the incompressible Navier–Stokes equations

\[
\begin{aligned}
\partial_t v_\nu + v_\nu \cdot \nabla v_\nu - \nu \Delta v_\nu &= -\nabla p_\nu, \\
\nabla \cdot v_\nu &= 0, \\
v_\nu|_{t=0} &= \nu^0,
\end{aligned}
\] (NS$_\nu$)

and the ones of the incompressible Euler equations

\[
\begin{aligned}
\partial_t v + v \cdot \nabla v &= -\nabla p, \\
\nabla \cdot v &= 0, \\
v|_{t=0} &= \nu^0,
\end{aligned}
\] (E)

when $\nu$ goes to 0.

The study of this problem of convergence (the so-called inviscid limit for incompressible fluids) has retained a lot of attention these last years. The answer strongly depends on the boundary conditions. In the case of a fluid evolving in some domain $\Omega$ with no empty boundary, there is a discrepancy between the natural boundary conditions for (NS$_\nu$) (Dirichlet boundary conditions) and the ones for (E) (tangential boundary conditions) so that the rate of convergence near the boundary may be quite bad.

We restrict ourselves to fluids evolving in the whole plane $\mathbb{R}^2$ so that we do not have to worry about boundary conditions. Besides, according to a celebrated result by V. Yudovich (see the original paper by

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Then we obviously have

\[ v \in L^2 \text{ for some } C^\infty \text{ stationary solenoidal vector field } \sigma \text{ whose vorticity } \omega(\sigma) \text{ is radial and belongs to } C_0^\infty(\mathbb{R}^2). \]

Let \( v^0 \in E_m \) with vorticity \( \omega^0 \) in \( L^2 \cap L^\infty \). Then (E) and (NS) have a unique global solution which belongs to \( C(\mathbb{R}^+; E_m) \) and whose vorticity belongs to \( L^\infty(\mathbb{R}^+; L^2 \cap L^\infty) \).

In the present work, we aim at finding the optimal rate of convergence to 0 in the energy space \( L^2 \) for \( (v_\nu - v) \).

Partial answers have been given by a number of authors. As a matter of fact, it is well known that the rate is bounded by \( \nu t \) for smooth enough data: the case \( v_0 \in H^s \) with \( s > 4 \) is proved in [7, 1], the case \( s > 2 \) is treated in [11] in the more general framework of nonhomogeneous fluids.

In [8], Constantin and Wu considered rougher initial data. They were in particular interested by vortex patches data (i.e., the initial vorticity \( \omega^0 \) is assumed to be the characteristic function of some bounded domain \( \Omega^0 \)). They stated the following estimate:

\[
\|v_\nu(t) - v(t)\|_{L^2} \lesssim \sqrt{2\nu t} \|\omega^0\|_{L^2} + \int_0^t \|\nabla v(\tau)\|_{L^\infty} \, d\tau.
\]

In the case of smooth vortex patches (i.e., \( \Omega^0 \) is a \( C^{1+\varepsilon} \) domain), the vorticity associated to the solution \( v \) of (E) remains a smooth vortex patch for all time and, besides, \( \nabla v \) belongs to \( L^\infty_{loc}(\mathbb{R}^+; L^\infty) \) (see, e.g., [4, Chapters 3 and 5]). Hence, according to inequality (1), the rate of convergence in \( L^2 \) for the inviscid limit in the case of smooth vortex patches is \( O(\sqrt{\nu t}) \) (see also [9] for the study of convergence in \( L^p \)).

In the conclusion of [8], the authors claim that taking circular patches provide lower bounds. As no computations are done there, the precise meaning of their remark is unclear. Indeed, we shall see later on that the rate of convergence is this case is not \( (\nu t)^{1/2} \); it is exactly of order \( (\nu t)^{3/4} \).

In [5], Chemin addressed a similar question for rougher initial data: \( v_0 \in E_m \) with vorticity \( \omega^0 \) in \( L^2 \cap L^\infty \) only. He proved that \( (v_\nu - v) \) still tends to 0 in \( L^2 \) when \( \nu \) goes to 0 but that the rate of convergence may coarsen exponentially with time: it is bounded by \( (\nu t)^{1/2} \exp(-Ct) \) with \( C \) depending only on \( v_0 \). Whether this coarsening does occur is not discussed there.

In the present paper, we rather aim at finding the optimal rate of convergence for vortex patches type solutions.

1. Statement of the main results

Before stating our main result of convergence, we need to define the Besov spaces \( B^s_{2,\infty} \).

Let us introduce the following (rough) spectral truncation:

\[
I_q u = \mathcal{F}^{-1}(1_{\{2^q \leq |\xi| < 2^{q+1}\}} \mathcal{F} u) \quad \text{for } q \geq 1, \quad \text{and} \quad I_0 u = \mathcal{F}^{-1}(1_{\{|\xi| \leq 1\}} \mathcal{F} u) \quad \text{for } q = 0.
\]

Then we obviously have

\[
u \in H^s \iff \|\nu\|_{H^s} \overset{\text{def}}{=} \left( \sum_{q \in \mathbb{N}} 2^{qs} \|I_q u\|_{L^2}^2 \right)^{1/2}.
\]
We define the inhomogeneous Besov space $B^{s}_{2,\infty}$ as the set of distributions $u \in S'(\mathbb{R}^2)$ such that

$$\|u\|_{B^{s}_{2,\infty}} \overset{\text{def}}{=} \sup_{q \in \mathbb{N}} 2^{sq} \|J_q u\|_{L^2} < +\infty.$$ 

Therefore $B^{s}_{2,\infty}$ is slightly larger than $H^s$.

We can now state a general result of convergence for rough data:

**Theorem 1.1.** Let $\omega^0 \in L^2 \cap L^\infty$ and $\omega^0_\eta \in L^\infty \cap B^{s}_{2,\infty}$ with $\eta \in [0, 1)$. Let $v^0$ (resp. $v^0_\eta$) be the divergence-free vector field in $\mathbb{R}^2$ with vorticity $\omega^0$ (resp. $\omega^0_\eta$) given by Biot–Savart’s law:

$$v^0_{(\eta)}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x - y)_y}{|x - y|^2} \omega^0_{(\eta)}(y) \, dy.$$  

For $\nu > 0$, let $v_\nu$ (resp. $v$) be the unique global solution of (NS$_\nu$) (resp. (E)) with initial data $v^0_\nu$ (resp. $v^0$) given by Theorem 0.1. Assume in addition that $\nabla v_\nu$ and $\nabla v$ belong to $L^\infty_{\text{loc}}(\mathbb{R}^+; L^\infty)$. Let $V(t) \overset{\text{def}}{=} \int_0^t \|\nabla v(\tau)\|_{L^\infty} \, d\tau$ and $V_\nu(t) \overset{\text{def}}{=} \int_0^t \|\nabla v_\nu(\tau)\|_{L^\infty} \, d\tau$.

Then we have the following estimate:

$$\|(v_\nu - v)(t)\|_{L^2} \leq C e^{C(V(t)+V_\nu(t))} \left(\|v^0_\nu - v^0\|_{L^2} + (\nu t)^{(1+\eta)/2} \left(\|\omega^0_\nu\|_{B^{s}_{2,\infty}} + \nu t \|\omega^0_\nu\|_{L^2}\right)\right),$$

where the constant $C$ depends only on $\eta$.

Of particular interest is the case of initial data which are smooth vortex patches. We have the following result:

**Proposition 1.2.** If $\Omega^0$ is a $C^1$ bounded domain, then the function $1_{\Omega^0}$ belongs to $B^{1/2}_{2,\infty}$.

Hence, smooth vortex patches belong to $B^{1/2}_{2,\infty}$.

On the other hand, in [10], it has been shown that smooth initial patches yield global solutions whose gradient is bounded in $L^\infty_{\text{loc}}(\mathbb{R}^+; L^\infty)$ uniformly with respect to $\nu$:

**Theorem 1.3.** Let $\Omega^0$ be a bounded domain with $C^{1+\varepsilon}$ boundary ($\varepsilon > 0$). Let $v^0$ be the solenoidal vector field with vorticity $\omega^0 = 1_{\Omega^0}$ given by the Biot–Savart law. Then for all $\nu > 0$, (NS$_\nu$) (resp. (E)) has a unique global solution $v_\nu$ (resp. $v$) in $L^\infty_{\text{loc}}(\mathbb{R}^+; \text{Lip}(\mathbb{R}^2))$ and we have

$$\forall t \in \mathbb{R}^+, \quad \sup_{\nu > 0} \|\nabla v_\nu(t)\|_{L^\infty} \leq C e^{C t} \quad \text{and} \quad \|\nabla v(t)\|_{L^\infty} \leq C e^{C t},$$

for some constant $C$ depending only on the initial domain $\Omega_0$.

Combining Proposition 1.2 with Theorems 1.1 and 1.3 enables us to improve the $\sqrt{\nu t}$ rate obtained in [8] for the convergence in $L^2$ norm:
Corollary 1.4. If \( \omega^0 = \omega^1_0 = 1 \) with \( \Omega^0 \) bounded domain with \( C^{1+\epsilon} \) boundary then there exists some constant \( C_0 \) depending only on \( \Omega_0 \) and such that for all \( t \in \mathbb{R}^+ \),

\[
\|v_\nu(t) - v(t)\|_{L^2} \leq C_0 e^{C_0 t} e^{C_0 (\nu t)^{3/4}}(1 + \nu t).
\]

Moreover the rate \((\nu t)^{3/4}\) is optimal:

Proposition 1.5. Assume that \( \omega^0 \) is a circular vortex patch. Then there exist two positive constants \( c_* \) and \( c^{*} \) independent of \( t \) and \( \nu \), and such that for \( \nu t \leq 1 \), we have

\[
c_*(\nu t)^{3/4} \leq \|v_\nu(t) - v(t)\|_{L^2} \leq c^*(\nu t)^{3/4}.
\]

The paper is structured as follows. Section 2 is devoted to a quick presentation of Littlewood–Paley decomposition and Besov spaces. In Section 3, we prove Theorem 1.1 and Proposition 1.2 (hence Corollary 1.4). In Section 4, we show the optimality of the rate \((\nu t)^{3/4}\) in the case of a circular vortex patch.

Notation. Throughout the paper, \( C \) stands for a “harmless constant” whose precise meaning will be clear from the context. We shall sometimes alternatively use the notation \( A \lesssim B \) instead of \( A \leq CB \), and \( A \approx B \) means that \( A \lesssim B \) and \( B \lesssim A \).

2. Basic Fourier analysis

The proof of some of the results presented in the paper requires a dyadic decomposition in Fourier variables, which is called Littlewood–Paley decomposition. Let us briefly explain what it consists of. Let \((\chi, \varphi)\) be a couple of \( C^\infty \) functions with

\[
\text{Supp } \chi \subset \left\{ |\xi| \leq \frac{4}{3} \right\}, \quad \text{Supp } \varphi \subset \left\{ \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \right\} \quad \text{and} \quad \chi(\xi) + \sum_{q \in \mathbb{N}} \varphi(2^{-q}\xi) = 1.
\]

Let \( \varphi_q(\xi) = \varphi(2^{-q}\xi) \), \( h_q = \mathcal{F}^{-1} \varphi_q \) and \( \hat{h} = \mathcal{F}^{-1} \chi \). The dyadic blocks are defined by

\[
\Delta_q u \overset{\text{def}}{=} 0 \quad \text{if } q \leq -1, \quad \Delta_{-1} u \overset{\text{def}}{=} \chi(D)u = \int_{\mathbb{R}^N} \hat{h}(y)u(x - y)\,dy,
\]

\[
\Delta_q u \overset{\text{def}}{=} \varphi(2^{-q}D)u = \int_{\mathbb{R}^N} h_q(y)u(x - y)\,dy \quad \text{if } q \geq 0.
\]

One can easily prove that

\[
\forall u \in \mathcal{S}'(\mathbb{R}^N), \quad u = \sum_{q \in \mathbb{Z}} \Delta_q u. \quad (2)
\]

A number of functional spaces may be characterized in terms of Littlewood–Paley decomposition. Let us give the definition of (nonhomogeneous) Besov spaces:
Definition 2.1. For \( s \in \mathbb{R}, (p, r) \in [1, +\infty]^2 \) and \( u \in \mathcal{S}'(\mathbb{R}^N) \), we set
\[
\|u\|_{B^s_{p,r}} = \left( \sum_{q \geq 1} 2^{qs} \|\Delta_qu\|_{L^p}^r \right)^{1/r}
\]
if \( r < +\infty \), and \( \|u\|_{B^s_{p,\infty}} = \sup_{q \geq 1} 2^{qs} \|\Delta_qu\|_{L^p} \).

We then define the Besov space \( B^s_{p,r} = \{ u \in \mathcal{S}' \mid \|u\|_{B^s_{p,r}} < +\infty \} \).

The definition of \( B^s_{p,r} \) does not depend on the choice of \((\chi, \varphi)\). One can further remark that \( H^s \) coincide with \( B^s_{2,2} \) and that the space \( B^s_{2,\infty} \) is the same that the one defined in Section 1. Let us now state some classical properties for these Besov spaces.

Proposition 2.2. The following properties hold true:

(i) Derivatives: we have
\[
\|\nabla u\|_{B^s_{p,r}^{-1}} \lesssim \|u\|_{B^s_{p,r}}.
\]

(ii) Sobolev embeddings: if \( p_1 \leq p_2 \) and \( r_1 \leq r_2 \) then \( B^s_{p_1,r_1} \hookrightarrow B^s_{p_2,r_2} \).

If \( s_1 > s_2 \) and \( 1 \leq p, r_1, r_2 \leq +\infty \), then \( B^s_{p,r_1} \hookrightarrow B^s_{p,r_2} \).

Notation. For \( X \) Banach space, we denote by \( C([0,T];X) \) the set of continuous functions on \([0,T]\) with values in \( X \). For \( p \in [1, +\infty] \), the notation \( L^p(0,T;X) \) stands for the set of measurable functions on \([0,T]\) with values in \( X \), such that \( t \mapsto \|f(t)\|_X \) belongs to \( L^p(0,T) \). The corresponding norm is denoted by \( \|f\|_{L^p_T(X)} \).

In the present paper, the space \( X \) will be a Sobolev space \( H^s \) or a Besov space. Hence, localizing the PDE’s we are interested in by mean Littlewood–Paley decomposition is the most straightforward way of getting estimates in spaces of type \( L^p_T(X) \). Indeed, energy methods yield \( L^p \) estimates for each dyadic block. Performing integration in time would be the next step. But in doing so, we obtain bounds in spaces which are not exactly of type \( L^p(0,T;X) \). That remark naturally leads to the following definition (introduced in [6] and generalized in [3]):

Definition 2.3. For \( (\rho, p, r) \in [1, +\infty]^3 \), \( s \in \mathbb{R} \) and \( T \in [0, +\infty] \), we set
\[
\|u\|_{\tilde{L}_{\rho}^p(B^s_{p,r})} \overset{\text{def}}{=} \left( \sum_{q \geq 1} 2^{qs} \|\Delta_qu\|_{L^p_T(L^p)}^r \right)^{1/r},
\]
with the usual change if \( r = +\infty \), and denote by \( \tilde{L}_{\rho}^p(B^s_{p,r}) \) the subset of distributions \( u \in \mathcal{S}'(0,T \times \mathbb{R}^N) \) with finite \( \|u\|_{\tilde{L}_{\rho}^p(B^s_{p,r})} \) norm. If \( p = r = 2 \), the space \( \tilde{L}_{\rho}^2(B^s_{p,r}) \) is alternately denoted by \( \tilde{H}^s(0,T) \).

Let us remark that by virtue of Minkowski inequality, we have
\[
\|u\|_{\tilde{L}_{\rho}^p(B^s_{p,r})} \leq \|u\|_{L_{\rho}^p(B^s_{p,r})} \quad \text{if } \rho \leq r \quad \text{and} \quad \|u\|_{\tilde{L}_{\rho}^p(B^s_{p,r})} \leq \|u\|_{\tilde{L}_{\rho}^p(B^s_{p,r})} \quad \text{if } \rho \geq r.
\]
and one can easily prove that, if \( \varepsilon > 0, 1 \leq r, r' , \rho \leq + \infty \),
\[
\| u \|_{L^p_T(B_{p,r}^s)} \lesssim \| u \|_{L^p_T(B_{p,r'}^{s+\varepsilon})} \quad \text{and} \quad \| u \|_{L^p_T(B_{p,r}^s)} \simeq \| u \|_{L^p_T(B_{p,r'}^{s+\varepsilon})}.
\]

Hence the spaces \( \tilde{L}^p_T(B_{p,r}^s) \) and \( L^p(0, T; B_{p,r'}^s) \) are very close to each other.

**Remark 2.4.** Proposition 2.2 may be easily extended to the spaces \( \tilde{L}^p_T(B_{p,r}^s) \).

In the proof of Theorem 1.1, the following result of real interpolation is needed:

**Lemma 2.5.** Let \( T > 0, 1 \leq p, r, \rho \leq + \infty \) and \( \sigma_1 < \sigma < \sigma_2 \). Let \( \theta \in (0, 1) \) satisfy \( \sigma = \theta \sigma_1 + (1 - \theta) \sigma_2 \).

The following interpolation inequality holds true
\[
\| u \|_{L^p_T(B_{p,r}^s)} \leq C \| u \|_{L^p_T(B_{p,r}^{\sigma_1})}^{\theta} \| u \|_{L^p_T(B_{p,r}^{\sigma_2})}^{1-\theta},
\]
for some constant \( C \) depending only on \( \theta \) and \( r \).

**Proof.** Let \( N \in \mathbb{N} \) be an integer to be fixed hereafter. We have
\[
\| u \|_{L^p_T(B_{p,r}^s)} \leq \left( \sum_{q=1}^{N-1} 2^{q \sigma_1} \| \Delta_q u \|_{L^p_T(L^\rho)} \right)^{1/r} + \left( \sum_{q \geq N} 2^{q \sigma_1} \| \Delta_q u \|_{L^p_T(L^\rho)} \right)^{1/r} \leq \| u \|_{\tilde{L}^p_T(B_{p,r}^{\sigma_1})} \left( \sum_{q=1}^{N-1} 2^{q (\sigma - \sigma_1)} \right)^{1/r} + \| u \|_{L^p_T(B_{p,r}^{\sigma_2})} \left( \sum_{q \geq N} 2^{q (\sigma - \sigma_2)} \right)^{1/r} \leq C (2^N (\sigma - \sigma_1) \| u \|_{L^p_T(B_{p,r}^{\sigma_1})} + 2^N (\sigma - \sigma_2) \| u \|_{L^p_T(B_{p,r}^{\sigma_2})}).
\]
Choosing \( N \) such that
\[
2^N (\sigma - \sigma_1) \| u \|_{L^p_T(B_{p,r}^{\sigma_1})} \approx 2^N (\sigma - \sigma_2) \| u \|_{L^p_T(B_{p,r}^{\sigma_2})}
\]
yields the desired inequality. \( \square \)

To finish with, let us state a result of continuity for the Biot–Savart law:

**Lemma 2.6.** The map \( B : \omega \mapsto \nabla v \) where \( v \) is the solenoidal vector field given by the Biot–Savart law, is continuous in \( \tilde{L}^1_T(H^1) \).

**Proof.** By definition of the space \( \tilde{L}^1_T(H^1) \), we have
\[
\| \nabla v \|_{L^1_T(H^1)} = \left( \sum_{q \geq 1} 2^{2q} \| \Delta_q \nabla v \|_{L^1_T(L^2)}^2 \right)^{1/2}.
\]
As the map $B$ is continuous in $L^2$ and maps $\Delta_q \omega$ on $\Delta_q \nabla v$, we get
\[
\||\nabla v||_{L^2_t(H^1, B^{2q})} \lesssim \left( \sum_{q \geq -1} 2^{2q} ||\Delta_q \omega||_{L^2_t(L^2)}^2 \right)^{1/2} = ||\omega||_{L^2_t(H^1)},
\]

hence the desired result.

3. Proof of the main results

3.1. An a priori estimate for transport–diffusion equations

**Proposition 3.1.** Let $\eta \in (-1, 1)$. Let $v \in L^\infty(0, T; \text{Lip})$ with $\text{div} v = 0$. Let $a$ solve
\[
\partial_t a + v \cdot \nabla a - \nu \Delta a = 0. \tag{T_v}
\]

Then the following estimate holds true for some universal constant $\kappa$ and $C = C(\eta)$:
\[
\max \left( ||a(t)||_{B^{2q}_{2,\infty}} , \kappa \nu ||a(t)||_{L^1_t(B^{2q + \eta}_{2,\infty})} \right) \leq e^{CV(t)} \left( ||a(0)||_{B^{2q}_{2,\infty}} + \kappa \nu \int_0^t e^{-CV(\tau)} ||\Delta_q a(\tau)||_{L^2} \ d\tau \right)
\]
with $V(t) \overset{\text{def}}{=} \int_0^t ||\nabla v(\tau)||_{L^\infty} \ d\tau$.

**Proof.** Applying operator $\Delta_q$ to Eq. (T_v) yields
\[
\partial_t \Delta_q a + v \cdot \nabla \Delta_q a - \nu \Delta \Delta_q a = [v \cdot \nabla, \Delta_q] a.
\]

Hence, taking the $L^2$ scalar product with $\Delta_q a$,
\[
\frac{1}{2} \frac{d}{dt} ||\Delta_q a||_{L^2}^2 + \nu ||\nabla \Delta_q a||_{L^2}^2 \leq ||\Delta_q a||_{L^2} ||v \cdot \nabla, \Delta_q] a||_{L^2}.
\]
(4)

According to Lemma 4.2 in [10], the commutator satisfies
\[
||[v \cdot \nabla, \Delta_q] a||_{L^2} \lesssim 2^{-\eta q} ||\nabla v||_{L^\infty} ||a||_{B^{2q}_{2,\infty}}.
\]
(5)

Bernstein inequality yields for some universal positive constant $\kappa$,
\[
\forall q \in \mathbb{N}, \quad ||\nabla \Delta_q a||_{L^2} \geq \kappa 2^q ||\Delta_q a||_{L^2}.
\]
(6)

Plugging inequalities (5) and (6) in (4) and multiplying both sides by $2^{2q\eta}$ yields
\[
\frac{1}{2} \frac{d}{dt} \left( 2^{2q\eta} ||\Delta_q a||_{L^2} \right)^2 + \nu \kappa (2^{2(\eta + 2)} ||\Delta_q a||_{L^2}) (2^{2q\eta} ||\Delta_q a||_{L^2})
\]
\[
\leq C ||\nabla v||_{L^\infty} ||a||_{B^{2q}_{2,\infty}} (2^{2q\eta} ||\Delta_q a||_{L^2}) + \nu \kappa \delta_q (2^{2(\eta + 2)} ||\Delta_q a||_{L^2}) (2^{2q\eta} ||\Delta_q a||_{L^2}).
\]
where \( \delta_{ij} \) stands for the Kronecker symbol on \( \mathbb{Z}^2 \).

Now, standard computations combined with a time integration eventually lead to

\[
2^{2q} \left\| \Delta_q a(t) \right\|_{L^2} + \kappa \nu 2^{2q} \int_{t_0}^t \left\| \Delta_q a(\tau) \right\|_{L^2} \, d\tau \\
\leq 2^{2q} \left\| \Delta_q a(0) \right\|_{L^2} + C \int_{t_0}^t \left\| \nabla v \right\|_{L^\infty} \| a \|_{B^q_{2,\infty}} \, d\tau + \kappa \nu \delta_{-1q} 2^{2q} \int_{t_0}^t \left\| \Delta_q a(\tau) \right\| \, d\tau,
\]

hence taking the supremum on \( q \geq -1, \)

\[
\left\| a(t) \right\|_{B^q_{2,\infty}} + \kappa \nu \left\| a \right\|_{L^1(B^{q+1}_{2,\infty})} \\
\leq \left\| a(0) \right\|_{B^q_{2,\infty}} + \kappa \nu \int_{t_0}^t \left\| \Delta_{-1} a(\tau) \right\|_{L^2} \, d\tau + C \int_{t_0}^t \left\| \nabla v(\tau) \right\|_{L^\infty} \| a(\tau) \|_{B^q_{2,\infty}} \, d\tau.
\]

Gronwall lemma completes the proof. \( \square \)

### 3.2. Proof of Theorem 1.1

The vorticity \( \omega_\nu \), associated to a two-dimensional incompressible viscous flow \( v_\nu \), satisfies

\[
\partial_t \omega_\nu + v_\nu \cdot \nabla \omega_\nu - \nu \Delta \omega_\nu = 0.
\]

Hence \( \left\| \omega_\nu(t) \right\|_{L^2} \leq \left\| \omega_\nu(0) \right\|_{L^2} \) for all \( t \in [0, T] \). Combining with the estimate given in Proposition 3.1 and defining \( V \) according to Theorem 1.1, we gather

\[
\left\| \omega_\nu \right\|_{L^\infty_t(B^q_{2,\infty})} + \kappa \nu \left\| \omega_\nu \right\|_{L^1_t(B^{q+1}_{2,\infty})} \leq e^{C_t V(t)} \left( \left\| \omega_\nu(0) \right\|_{B^q_{2,\infty}} + \kappa \nu t \left\| \omega_\nu(0) \right\|_{L^2} \right).
\]

Denote \( w_\nu \overset{\text{def}}{=} v_\nu - v \). Let us admit for a while the following inequality:

\[
\left\| w_\nu(t) \right\|_{L^2} \leq e^{C(V(t) + V_\nu(t))} \left( \left\| w_\nu(0) \right\|_{L^2} + \nu \left\| \Delta v_\nu \right\|_{L^1_t(H^0)} \right).
\]

Combining Proposition 2.2, Remark 2.4 and Lemma 2.6 yields

\[
\left\| \Delta v_\nu \right\|_{L^1_t(H^0)} \lesssim \left\| \nabla v_\nu \right\|_{L^1_t(H^1)} \lesssim \left\| \omega_\nu \right\|_{L^1_t(H^1)}.
\]

On the other hand, according to Lemma 2.5,

\[
\left\| \omega_\nu \right\|_{L^1_t(H^0)} \lesssim \left\| \omega_\nu \right\|_{L^1_t(B^q_{2,\infty})} \left\| \omega_\nu \right\|_{L^1_t(B^{q+1}_{2,\infty})}.
\]

Hence, using Hölder and Young inequality,

\[
\kappa \nu \left\| \omega_\nu \right\|_{L^1_t(H^0)} \lesssim \kappa \nu t^{(1+\eta)/2} \left( \left\| \omega_\nu \right\|_{L^\infty_t(B^q_{2,\infty})} \right)^{(1-\eta)/2} \left( \left\| \omega_\nu \right\|_{L^1_t(B^{q+1}_{2,\infty})} \right)^{(1-\eta)/2} \\
\lesssim (\kappa \nu t)^{(1+\eta)/2} \left( \left\| \omega_\nu \right\|_{L^\infty_t(B^q_{2,\infty})} + \kappa \nu \left\| \omega_\nu \right\|_{L^1_t(B^{q+1}_{2,\infty})} \right).
\]
Plugging this later inequality in (9), we get by virtue of (7),
\[ \nu \| \Delta w_\nu \|_{L_t^1(H^0)} \leq C (\nu t)^{(1 + \alpha)/2} e^{CV(t)} (\| \omega_\nu(0) \|_{B^2_{\infty}} + \nu t \| \omega_\nu(0) \|_{L_t^1}). \]
Inserting this in (8) gives the desired inequality.

We still have to prove inequality (8). Obviously, \( w_\nu \) satisfies
\[ \partial_t w_\nu + v \cdot \nabla w_\nu + w_\nu \cdot \nabla v_\nu + \nabla (p_\nu - p) = \nu \Delta v_\nu. \]

Hence, applying operator \( \Delta_q \),
\[ \partial_t \Delta_q w_\nu + v \cdot \nabla \Delta_q w_\nu + \nabla (\Delta_q p_\nu - \Delta_q p) = \nu \Delta \Delta_q v_\nu - \Delta_q (w_\nu \cdot \nabla v_\nu) + [v \cdot \nabla, \Delta_q] w_\nu. \]

Taking the \( L^2 \) scalar product with \( w_\nu \) and using that \( \text{div} \ v = \text{div} \ w_\nu = 0 \) yields
\[ \frac{1}{2} \frac{d}{dt} \| \Delta_q w_\nu(t) \|_{L^2}^2 \leq \| \Delta_q w_\nu(0) \|_{L^2}^2 + \int_0^t (\nu \| \Delta \Delta_q v_\nu \|_{L^2} + \| \Delta_q (w_\nu \cdot \nabla v_\nu) \|_{L^2} + \| [v \cdot \nabla, \Delta_q] w_\nu \|_{L^2}^2) \, d\tau. \]

hence, performing a time integration,
\[ \| \Delta_q w_\nu(t) \|_{L^2} \leq \| \Delta_q w_\nu(0) \|_{L^2} + \int_0^t \left( \sum_{q \geq 1} \left\| [v \cdot \nabla, \Delta_q] w_\nu(\tau) \right\|_{L^2}^2 \right)^{1/2} \, d\tau. \]

Taking the \( l^2 \) norm of both sides and applying Minkowski inequality where needed yields
\[ \| w_\nu(t) \|_{L^2} \leq \| w_\nu(0) \|_{L^2} + \int_0^t \left( \sum_{q \geq 1} \left\| [v \cdot \nabla, \Delta_q] w_\nu(\tau) \right\|_{L^2}^2 \right)^{1/2} \, d\tau. \]

Of course, one has \( \left\| [v \cdot \nabla, \Delta_q] w_\nu \right\|_{L^2} \leq \| \nabla v_\nu \|_{L^\infty} \). On the other hand, straightforward changes in the proof of Lemma 4.2 in [10] would give
\[ \left( \sum_{q \geq 1} \left\| [v \cdot \nabla, \Delta_q] w_\nu \right\|_{L^2}^2 \right)^{1/2} \leq \| \nabla v \|_{L^\infty} \| w_\nu \|_{L^2}. \]

Coming back to (10), we eventually conclude that
\[ \| w_\nu(t) \|_{L^2} \leq \| w_\nu(0) \|_{L^2} + \nu \| \Delta v_\nu \|_{L_t^1(H^0)} + C \int_0^t (\| \nabla v \|_{L^\infty} + \| \nabla v_\nu \|_{L^\infty}) \| w_\nu \|_{L^2} \, d\tau. \]

Gronwall lemma completes the proof of (8).
3.3. Proof of Proposition 1.2

Let \( \omega^0 \overset{\text{def}}{=} 1_{\Omega^0} \). We want to show that \( \omega^0 \) belongs to \( B^{1/2}_{2,\infty} \).

Assume first that \( \Omega^0 \) is a square. With no loss of generality, one may restrict oneself to the case of the unit square centered at 0:

\[ \Omega^0 = \{(x_1, x_2) \in \mathbb{R}^2; \ -1 \leq x_1, x_2 \leq 1 \} . \]

A simple calculation leads to

\[ \mathcal{F}_q \omega^0(\xi_1, \xi_2) = \frac{4}{\xi_1 \xi_2} \sin \xi_1 \sin \xi_2 , \]

so that for \( q \geq 1 \),

\[ \| \mathcal{F}_q \omega^0 \|_{L^2} \leq 16 \int_{2^{q-1} \leq |\xi| \leq 2^q} \left[ \frac{\sin \xi_1}{\xi_1} \right]^2 \left[ \frac{\sin \xi_2}{\xi_2} \right]^2 d\xi_1 d\xi_2 \]

\[ \lesssim \int_{|\eta| \leq 2^q} \left[ \frac{\sin \eta}{\eta} \right]^2 d\eta \lesssim 2^{-q} . \]

Of course, we have

\[ \| I_0 \omega^0 \|_{L^2} \leq \| \omega^0 \|_{L^2} < +\infty . \]

Hence \( \omega^0 \) belongs to \( B^{1/2}_{2,\infty} \) in this particular case.

Let us assume now that \( \Omega^0 \) is a general bounded domain with \( C^1 \) boundary. For all \( x \in \partial \Omega^0 \) there exists a neighbourhood \( U \) of \( x \) in \( \mathbb{R}^2 \) and a \( C^1 \) diffeomorphism \( H : Q \rightarrow U \) such that \( 1_{U \cap \Omega^0} = 1_{Q_+} \circ H^{-1} \) (see, e.g., [2]) where we denote:

\[ Q = \{ x = (x_1, x_2); \ |x_1| < 1 \text{ and } |x_2| < 1 \} , \]

\[ Q_+ = Q \cap \mathbb{R}_+^2 , \]

\[ \mathbb{R}_+^2 = \{ x = (x_1, x_2); \ x_2 > 0 \} . \]

As we have \( 1_{Q_+} \in B^{1/2}_{2,\infty} \), the composition Lemma 1.2 given in [10] enables us to conclude that \( 1_{U \cap \Omega^0} \in B^{1/2}_{2,\infty} \). Next, introducing a convenient partition of unity yields \( \omega^0 \in B^{1/2}_{2,\infty} \). \( \square \)

4. The case of circular vortex patches

In this section, we show the optimality of the rate \((\nu t)^{3/4}\) in the case of a circular vortex patch. Of course such a patch satisfies the assumptions of corollary 1.4 so that the rate of convergence is at most \((\nu t)^{3/4}\) for \( \nu t \geq 1 \).
Let us now get a bound by below. Note that as the initial vorticity is radial, the solution $v_\nu$ (resp. $v$) of $(\text{NS}_\nu)$ (resp. (E)) is also radial. Therefore

$$v_\nu \cdot \nabla \omega_\nu = 0 \quad \text{and} \quad v \cdot \nabla \omega = 0,$$

where $\omega_\nu$ (resp. $\omega$) is the vorticity associated to $(\text{NS}_\nu)$ (resp. (E)).

Hence, $\omega$ is independent of $t$ whereas the equation for $\omega_\nu$ reduces to a mere heat equation:

$$\partial_t \omega_\nu - \nu \Delta \omega_\nu = 0 \quad \text{for} \quad \nu t \leq 1.$$  \hspace{1cm} (11)

Now using Biot–Savart formula, one can give an explicit formula for $v_\nu$ and $v$ from which the optimality of the rate $(\nu t)^{3/4}$ (when $\nu t \leq 1$) easily stems. Indeed

$$F\omega(t, \xi) = F\omega^0(\xi) \quad \text{and} \quad F\omega_\nu(t, \xi) = e^{-\nu t |\xi|^2} F\omega^0(\xi).$$

Besides,

$$F\omega^0(\xi) = \frac{2}{|\xi|} J_1(|\xi|),$$

where $J_1$ is the Bessel function of order 1, given by

$$J_1(x) = \left(\frac{x}{2}\right) \sum_{n=0}^{\infty} \frac{(-1)^n}{n! (n+2)} \left(\frac{x}{2}\right)^{2n}.$$

From Biot–Savart’s law we get,

$$Fv(t, \xi) = \frac{1}{2\pi |\xi|^2} F\omega(t, \xi) \begin{pmatrix} \xi_2 \\ -\xi_1 \end{pmatrix},$$

$$Fv_\nu(t, \xi) = \frac{1}{2\pi |\xi|^2} F\omega_\nu(t, \xi) \begin{pmatrix} \xi_2 \\ -\xi_1 \end{pmatrix}.$$

Therefore,

$$\| (Fv_\nu - Fv)(t) \|_{L^2}^2 = \int_0^{+\infty} \frac{(1 - e^{-\nu tr^2})^2}{r^3} J_1^2(r) \, dr.$$

Fix a large constant $A > 0$ independent of $\nu$ and $t$. From the above equality, we infer

$$\| (Fv_\nu - Fv)(t) \|_{L^2}^2 \geq (1 - e^{-A})^2 \int_0^{+\infty} \frac{J_1^2(r)}{\sqrt{A/(\nu t)}} \, dr.$$

On the other hand, we have the following classical result (see, e.g., [13, p. 195]):

$$J_1^2(r) \approx +\infty \frac{\sin^2 r}{r}.$$  \hspace{1cm} (12)
Hence, if \( A \) has been chosen large enough and if \( \nu t \leq 1 \),

\[
\left\| (F_{\nu}v - Fv)(t) \right\|_{L^2}^2 \gtrsim \int_{\infty}^{+\infty} \frac{\sin^2 r}{A/(\nu t)} \frac{dr}{r^4},
\]

whence

\[
\left\| (F_{\nu}v - Fv)(t) \right\|_{L^2}^2 \gtrsim \sum_{k \geq (1/\pi)\sqrt{A/(\nu t)}} \int_{\pi/4 + k\pi}^{\pi/4 + k\pi} \frac{dr}{r^4} \gtrsim \left( \frac{\nu t}{A} \right)^3.
\]

\[\square\]

Acknowledgements

A proof of the optimality of the rate \( (\nu t)^{3/4} \) for circular vortex patches has been obtained independently by Wu [14].

References