Navier-Stokes equations with variable density

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Abstract. We study the unique solvability of density-dependent incompressible Navier-Stokes equations in the whole space $\mathbb{R}^N$ or in the torus $\mathbb{T}^N$ ($N \geq 2$). We aim at extending to this new context the celebrated results of well-posedness in $H^{N-1}$ for Navier-Stokes equations with constant density (proved by H. Fujita and T. Kato in [13]). In the present paper, we show that local well-posedness holds for viscous fluids with non vanishing density. We further state global well-posedness in dimension $N = 2$. In large dimension, we show that global unique solutions do exist for small perturbations of an initial state with zero velocity and constant density. Our functional setting is very close to the one used by H. Fujita and T. Kato.

Introduction

These last two decades, an increasing number of papers have been published on the so called Navier-Stokes equations

\begin{equation}
\begin{aligned}
&\partial_t v + v \cdot \nabla v - \mu \Delta v + \nabla \Pi = 0, \\
&\text{div} v = 0, \\
&v|_{t=0} = v_0.
\end{aligned}
\end{equation}

(0.1)

Roughly, two main approaches have been used for solving the initial value problem associated to (0.1). The first one goes back to the pioneering work by J. Leray in 1934 (see [15]) and takes advantage of the conservation of energy, namely

\begin{equation}
E(t) \equiv \|u(t)\|_{L^2}^2 + 2\mu \int_0^t \|\nabla u(\tau)\|_{L^2}^2 \, d\tau \leq \|v_0\|_{L^2}^2,
\end{equation}

(0.2)

to build global (weak) solutions with finite energy. This approach is very satisfactory in dimension $N = 2$ as uniqueness may be proved. On the other hand, the question of uniqueness for finite energy solutions when $N \geq 3$ has remained an open problem.

A second kind of approach has been introduced by H. Fujita and T. Kato in the sixties (see e.g [13]). It is based on contractive mapping arguments and on the smoothing properties of the semi-group of the heat equation. That approach is particularly efficient in functional spaces which have the same scaling invariance as (0.1), in the sense that their norm is invariant for all $\ell > 0$ by the transformation

\begin{equation}
v_0(x) \mapsto \ell v_0(\ell x), \quad v(t, x) \mapsto \ell v(\ell^2 t, \ell x).
\end{equation}

(0.3)

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It leads to the following type of statement:

**“Typical statement”:** Let $E$ and $F \subset C(\mathbb{R}^+; E)$ be two functional spaces whose norm is invariant by (0.3). For $T > 0$, denote by $F_T$ the local version of $F$ pertaining to functions defined on $[0, T]$. Under appropriate compatibility conditions on $E$ and $F$, the following result holds true: for any data $v_0 \in E$, there exists $T > 0$ such that (NSI) has a unique local solution $v \in F_T$. If in addition $\|v_0\|_E < \mu$, then that solution is global.

In dimension $N = 3$, the first example of spaces $(E, F)$ for which the “typical statement” holds has been given by H. Fujita and T. Kato in [13]. There, $E$ is the homogeneous Sobolev space $\dot{H}^\frac{N-3}{2}$ and

$$F = \left\{ u \in C(\mathbb{R}^+; \dot{H}^\frac{N}{2}) : \right. \left. t^\frac{1}{2} \nabla u \in C(\mathbb{R}^+; L^2) \quad \text{and} \quad t^\frac{1}{2} \nabla u \longrightarrow_{t \to 0} 0 \right\}.$$  

It turns out that in dimension $N = 2$, the spaces $E = L^2$ and $F = L^2(\mathbb{R}^+; \dot{H}^1)$ are adapted so that two-dimensional Kato’s and Leray’s solutions are actually the same.

Since then, (0.1) has been shown to be well-posed in a number of functional spaces: Lebesgue spaces $L^N$, Besov spaces $\dot{B}_{p,r}^k$, and so on. Yet, the fundamental problem of proving the existence of global smooth solutions for any large smooth data has not been solved.

Though exciting from a mathematical viewpoint, studying (0.1) is somewhat disconnected to applications in fluid mechanics. Indeed a “real fluid” is hardly homogeneous or incompressible. We here aim at investigating the robustness of the “typical statement” for incompressible fluids with variable density. A similar concern is also relevant for compressible fluids (see e.g [4]).

The equations we are interested in read:

$$\begin{align*}
\partial_t \rho + \text{div } \rho u &= 0, \\
\partial_t (\rho u) + \text{div } (\rho u \otimes u) - \mu \Delta u + \nabla \Pi &= 0, \\
\text{div } u &= 0, \\
(\rho, u)|_{t=0} &= (\rho_0, u_0),
\end{align*}$$

(0.4)

where $\rho = \rho(t, x) \in \mathbb{R}^+$ stands for the density, and $u = u(t, x) \in \mathbb{R}^N$, for the velocity field. The term $\nabla \Pi$ (the gradient of the pressure) may be seen as the Lagrange multiplier associated to the constraint $\text{div } u = 0$. The initial conditions $(\rho_0, u_0)$ are prescribed. Unless otherwise specified, we shall assume throughout that $x$ belongs to the whole space $\mathbb{R}^N$ or to the torus $T^N$.

Leray’s approach is still relevant for (0.4): assuming that $\rho_0 \in L^\infty$ is non-negative and that $\sqrt{\rho_0}u_0 \in L^2$, one can prove the existence of global weak solutions $(\rho, u)$ with finite energy:

$$\|\sqrt{\rho}u(t)\|_{L^2}^2 + 2\mu \int_0^t \|\nabla u(\tau)\|_{L^2}^2 \, d\tau \leq \|\sqrt{\rho_0}u_0\|_{L^2}^2.$$  

(0.5)

In dimension $N = 2$ and under the additional assumption $\nabla u_0 \in L^2$, one can further get a pseudo conservation law involving the norm of $\nabla u$ in $L^\infty(0, T; L^2)$, and of $\partial_t u$, $\nabla \Pi$ and $\nabla^2 u$ in $L^2(0, T; L^2)$. This provides smoother weak solutions. Even in this latter case however, the problem of uniqueness has not been solved. We refer to [1] and to [16] for an overview of results on weak solutions. Some recent improvements have been obtained by B. Desjardins in [10], [11] and [12].
Very few works address the question of unique solvability for (0.4). The main paper in this direction is probably the one written by O. Ladyzhenskaia and V. Solonnikov in the seventies (see [14]). There, the authors consider system (0.4) in a bounded domain \( \Omega \) with homogeneous Dirichlet boundary conditions. Under the assumption that \( u_0 \in W^{2-\frac{2}{q}, q} \) (\( q > N \)) and \( \rho_0 \in C^1(\overline{\Omega}) \) is bounded away from zero, the results are the following:

- In dimension \( N = 2 \), global well-posedness holds.
- In dimension \( N = 3 \), local well-posedness holds. If in addition \( u_0 \) is small in \( W^{2-\frac{2}{q}, q} \), then global well-posedness holds.

We aim at finding a framework as general as possible in which unique solvability holds true. We also would like this framework to be compatible with the “typical statement” in the case where \( \rho \) is a positive constant.

Scaling considerations should help us to find an adapted functional framework. Remark that system (0.4) is invariant under the transformation

\[
(\rho(t), u(x)) \mapsto (\rho(\ell^2 t, \ell^2 x), \ell^2 u(\ell^2 t, \ell^2 x)).
\]

Therefore choosing initial data \( (\rho_0, u_0) \) such that \( \nabla \rho_0 \) and \( u_0 \) belong to an adapted space \( E \) for the “ordinary” Navier-Stokes equations should give satisfactory results. Taking \( (\rho_0, u_0) \) in \( W^{1,N} \times (L^N)^N \) seems to be a possible choice. This should be compared with the spaces used by Ladyzhenskaia and Solonnikov. As a matter of fact, in a work in progress devoted to the case of bounded domains, we extend the results by Ladyzhenskaia and Solonnikov to such rough initial data (see [8]).

In the present work, we consider fluids in \( \mathbb{R}^N \) or \( T^N \) and use Sobolev spaces. Therefore, (0.6) suggests to choose initial data \( (\rho_0, u_0) \) such that \( \nabla \rho_0 \) and \( u_0 \) belong to \( \dot{H}^{\frac{N}{2}} \). As system (0.4) degenerates if \( \rho \) vanishes or becomes unbounded, we further assume that \( \rho_0^{\frac{1}{2}} \in L^\infty \). We also suppose that \( \rho \) tends to some positive constant (say 1) at infinity (or has average 1 in the periodic case).

From now on, we make the change of unknown \( a \equiv 1/\rho - 1 \), which leads to the following system:

\[
\begin{align*}
\partial_t a + u \cdot \nabla a &= 0, \\
\partial_t u + u \cdot \nabla u - \mu(1+a)\Delta u + (1+a)\nabla \Pi &= 0, \\
div u &= 0.
\end{align*}
\]

For (at least) technical reasons however, we did not obtain well-posedness in the critical Sobolev spaces described above. Either the data have to be (slightly) more regular, or \( \rho_0 \), close to a constant. As a matter of fact, the critical regularity exponent may be achieved in the framework of Besov spaces \( \dot{B}_{2,1}^{\frac{N}{2}} \) (see the definition in section 1), and under a smallness hypothesis.

Our first well-posedness result reads:

**Theorem 0.1.** There exists a constant \( c \) depending only on \( N \), and such that for all \( u_0 \in \dot{B}_{2,1}^{\frac{N}{2}} \) with \( \div u_0 = 0 \), and \( a_0 \in \dot{B}_{2,1}^{\frac{N}{2}} \) with \( \|a_0\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \leq c \), there is a \( T \in (0, \infty) \) such that system (0.7) has a unique solution \( (a, u, \nabla \Pi) \) with \( a \in C_0([0, T]; \dot{B}_{2,1}^{\frac{N}{2}}) \), \( u \in C_0([0, T]; \dot{B}_{2,1}^{\frac{N}{2}}) \cap L^1(0, T; \dot{B}_{2,1}^{\frac{N}{2}}) \), \( \nabla \Pi \in L^1(0, T; \dot{B}_{2,1}^{\frac{N}{2}}) \). If in addition \( \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \leq c \mu \), one can take \( T = +\infty \).
Remark 0.2. As $\dot{B}^2_{2,1} \hookrightarrow L^\infty$, we have a $L^\infty$ control on the density for free!

Remark 0.3. In dimension $N=2$, the above result means that system (0.4) is “almost” well-posed for $u_0 \in L^2$ and $\rho_0$ close to a constant in $H^1 \cap L^\infty$, a result which may be compared with the assumptions needed for getting global weak solutions à la Leray.

For more drastic perturbations of homogeneous fluids, we obtain the following result:

Theorem 0.4. Let $\alpha \in (0,1)$, $\beta \in (0,1+\alpha)$, $a_0 \in H^{\frac{N}{2}+\alpha}$ with $0 < \frac{N}{2} \leq 1 + \alpha$, and $u_0 \in H^{\frac{N}{2}-1+\beta}$ with div $u_0 = 0$. There exists a $T > 0$ such that system (0.7) has a unique solution $(a,u,\nabla u)$ in the space

$$E^\alpha,\beta_T \defeq \tilde{C}_T(H^{\frac{N}{2}+\alpha}) \times \left( \tilde{C}_T(H^{\frac{N}{2}+\beta-1}) \cap \mathring{L}^1_T(H^{\frac{N}{2}+\beta+1}) \right)^N \times \left( \mathring{L}^1_T(H^{\frac{N}{2}+\beta-1}) \right)^N.$$

Moreover, $0 < \frac{N}{2} \leq 1 + \alpha$ and the following energy equality is satisfied:

$$(0.8) \quad \forall t \in [0,T], \| (\sqrt{\rho}u)(t) \|^2_{L^2} + 2\mu \int_0^t \| \nabla u(\tau) \|^2_{L^2} \, d\tau = \| \sqrt{\rho_0} u_0 \|^2_{L^2}.$$

Above, $\tilde{C}_T(H^s)$ stands for a (large) subspace of $C([0,T];H^s)$, and $\mathring{L}^1_T(H^s)$ is slightly larger than $L^1(0,T;H^s)$. The reader is referred to definition 1.5 for more details.

Remark 0.5. Higher orders of regularity may be considered.

In dimension $N=2$, the system with variable non-vanishing density is globally well-posed, a result which is in agreement with the homogeneous case and with those obtained in [14] in a bounded domain:

Theorem 0.6. Assume that $N=2$, $\alpha \in (0,1)$ and $\beta \in (0,1+\alpha)$. Let $a_0 \in H^{1+\alpha}$ with $0 < \frac{N}{2} \leq 1 + \alpha$, and $u_0 \in H^\beta$ with div $u_0 = 0$. Then system (0.7) has a unique global solution $(a,u,\nabla u)$ which belongs to $E^\alpha,\beta_T$ for all $T > 0$ and satisfies $0 < \frac{N}{2} \leq 1 + \alpha$ and the energy equality.

Remark 0.7. For the sake of simplicity, we did not consider any external force $f$ in the right-hand side of the momentum equation in (0.7). It is not hard to extend theorem 0.1 to the case $f \in L^1(0,T;\dot{B}^{\frac{N}{2}-1}_{2,1})$ whereas theorem 0.4 holds with $f \in \mathring{L}^1_T(H^{\frac{N}{2}+\beta})$.

Remark 0.8. In dimension $N=2$, the situation is somewhat strange. On one hand, there exists global weak solutions (whose uniqueness has not been stated) for rather smooth initial velocity $u_0 \in H^1$ but rough initial density $(\rho_0 \in L^\infty$ only). On the other hand, we have just stated that unique global strong solutions do exist for rough velocity $(u_0 \in H^\beta$ only) but smooth density $(\rho_0 \in H^{1+\alpha})$.

Open questions:

- Even in the Besov spaces framework, we do not know how to get the critical regularity exponent if $a_0$ is not small.
- Global existence in $\mathbb{R}^3$ or $\mathbb{T}^3$ for small $u_0$ but large $a_0$ has still to be proved. Yet this qualitative result is known to be true in bounded domains (see [14]).
The paper is structured as follows. In the first section, we define some notation and functional spaces. Section 2 is devoted to the study of well-posedness under the assumption that $a_0$ is small. In the third section, we remove the smallness assumption on $a_0$ and sketch the proof of theorem 0.4. Global existence of strong solutions in dimension $N = 2$ is studied in the last section.

1. Definitions and notation

Throughout the paper, $C$ stands for a “harmless constant” whose precise meaning will be clear from the context. We shall sometimes alternatively use the notation $A \lesssim B$ instead of $A \leq CB$, and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

To define the Besov spaces that we shall use, and to localize system (0.7), a Littlewood-Paley decomposition is needed. An (homogeneous) Littlewood-Paley decomposition relies upon a dyadic partition of unity: let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ be supported in, say, $\mathcal{C} = \{\xi \in \mathbb{R}^N, 3/4 \leq |\xi| \leq 8/3\}$ and such that $\sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1$ if $\xi \neq 0$.

Denoting $h = \mathcal{F}^{-1}\varphi$, we then define the dyadic blocks as follows

$$\dot{\Delta}_q u \overset{\text{def}}{=} \varphi(2^{-q}D)u = 2^{qN} \int_{\mathbb{R}^N} h(2^{q}y)u(x-y) \, dy.$$  

The formal decomposition $u = \sum_{q \in \mathbb{Z}} \dot{\Delta}_q u$ holds true modulo polynomials: if $u \in S'(\mathbb{R}^N)$ then $\sum_{q \in \mathbb{Z}} \dot{\Delta}_q u$ converges modulo $\mathcal{P}[\mathbb{R}^N]$ and (1.1) holds in $S'(\mathbb{R}^N)/\mathcal{P}[\mathbb{R}^N]$.

One can now define homogeneous Besov spaces:

**Definition 1.1.** For $s \in \mathbb{R}, (p,r) \in [1, +\infty]^2$ and $u \in S'(\mathbb{R}^N)$, we set

$$\|u\|_{\dot{B}^s_{p,r}} = \left(\sum_{q \in \mathbb{Z}} 2^{rsq} \|\dot{\Delta}_q u\|_{L^p}^r\right)^{\frac{1}{r}}$$

with the usual change if $r = +\infty$, and we denote

$$\dot{B}^s_{p,r} = \left\{ u \in S \mid \|u\|_{\dot{B}^s_{p,r}} < +\infty \right\}.$$ 

For $s < N/p$ ($s \leq N/p$ if $r = 1$), we then define $\dot{B}^s_{p,r}$ as the completion of $\dot{B}^s_{p,r}$ for $\|\cdot\|_{\dot{B}^s_{p,r}}$. If $m \in \mathbb{N}$ and $N/p + m \leq s < N/p + m + 1$ ($N/p + m < s \leq N/p + m + 1$ if $r = 1$), $\dot{B}^s_{p,r}$ is defined as the subset of distributions $u \in S'$ modulo a polynomial of degree $m$, and such that $\partial^\alpha u \in \dot{B}^{s-m}_{p,r}$ whenever $|\alpha| = m$.

The definition of $\dot{B}^s_{p,r}$ does not depend on the choice of the Littlewood-Paley. We shall make an extensive use of spaces $\dot{B}^s_{2,2}$ which are nothing but the “usual” Sobolev spaces $H^s$, and of the slightly smaller spaces $\dot{B}^s_{2,1}$ which have some nice embedding and product properties.

Let us state some classical properties for those Besov spaces.
A more generally, if $A$ is an homogeneous PDO of degree $m$ then

$$
\|A(D)u\|_{B^m_{p,r}} \lesssim \|u\|_{B^m_{p,r}}
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Proposition 1.2. The following properties hold:

i) Derivatives: there exists a universal constant $C$ such that

$$
C^{-1}\|u\|_{B^m_{p,r}} \leq \|\nabla u\|_{B^{m-1}_{p,r}} \leq C\|u\|_{B^m_{p,r}}.
$$

More generally, if $A$ is an homogeneous PDO of degree $m$ then

$$
\|A(D)u\|_{B^m_{p,r}} \lesssim \|u\|_{B^m_{p,r}}
$$

ii) Sobolev embeddings: if $p_1 < p_2$ and $r_1 \leq r_2$ then $B^m_{p_1,r_1} \hookrightarrow B^m_{p_2,r_2}$.

In particular, we have

$$
\dot{B}^{s}_{2,1} \hookrightarrow \dot{H}^s \hookrightarrow \dot{B}^s_{2,\infty} \quad \text{and} \quad \dot{B}^{s}_{2,1} \hookrightarrow L^\infty.
$$

iii) Algebraic properties: for $s > 0$, $\dot{B}^{s}_{p,r} \cap L^\infty$ is an algebra. So does $\dot{B}^\infty_{2,1}$.

iv) Complex interpolation: $[\dot{B}^{s}_{p_1,r_1}, \dot{B}^{s}_{p_2,r_2}]_\theta = \dot{B}^{\theta s + (1-\theta)s_2}_{2,1}$.

Let us now state some results of continuity in Besov spaces for the usual product

(see [17], section 4.4, for more details):

Proposition 1.3. The following inequalities hold true:

$$
\|uv\|_{\dot{B}^s_{2,1}} \lesssim \|u\|_{L^\infty} \|v\|_{\dot{B}^s_{2,1}} + \|v\|_{L^\infty} \|u\|_{\dot{B}^s_{2,1}}, \quad \text{if} \quad s > 0,
$$

(1.3) $\|uv\|_{\dot{B}^{s_1+s_2}_{2,1}} \lesssim \|u\|_{\dot{B}^{s_1}_{2,1}} \|v\|_{\dot{B}^{s_2}_{2,1}}$ if $s_1, s_2 \leq N/2$ and $s_1 + s_2 > 0$.

Remark 1.4. We shall make an extensive use of the estimate

$$
\|uv\|_{\dot{B}^s_{2,1}} \lesssim \|u\|_{\dot{B}^\infty_{2,1}} \|v\|_{\dot{B}^s_{2,1}}.
$$

For treating the case when the density may be a large perturbation of a constant, the usual nonhomogeneous Sobolev spaces $H^s$ will be needed. We just want to emphasize here that they may be alternately defined by means of a nonhomogeneous Littlewood-Paley decomposition:

$$
\Delta_q = \dot{\Delta}_q \quad \text{if} \quad q \geq 0,
$$

$$
\Delta_{-1} = \chi(D) \quad \text{with} \quad \chi(\xi) = 1 - \sum_{q \geq 0} \varphi(2^{-q} \xi).
$$

Now, it is easy to check that

$$
\|u\|_{H^s} \overset{def}{=} \left( \left( 1 + |\xi|^2 \right)^s \left| \hat{u}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}} \approx \left( \sum_{q \geq -1} 2^{qs} \|\Delta_q u\|_{L^2}^2 \right)^{\frac{1}{2}}.
$$

For $X$ Banach space, we denote by $C([0,T]; X)$ the set of continuous functions on $[0,T]$ with values in $X$, and by $C_b([0,T]; X)$ the subset of bounded functions of $C([0,T]; X)$. For $p \in [1, +\infty]$, the notation $L^p(0,T; X)$ stands for the set of measurable functions on $[0,T]$ with values in $X$, such that $t \mapsto \|f(t)\|_X$ belongs to $L^p(0,T)$.

In the present paper, $X$ will be a (homogeneous) Besov space $\dot{B}^s_{2,1}$ or a nonhomogeneous Sobolev space $H^s$. In the former case, we will be induced to localize the equations through Littlewood-Paley decomposition, therefore ending up with $L^2$ estimates for each dyadic block. Performing a time integration should be the next step, but in doing so, we obtain bounds in spaces which are not of type $L^p(0,T; H^s)$ (except if $p = 2$). That remark naturally leads to the following definition:
Definition 1.5. Let $\rho \in [1, +\infty]$, $T \in [0, +\infty]$ and $s \in \mathbb{R}$. We set

$$
\|u\|_{L^p_s(H^r)} \overset{\text{def}}{=} \left( \sum_{q \geq -1} 2^{2qs} \left( \int_0^T \|\Delta_q u(t)\|_{L^p}^p \, dt \right) \right)^{\frac{1}{p}}.
$$

Let us stress that by virtue of Minkowski inequality, we have

$$
\|u\|_{L^p_s(H^r)} \leq \|u\|_{L^p_s(H^r)} \quad \text{if} \quad \rho \leq 2,
$$

$$
\|u\|_{L^p_s(H^r)} \leq \|u\|_{L^p_s(H^r)} \quad \text{if} \quad \rho \geq 2.
$$

We will often use the following interpolation inequality:

$$(1.4)\quad \|u\|_{L^p_s(H^r)} \leq \|u\|_{L^p_s(H^{r+1})} \|u\|_{L^p_s(H^{r-2})} \quad \text{with} \quad \frac{1}{\rho} = \frac{\theta}{\rho_1} + \frac{1-\theta}{\rho_2}, \quad s = \theta s_1 + (1-\theta)s_2.
$$

Let us state some estimates for the product in those spaces, the proof of which is straightforward adaptation of the one for ordinary Sobolev spaces (see e.g. [17]):

**Proposition 1.6.** If $s > 0$ and $1/\rho_2 + 1/\rho_3 = 1/\rho_1 + 1/\rho_4 = 1/\rho \leq 1$ then

$$
\|uv\|_{L^p_s(H^r)} \lesssim \|u\|_{L^p_s(L^{\infty})} \|v\|_{L^p_s(H^r)} + \|v\|_{L^p_s(L^{\infty})} \|u\|_{L^p_s(H^r)}.
$$

If $s_1, s_2 < N/2$, $s_1 + s_2 > 0$ and $1/\rho_1 + 1/\rho_2 = 1/\rho \leq 1$ then

$$
\|uv\|_{L^p_s(H^{r+1}+s_2-s_1)} \lesssim \|u\|_{L^p_s(H^{r+1})} \|v\|_{L^p_s(H^{r-2})}.
$$

2. Small perturbation of an homogeneous fluid

In this section, we give an insight into the proof of theorem 0.1.

2.1. A priori estimates for the solution. Introducing $\mathcal{P}$ the Leray projector on solenoidal vector fields, system (0.7) rewrites

$$
\begin{cases}
\partial_t a + u \cdot \nabla a = 0, \\
\partial_t u - \mu\Delta u = \mathcal{P} \left( a(\mu\Delta u - \nabla \Pi) - u \cdot \nabla u \right), \\
\Delta \Pi = \text{div} \left( a(\mu\Delta u - \nabla \Pi) \right) - \text{div}(u \cdot \nabla u).
\end{cases}
$$

As $a$ is merely advected by the vector field $u$ whose gradient belongs to $L^1(0, T; \mathbb{L}^\infty)$ ($\nabla u$ is expected to belong to $L^1(0, T; \mathring{B}_{2,1}^\infty)$, and $\mathring{B}_{2,1}^\infty \hookrightarrow \mathbb{L}^\infty$), the initial regularity of $a$ is preserved. More precisely, we have (see the proof in the appendix of [8]):

**Proposition 2.1.** Let $s \in (-1-N/2, 1+N/2)$ and $v$ be a solenoidal vector field such that $\nabla v$ belongs to $L^1(0, T; \mathring{B}_{2,1}^s)$. Suppose that $f_0 \in \mathring{B}_{2,1}^s$, $F \in L^1(0, T; \mathring{B}_{2,1}^s)$ and that $f \in L^\infty(0, T; \mathring{B}_{2,1}^s) \cap C([0, T]; S')$ solves

$$
\begin{cases}
\partial_t f + v \cdot \nabla f = F, \\
f_{t=0} = f_0.
\end{cases}
$$

There exists a constant $C$ depending only on $s$ and $N$, and such that the following inequality holds true:

$$(2.2)\quad \|f(t)\|_{\mathring{B}_{2,1}^s} \leq e^{CV(t)} \left( \|f_0\|_{\mathring{B}_{2,1}^s} + \int_0^t e^{-CV(\tau)} \|F(\tau)\|_{\mathring{B}_{2,1}^s} \, d\tau \right)
$$

with $V(t) = \int_0^t \|\nabla v(\tau)\|_{\mathring{B}_{2,1}^s} \, d\tau$. Moreover, $f$ belongs to $C([0, T]; \mathring{B}_{2,1}^s)$. 

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Therefore, we have

\[(2.3) \quad \|a\|_{L_T^\infty(B_{2,1}^\infty)} \leq e^{C\|\nabla u\|_{L_T^\infty(a_{x,1}^N)}} \|a_0\|_{B_{2,1}^N}^N.\]

Now, two types of terms appear in the right-hand side of the remaining equations in (2.1). We do not have to worry about the quadratic term $u \cdot \nabla u$ which is also present in the “ordinary” Navier-Stokes equations. On the other hand, taking a small enough should enable us to get rid of the “new term” $a(\mu \Delta u - \nabla \Pi)$ (as it is going to be absorbed in the estimates).

Of course, a functional framework has to be found in which those arguments are rigorous. For that aim, we shall use the following estimates for the heat equation (due to J.-Y. Chemin in [3]) :

**Proposition 2.2.** Let $s \in \mathbb{R}$ and $v_0$ be a divergence-free vector field with coefficients in $B_{2,1}^{s-1}$ and $f$ be a time dependent vector field with coefficients in $L^1(0,T; \dot{B}_{2,1}^{s-1})$. Let $v \in C([0,T[; \dot{B}_{2,1}^{s-1}) \cap L^1(0,T; \dot{B}_{2,1}^{s+1})$ be a solution to the heat equation

\[
(2.4) \quad \begin{cases}
\partial_t v - \mu \Delta v = f, \\
v_{|t=0} = v_0.
\end{cases}
\]

There exists $c = c(N)$ such that the following estimate holds:

\[
\|v\|_{L_T^\infty(B_{2,1}^{s+1})} + c\|v\|_{L_T^\infty(B_{2,1}^{s+1})} \leq \|v_0\|_{B_{2,1}^{s-1}} + \|f\|_{L_T(B_{2,1}^{s-1})}.
\]

Applying the proposition above to the second equation in (3.2) and using that $\mathcal{P}$ is an homogeneous PDO of degree 0 yields

\[
\|u\|_{L_T^\infty(B_{2,1}^{s-1})} + \mu\|u\|_{L_T^1(B_{2,1}^{s+1})} \lesssim \|u_0\|_{B_{2,1}^{s-1}} + \|u \cdot \nabla u\|_{L_T^1(B_{2,1}^{s-1})} + \|a(\mu \Delta u - \nabla \Pi)\|_{L_T^1(B_{2,1}^{s-1})}.
\]

By virtue of (1.3), we get

\[
(2.5) \quad \|a\|_{L_T^\infty(B_{2,1}^{s-1})} + \mu\|a\|_{L_T^1(B_{2,1}^{s+1})} \lesssim \|u_0\|_{B_{2,1}^{s-1}} + \|u \cdot \nabla u\|_{L_T^1(B_{2,1}^{s-1})} + \|\Delta u\|_{L_T^p(B_{2,1}^{s-1})} + \|\nabla \Pi\|_{L_T^1(B_{2,1}^{s-1})}.
\]

From the last equation in (2.1), we gather

\[
\|\nabla \Pi\|_{L_T^1(B_{2,1}^{s-1})} \lesssim \|a\|_{L_T^\infty(B_{2,1}^{s-1})} \left( \mu\|\Delta u\|_{L_T^1(B_{2,1}^{s-1})} + \|\nabla \Pi\|_{L_T^\infty(B_{2,1}^{s-1})} \right) + \|u\|_{L_T^\infty(B_{2,1}^{s-1})} \|\nabla u\|_{L_T^1(B_{2,1}^{s-1})}.
\]

Adding that latter inequality to (2.5) yields

\[
\|u\|_{L_T^\infty(B_{2,1}^{s-1})} + \mu\|u\|_{L_T^1(B_{2,1}^{s+1})} + \|\nabla \Pi\|_{L_T^1(B_{2,1}^{s-1})} \lesssim \|u_0\|_{B_{2,1}^{s-1}} + \|a\|_{L_T^\infty(B_{2,1}^{s-1})} \left( \mu\|u\|_{L_T^1(B_{2,1}^{s+1})} + \|\nabla \Pi\|_{L_T^1(B_{2,1}^{s+1})} \right) + \|u\|_{L_T^\infty(B_{2,1}^{s-1})} \|\nabla u\|_{L_T^1(B_{2,1}^{s-1})}.
\]

Now, it is clear that if $a_0$ has been chosen small enough, $a$ will remains small on some non empty time interval (see (2.3). If in addition, $u_0$ is also small, the last line in the above inequality may be absorbed by the left-hand side, which enables us to close the estimates globally in time.
A similar result holds on a bounded time interval even if \( u_0 \) is not small. It is only a matter of splitting \( u \) into \( u = \bar{u} + u_L \), where \( u_L \) is the free solution to the heat equation (2.4) with right-hand side 0 and initial datum \( u_0 \), then going along the same lines as for the case \( u \) small (see [6] for more details).

Once the above estimates stated, the existence of solution may be easily proved by mean of a standard iterative scheme.

### 2.2. Remarks on uniqueness.

The equations for

\[
(\delta a, \delta u, \delta \Pi) \overset{\text{def}}{=} (a^2 - a^1, u^2 - u^1, \Pi^2 - \Pi^1)
\]

read

\[
\begin{align*}
\partial_t \delta a + u^2 \cdot \nabla \delta a &= -\delta a \cdot \nabla a^1, \\
\partial_t \delta u + u^2 \cdot \nabla \delta u - \mu \Delta \delta u + \nabla \delta \Pi &= -\delta a \cdot \nabla u^1 + a^1 (\mu \Delta \delta u - \nabla \delta \Pi) + \delta a (\mu \Delta u^2 - \nabla \Pi^2).
\end{align*}
\]

(2.6)

Obviously, the right-hand side of the first equation is responsible for the loss of one derivative in the estimates involving \( \delta a \). This induces us to bound \((\delta a, \delta u, \nabla \delta \Pi)\) in the space

\[
C([0, T]; B^{2\frac{N}{N+1}}_2) \times \left( L^1_t (B^{N+1}_{2,1}) \cap C([0, T]; B^{N-2}_{2,\infty}) \right)^N \times \left( L^1_t (B^{N-2}_{2,1}) \right)^N.
\]

This loss of one derivative is not going to be much a trouble in dimension \( N \geq 3 \). In the two-dimensional case however, we are induced to use a (forbidden) endpoint estimate in some nonlinear terms in (2.6) : if we assume that \( \delta u \) belongs to \( L^1(0, T; \dot{B}^{1}_{2,1}) \) and \( a^1 \in L^\infty(0, T; \dot{B}^{1}_{2,1}) \), the term \( a^1 \Delta \delta u \) does not belong to \( L^1(0, T; \dot{B}^{1}_{2,1}) \) (as required to close the estimates according to proposition 2.2) but rather to the larger space \( L^1(0, T; \dot{B}^{2}_{2,\infty}) \). The reason why is that the product is continuous from \( \dot{B}^{1}_{2,1} \times \dot{B}^{1}_{2,1} \) to \( \dot{B}^{1}_{2,\infty} \) only. Therefore, the right-hand side of the second equation in (2.6) may be estimated in \( L^1(0, T; \dot{B}^{2}_{2,\infty}) \) only. A straightforward generalization of proposition 2.2 yields bounds for \( \delta a \) in the (larger) space \( L^1(0, T; \dot{B}^{1}_{2,\infty}) \cap L^\infty(0, T; \dot{B}^{1}_{2,\infty}) \). Now, assuming \( a^1 \in L^\infty(0, T; \dot{B}^{1}_{2,1}) \) and \( \delta a \in L^1(0, T; \dot{B}^{1}_{2,\infty}) \) suffices to get an estimate for \( a^1 \Delta \delta a \) in \( L^1(0, T; \dot{B}^{1}_{2,\infty}) \).

Yet, we still are in trouble for the bound for having \( \delta u \) in \( L^1(0, T; \dot{B}^{2}_{2,\infty}) \) does not yield appropriate estimates for the right-hand side of the first equation in (2.6). We lack an additional bound in \( L^1(0, T; L^\infty) \) for \( \delta a \). The key to that ultimate difficulty is given by some logarithmic interpolation inequality. Then a (generalized) Gronwall estimate yields the stability (see [6] for more details).

### 3. Local well-posedness in the general case

In this section, we aim at proving theorem 0.4. Compare to the previous section, we have to get rid of the smallness condition. The price to pay for that is some extra regularity requirement on the data : \( a_0 \) is in \( H^{\frac{N}{2} + \alpha} \) and \( H^{\frac{N}{2} - 1 + \beta} \) for some small positive \( \alpha \) and \( \beta \). We expect to glean some smallness from that extra smoothness so that it will be possible to close estimates even though \( a \) may be large.

Once again, as \( \nabla u \) is expected to be (at least) in \( L^1(0, T; L^\infty) \), there will be no difficulty coming from the transport equation for \( u \). Indeed, slight changes to the
Indeed, if \( s_1 + \alpha \), and \( s_2 \), we are left with the momentum equation.

### 3.1. The linearized momentum equation.

This section is devoted to the proof of estimates for the following linear system:

\[
\begin{align*}
\partial_t u + v \cdot \nabla u + b(\nabla u - \mu \Delta u) &= f, \\
\text{div } u &= 0, \\
u(0) &= u_0,
\end{align*}
\]

where \( b, f, v \) and \( u_0 \) are given. We shall further assume that \( b \overset{\text{def}}{=} \inf_x b(x) > 0 \) and that \( b \) tends to some positive constant (that we can assume to be 1 with no loss of generality) at infinity. We shall denote \( a \overset{\text{def}}{=} b - 1 \).

**Proposition 3.1.** Let \( \alpha \in [0, 1) \), \( p \in [1, +\infty) \), \( p' \) conjugate exponent of \( p \), and \( s \in (1, \min(1, 2/p) + \alpha + N/2) \). Denote \( \bar{\alpha} \overset{\text{def}}{=} \min(\alpha, (s - 1)/2) \) and \( \bar{A}_T \overset{\text{def}}{=} 1 + \frac{1}{2} \tau^{1/2} \). There exists \( C = C(s, N, \alpha, p) \) and a positive exponent \( \kappa = \kappa(s, \alpha) \) such that

\[
\|u\|_{L^\infty_t L^{\frac{\alpha}{1-\alpha}}(H^{\frac{s}{2} - \alpha})} + \mu \|u\|_{L^\infty_t L^{\frac{\alpha}{1-\alpha}}(H^{\frac{s}{2} + \frac{\alpha}{2}})} + \|\nabla u\|_{L^\infty_t L^{\frac{\alpha}{1-\alpha}}(H^{\frac{s}{2} - \alpha})} \leq C A_T^{\bar{A}_T} e^{-C A_T^{\bar{A}_T} V(T)} \times \left( \|u_0\|_{H^{\frac{s}{2} - \alpha}} + \|f\|_{L^1_t L^{\frac{\alpha}{1-\alpha}}(H^{\frac{s}{2} - \alpha})} + \mu \bar{A}_T \|u\|_{L^1_t L^{\frac{\alpha}{1-\alpha}}(H^{\frac{s}{2} + \frac{\alpha}{2}})} \right).
\]

where

\[
V(T) \overset{\text{def}}{=} \begin{cases} 
\int_0^T \|\nabla u(t)\|_{H^{\frac{s}{2} - \alpha}} \, dt & \text{if } p = 1, \\
\int_0^T \|\nabla u(t)\|_{H^{\frac{s}{2} - \alpha}}^{p-1} \, dt & \text{if } p > 1.
\end{cases}
\]

**Sketchy proof of proposition 3.1 in the case \( p = 1 \):** It lies on spectral localization of (3.2) by mean of the nonhomogeneous Littlewood-Paley decomposition defined in section 1.

With no loss of generality, one can assume that \( s \in [1 + 2\alpha, 1 + \alpha + N/2] \). Indeed, if \( s \in (1, 1 + 2\alpha) \), it is only a matter of changing \( \alpha \) into \((s - 1)/2\). Applying \( \Delta_q \) to (3.2), we get

\[
\partial_t \Delta_q u - \mu \text{div}(b \Delta_q \nabla u) + v \cdot \nabla \Delta_q u + \Delta_q \nabla u = \Delta_q f - \Delta_q (a \nabla u) + [v, \Delta_q] \cdot \nabla u + R_q
\]

with \( R_q \overset{\text{def}}{=} \Delta_q (b \nabla u^i) - \text{div}(b \Delta_q \nabla u^i) \).

Taking the \( L^2 \)-scalar product with \( \Delta_q u \) and doing elementary computations yields

\[
\frac{1}{2} \frac{d}{dt} \|\Delta_q u\|_{L^2}^2 + \mu b \|\nabla \Delta_q u\|_{L^2}^2 \leq \|\Delta_q u\|_{L^2} \left( \mu \|R_q\|_{L^2} + \|v, \Delta_q\|_{L^2} + \|\Delta_q f\|_{L^2} \right) + \left( \Delta_q (a \nabla u) \cdot \nabla u \right).
\]
By virtue of Bernstein inequality, we have \( \| \Delta_q \nabla u \|_{L^2} \leq k^{q!} \| \Delta_q u \|_{L^2} \) for some universal constant \( k > 0 \) whenever \( q \geq 0 \). Next, as \( s \in (1 - N/2, 1 + \alpha + N/2) \), the commutators may be bounded according to the appendix of [7]:

\[
\left( \sum_{q \geq -1} 2^{2q(s-1)} \|[v, \Delta_q] \cdot \nabla u \|_{L^2(L^2)}^2 \right)^{\frac{1}{2}} \lesssim \int_0^T \| \nabla v(t) \|_{L^\infty} \| u(t) \|_{H^{s-1}} \, dt,
\]

\[
\left( \sum_{q \geq -1} 2^{2q(s-1)} \| \Delta_q (a \div w) - \div (a \Delta_q w) \|_{L^2(L^2)}^2 \right)^{\frac{1}{2}} \lesssim \| \nabla a \|_{L^\infty(H^{\frac{2}{3} + \alpha}, H^{s-1})} \| w \|_{L^\infty(H^{s-\alpha})}.
\]

Finally, the term \( \langle \Delta_q (a \nabla \Pi) | \Delta_q u \rangle \) may be bounded by mean of elementary paradifferential calculus (namely Bony’s decomposition introduced in [2]) combined with integration by parts. Using also proposition 1.6, one ends up with

\[
\| u \|_{L^\infty(T; H^{s-1})} + \mu b \| u \|_{L^\infty(T; H^{s-1})} \lesssim \| u_0 \|_{H^{s-1}} + \| f \|_{L^\infty(T; H^{s-1})} + \mu b \| u \|_{L^\infty(T; H^{s+\alpha})} + \mu b \| u \|_{L^\infty(T; H^{s+\alpha})} \int_0^T \| \nabla v \|_{H^{\frac{2}{3} + \alpha}, L^\infty} \| u \|_{H^{s-1}} \, dt
\]

\[
(3.3)
\]

The pressure \( \Pi \) verifies an elliptic equation: \( \div (v \nabla \Pi) = \div F \) with

\[
F \overset{\text{def}}{=} f + \mu a \Delta u - v \cdot \nabla u.
\]

In [7], we prove that

\[
\| \nabla \Pi \|_{L^\infty(T; H^{s-1})} \lesssim \left( 1 + \frac{\| \nabla a \|_{L^\infty(T; H^{s+\alpha})}}{b} \right) \| \Pi F \|_{L^\infty(T; H^{s+\alpha})},
\]

where \( Q \overset{\text{def}}{=} I - P \) denotes the projector on gradient vector fields.

According to proposition 1.6, we have

\[
\| \Pi F \|_{L^\infty(T; H^{s-1})} \lesssim \| \Pi F \|_{L^\infty(T; H^{s-1})} + \mu \| \nabla a \|_{L^\infty(T; H^{s+\alpha})} \| \Delta u \|_{L^\infty(T; H^{s-1})} + \int_0^T \| u \|_{H^{s-1}} \| \nabla v \|_{H^{\frac{2}{3} + \alpha}, L^\infty} \, dt.
\]

Now, coming back to (3.3), one concludes that

\[
\| u \|_{L^\infty(T; H^{s-1})} + \mu b \| u \|_{L^\infty(T; H^{s+1})} \lesssim \| u_0 \|_{H^{s-1}} + \| f \|_{L^\infty(T; H^{s-1})} + \mu b A_T^{s+\alpha} \| u \|_{L^\infty(T; H^{s+\alpha})} + A_T^{s+\alpha} \int_0^T \| u \|_{H^{s-1}} \| \nabla v \|_{H^{\frac{2}{3} + \alpha}, L^\infty} \, dt.
\]

Gronwall lemma completes the proof of the desired inequality. \( \square \)

**3.2. Outline of the proof of theorem 0.4.** It suffices to show that the estimates may be closed for small enough time. Indeed, the existence will then stem from some standard iterative scheme, and uniqueness is not going to be a trouble (even in dimension \( N = 2 \)) as we are given some extra smoothness compared to the previous section.

According to inequality (3.1), a control on \( u \) in \( L^1(0, T; H^{\frac{2}{3} + \alpha}) \) provides us with a bound of the type

\[
\| a \|_{L^\infty(T; H^{\frac{2}{3} + \alpha})} \leq M \| a_0 \|_{H^{\frac{2}{3} + \alpha}}.
\]
for some $M \geq 1$.

Applying proposition 3.1 with $p = 1$ and $s = \beta + N/2$ to the momentum equation in (0.7), and using Sobolev embeddings yields

$$U(T) \overset{\text{def}}{=} \| u \|_{L^\infty_T(H^\frac{N}{2} + \beta - 1)} + \mu \| u \|_{L^\infty_T(H^\frac{N}{2} + \beta + 1)} + \| \nabla u \|_{L^1_T(H^\frac{N}{2} + \beta - 1)} \leq CA_T^\beta \| u \|_{L^\infty_T(H^\frac{N}{2} + \beta + 1)} \left( \| u_0 \|_{H^\frac{N}{2} + \beta - 1} + \mu b A_T \| u \|_{L^1_T(H^\frac{N}{2} + \beta + 1 - 1 - \beta)} \right).$$

For closing the estimates, we lack a bound for the last term.

On the other hand, by interpolation, we have

$$\| u \|_{L^\infty_T(H^\frac{N}{2} + \beta + 1 - \beta)} \leq \left( \| u \|_{L^\infty_T(H^\frac{N}{2} + \beta - 1)} \right)^{\frac{2}{3}} \left( \| u \|_{L^1_T(H^\frac{N}{2} + \beta + 1)} \right)^{\frac{1}{3}}$$

so that

$$\mu b \| u \|_{L^1_T(H^\frac{N}{2} + \beta + 1 - \beta)} \leq (\mu b T)^{\frac{2}{3}} U(T).$$

Now, it is clear that this term becomes negligible for small $T$, thus estimates may be closed on a small time interval.

4. Global well-posedness in dimension $N = 2$

Obviously, the proof used hitherto cannot yield global well-posedness even under some smallness condition on the data. The reason why is that a time dependence appears when bounding the last term in the estimate of proposition 3.1. A Poincaré inequality would enable us to avoid the appearance of that bad term. As a matter of fact, it was a key argument for proving global results in bounded two-dimensional domains (see [14]). In the whole space however, no inequality of this type is expected. More surprisingly, the situation is not better in the torus as we do not have a time dependence involving the law involving the $H^1$ norm of the velocity. At first sight, this seems to enforce us to make quite strong an assumption on the velocity, namely that $u_0 \in H^1$. We shall see later on how it can be removed.

4.1. Proof in the case $\beta = 1$.

First step : Local well-posedness. Theorem 0.4 provides us with a local strong solution $(a, u, \nabla u)$ which belongs to $E_{T^*}^{\alpha, 1}$ (see the statement of theorem 0.4). We denote by $T^* \in (0, +\infty)$ the maximum time of existence of $(a, u, \nabla u)$, i.e. the supremum of all $T$ such that $(a, u, \nabla u)$ belongs to $E_{T^*}^{\alpha, 1}$. A standard continuation argument shows that $(a, u, \nabla u)$ cannot belong to $E_{T^*}^{\alpha, 1}$ unless $T^* = +\infty$.

Second step : Existence of a global weak solution. In dimension $N = 2$, the existence of a global weak solution $(\tilde{u}, \tilde{u}, \nabla \tilde{u})$ under the hypotheses that $u_0 \in H^1$ and $\rho_0 \in L^\infty$ is bounded away from vacuum is well known (see e.g. [1] or [16]). Actually besides the energy inequality (0.5) and the conservation of $L^\infty$ norm for $\tilde{\rho}^{\pm}$, we have $\nabla \tilde{u} \in L^\infty_{tloc}(\mathbb{R}^+; H^1) \cap L^2_{tloc}(\mathbb{R}^+; H^1)$, $\partial_t \tilde{u}, \nabla \tilde{u} \in L^2_{tloc}(\mathbb{R}^+; L^2)$ and the following inequality with some universal constant $C$

$$\mu \| \nabla \tilde{u}(t) \|_{L^2}^2 + \int_0^t \| \sqrt{\rho} \partial_t \tilde{u} \|_{L^2}^2 \, dt + C^{-1} \int_0^t \left( \| \nabla \tilde{u} \|_{L^2}^2 + \mu^2 \| \Delta \tilde{u} \|_{L^2}^2 \right) \, dt \leq \mu \| \nabla u_0 \|_{L^2}^2 + e^{C \| u_0 \|_{L^4}^4} \int_0^t \| \tilde{u} \|_{L^4}^4 \, dt.$$
Of course, the standard energy inequality (0.5) combined with Gagliardo-Nirenberg inequality yields bounds for the argument of the exponential.

Therefore, we are given with the local smooth unique solution \((a, u, \nabla \Pi)\) and with (at least) another solution \((\tilde{a}, \tilde{u}, \nabla \tilde{\Pi})\) which is global and satisfies the properties described above.

Third step: Loosening a priori estimates for transport equations. At this point, there is no (known) way of proving uniqueness of \((\tilde{a}, \tilde{u}, \nabla \tilde{\Pi})\). Yet, only a part of the assumptions on the data have been used. Indeed, we used that \(\rho_0 \in L^\infty\) while \(a_0 = \rho_0 - 1\) is actually in \(H^{1+\alpha}\). Does anything persist of the initial smoothness?

To answer this question, we shall take advantage of the fact that \(\tilde{a}\) solves a transport equation by the vector field \(\tilde{u}\) which belongs to \(L^2_{\text{loc}}(\mathbb{R}^+; H^2)\). In view of proposition (3.1), the \(H^{1+\alpha}\) regularity of \(\tilde{a}\) would be preserved if \(\nabla \tilde{u}\) were also in \(L^2_{\text{loc}}(\mathbb{R}^+; L^\infty)\) but \(H^1\) fails to be embedded in \(L^\infty\) in dimension \(N = 2\) ! Nevertheless, the following statement holds true:

**Proposition 4.1.** Assume \(N = 2\) and \(0 \leq s < 2\). Let \(f_0 \in H^s\) and \(v \in L^1(0, T; H^2)\). Then the transport equation

\[
\begin{aligned}
\partial_t f + v \cdot \nabla f &= 0, \\
|t = 0 &= f_0,
\end{aligned}
\]

has a unique solution \(f\) which belongs to \(C([0, T]; H^{s-\epsilon})\) for all \(\epsilon > 0\). Moreover, \(\|f(t)\|_{H^{s-\epsilon}}\) may be bounded in terms of \(\epsilon\), \(\|f_0\|_{H^s}\) and \(\|\nabla v\|_{L^1_t(\mathcal{H}^s)}\).

This proposition has been first proved by B. Desjardins in [10] and [12] under a slightly stronger assumption: \(v \in L^2(0, T; H^2)\) and in the case of a bounded domain or in \(\mathbb{T}^2\). In [9], we do a systematic study of loosing a priori estimates for transport equations, and proposition 4.1 comes up as a particular case.

Fourth step: Weak=strong uniqueness. According to the previous step, we have for all \(\alpha' < \alpha\),

\[
\tilde{a} \in C(\mathbb{R}^+; H^{1+\alpha'}), \quad \tilde{u} \in L^\infty_{\text{loc}}(\mathbb{R}^+; H^1) \cap L^2_{\text{loc}}(\mathbb{R}^+; H^2) \quad \text{and} \quad \nabla \tilde{\Pi} \in L^2_{\text{loc}}(\mathbb{R}^+; L^2).
\]

Slight modifications to the proof of uniqueness in section 3 enable us to get

\[
(\tilde{a}, \tilde{u}, \nabla \tilde{\Pi}) = (a, u, \nabla \Pi)
\]

as long as \((a, u, \nabla \Pi)\) belongs to \(E^{\alpha, \beta}_{a, u, \nabla \Pi}\), thus on the time interval \([0, T^*]\).

Fifth step: Continuation. Assume that \(T^* < +\infty\). Applying proposition 3.1 with \(s = 2\), \(p = 2\), and using (0.5) yields

\[
\|u\|_{L^\infty_T(\mathcal{H}^1)} + \mu \|u\|_{L^2_T(\mathcal{H}^3)} + \|\nabla \Pi\|_{L^1_T(\mathcal{H}^1)} \leq C\mathcal{A}_T e^{C\mathcal{A}^{\beta} \int_{\tau}^{T} |v_{a, u, \nabla \Pi(t)}|^2 \alpha dt} \times \left(\|u_0\|_{\mathcal{H}^1} + \mu \mathcal{A}_T \|u\|_{L^2_T(\mathcal{H}^{3-\frac{\alpha}{2}})}\right)
\]

with \(\mathcal{A}_T = 1 + \frac{1}{T_0} \|a\|_{L^\infty_T(\mathcal{H}^{\frac{3}{2}+\epsilon})}\). Now, in view of proposition 4.1 and second step, \(\mathcal{A}_T\) may be bounded by some number involving only the initial data (as \(T^*\) is assumed to be finite). On the other hand, the energy inequality (0.5) yields a uniform bound for \(\nabla u\) in \(L^2(0, T^* \times \mathbb{R}^2)\). Therefore, for some constant \(C_0\) depending only on the initial data, we have

\[
(4.1) \quad \|u\|_{L^\infty_T(\mathcal{H}^1)} + \mu \|u\|_{L^2_T(\mathcal{H}^3)} + \|\nabla \Pi\|_{L^1_T(\mathcal{H}^1)} \leq C_0 \left(\|u_0\|_{\mathcal{H}^1} + \mu \|u\|_{L^2_T(\mathcal{H}^{3-\frac{\alpha}{2}})}\right).
\]
As \( \tilde{L}_T^1(H^{1-\frac{2}{\beta}}) = [\tilde{L}_T^1(H^2), \tilde{L}_T^1(H^3)] \), Young inequality yields
\[
\|u\|_{\tilde{L}_T^1(H^{1-\frac{2}{\beta}})} \leq K \|u\|_{\tilde{L}_T^1(H^2)} + K^{1-\frac{2}{\beta}} \|u\|_{\tilde{L}_T^1(H^3)},
\]
Note that the last term is bounded in terms of the initial data and \( T^* \). Therefore, plugging the above inequality in (4.1) with a small enough \( K \) shows that \((a, u, \nabla \Pi)\) is unique globally strong solution \((0, T)\). Uniqueness ensures that \((\tilde{a}, \tilde{u}, \nabla \Pi)\) (on the time interval \([T_0, +\infty]\)) with data \((a_{T_0}, u_{T_0})\). Uniqueness ensures that \((\tilde{a}, \tilde{u}, \nabla \Pi) \equiv (a, u, \nabla \Pi)\) on \([T_0, T]\). Therefore \((a, u, \nabla \Pi)\) may be continued globally.

4.2. General case \( \beta > 0 \). Let us now treat the case where \( \beta \in (0, 1) \). Once again, theorem 0.4 provides us with a strong unique solution \((a, u, \nabla \Pi)\) in \( \mathcal{E}_T^{\alpha, \beta} \).

In particular, a straightforward interpolation argument show that \( u \) belongs to \( L^2(0, T; H^{1+\beta}) \), thus to \( L^2(0, T; H^1) \). This ensures that \( u(T_0) \) belongs to \( H^1 \) for some \( T_0 < T \). Now, one may go along the proof in the case \( \beta = 1 \) and get a unique global strong solution \((\tilde{a}, \tilde{u}, \nabla \Pi)\) (on the time interval \([T_0, +\infty]\)) with data \((a_{T_0}, u_{T_0})\).

Indeed, it does not give much information on the possible growth of the solution.

Yet it works!

References


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