FEFFERMAN-STEIN INEQUALITIES FOR THE $\mathbb{Z}_2^d$ DUNKL MAXIMAL OPERATOR

LUC DELEAVAL

ABSTRACT. In this article, we establish the Fefferman-Stein inequalities for the Dunkl maximal operator associated with a finite reflection group generated by the sign changes. Similar results are also given for a large class of operators related to Dunkl's analysis.

1. INTRODUCTION

In the early seventies, C. Fefferman and E. M. Stein have proved in [6] the following extension of the Hardy-Littlewood maximal theorem.

Theorem 1.1. *Let $(f_n)_{n\geqslant 1}$ be a sequence of measurable functions defined on $\mathbb{R}^d$ and let M be the well-known maximal operator given by*

$$Mf(x) = \sup \frac{1}{m(Q)} \int_Q |f(y)| \, \mathrm{d}y, \quad x \in \mathbb{R}^d,$$

where the sup *is taken over all cubes Q centered at x and $m(X)$ is the Lebesgue measure of X.*

(1) *If $1 < r < +\infty$, $1 < p < +\infty$ and if $\left(\sum_{n=1}^{\infty} |f_n(\cdot)|^r\right)^{\frac{1}{r}} \in L^p(\mathbb{R}^d; \mathrm{d}m)$, then we have*

$$\left\| \left(\sum_{n=1}^{\infty} |Mf_n(\cdot)|^r \right)^{\frac{1}{r}} \right\|_p \leqslant C \left\| \left(\sum_{n=1}^{\infty} |f_n(\cdot)|^r \right)^{\frac{1}{r}} \right\|_p,$$

where $C = C(r,p)$ is independent of $(f_n)_{n\geqslant 1}$.

(2) *If $1 < r < +\infty$ and if $\left(\sum_{n=1}^{\infty} |f_n(\cdot)|^r\right)^{\frac{1}{r}} \in L^1(\mathbb{R}^d; \mathrm{d}m)$, then for every $\lambda > 0$ we have*

$$m\left(\left\{ x \in \mathbb{R}^d : \left(\sum_{n=1}^{\infty} |Mf_n(x)|^r \right)^{\frac{1}{r}} > \lambda \right\} \right) \leqslant \frac{C}{\lambda} \left\| \left(\sum_{n=1}^{\infty} |f_n(\cdot)|^r \right)^{\frac{1}{r}} \right\|_1,$$

where $C = C(r)$ is independent of $(f_n)_{n\geqslant 1}$ and λ.

2000 *Mathematics Subject Classification.* 42B10, 42B25.

Key words and phrases. Dunkl maximal operator, Dunkl transform, Fefferman-Stein inequalities, Harmonic analysis.

The author is pleased to express his respectful thanks to the referee for his/her careful reading of the manuscript and for his/her comments which contributed to the improvement of the quality of the paper. He also wishes to thank his supervisor Sami Mustapha for sharing his ideas with him.

One would like to extend this result to the case of the Dunkl maximal operator M_κ which is defined according to S. Thangavelu and Y. Xu (see [16]) by

$$M_\kappa f(x) = \sup_{r>0} \frac{1}{\mu_\kappa(B_r)} \left|(f *_\kappa \chi_{B_r})(x)\right|, \quad x \in \mathbb{R}^d,$$

where we denote by χ_X the characteristic function of the set X, by B_r the Euclidean ball centered at the origin and whose radius is r, by μ_κ a weighted Lebesgue measure invariant under the action of a finite reflection group and by $*_\kappa$ the Dunkl convolution operator (see Section 2 for more details).

However, the lack of information on this convolution, which is defined through a generalized translation operator (also called Dunkl translation), prevents from stating a general result. Just as in the study of the weighted Riesz transform associated with the Dunkl transform (see [17]), we can only establish a complete result for the finite reflection group $G \simeq \mathbb{Z}_2^d$ with the associated measure μ_κ given for every $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ by

$$\mathrm{d}\mu_\kappa(x) = h_\kappa^2(x)\,\mathrm{d}x, \tag{1.1}$$

with h_κ the $\mathbb{Z}_2^d$-invariant function defined by

$$h_\kappa(x) = \prod_{j=1}^d |x_j|^{\kappa_j} = \prod_{j=1}^d h_{\kappa_j}(x_j),$$

where $\kappa_1, \dots, \kappa_d$ are nonnegative real numbers (let us note that h_κ is homogeneous of degree $\gamma_\kappa = \sum_{j=1}^d \kappa_j$).

To become more precise, the aim of this paper is to prove the following Fefferman-Stein inequalities, where we denote by $L^p(\mu_\kappa)$ the space $L^p(\mathbb{R}^d; \mathrm{d}\mu_\kappa)$ and we use the shorter notation $\|\cdot\|_{\kappa,p}$ instead of $\|\cdot\|_{L^p(\mu_\kappa)}$. For $p \in [1, +\infty]$, the space $L^p(\mu_\kappa)$ is of course the space of measurable functions on $\mathbb{R}^d$ such that

$$\begin{aligned} \|f\|_{\kappa,p} &= \left(\int_{\mathbb{R}^d} |f(y)|^p \,\mathrm{d}\mu_\kappa(y)\right)^{\frac{1}{p}} < +\infty \quad && \text{if } 1 \leqslant p < +\infty, \\ \|f\|_{\kappa,\infty} &= \operatorname*{ess\,sup}_{y \in \mathbb{R}^d} |f(y)| < +\infty && \text{otherwise.} \end{aligned}$$

Theorem 1.2. *Let $G \simeq \mathbb{Z}_2^d$ and let μ_κ be the measure given by (1.1). Let $(f_n)_{n\geqslant 1}$ be a sequence of measurable functions defined on $\mathbb{R}^d$.*

(1) *If $1 < r < +\infty$, $1 < p < +\infty$ and if $\left(\sum_{n=1}^\infty |f_n(\cdot)|^r\right)^{\frac{1}{r}} \in L^p(\mu_\kappa)$, then we have*

$$\left\| \left(\sum_{n=1}^\infty |M_\kappa f_n(\cdot)|^r \right)^{\frac{1}{r}} \right\|_{\kappa,p} \leqslant C \left\| \left(\sum_{n=1}^\infty |f_n(\cdot)|^r \right)^{\frac{1}{r}} \right\|_{\kappa,p},$$

where $C = C(\kappa_1, \dots, \kappa_d, r, p)$ is independent of $(f_n)_{n\geqslant 1}$.

(2) *If $1 < r < +\infty$ and if $\left(\sum_{n=1}^\infty |f_n(\cdot)|^r\right)^{\frac{1}{r}} \in L^1(\mu_\kappa)$, then for every $\lambda > 0$ we have*

$$\mu_\kappa\left(\left\{ x \in \mathbb{R}^d : \left(\sum_{n=1}^\infty |M_\kappa f_n(x)|^r \right)^{\frac{1}{r}} > \lambda \right\} \right) \leqslant \frac{C}{\lambda} \left\| \left(\sum_{n=1}^\infty |f_n(\cdot)|^r \right)^{\frac{1}{r}} \right\|_{\kappa,1},$$

where $C = C(\kappa_1, \dots, \kappa_d, r)$ is independent of $(f_n)_{n\geqslant 1}$ and λ.

The proof of Theorem 1.1 is mainly based on a maximal theorem, a Calderón-Zygmund decomposition and a weighted inequality. Nevertheless, the Dunkl maximal operator cannot be treated by this method even if a maximal theorem has been established for this one in [16]. This is closely related to the fact that a theory of singular integrals associated with the Dunkl transform seems to be out of reach at the moment.

In order to bypass this problem, we will construct a weighted maximal operator M_κ^R of Hardy-Littlewood type which satisfies the classical Fefferman-Stein inequalities and which controls M_κ in the sense that for every $x \in \mathbb{R}^d_{\text{reg}}$

$$M_\kappa f(x) \leqslant C M_\kappa^R f(x), \tag{1.2}$$

where C is a positive constant independent of x and f and where we set

$$\mathbb{R}^d_{\text{reg}} = \mathbb{R}^d \setminus \bigcup_{j=1}^{d} \big\{x = (x_1, \dots, x_d) \in \mathbb{R}^d : x_j = 0\big\}.$$

The paper is organised as follows.
In the next section, we collect some definitions and results related to Dunkl's analysis. In particular, we list the properties of the Dunkl transform (and the associated tools) which will be relevant for the sequel.
Section 3 is devoted to the proof of Theorem 1.2. In view of this, we will prove the inequality (1.2) thanks to a more convenient Dunkl maximal operator M_κ^Q and we will explain why the classical Fefferman-Stein inequalities hold for the operator M_κ^R. Therefore, there will be nothing more to do to conclude that Theorem 1.2 is true.
An application of our Fefferman-Stein inequalities is given in Section 4.

Throughout this paper, C denotes a positive constant, which depends only on fixed parameters, and whose value may vary from line to line.

2. Preliminaries

This section is devoted to the preliminaries and background. These concern in particular the intertwining operator, the Dunkl transform, the Dunkl translation and the Dunkl convolution. We restrict the statement from Dunkl's analysis to the special case considered in this article. For a large survey about this theory, the reader may especially consult [3, 5, 10, 11, 16, 18].

Let $e_1, \dots, e_d$ be the standard basis of $\mathbb{R}^d$. We denote by σ_j (for each j from 1 to d) the reflection with respect to the hyperplane perpendicular to e_j, that is to say for every $x = (x_1, \dots, x_d) \in \mathbb{R}^d$

$$\sigma_j(x) = x - 2\tfrac{\langle x, e_j\rangle}{\|e_j\|^2} e_j = (x_1, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_d).$$

Of course $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^d \times \mathbb{R}^d$ and $\|\cdot\|$ is the associated norm. Let G be the finite reflection group generated by $\{\sigma_j : j = 1, \dots, d\}$, so G is isomorphic to $\mathbb{Z}_2^d$. Let $\kappa_1, \kappa_2, \dots, \kappa_d$ be nonnegative real numbers.

Associated with these objects are the Dunkl operators $\mathcal{D}_k$ (for $k = 1, \dots, d$) which have been introduced in [4] by C. F. Dunkl. They are given for $x \in \mathbb{R}^d$ by

$$\mathcal{D}_k f(x) = \partial_k f(x) + \sum_{j=1}^{d} \kappa_j \frac{f(x) - f\big(\sigma_j(x)\big)}{\langle x, e_j\rangle} \langle e_k, e_j\rangle = \partial_k f(x) + \kappa_k \frac{f(x) - f\big(\sigma_k(x)\big)}{x_k},$$

where ∂_k denotes the usual partial derivative. A fundamental property of these differential-difference operators is their commutativity, that is to say $\mathcal{D}_k\mathcal{D}_l = \mathcal{D}_l\mathcal{D}_k$.

Closely related to them is the so-called intertwining operator V_κ (the subscript means that the operator depends on the parameters κ_j, except in the rank-one case where the subscript is then a single parameter) which is the unique linear isomorphism of $\bigoplus_{n\geqslant 0}\mathcal{P}_n$ such that

$$V_\kappa(\mathcal{P}_n) = \mathcal{P}_n, \quad V_\kappa(1) = 1, \quad \mathcal{D}_k V_\kappa = V_\kappa \partial_k \text{ for } k = 1, \ldots, d,$$

with $\mathcal{P}_n$ the subspace of homogeneous polynomials of degree n in d variables. Even if the positivity of the intertwining operator has been established in [9] by M. Rösler, an explicit formula of V_κ is not known in general. However, in our setting, the operator V_κ is given according to [20] by the following integral representation

$$V_\kappa f(x) = \int_{[-1,1]^d} f(x_1t_1, \ldots, x_dt_d) \prod_{j=1}^d M_{\kappa_j}(1+t_j)(1-t_j^2)^{\kappa_j-1}\,\mathrm{d}t,$$

with $M_{\kappa_j} = \frac{\Gamma(\kappa_j+\frac{1}{2})}{\Gamma(\kappa_j)\Gamma(\frac{1}{2})}$ (where Γ is the well-known Gamma function).

In order to define the Dunkl transform, we also need to introduce the Dunkl kernel E_κ which is given for $x \in \mathbb{C}^d$ by

$$E_\kappa(\cdot, x)(y) = V_\kappa\big(\mathrm{e}^{\langle \cdot, x\rangle}\big)(y), \quad y \in \mathbb{R}^d.$$

It has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$ and it satisfies the following basic properties: $E_\kappa(x,y) = E_\kappa(y,x)$ for $x, y \in \mathbb{C}^d$, $E_\kappa(x,0) = 1$ for $x \in \mathbb{C}^d$ and $|E_\kappa(ix,y)| \leqslant 1$ for $x, y \in \mathbb{R}^d$. Considering the definition of E_κ together with the explicit formula for V_κ gives us

$$E_\kappa(x,y) = \prod_{j=1}^d E_{\kappa_j}(x_j, y_j).$$

In the rank-one case, E_κ is explicitly known. More precisely, it is given for both x and y in $\mathbb{C}$ by

$$E_\kappa(x,y) = j_{\kappa-\frac{1}{2}}(ixy) + \frac{xy}{2\kappa+1} j_{\kappa+\frac{1}{2}}(ixy),$$

where j_κ is the normalized Bessel function of the first kind and of order κ (see [19]). Moreover, we have a crucial one-dimensional product formula for this kernel. Before formulating it, let us introduce some notations.

Notations. (1) For $x, y, z \in \mathbb{R}$, we put

$$\sigma_{x,y,z} = \begin{cases} \frac{1}{2xy}(x^2+y^2-z^2) & \text{if } x, y \neq 0, \\ 0 & \text{if } x = 0 \text{ or } y = 0, \end{cases}$$

as well as

$$\varrho(x,y,z) = \frac{1}{2}(1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x}).$$

(2) For $x, y, z > 0$, we put

$$K_\kappa(x,y,z) = 2^{2\kappa-2} M_\kappa \frac{\Delta(x,y,z)^{2\kappa-2}}{(xyz)^{2\kappa-1}} \chi_{[|x-y|, x+y]}(z),$$

where $\Delta(x,y,z)$ denotes the area of the triangle (perhaps degenerated) with sides x, y, z.

With these notations in mind, we can now state the product formula for the Dunkl kernel (this formula has been proved in [7] in the more general setting of signed hypergroups).

Proposition 2.1. *Let $x, y \in \mathbb{R}$.*

(1) *For every $\lambda \in \mathbb{R}$ we have*

$$E_\kappa(ix,\lambda)E_\kappa(iy,\lambda) = \int_{\mathbb{R}} E_\kappa(i\lambda, z)\,\mathrm{d}\nu^\kappa_{x,y}(z),$$

where the measure $\nu^\kappa_{x,y}$ is given by

$$\mathrm{d}\nu^\kappa_{x,y}(z) = \begin{cases} \mathcal{K}_\kappa(x,y,z)\,\mathrm{d}\mu_\kappa(z) & \text{if } x,y \neq 0,\\ \mathrm{d}\delta_x(z) & \text{if } y = 0,\\ \mathrm{d}\delta_y(z) & \text{if } x = 0,\end{cases}$$

with

$$\mathcal{K}_\kappa(x,y,z) = K_\kappa\big(|x|,|y|,|z|\big)\varrho(x,y,z).$$

(2) *The measure $\nu^\kappa_{x,y}$ satisfies*

(a) $\operatorname{supp}\nu^\kappa_{x,y} = \Big[-|x|-|y|, -\big||x|-|y|\big|\Big] \bigcup \Big[\big||x|-|y|\big|, |x|+|y|\Big]$ *for $x, y \neq 0$.*

(b) $\nu^\kappa_{x,y}(\mathbb{R}) = 1$ *and* $\|\nu^\kappa_{x,y}\| \leqslant 4$*, for $x, y \in \mathbb{R}$.*

We are now in a position to introduce the Dunkl transform which is taken with respect to the measure μ_κ defined by (1.1). For $f \in L^1(\mu_\kappa)$, the Dunkl transform of f, denoted by $\mathcal{F}_\kappa(f)$, is given by

$$\mathcal{F}_\kappa(f)(x) = c_\kappa \int_{\mathbb{R}^d} f(y)E_\kappa(x,-iy)\,\mathrm{d}\mu_\kappa(y), \quad x \in \mathbb{R}^d,$$

where c_κ is the following constant

$$c_\kappa^{-1} = \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{\|x\|^2}{2}}\,\mathrm{d}\mu_\kappa(x) = \prod_{j=1}^d c_{\kappa_j}^{-1}.$$

If $\kappa_1 = \cdots = \kappa_d = 0$, then $V_\kappa = \mathrm{id}$ and the Dunkl transform coincides with the Euclidean Fourier transform. In the rank-one case, it is more or less a Hankel transform (see [19]). The following proposition (see [3]) gives us a Plancherel theorem and an inversion formula.

Proposition 2.2. (1) *The Dunkl transform extends uniquely to an isometric isomorphism of $L^2(\mu_\kappa)$.*

(2) *If both f and $\mathcal{F}_\kappa(f)$ are in $L^1(\mu_\kappa)$ then*

$$f(x) = c_\kappa \int_{\mathbb{R}^d} \mathcal{F}_\kappa(f)(y)E_\kappa(ix,y)\,\mathrm{d}\mu_\kappa(y).$$

The Dunkl transform shares many other properties with the Fourier transform. Therefore, it is natural to associate a generalized translation operator and a generalized convolution operator with this transform.

There are many ways to define the Dunkl translation. We use the definition which most underlines the analogy with the Fourier transform. It is the definition given in [16] with a different convention.

Let $x \in \mathbb{R}^d$. The Dunkl translation operator τ^κ_x is given for $f \in L^2(\mu_\kappa)$ by

$$\mathcal{F}_\kappa\big(\tau^\kappa_x(f)\big)(y) = E_\kappa(ix,y)\mathcal{F}_\kappa(f)(y), \quad y \in \mathbb{R}^d.$$

It plays the role of $f \mapsto f(\cdot + x)$ in Fourier analysis. It is important to note that it is not a positive operator. The following explicit formula for τ_x^κ is due to Rösler (see [7]). In the case $G \simeq \mathbb{Z}_2$, we have for a continuous function f on $\mathbb{R}$ and for $x, y \in \mathbb{R}$

$$\tau_x^\kappa(f)(y) = \frac{1}{2}\int_{-1}^{1} f\Big(\sqrt{x^2+y^2+2xyt}\Big)\Big(1+\frac{x+y}{\sqrt{x^2+y^2+2xyt}}\Big)\Phi_\kappa(t)\,\mathrm{d}t + \frac{1}{2}\int_{-1}^{1} f\Big(-\sqrt{x^2+y^2+2xyt}\Big)\Big(1-\frac{x+y}{\sqrt{x^2+y^2+2xyt}}\Big)\Phi_\kappa(t)\,\mathrm{d}t, \tag{2.1}$$

where $\Phi_\kappa(t) = M_\kappa(1+t)(1-t^2)^{\kappa-1}$. It follows from (2.1) a formula for τ_x^κ in the case $G \simeq \mathbb{Z}_2^d$ and this formula implies the boundedness of τ_x^κ (it is still a challenging problem for a general reflection group).

Proposition 2.3. *Let $x \in \mathbb{R}^d$. The operator τ_x^κ extends to $L^p(\mu_\kappa)$ for $p \in [1, +\infty]$ and for $f \in L^p(\mu_\kappa)$ we have*

$$\big\|\tau_x^\kappa(f)\big\|_{\kappa,p} \leqslant C\|f\|_{\kappa,p},$$

where C is independent of x and f.

The last result we mention about the generalized translation is the following one-dimensional inequality which has been recently proved by C. Abdelkefi and M. Sifi in [1] (see also [2]).

Proposition 2.4. *There exists a positive constant C such that for $x, y \in \mathbb{R}$ and for every $r > 0$ we have*

$$\big|\tau_x^\kappa(\chi_{[-r,r]})(y)\big| \leqslant C\,\frac{\mu_\kappa\big(]-r,r[\big)}{\mu_\kappa\big(I(x,r)\big)},$$

where we denote by $I(x,r)$ the following set

$$I(x,r) = \big[\max\{0; |x|-r\}, |x|+r\big[.$$

We conclude this section with the definition and the basic properties of the Dunkl convolution operator. According to [16], this operator is defined for both f and g in $L^2(\mu_\kappa)$ by

$$(f *_\kappa g)(x) = c_\kappa \int_{\mathbb{R}^d} f(y)\tau_x^\kappa(g)(-y)\,\mathrm{d}\mu_\kappa(y), \quad x \in \mathbb{R}^d.$$

Thanks to Proposition 2.3, the usual Young's inequality holds (for the proof, see for instance [21]).

Proposition 2.5. *Assume that $p^{-1} + q^{-1} = 1 + r^{-1}$ with $p, q, r \in [1, +\infty]$. Then, the map $(f, g) \mapsto f *_\kappa g$ defined on $L^2(\mu_\kappa) \times L^2(\mu_\kappa)$ extends to a continuous map from $L^p(\mu_\kappa) \times L^q(\mu_\kappa)$ to $L^r(\mu_\kappa)$ and we have*

$$\|f *_\kappa g\|_{\kappa,r} \leqslant C\|f\|_{\kappa,p}\,\|g\|_{\kappa,q},$$

where C is independent of f and g.

We finally note that the Dunkl convolution satisfies the properties $f *_\kappa g = g *_\kappa f$ and $\mathcal{F}_\kappa(f *_\kappa g) = \mathcal{F}_\kappa(f) \cdot \mathcal{F}_\kappa(g)$.

3. Fefferman-Stein inequalities

This section is concerned with the proof of our Fefferman-Stein inequalities, that is to say Theorem 1.2. In fact, as we have already claimed, the proof is straightforward once we have constructed an operator M_κ^R which controls M_κ and which satisfies the classical Fefferman-Stein inequalities. What we have in mind for the construction of M_κ^R is that we want to use the sharp inequality of Proposition 2.4 because it is a key argument to bypass the lack of information on the Dunkl translation operator. Nevertheless, this proposition is one-dimensional. This is the reason for which we shall introduce a Dunkl maximal operator M_κ^Q defined with cubes. Indeed, the basic observation $\chi_{\overline{Q}_r}(x) = \prod_{j=1}^d \chi_{[-r,r]}(x_j)$ (together with the fact that $E_\kappa(x,y) = \prod_{j=1}^d E_{\kappa_j}(x_j, y_j)$) will allow us to prove the formula

$$\tau_x^\kappa(\chi_{\overline{Q}_r})(y) = \prod_{j=1}^d \tau_{x_j}^{\kappa_j}(\chi_{[-r,r]})(y_j),$$

from which we will deduce not only the definition of the operator M_κ^R but also the inequality $M_\kappa^Q f \leqslant M_\kappa^R f$. Therefore, in order to prove the inequality (1.2), it will be enough to prove that M_κ^Q controls M_κ. Since τ_x^κ is not a positive operator, it is not at all obvious that they are connected. Thus, we shall study how they are related to each other.

First of all, we introduce the auxiliary operator M_κ^Q.

Definition. Let M_κ^Q be the Dunkl maximal operator defined with cubes centered at the origin and whose sides are parallel to the axes by

$$M_\kappa^Q f(x) = \sup_{r>0} \frac{1}{\mu_\kappa(Q_r)} \left| \int_{\mathbb{R}^d} f(y) \tau_x^\kappa(\chi_{Q_r})(-y)\,\mathrm{d}\mu_\kappa(y) \right|, \quad x \in \mathbb{R}^d,$$

where for every $r > 0$ we set $Q_r = \{x \in \mathbb{R}^d : |x_j| < r,\ j = 1, \ldots, d\}$.

Our first aim is to prove that this maximal operator controls M_κ. In view of this, we need the following lemma. Before stating it, we have to introduce a notation.

Notation. For $x, y \in \mathbb{R} \setminus \{0\}$, we denote by $\nu_{x,y}^{\kappa,+}$ the measure given for every $z \in \mathbb{R}$ by

$$\mathrm{d}\nu_{x,y}^{\kappa,+}(z) = \frac{1}{2} K_\kappa\big(|x|, |y|, |z|\big)(1 - \sigma_{x,y,z})\,\mathrm{d}\mu_\kappa(z).$$

Let us point out that this measure is positive. Indeed, it is a simple consequence of the following observation

$$|z| \in \Big[\big||x| - |y|\big|, |x| + |y|\Big] \Longrightarrow |\sigma_{x,y,z}| \leqslant 1.$$

With this notation in mind, we can now formulate the lemma.

Lemma 3.1. *Let $x = (x_1, \ldots, x_d) \in \mathbb{R}^d_{\mathrm{reg}}$. Then $\tau_x^\kappa(\chi_{\overline{Q}_r})$ is a positive function on $\mathbb{R}^d_{\mathrm{reg}}$ and for $y = (y_1, \ldots, y_d) \in \mathbb{R}^d_{\mathrm{reg}}$ we have*

$$\tau_x^\kappa(\chi_{\overline{Q}_r})(y) = \int_{\mathbb{R}^d} \chi_{\overline{Q}_r}(z)\,\mathrm{d}\upsilon_{x,y}(z),$$

where the measure $\upsilon_{x,y}^\kappa$ is given by

$$\mathrm{d}\upsilon_{x,y}^\kappa(z) = \mathrm{d}\nu_{x_1,y_1}^{\kappa_1,+}(z_1) \cdots \mathrm{d}\nu_{x_d,y_d}^{\kappa_d,+}(z_d).$$

Before we come to the proof of this lemma, let us introduce the so-called Dunkl heat kernel q_κ^t which is associated with the Dunkl Laplacian $\Delta_\kappa = \sum_{j=1}^d \mathcal{D}_j^2$. This kernel is given for every $t > 0$ by

$$q_\kappa^t(\cdot) = \frac{1}{(2t)^{\gamma_\kappa + \frac{d}{2}}} \, \mathrm{e}^{-\frac{\|\cdot\|^2}{4t}}.$$

It satisfies $\mathcal{F}_\kappa(q_\kappa^t)(\cdot) = \mathrm{e}^{-t\|\cdot\|^2}$ and the following equality

$$\tau_x^\kappa(q_\kappa^t)(y) = \frac{1}{(2t)^{\gamma_\kappa + \frac{d}{2}}} \, \mathrm{e}^{-\frac{\|x\|^2+\|y\|^2}{4t}} \, E_\kappa\left(\frac{x}{\sqrt{2t}}, -\frac{y}{\sqrt{2t}}\right), \quad x, y \in \mathbb{R}^d. \tag{3.1}$$

Moreover, we know that $\tau_x^\kappa(q_\kappa^t)(y) > 0$ for x and y in $\mathbb{R}^d$ and that

$$\int_{\mathbb{R}^d} \tau_x^\kappa(q_\kappa^t)(y) \, \mathrm{d}\mu_\kappa(y) = \frac{1}{c_\kappa}. \tag{3.2}$$

For all these results (and for more details), the reader may consult [8] or [10].

We now turn to the proof of Lemma 3.1.

Proof. One begins with the proof of the following one-dimensional equality

$$\tau_x^\kappa(\chi_{[-r,r]})(y) = \int_{\mathbb{R}} \chi_{[-r,r]}(z) \, \mathrm{d}\nu_{x,y}^{\kappa,+}(z), \quad x, y \in \mathbb{R} \setminus \{0\}. \tag{3.3}$$

Let q_κ^t be the Dunkl heat kernel defined above.
We readily observe that $\chi_{[-r,r]} *_\kappa q_\kappa^t \in L^1(\mu_\kappa)$, which implies, on account of Proposition 2.3, that $\tau_x^\kappa(\chi_{[-r,r]} *_\kappa q_\kappa^t) \in L^1(\mu_\kappa)$. Moreover, we have by Hölder's inequality and Plancherel's theorem

$$\big\|\mathcal{F}_\kappa(\chi_{[-r,r]}) \cdot \mathcal{F}_\kappa(q_\kappa^t)\big\|_{\kappa,1} \leqslant \big\|\chi_{[-r,r]}\big\|_{\kappa,2} \big\|q_\kappa^t\big\|_{\kappa,2},$$

from which we deduce that

$$\mathcal{F}_\kappa(\chi_{[-r,r]} *_\kappa q_\kappa^t) = \mathcal{F}_\kappa(\chi_{[-r,r]}) \cdot \mathcal{F}_\kappa(q_\kappa^t) \in L^1(\mu_\kappa).$$

Since we have by definition

$$\mathcal{F}_\kappa\big(\tau_x^\kappa(\chi_{[-r,r]} *_\kappa q_\kappa^t)\big)(\cdot) = E_\kappa(ix, \cdot)\mathcal{F}_\kappa(\chi_{[-r,r]} *_\kappa q_\kappa^t)(\cdot),$$

then $\mathcal{F}_\kappa\big(\tau_x^\kappa(\chi_{[-r,r]} *_\kappa q_\kappa^t)\big) \in L^1(\mu_\kappa)$ and we can apply the inversion formula to obtain

$$\tau_x^\kappa(\chi_{[-r,r]} *_\kappa q_\kappa^t)(y) = c_\kappa \int_{\mathbb{R}} E_\kappa(ix, z) E_\kappa(iy, z) \mathcal{F}_\kappa(\chi_{[-r,r]})(z) \mathrm{e}^{-tz^2} \, \mathrm{d}\mu_\kappa(z).$$

If we now use the product formula of Proposition 2.1 we get

$$\begin{aligned}
\tau_x^\kappa(\chi_{[-r,r]} *_\kappa q_\kappa^t)(y) &= c_\kappa \int_{\mathbb{R}} \left(\int_{\mathbb{R}} E_\kappa(iz, z') \, \mathrm{d}\nu_{x,y}^\kappa(z') \right) \mathcal{F}_\kappa(\chi_{[-r,r]})(z) \mathrm{e}^{-tz^2} \, \mathrm{d}\mu_\kappa(z) \\
&= c_\kappa \int_{\mathbb{R}} \left(\int_{\mathbb{R}} E_\kappa(iz, z') \mathcal{F}_\kappa(\chi_{[-r,r]})(z) \mathrm{e}^{-tz^2} \, \mathrm{d}\mu_\kappa(z) \right) \mathrm{d}\nu_{x,y}^\kappa(z'),
\end{aligned}$$

from which we deduce thanks to the inversion formula

$$\tau_x^\kappa(\chi_{[-r,r]} *_\kappa q_\kappa^t)(y) = \int_{\mathbb{R}} (\chi_{[-r,r]} *_\kappa q_\kappa^t)(z') \, \mathrm{d}\nu_{x,y}^\kappa(z'). \tag{3.4}$$

But we claim that $\chi_{[-r,r]} *_\kappa q_\kappa^t$ is an even function. Indeed

$$\begin{aligned}(\chi_{[-r,r]} *_\kappa q_\kappa^t)(-\xi) &= c_\kappa \int_{\mathbb{R}} \chi_{[-r,r]}(\xi')\tau_{-\xi}^\kappa(q_\kappa^t)(-\xi')\,\mathrm{d}\mu_\kappa(\xi') \\ &= c_\kappa \int_{\mathbb{R}} \chi_{[-r,r]}(\xi')\tau_{\xi}^\kappa(q_\kappa^t)(\xi')\,\mathrm{d}\mu_\kappa(\xi') = (\chi_{[-r,r]} *_\kappa q_\kappa^t)(\xi),\end{aligned}$$

where we have used the definition of $*_\kappa$ in the first step, the formula (3.1) in the second step (in order to prove that $\tau_{-\xi}^\kappa(q_\kappa^t)(-\xi') = \tau_\xi^\kappa(q_\kappa^t)(\xi')$) and a change of variables and the definition of the Dunkl convolution in the last step.

Since both $z \mapsto \sigma_{z,x,y}$ and $z \mapsto \sigma_{z,y,x}$ are odd functions, the equality (3.4) is therefore equivalent to the following one

$$\tau_x^\kappa(\chi_{[-r,r]} *_\kappa q_\kappa^t)(y) = \int_{\mathbb{R}} (\chi_{[-r,r]} *_\kappa q_\kappa^t)(z')\,\mathrm{d}\nu_{x,y}^{\kappa,+}(z'). \tag{3.5}$$

In order to prove (3.3) we will take limit in (3.5) as t goes to 0. Observe that, by Plancherel's theorem

$$\begin{aligned}\left\|\chi_{[-r,r]} *_\kappa q_\kappa^t - \chi_{[-r,r]}\right\|_{\kappa,2}^2 &= \left\|\mathcal{F}_\kappa(\chi_{[-r,r]}) \cdot \mathcal{F}_\kappa(q_\kappa^t) - \mathcal{F}_\kappa(\chi_{[-r,r]})\right\|_{\kappa,2}^2 \\ &= \int_{\mathbb{R}} \left|\mathcal{F}_\kappa(\chi_{[-r,r]})(\xi)\right|^2 \left(1 - \mathrm{e}^{-t\xi^2}\right)^2 \mathrm{d}\mu_\kappa(\xi).\end{aligned}$$

Thus, $\chi_{[-r,r]} *_\kappa q_\kappa^t \to \chi_{[-r,r]}$ in $L^2(\mu_\kappa)$ as $t \to 0$. Since τ_x^κ is a bounded operator on $L^2(\mu_\kappa)$ we also have $\tau_x^\kappa(\chi_{[-r,r]} *_\kappa q_\kappa^t) \to \tau_x^\kappa(\chi_{[-r,r]})$ in $L^2(\mu_\kappa)$ as $t \to 0$. By passing to a subsequence if necessary we can therefore assume that the convergence is also almost everywhere. Taking limit as t goes to 0 in (3.5) gives us

$$\tau_x^\kappa(\chi_{[-r,r]})(y) = \lim_{t\to 0} \int_{\mathbb{R}} (\chi_{[-r,r]} *_\kappa q_\kappa^t)(z')\,\mathrm{d}\nu_{x,y}^{\kappa,+}(z').$$

Then (3.3) is proved if we show the following equality

$$\lim_{t\to 0} \int_{\mathbb{R}} (\chi_{[-r,r]} *_\kappa q_\kappa^t)(z')\,\mathrm{d}\nu_{x,y}^{\kappa,+}(z') = \int_{\mathbb{R}} \chi_{[-r,r]}(z')\,\mathrm{d}\nu_{x,y}^{\kappa,+}(z'). \tag{3.6}$$

In view of this, we shall use the Lebesgue dominated convergence theorem. Since the almost everywhere convergence of $\chi_{[-r,r]} *_\kappa q_\kappa^t$ to $\chi_{[-r,r]}$ has been already proved above, it suffices to majorize $|\chi_{[-r,r]} *_\kappa q_\kappa^t|$ by a function independent of t and which is integrable with respect to $\nu_{x,y}^{\kappa,+}$.

By the definition of the Dunkl convolution

$$(\chi_{[-r,r]} *_\kappa q_\kappa^t)(z') = c_\kappa \int_{\mathbb{R}} \chi_{[-r,r]}(\xi)\tau_{z'}^\kappa(q_\kappa^t)(-\xi)\,\mathrm{d}\mu_\kappa(\xi),$$

from which we deduce that

$$\left|(\chi_{[-r,r]} *_\kappa q_\kappa^t)(z')\right| \leqslant c_\kappa \int_{\mathbb{R}} \left|\tau_{z'}^\kappa(q_\kappa^t)(-\xi)\right| \mathrm{d}\mu_\kappa(\xi) = c_\kappa \int_{\mathbb{R}} \tau_{z'}^\kappa(q_\kappa^t)(\xi)\,\mathrm{d}\mu_\kappa(\xi),$$

where we have used the positivity of $\tau_{z'}^\kappa(q_\kappa^t)$ and a change of variables in the last step.

On account of (3.2) we then obtain

$$\left|(\chi_{[-r,r]} *_\kappa q_\kappa^t)(z')\right| \leqslant 1.$$

Since the function equal to 1 is integrable with respect to $\nu_{x,y}^{\kappa,+}$, the Lebesgue dominated convergence theorem allows us to complete the proof of (3.6) and then (3.3) is proved.

Let us point out that we deduce from (3.3) the positivity of $\tau_x^\kappa(\chi_{[-r,r]})$.

We next prove the following equality

$$\tau_x^\kappa(\chi_{\overline{Q}_r})(y) = \prod_{j=1}^{d} \tau_{x_j}^{\kappa_j}(\chi_{[-r,r]})(y_j), \quad x, y \in \mathbb{R}^d_{\mathrm{reg}}. \tag{3.7}$$

We can apply the inversion formula (by a reprise of the argument given above) to obtain

$$\tau_x^\kappa(\chi_{\overline{Q}_r} *_\kappa q_\kappa^t)(y) = c_\kappa \int_{\mathbb{R}^d} E_\kappa(ix,z)E_\kappa(iy,z)\mathcal{F}_\kappa(\chi_{\overline{Q}_r})(z)\mathrm{e}^{-t\|z\|^2}\,\mathrm{d}\mu_\kappa(z). \tag{3.8}$$

Let us notice that we have the following product formula

$$\mathcal{F}_\kappa(\chi_{\overline{Q}_r})(z) = \prod_{j=1}^{d} \mathcal{F}_{\kappa_j}(\chi_{[-r,r]})(z_j), \quad z \in \mathbb{R}^d. \tag{3.9}$$

Indeed, by the definition of the Dunkl transform we have

$$\mathcal{F}_\kappa(\chi_{\overline{Q}_r})(z) = c_\kappa \int_{\mathbb{R}^d} E_\kappa(z,-iz')\chi_{\overline{Q}_r}(z')\,\mathrm{d}\mu_\kappa(z').$$

Since we can separate the variables we get

$$\mathcal{F}_\kappa(\chi_{\overline{Q}_r})(z) = \prod_{j=1}^{d}\left(\int_{\mathbb{R}} c_{\kappa_j}E_{\kappa_j}(z_j,-iz_j')\chi_{[-r,r]}(z_j')h_{\kappa_j}^2(z_j')\,\mathrm{d}z_j'\right),$$

from which (3.9) follows. We combine (3.9) with (3.8) to obtain

$$\begin{aligned}\tau_x^\kappa&(\chi_{\overline{Q}_r} *_\kappa q_\kappa^t)(y)\\ &= \prod_{j=1}^{d}\left(\int_{\mathbb{R}} c_{\kappa_j}E_{\kappa_j}(ix_j,z_j)E_{\kappa_j}(iy_j,z_j)\mathcal{F}_{\kappa_j}(\chi_{[-r,r]})(z_j)\mathrm{e}^{-tz_j^2}h_{\kappa_j}^2(z_j)\,\mathrm{d}z_j\right),\end{aligned}$$

that is to say

$$\tau_x^\kappa(\chi_{\overline{Q}_r} *_\kappa q_\kappa^t)(y) = \prod_{j=1}^{d}\tau_{x_j}^{\kappa_j}(\chi_{[-r,r]} *_{\kappa_j} q_{\kappa_j}^t)(y_j),$$

from which we deduce (3.7) by taking limit.
The proof of the lemma is now obvious. Indeed, using the equality (3.3) in (3.7) gives us

$$\tau_x^\kappa(\chi_{\overline{Q}_r})(y) = \prod_{j=1}^{d}\int_{\mathbb{R}} \chi_{[-r,r]}(z_j)\,\mathrm{d}\nu_{x_j,y_j}^{\kappa_j,+}(z_j),$$

which is precisely what we wanted to prove. □

We are now in a position to prove that M_κ^Q controls M_κ. More precisely, we have the following proposition.

Proposition 3.1. *There exists a positive constant C such that for every $x \in \mathbb{R}^d_{\mathrm{reg}}$ we have*

$$0 \leqslant M_\kappa f(x) \leqslant C M_\kappa^Q|f|(x).$$

Proof. Thanks to the definition of M_κ there is nothing to do for the first inequality. We now turn to the second one.

Let $x \in \mathbb{R}^d_{\mathrm{reg}}$ and $r > 0$. Let us remark that we readily have

$$\int_{\mathbb{R}^d} f(y)\tau_x^\kappa(\chi_{B_r})(-y)\,\mathrm{d}\mu_\kappa(y) = \int_{\mathbb{R}^d_{\mathrm{reg}}} f(y)\tau_x^\kappa(\chi_{\overline{B}_r})(-y)\,\mathrm{d}\mu_\kappa(y). \tag{3.10}$$

The key argument for the proof is that we can show, even if τ_x^κ is not a positive operator, the following inequality

$$0 \leqslant \tau_x^\kappa(\chi_{\overline{B}_r})(y) \leqslant \tau_x^\kappa(\chi_{\overline{Q}_r})(y), \quad x, y \in \mathbb{R}^d_{\mathrm{reg}}. \tag{3.11}$$

Thanks to the explicit formula of $\tau_x^\kappa(\chi_{\overline{Q}_r})$ given in the previous lemma, it is enough to show that

$$\tau_x^\kappa(\chi_{\overline{B}_r})(y) = \int_{\mathbb{R}^d} \chi_{\overline{B}_r}(z)\,\mathrm{d}\upsilon_{x,y}^\kappa(z), \quad x, y \in \mathbb{R}^d_{\mathrm{reg}}, \tag{3.12}$$

in order to prove (3.11). Therefore, we now turn to the proof of (3.12). By a reprise of the argument given in the proof of Lemma 3.1, we can apply the inversion formula to write for both x and y in $\mathbb{R}^d_{\mathrm{reg}}$

$$\tau_x^\kappa(\chi_{\overline{B}_r} *_\kappa q_\kappa^t)(y) = c_\kappa \int_{\mathbb{R}^d} E_\kappa(ix,z)E_\kappa(iy,z)\mathcal{F}_\kappa(\chi_{\overline{B}_r})(z)\mathrm{e}^{-t\|z\|^2}\,\mathrm{d}\mu_\kappa(z).$$

Since $E_\kappa(x,y) = \prod_{j=1}^d E_{\kappa_j}(x_j,y_j)$, we have thanks to Proposition 2.1

$$\begin{aligned}&\tau_x^\kappa(\chi_{\overline{B}_r} *_\kappa q_\kappa^t)(y)\\ &= c_\kappa \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} E_\kappa(iz,z')\,\mathrm{d}\nu_{x_1,y_1}^{\kappa_1}(z_1')\cdots\mathrm{d}\nu_{x_d,y_d}^{\kappa_d}(z_d')\right)\mathcal{F}_\kappa(\chi_{\overline{B}_r})(z)\mathrm{e}^{-t\|z\|^2}\,\mathrm{d}\mu_\kappa(z),\end{aligned}$$

from which it follows

$$\begin{aligned}&\tau_x^\kappa(\chi_{\overline{B}_r} *_\kappa q_\kappa^t)(y)\\ &= c_\kappa \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} E_\kappa(iz,z')\mathcal{F}_\kappa(\chi_{\overline{B}_r})(z)\mathrm{e}^{-t\|z\|^2}\,\mathrm{d}\mu_\kappa(z)\right)\mathrm{d}\nu_{x_1,y_1}^{\kappa_1}(z_1')\cdots\mathrm{d}\nu_{x_d,y_d}^{\kappa_d}(z_d').\end{aligned}$$

We apply the inversion formula to get

$$\tau_x^\kappa(\chi_{\overline{B}_r} *_\kappa q_\kappa^t)(y) = \int_{\mathbb{R}^d}(\chi_{\overline{B}_r} *_\kappa q_\kappa^t)(z')\,\mathrm{d}\nu_{x_1,y_1}^{\kappa_1}(z_1')\cdots\mathrm{d}\nu_{x_d,y_d}^{\kappa_d}(z_d'),$$

and we obtain thanks to the Fubini theorem

$$\tau_x^\kappa(\chi_{\overline{B}_r} *_\kappa q_\kappa^t)(y) = \int_{\mathbb{R}^{d-1}}\left(\int_{\mathbb{R}}(\chi_{\overline{B}_r} *_\kappa q_\kappa^t)(z')\,\mathrm{d}\nu_{x_1,y_1}^{\kappa_1}(z_1')\right)\mathrm{d}\nu_{x_2,y_2}^{\kappa_2}(z_2')\cdots\mathrm{d}\nu_{x_d,y_d}^{\kappa_d}(z_d'). \tag{3.13}$$

Since $\chi_{\overline{B}_r}$ is radial, $\chi_{\overline{B}_r} *_\kappa q_\kappa^t$ is also radial. Therefore, it is even with respect to each of its variables, that is to say $(\chi_{\overline{B}_r} *_\kappa q_\kappa^t)(\varepsilon_1 z_1,\dots,\varepsilon_d z_d) = (\chi_{\overline{B}_r} *_\kappa q_\kappa^t)(z_1,\dots,z_d)$ with $\varepsilon_j = \pm 1$. Then (3.13) is equivalent to

$$\tau_x^\kappa(\chi_{\overline{B}_r} *_\kappa q_\kappa^t)(y) = \int_{\mathbb{R}^{d-1}}\left(\int_{\mathbb{R}}(\chi_{\overline{B}_r} *_\kappa q_\kappa^t)(z')\,\mathrm{d}\nu_{x_1,y_1}^{\kappa_1,+}(z_1')\right)\mathrm{d}\nu_{x_2,y_2}^{\kappa_2}(z_2')\cdots\mathrm{d}\nu_{x_d,y_d}^{\kappa_d}(z_d').$$

By successive uses of the Fubini theorem we are readily led to

$$\tau_x^\kappa(\chi_{\overline{B}_r} *_\kappa q_\kappa^t)(y) = \int_{\mathbb{R}^d} (\chi_{\overline{B}_r} *_\kappa q_\kappa^t)(z')\,\mathrm{d}\nu_{x_1,y_1}^{\kappa_1,+}(z_1') \cdots \mathrm{d}\nu_{x_d,y_d}^{\kappa_d,+}(z_d'). \tag{3.14}$$

Taking limit as t tends to 0 in (3.14) gives us (3.12) which in turn implies (3.11). Consequently, if we apply (3.11) in (3.10) we are led to the following inequality

$$\left|\int_{\mathbb{R}^d} f(y)\tau_x^\kappa(\chi_{B_r})(-y)\,\mathrm{d}\mu_\kappa(y)\right| \leqslant \int_{\mathbb{R}^d_{\mathrm{reg}}} |f(y)|\tau_x^\kappa(\chi_{\overline{Q}_r})(-y)\,\mathrm{d}\mu_\kappa(y).$$

Since it is obvious that

$$\int_{\mathbb{R}^d_{\mathrm{reg}}} |f(y)|\tau_x^\kappa(\chi_{\overline{Q}_r})(-y)\,\mathrm{d}\mu_\kappa(y) = \int_{\mathbb{R}^d} |f(y)|\tau_x^\kappa(\chi_{Q_r})(-y)\,\mathrm{d}\mu_\kappa(y),$$

we can therefore write

$$\left|\int_{\mathbb{R}^d} f(y)\tau_x^\kappa(\chi_{B_r})(-y)\,\mathrm{d}\mu_\kappa(y)\right| \leqslant \int_{\mathbb{R}^d} |f(y)|\tau_x^\kappa(\chi_{Q_r})(-y)\,\mathrm{d}\mu_\kappa(y),$$

from which it follows at once that

$$\frac{1}{\mu_\kappa(B_r)}\left|\int_{\mathbb{R}^d} f(y)\tau_x^\kappa(\chi_{B_r})(-y)\,\mathrm{d}\mu_\kappa(y)\right| \leqslant \frac{1}{\mu_\kappa(B_r)}\int_{\mathbb{R}^d} |f(y)|\tau_x^\kappa(\chi_{Q_r})(-y)\,\mathrm{d}\mu_\kappa(y). \tag{3.15}$$

Let us notice that $\mu_\kappa(Q_r) = C\mu_\kappa(B_r)$ with

$$C = \frac{2^d(2\gamma_\kappa + d)}{\prod_{j=1}^d (2\kappa_j + 1)}\left(\int_{S^{d-1}} h_\kappa^2(y)\,\mathrm{d}y\right)^{-1}.$$

Indeed, we have on one hand

$$\mu_\kappa(Q_r) = \prod_{j=1}^d \mu_{\kappa_j}(]-r,r[) = 2^d\prod_{j=1}^d\Big(\frac{1}{2\kappa_j+1}\Big) r^{2\gamma_\kappa+d},$$

and on the other hand, changing to polar coordinates gives

$$\mu_\kappa(B_r) = \int_0^r u^{2\gamma_\kappa+d-1}\,\mathrm{d}u \int_{S^{d-1}} h_\kappa^2(y)\,\mathrm{d}y = \frac{1}{2\gamma_\kappa+d}\left(\int_{S^{d-1}} h_\kappa^2(y)\,\mathrm{d}y\right) r^{2\gamma+d},$$

where we have used the fact that h_κ^2 is homogeneous of degree $2\gamma_\kappa$.
We can therefore reformulate (3.15) as follows

$$\frac{1}{\mu_\kappa(B_r)}\left|\int_{\mathbb{R}^d} f(y)\tau_x^\kappa(\chi_{B_r})(-y)\,\mathrm{d}\mu_\kappa(y)\right| \leqslant \frac{C}{\mu_\kappa(Q_r)}\int_{\mathbb{R}^d} |f(y)|\tau_x^\kappa(\chi_{Q_r})(-y)\,\mathrm{d}\mu_\kappa(y),$$

from which we deduce that

$$\frac{1}{\mu_\kappa(B_r)}\left|\int_{\mathbb{R}^d} f(y)\tau_x^\kappa(\chi_{B_r})(-y)\,\mathrm{d}\mu_\kappa(y)\right| \leqslant CM_\kappa^Q|f|(x),$$

and then the result. □

Thanks to this proposition, it is enough to construct an operator M_κ^R which controls M_κ^Q in order to prove the inequality (1.2). Before we come to the definition of M_κ^R we give some notations.

Notations. For $z = (z_1, \ldots, z_d) \in \mathbb{R}^d$ we put $\tilde{z} = \big(|z_1|, \ldots, |z_d|\big)$ and we denote by $R(z, r)$ (for every $r > 0$) the following set

$$R(z,r) = I(z_1, r) \times \cdots \times I(z_d, r).$$

Recall that we have defined for $x \in \mathbb{R}$ and $r > 0$ the set $I(x, r)$ by

$$I(x,r) = \big[\max\{0; |x| - r\}, |x| + r\big[.$$

Since we want to use the sharp inequality of Poposition 2.4 together with the fact that $\tau_x^\kappa(\chi_{\overline{Q}_r})(y) = \prod_{j=1}^d \tau_{x_j}^{\kappa_j}(\chi_{[-r,r]})(y_j)$, we are naturally led to introduce the following operator.

Definition. Let M_κ^R be the weighted maximal operator defined by

$$M_\kappa^R f(x) = \sup_{r>0} \frac{1}{\mu_\kappa\big(R(x,r)\big)} \int_{\tilde{y} \in R(x,r)} |f(y)| \,\mathrm{d}\mu_\kappa(y), \quad x \in \mathbb{R}^d.$$

This operator satisfies the classical properties of maximal operators. Let us clarify our statement.
Since μ_κ is a doubling weight, we have the following covering lemma (a one-dimensional result for $I(x, r)$ can be found in [1] or [2]).

Lemma 3.2. *Let E be a measurable (with respect to μ_κ) subset of $\mathbb{R}_+ \times \cdots \times \mathbb{R}_+$. Suppose $E \subset \cup_{j \in J} R_j$ with $R_j = R(z_j, r_j)$ bounded for every $j \in J$ (where $z_j \in \mathbb{R}^d$ and $r_j > 0$). Then, from this family, we can choose a sequence (which may be finite) of disjoint sets $R_1, \ldots, R_n, \ldots$, such that*

$$\mu_\kappa(E) \leqslant C \sum_n \mu_\kappa(R_n),$$

where C is a positive constant which depends only on $\kappa_1, \ldots, \kappa_d$.

Thanks to this lemma, a weak-type $(1,1)$ result for M_κ^R can be easily proved. Indeed, if we set

$$E_+ = \Big\{x \in \mathbb{R}_+^* \times \cdots \times \mathbb{R}_+^* : M_\kappa^R f(x) > \lambda\Big\},$$

we can choose (thanks to the definition of M_κ^R and the covering lemma) a suitable sequence of disjoint sets R_n such that $\mu_\kappa(E_+) \leqslant C \sum_n \mu_\kappa(R_n)$, where C depends only on $\kappa_1, \ldots, \kappa_d$. We can then follow the standard techniques (see for instance [13]) in order to prove that $\mu_\kappa(E_+) \leqslant \frac{C}{\lambda} \|f\|_{\kappa,1}$.
Finally, the basic but crucial observation

$$M_\kappa^R f(x) = M_\kappa^R f(\varepsilon_1 x_1, \ldots, \varepsilon_d x_d), \tag{3.16}$$

with $\varepsilon_j = \pm 1$, allows us to deduce the weak-type inequality, that is

$$\mu_\kappa\Big(\Big\{x \in \mathbb{R}^d : M_\kappa^R f(x) > \lambda\Big\}\Big) \leqslant \frac{C}{\lambda} \|f\|_{\kappa,1}.$$

Since M_κ^R is obviously bounded on L^∞, the weak-type $(1,1)$ inequality implies the strong-type (p,p) inequality by the Marcinkiewicz interpolation theorem (see [13]). Thus, we have proved the following maximal theorem for M_κ^R.

Theorem 3.1. *Let f be a function defined on $\mathbb{R}^d$.*

(1) *If* $f \in L^1(\mu_\kappa)$, *then for every* $\lambda > 0$ *we have*

$$\mu_\kappa\Big(\Big\{x \in \mathbb{R}^d : M_\kappa^R f(x) > \lambda\Big\}\Big) \leqslant \frac{C}{\lambda}\|f\|_{\kappa,1},$$

where C *is a positive constant independent of* f *and* λ.

(2) *If* $f \in L^p(\mu_\kappa)$, $1 < p \leqslant +\infty$, *then* $M_\kappa^R f \in L^p(\mu_\kappa)$ *and we have*

$$\|M_\kappa^R f\|_{\kappa,p} \leqslant C\|f\|_{\kappa,p},$$

where C *is a positive constant independent of* f.

Moreover, we claim that the following weighted inequality is true.

Lemma 3.3. *Let* W *be a positive and locally integrable (with respect to* μ_κ*) function defined on* $\mathbb{R}^d$. *For* $1 < q < +\infty$, *there exists a positive constant* C *which depends only on* $\kappa_1, \dots, \kappa_d$ *and* q *and such that*

$$\int_{\mathbb{R}^d} \big(M_\kappa^R f(y)\big)^q W(y)\,\mathrm{d}\mu_\kappa(y) \leqslant C\int_{\mathbb{R}^d} |f(y)|^q M_\kappa^R W(y)\,\mathrm{d}\mu_\kappa(y).$$

Indeed, by the Marcinkiewicz interpolation theorem, this lemma is an immediate consequence of the trivial fact that M_κ^R is bounded on L^∞ together with the following inequality

$$\tilde{\mu}_\kappa\Big(\Big\{x \in \mathbb{R}^d : M_\kappa^R f(x) > \lambda\Big\}\Big) \leqslant \frac{C}{\lambda}\int_{\mathbb{R}^d} |f(y)| M_\kappa^R W(y)\,\mathrm{d}\mu_\kappa(y), \tag{3.17}$$

where $\tilde{\mu}_\kappa(X) = \int_X W(y)\,\mathrm{d}\mu_\kappa(y)$ and where C is a positive constant which depends only on $\kappa_1, \dots, \kappa_d$. The just-written inequality is easy to prove. Indeed, we can show the key inequality

$$\tilde{\mu}_\kappa(K) \leqslant \frac{C}{\lambda}\int_{\mathbb{R}^d} |f(y)| M_\kappa^R W(y)\,\mathrm{d}\mu_\kappa(y)$$

for any compact set K in E_+ just as in the proof for the classical maximal operator (see [15]). Therefore

$$\tilde{\mu}_\kappa(E_+) \leqslant \frac{C}{\lambda}\int_{\mathbb{R}^d} |f(y)| M_\kappa^R W(y)\,\mathrm{d}\mu_\kappa(y)$$

and we then deduce (3.17) on account of (3.16).

To conclude, we claim that we can combine the maximal theorem and the weighted inequality for M_κ^R with a Calderón-Zygmund decomposition of f (see for instance [13]) to obtain the Fefferman-Stein inequalities for M_κ^R following almost verbatim the proof in [6].

Theorem 3.2. *Let* $(f_n)_{n\geqslant 1}$ *be a sequence of measurable functions defined on* $\mathbb{R}^d$.

(1) *If* $1 < r < +\infty$, $1 < p < +\infty$ *and if* $\big(\sum_{n=1}^\infty |f_n(\cdot)|^r\big)^{\frac{1}{r}} \in L^p(\mu_\kappa)$, *then we have*

$$\bigg\|\Big(\sum_{n=1}^\infty |M_\kappa^R f_n(\cdot)|^r\Big)^{\frac{1}{r}}\bigg\|_{\kappa,p} \leqslant C\bigg\|\Big(\sum_{n=1}^\infty |f_n(\cdot)|^r\Big)^{\frac{1}{r}}\bigg\|_{\kappa,p},$$

where $C = C(\kappa_1, \dots, \kappa_d, r, p)$ *is independent of* $(f_n)_{n\geqslant 1}$.

(2) *If* $1 < r < +\infty$ *and if* $\left(\sum_{n=1}^{\infty} |f_n(\cdot)|^r\right)^{\frac{1}{r}} \in L^1(\mu_\kappa)$, *then for every* $\lambda > 0$ *we have*

$$\mu_\kappa\left(\left\{x \in \mathbb{R}^d : \left(\sum_{n=1}^{\infty} |M_\kappa^R f_n(x)|^r\right)^{\frac{1}{r}} > \lambda\right\}\right) \leqslant \frac{C}{\lambda}\left\|\left(\sum_{n=1}^{\infty} |f_n(\cdot)|^r\right)^{\frac{1}{r}}\right\|_{\kappa,1},$$

where $C = C(\kappa_1, \ldots, \kappa_d, r)$ *is independent of* $(f_n)_{n\geqslant 1}$ *and* λ.

Therefore, in order to prove Theorem 1.2, it remains to show that the operator M_κ^R controls M_κ^Q. More precisely, we have the following proposition.

Proposition 3.2. *There exists a positive constant* C *such that for every* $x \in \mathbb{R}^d_{\mathrm{reg}}$ *we have*

$$M_\kappa^Q f(x) \leqslant C M_\kappa^R f(x).$$

Proof. Let $x \in \mathbb{R}^d_{\mathrm{reg}}$ and $r > 0$. By the definition of the Dunkl convolution we have

$$\left|(f *_\kappa \chi_{Q_r})(x)\right| = c_\kappa \left|\int_{\mathbb{R}^d} f(y)\tau_x^\kappa(\chi_{Q_r})(-y)\,\mathrm{d}\mu_\kappa(y)\right|,$$

from which we deduce at once that

$$\left|(f *_\kappa \chi_{Q_r})(x)\right| = c_\kappa \left|\int_{\mathbb{R}^d_{\mathrm{reg}}} f(y)\tau_x^\kappa(\chi_{\overline{Q}_r})(-y)\,\mathrm{d}\mu_\kappa(y)\right|.$$

Using the positivity of $\tau_x^\kappa(\chi_{\overline{Q}_r})$ gives us

$$\left|(f *_\kappa \chi_{Q_r})(x)\right| \leqslant c_\kappa \int_{\mathbb{R}^d_{\mathrm{reg}}} |f(y)|\tau_x^\kappa(\chi_{\overline{Q}_r})(-y)\,\mathrm{d}\mu_\kappa(y).$$

On account of (3.7) we then obtain

$$\left|(f *_\kappa \chi_{Q_r})(x)\right| \leqslant c_\kappa \int_{\mathbb{R}^d_{\mathrm{reg}}} |f(y)| \prod_{j=1}^{d} \tau_{x_j}^{\kappa_j}(\chi_{[-r,r]})(-y_j)\,\mathrm{d}\mu_\kappa(y).$$

Since we can readily deduce from (3.3) the following property

$$|y_j| \notin I(x_j, r) \Longrightarrow \tau_{x_j}^{\kappa_j}(\chi_{[-r,r]})(y_j) = 0,$$

we can write

$$\left|(f *_\kappa \chi_{Q_r})(x)\right| \leqslant c_\kappa \int_{A_x} |f(y)| \prod_{j=1}^{d} \tau_{x_j}^{\kappa_j}(\chi_{[-r,r]})(-y_j)\,\mathrm{d}\mu_\kappa(y),$$

where A_x is the following set

$$A_x = \mathbb{R}^d_{\mathrm{reg}} \cap \left\{y \in \mathbb{R}^d : \tilde{y} \in R(x,r)\right\}.$$

If we now apply the inequality of Proposition 2.4 we get

$$\left|(f *_\kappa \chi_{Q_r})(x)\right| \leqslant C \int_{A_x} |f(y)| \prod_{j=1}^{d} \frac{\mu_{\kappa_j}\left(]-r,r[\right)}{\mu_{\kappa_j}\left(I(x_j,r)\right)}\,\mathrm{d}\mu_\kappa(y).$$

The following obvious equalities

$$\prod_{j=1}^{d} \mu_{\kappa_j}\left(]-r,r[\right) = \mu_\kappa(Q_r), \quad \prod_{j=1}^{d} \mu_{\kappa_j}\left(I(x_j,r)\right) = \mu_\kappa\left(R(x,r)\right),$$

imply that

$$\left|(f *_\kappa \chi_{Q_r})(x)\right| \leqslant \frac{C\mu_\kappa(Q_r)}{\mu_\kappa\big(R(x,r)\big)} \int_{A_x} |f(y)| \, \mathrm{d}\mu_\kappa(y),$$

from which we deduce that

$$\frac{1}{\mu_\kappa(Q_r)}\left|(f *_\kappa \chi_{Q_r})(x)\right| \leqslant \frac{C}{\mu_\kappa\big(R(x,r)\big)} \int_{\tilde{y}\in R(x,r)} |f(y)| \, \mathrm{d}\mu_\kappa(y).$$

It follows that

$$\frac{1}{\mu_\kappa(Q_r)}\left|(f *_\kappa \chi_{Q_r})(x)\right| \leqslant C M_\kappa^R f(x),$$

and then the result. □

This result, combined with Proposition 3.1, leads immediately to the following corollary.

Corollary 3.1. *There exists a positive constant C such that for every $x \in \mathbb{R}^d_{\mathrm{reg}}$ we have*

$$0 \leqslant M_\kappa f(x) \leqslant C M_\kappa^R f(x).$$

Then, Theorem 1.2 is true thanks to this corollary and the Fefferman-Stein inequalities for M_κ^R (Theorem 3.2).

Remark. Let us point out that Corollary 3.1, together with the maximal result for M_κ^R (Theorem 3.1), implies a maximal theorem for M_κ (proved in [16]) without using the Hopf-Dunford-Schwartz ergodic theorem (which is a general method given in [14]).

4. Application

Since the Fefferman-Stein inequalities are an important tool in Harmonic analysis, we would like to define a large class of operators such that each operator of this class satisfies these inequalities, and such that, in particular, the maximal operator associated with the Dunkl heat semigroup and the maximal operator associated with the Dunkl-Poisson semigroup belong to this class (see [14] for details about the classical heat semigroup and the classical Poisson semigroup).

To become more precise, let us now introduce this class of operators.

Definition. Let $\phi \in L^1(\mu_\kappa)$ be a radial function, that is $\phi(x) = \tilde{\phi}\big(\|x\|\big)$ for every $x \in \mathbb{R}^d$, such that $\tilde{\phi}$ is differentiable and satisfies the following properties

$$\lim_{r\to\infty} \tilde{\phi}(r) = 0, \quad \int_0^\infty r^{2\gamma_\kappa+d}\left|\frac{\mathrm{d}}{\mathrm{d}r}\tilde{\phi}(r)\right| \mathrm{d}r < +\infty.$$

Then we denote by M_κ^ϕ the following operator

$$M_\kappa^\phi f(x) = \sup_{t>0}\left|(f *_\kappa \phi_t)(x)\right|, \quad x \in \mathbb{R}^d,$$

where ϕ_t is for every $t > 0$ the dilation of ϕ given by

$$\phi_t(x) = \frac{1}{t^{2\gamma_\kappa+d}}\,\phi\Big(\frac{x}{t}\Big), \quad x \in \mathbb{R}^d.$$

Let us present two important examples of functions which satisfy the conditions of the previous definition.
The first one is concerned with the Dunkl heat kernel q_κ^t. Indeed if we let

$$\phi(x) = \mathrm{e}^{-\frac{\|x\|^2}{2}}, \quad x \in \mathbb{R}^d,$$

then for every $t > 0$ we have

$$\phi_{\sqrt{2t}}(x) = \frac{1}{(2t)^{\gamma_\kappa + \frac{d}{2}}} \, \mathrm{e}^{-\frac{\|x\|^2}{4t}} = q_\kappa^t(x).$$

In this case, M_κ^ϕ is therefore the maximal function of the Dunkl heat semigroup. Our second example deals with the Dunkl-Poisson kernel. If we define the function ϕ for every $x \in \mathbb{R}^d$ by

$$\phi(x) = \frac{a_\kappa}{(1 + \|x\|^2)^{\gamma_\kappa + \frac{d+1}{2}}}, \quad \text{with} \quad a_\kappa = \frac{c_\kappa \, 2^{\gamma_\kappa + \frac{d}{2}} \, \Gamma\big(\gamma_\kappa + \frac{d+1}{2}\big)}{\sqrt{\pi}},$$

then for every $t > 0$ we have

$$\phi_t(x) = \frac{a_\kappa \, t}{(t^2 + \|x\|^2)^{\gamma_\kappa + \frac{d+1}{2}}} = P_\kappa^t(x),$$

which is the Dunkl-Poisson kernel (for more details about this kernel, the reader is referred to [12] and [16]). Thus, in this case, M_κ^ϕ is the maximal function associated with the Dunkl-Poisson semigroup.

We now state the Fefferman-Stein inequalities for M_κ^ϕ (for $\phi, \tilde{\phi}$ and ϕ_t as above).

Theorem 4.1. *Let $(f_n)_{n\geqslant 1}$ be a sequence of measurable functions defined on $\mathbb{R}^d$.*

(1) *If $1 < r < +\infty$, $1 < p < +\infty$ and if $\big(\sum_{n=1}^\infty |f_n(\cdot)|^r\big)^{\frac{1}{r}} \in L^p(\mu_\kappa)$, then we have*

$$\Big\|\Big(\sum_{n=1}^\infty |M_\kappa^\phi f_n(\cdot)|^r\Big)^{\frac{1}{r}}\Big\|_{\kappa,p} \leqslant C \Big\|\Big(\sum_{n=1}^\infty |f_n(\cdot)|^r\Big)^{\frac{1}{r}}\Big\|_{\kappa,p},$$

where $C = C(\phi, \kappa_1, \ldots, \kappa_d, r, p)$ is independent of $(f_n)_{n\geqslant 1}$.

(2) *If $1 < r < +\infty$ and if $\big(\sum_{n=1}^\infty |f_n(\cdot)|^r\big)^{\frac{1}{r}} \in L^1(\mu_\kappa)$, then for every $\lambda > 0$ we have*

$$\mu_\kappa\bigg(\bigg\{x \in \mathbb{R}^d : \Big(\sum_{n=1}^\infty |M_\kappa^\phi f_n(x)|^r\Big)^{\frac{1}{r}} > \lambda\bigg\}\bigg) \leqslant \frac{C}{\lambda}\Big\|\Big(\sum_{n=1}^\infty |f_n(\cdot)|^r\Big)^{\frac{1}{r}}\Big\|_{\kappa,1},$$

where $C = C(\phi, \kappa_1, \ldots, \kappa_d, r)$ is independent of $(f_n)_{n\geqslant 1}$ and λ.

Proof. The proof is nearly obvious. Indeed, according to the proof of Theorem 7.5 in [16], we have for such a function ϕ and for $x \in \mathbb{R}^d$

$$\big|(f *_\kappa \phi)(x)\big| \leqslant C M_\kappa f(x) \int_0^\infty r^{2\gamma_\kappa + d}\Big|\frac{\mathrm{d}}{\mathrm{d}r}\tilde{\phi}(r)\Big| \,\mathrm{d}r,$$

where C depends only on $\kappa_1, \ldots, \kappa_d$. Therefore, for every $t > 0$ we get

$$\big|(f *_\kappa \phi_t)(x)\big| \leqslant C M_\kappa f(x) \int_0^\infty r^{2\gamma_\kappa + d}\Big|\frac{\mathrm{d}}{\mathrm{d}r}\widetilde{\phi_t}(r)\Big| \,\mathrm{d}r,$$

with C independent of t. Since we have

$$\frac{\mathrm{d}}{\mathrm{d}r}\widetilde{\phi_t}(r) = \frac{1}{t^{2\gamma_\kappa + d + 1}} \frac{\mathrm{d}}{\mathrm{d}r}\tilde{\phi}\Big(\frac{r}{t}\Big),$$

we can write

$$\left|(f *_\kappa \phi_t)(x)\right| \leqslant CM_\kappa f(x) \int_0^\infty \frac{r^{2\gamma_\kappa+d}}{t^{2\gamma_\kappa+d+1}} \left|\frac{\mathrm{d}}{\mathrm{d}r}\tilde{\phi}\Big(\frac{r}{t}\Big)\right| \mathrm{d}r.$$

A change of variables gives us

$$\left|(f *_\kappa \phi_t)(x)\right| \leqslant CM_\kappa f(x) \int_0^\infty r^{2\gamma_\kappa+d} \left|\frac{\mathrm{d}}{\mathrm{d}r}\tilde{\phi}(r)\right| \mathrm{d}r,$$

from which we deduce that

$$\sup_{t>0}\left|(f *_\kappa \phi_t)(x)\right| \leqslant CM_\kappa f(x),$$

where C depends only on $\kappa_1, \ldots, \kappa_d$ and ϕ. If we now apply Theorem 1.2 we obtain the desired result. □

References

[1] Chokri Abdelkefi and Mohamed Sifi. Dunkl translation and uncentered maximal operator on the real line. *Int. J. Math. Math. Sci.*, pages Art. ID 87808, 9, 2007.

[2] Walter R. Bloom and Zeng Fu Xu. The Hardy-Littlewood maximal function for Chébli-Trimèche hypergroups. In *Applications of hypergroups and related measure algebras (Seattle, WA, 1993)*, volume 183 of *Contemp. Math.*, pages 45–70. Amer. Math. Soc., Providence, RI, 1995.

[3] M. F. E. de Jeu. The Dunkl transform. *Invent. Math.*, 113:147–162, 1993.

[4] Charles F. Dunkl. Differential-difference operators associated to reflection groups. *Trans. Amer. Math. Soc.*, 311:167–183, 1989.

[5] Charles F. Dunkl. Hankel transforms associated to finite reflection groups. In *Hypergeometric functions on domains of positivity, Jack polynomials, and applications (Tampa, FL, 1991)*, volume 138 of *Contemp. Math.*, pages 123–138. Amer. Math. Soc., Providence, RI, 1992.

[6] C. Fefferman and E. M. Stein. Some maximal inequalities. *Amer. J. Math.*, 93:107–115, 1971.

[7] Margit Rösler. Bessel-type signed hypergroups on **R**. In *Probability measures on groups and related structures, XI (Oberwolfach, 1994)*, pages 292–304. World Sci. Publ., River Edge, NJ, 1995.

[8] Margit Rösler. Generalized Hermite polynomials and the heat equation for Dunkl operators. *Comm. Math. Phys.*, 192:519–542, 1998.

[9] Margit Rösler. Positivity of Dunkl's intertwining operator. *Duke Math. J.*, 98:445–463, 1999.

[10] Margit Rösler. Dunkl operators: theory and applications. In *Orthogonal polynomials and special functions (Leuven, 2002)*, volume 1817 of *Lecture Notes in Math.*, pages 93–135. Springer, Berlin, 2003.

[11] Margit Rösler. A positive radial product formula for the Dunkl kernel. *Trans. Amer. Math. Soc.*, 355:2413–2438, 2003.

[12] Margit Rösler and Michael Voit. Markov processes related with Dunkl operators. *Adv. in Appl. Math.*, 21:575–643, 1998.

[13] Elias M. Stein. *Singular integrals and differentiability properties of functions.* Princeton University Press, Princeton, N.J., 1970.

[14] Elias M. Stein. *Topics in harmonic analysis related to the Littlewood-Paley theory.* Princeton University Press, Princeton, N.J., 1970.

[15] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals.* Princeton University Press, Princeton, NJ, 1993.

[16] Sundaram Thangavelu and Yuan Xu. Convolution operator and maximal function for the Dunkl transform. *J. Anal. Math.*, 97:25–55, 2005.

[17] Sundaram Thangavelu and Yuan Xu. Riesz transform and Riesz potentials for Dunkl transform. *J. Comput. Appl. Math.*, 199:181–195, 2007.

[18] Khalifa Trimèche. Paley-Wiener theorems for the Dunkl transform and Dunkl translation operators. *Integral Transforms Spec. Funct.*, 13:17–38, 2002.

[19] G. N. Watson. *A Treatise on the Theory of Bessel Functions.* Cambridge University Press, Cambridge, England, 1944.

[20] Yuan Xu. Orthogonal polynomials for a family of product weight functions on the spheres. *Canad. J. Math.*, 49:175–192, 1997.

[21] A. Zygmund. *Trigonometric series. 2nd ed. Vols. I, II.* Cambridge University Press, New York, 1959.

Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie, 175, rue du Chevaleret, 75013 Paris, France

E-mail address: deleaval@math.jussieu.fr