ON A VECTOR-VALUED HOPF-DUNFORD-SCHWARTZ LEMMA

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Abstract. In this paper, we state as a conjecture a vector-valued Hopf-Dunford-Schwartz lemma and give a partial answer to it. As an application of this powerful result, we prove some Fefferman-Stein inequalities in the setting of Dunkl analysis where covering methods are not available.

1. Introduction

A fundamental object in real and harmonic analysis is the Hardy-Littlewood maximal operator. Originally introduced by Hardy and Littlewood for functions defined on the circle ([13]), it was later extended for homogeneous spaces, and even in some non-homogeneous setting like noncompact symmetric spaces ([23]). Moreover, Fefferman and Stein extended it in a vector-valued setting and showed the following generalization of the Hardy-Littlewood theorem ([10]), where we denote by M the Hardy-Littlewood maximal operator given by

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| \, \mathrm{d}y, \quad x \in \mathbb{R}^d,$$

where $|X|$ is the Lebesgue measure of X and $B_r(x)$ is the Euclidean ball of radius r centered at x.

Theorem 1.1. *Let $f = (f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions on $\mathbb{R}^d$ and let $\overline{M}$ be the Fefferman-Stein operator given by $\overline{M}f = (Mf_n)_{n\in\mathbb{N}}$. Let $1 < q < +\infty$.*

(1) *If $\|f\|_{\ell^q} \in L^1(\mathbb{R}^d)$, then for every $\lambda > 0$ we have*
$$\Big|\Big\{x \in \mathbb{R}^d : \|\overline{M}f(x)\|_{\ell^q} > \lambda\Big\}\Big| \leqslant \frac{C}{\lambda}\big\|\| f\|_{\ell^q}\big\|_{L^1(\mathbb{R}^d)},$$
where $C = C(q)$ is independent of $(f_n)_{n\in\mathbb{N}}$ and λ.

(2) *If $\|f\|_{\ell^q} \in L^p(\mathbb{R}^d)$, with $1 < p < +\infty$, then we have*
$$\big\|\| \overline{M}f\|_{\ell^q}\big\|_{L^p(\mathbb{R}^d)} \leqslant C\big\|\| f\|_{\ell^q}\big\|_{L^p(\mathbb{R}^d)},$$
where $C = C(p, q)$ is independent of $(f_n)_{n\in\mathbb{N}}$.

In contrast with the scalar case where very different proofs may be given (combinatorics, covering methods together with interpolation, operator theory with the scalar Hopf-Dunford-Schwartz lemma,...), the vector-valued case is more intricate and relies heavily on three powerful tools of real analysis: the Hardy-Littlewood maximal theorem, a Calderón-Zygmund decomposition and a suitable weighted inequality for the Hardy-Littlewood operator. This last tool is closely related to the theory of A_p weights (in [20], the author proves in this framework some vector-valued maximal inequalities when the weight satisfies various conditions). However, it seems to be important to give another proof, relying on a vector-valued version of the Hopf-Dunford-Schwartz lemma. Indeed, such a lemma should be very helpful in abstract setting, like Dunkl analysis, where a theory of singular integral is out of reach for the moment. More precisely, consider the Dunkl maximal operator M_κ^W defined by

$$M_\kappa^W f(x) = \sup_{r>0} \frac{1}{\mu_\kappa^W(B_r)} \left| \int_{\mathbb{R}^d} f(y)\tau_x^W(\chi_{B_r})(-y) \, \mathrm{d}\mu_\kappa^W(y)\right|, \quad x \in \mathbb{R}^d, \tag{1.1}$$

where χ_{B_r} is the characteristic function of the Euclidean ball of radius r centered at the origin, τ_x^W is the Dunkl translation and μ_κ^W is a weighted Lebesgue measure invariant under the action of a reflection group W (see Section 3 for more details). This operator, which reduces to the

2000 *Mathematics Subject Classification.* 47A35, 42B25.

Key words and phrases. maximal ergodic theorem, semi-group, Dunkl analysis.

Hardy-Littlewood maximal operator in the case where the multiplicity function κ is equal to 0, is of particular interest in view of developing harmonic analysis associated with root systems and reflection groups. Nevertheless, the structure of the Dunkl translation prevents us from using the tools of real analysis (covering lemma, weighted inequalities...) and makes it difficult to study M_κ^W, even if the measure μ_κ^W is doubling. Whereas a scalar maximal theorem has been proved via the Hopf-Dunford-Schwartz lemma (see [5, 25]), currently no vector-valued extension is available from this result and the other ingredients mentioned above. Nevertheless, Fefferman-Stein type inequalities have been established in [4] when W is isomorphic to $\mathbb{Z}_2^d$. This result was achieved thanks to explicit formulas which were available in this particular case, hence one could bypass the lack of information about the Dunkl translation and thus constuct a more convenient type of Hardy-Littlewood operator which controls $M_\kappa^{\mathbb{Z}_2^d}$ pointwise.

The paper is organized as follows. In the next section, we make a brief review on the classical Hopf-Dunford-Schwartz lemma and we give a partial improvement of it in the vector-valued setting. Moreover, we state as a conjecture the general expected result. Section 3 is devoted to an application in Dunkl analysis.

Throughout the paper, we use $X \lesssim Y$ to denote the estimate $X \leqslant CY$ for some constant C, without further precision; if C may depend on parameters that we want to specify, we will explicitly mention it; for example, we will use the notation $X \leqslant C(p,q)\,Y$ to specify the fact that the constant C may depend on p and q; in some computations, we will possibly let C change from one line to the next, without possible confusions.

2. A vector-valued Hopf-Dunford-Schwartz lemma

2.1. **Background on Hopf-Dunford-Schwartz lemma.** Let (Ω, m) be a positive measure space. We say that a one-parameter semi-group of linear operators $(T_t)_{t\geqslant 0}$ satisfies the contraction property if each T_t is a contraction from $L^1(\Omega)$ into itself (for convenience, we will just say T_t is a contraction in $L^1(\Omega)$) and in $L^\infty(\Omega)$. By Riesz convexity theorem, T_t is also a contraction in $L^p(\Omega)$, for any $1 \leqslant p \leqslant +\infty$ and any $t \geqslant 0$. Moreover, if such a semi-group $(T_t)_{t\geqslant 0}$ is assumed to be strongly measurable, then $(T_t)_{t\geqslant 0}$ is strongly integrable over every finite interval, i.e. for any $f \in L^p(\Omega)$, $t \mapsto T_t(f)$ is integrable with respect to the Lebesgue measure along every interval $0 \leqslant t \leqslant \alpha$. This allows to consider the averages

$$A_\alpha f := \frac{1}{\alpha}\int_0^\alpha T_t(f)\,\mathrm{d}t, \quad \alpha > 0.$$

We set $A_0(f) = f$ for any $f \in L^p(\Omega)$. Because the linear map $f \mapsto A_\alpha f$ is closed, A_α is a contraction in each $L^p(\Omega)$, $1 \leqslant p \leqslant +\infty$. In particular, the function $x \mapsto \frac{1}{\alpha}\int_0^\alpha T_t(f)(x)\,\mathrm{d}t$ is m-measurable as well as the maximal function $x \mapsto M_T f(x)$ defined by

$$M_T f(x) := \sup_{\alpha>0} \frac{1}{\alpha}\left|\int_0^\alpha T_t(f)(x)\,\mathrm{d}t\right|, \tag{2.1}$$

for every $x \in \Omega$ except those in a m-null subset of Ω. For all the details, we refer to [6, Chapter VIII.7].

The scalar Hopf-Dunford-Schwartz ergodic lemma may be stated as follows:

Theorem 2.1 (Lemma VIII.7.6 and Theorem VIII.7.7 of [6])**.** *Let $(T_t)_{t\geqslant 0}$ be a strongly measurable semi-group which satisfies the contraction property. Let f be a measurable function on Ω.*

(1) *If $f \in L^1(\Omega)$, then for every $\lambda > 0$ we have*

$$m(\{x \in \Omega : M_T f(x) > \lambda\}) \leqslant \frac{2}{\lambda}\|f\|_{L^1(\Omega)}. \tag{2.2}$$

(2) *If $f \in L^p(\Omega)$, with $1 < p < +\infty$, then we have*

$$\|M_T f\|_{L^p(\Omega)} \leqslant 2\left(\frac{p}{p-1}\right)^{1/p}\|f\|_{L^p(\Omega)}. \tag{2.3}$$

The first version of this theorem goes back to E. Hopf [14] and concerned ergodic means. A generalization was given by Dunford and Schwartz in [7, 21] leading to the above statement. We also refer to [15] for a nice presentation of ergodic theorems.

As mentioned in [22], the Hopf-Dunford-Schwartz lemma is a powerful tool in harmonic analysis. For instance, it allows to prove in abstract setting the boundedness of some maximal operators as soon as the relationship between averages on balls and the heat semi-group (or Poisson semi-group) could be exploited. Because difficulties arise when formulating estimates for operators in the context of vector-valued functions, an extension of an abstract tool such as Hopf-Dunford-Schwartz lemma would be very useful.

Actually, several vector-valued versions of this result have been given by various authors (for e.g. [2, 24, 26] and the references therein). They all consist of generalizing the maximal function operator (2.1) by replacing the absolute value by a norm in a Banach space. Different approaches to this problem were considered: in [2], G. Chacon gave a tricky but direct proof to obtain such a vector-valued version of the weak-type $(1, 1)$ inequality (2.2) (see (1) of Theorem 2.2 below). In fact, his result is not stated in terms of averages of semi-group but in terms of ergodic means, yet one can obtain it in full generality arguing as in [6]. Recently, the main contribution of [24] was to take Inequality (2.3) and improve it to get (2) of Theorem 2.2 ([24, Corollary 4.2]), by using -and proving- that a strongly continuous semi-group of contractions in L^p is dominated by a *positive* strongly continuous semi-group of contractions in L^p (see Theorem 2.6 below). The rest of Taggart's proof relies on classical Banach-valued extension results for positive operators (see [11, 12]).

We mention that another approach, based on transference theory as presented in [1], could possibly lead to the same kind of vector-valued extension of the Hopf-Dunford-Schwartz lemma as above.

Let us summarize the contribution of Chacon and Taggart in the following theorem.

Theorem 2.2. *Assume that B is a Banach space and that $(T_t)_{t\geqslant 0}$ is a strongly continuous semi-group which satisfies the contraction property. Define*

$$\widetilde{M_T} f := \sup_{\alpha>0} \frac{1}{\alpha} \left\| \int_0^\alpha \widetilde{T}_t (f) \, \mathrm{d}t \right\|_B ,$$

where $\widetilde{T}_t$ is the linear extension of T_t to the Banach space $L^p(\Omega, B)$ of vector-valued functions f such that $x \mapsto \|f(x)\|_B$ is measurable on Ω and belongs to $L^p(\Omega)$.

(1) *If $f \in L^1(\Omega, B)$, then for every $\lambda > 0$ we have*

$$m\left(\left\{x \in \Omega : \widetilde{M_T} f(x) > \lambda\right\}\right) \leqslant \frac{2}{\lambda} \|f\|_{L^1(\Omega,B)} .$$

(2) *If $f \in L^p(\Omega, B)$, with $1 < p < +\infty$, then we have*

$$\left\|\widetilde{M_T} f\right\|_{L^p(\Omega,B)} \leqslant 2 \left(\frac{p}{p-1}\right)^{1/p} \|f\|_{L^p(\Omega,B)} .$$

Because of the definition of the maximal operator $\widetilde{M_T}$ under consideration, this result is insufficient to obtain, in particular, Fefferman-Stein type inequalities. In the next paragraph, we introduce another extension which will fill this lack.

2.2. **A partial improvement of Hopf-Dunford-Schwartz lemma.** As in the previous paragraph, M_T stands for the maximal operator defined by (2.1). We consider the Banach space $L^p(\Omega, l^q)$ ($L^p(l^q)$ for short), $1 \leqslant p, q \leqslant +\infty$, consisting of all l^q-valued functions $f = (f_n)_{n\in\mathbb{N}}$ (each f_n is a measurable real or complex valued function) for which $x \mapsto \|f(x)\|_{l^q} := \left(\sum_{n=0}^{+\infty} |f_n(x)|^q\right)^{1/q}$ is finite m-a.e. and such that

$$\|f\|_{L^p(l^q)} := \left(\int_\Omega \left(\sum_{n=0}^{+\infty} |f_n(x)|^q \right)^{p/q} \mathrm{d}m(x) \right)^{1/q} < +\infty$$

(with the usual modification for p or $q = +\infty$). Now, for $f = (f_n)_{n\in\mathbb{N}} \in L^p(l^q)$, we define the vector-valued maximal function operator $\overline{M_T}$ by

$$\overline{M_T}f(x) := (M_T f_n(x))_{n\in\mathbb{N}} = \left(\sup_{\alpha>0}\frac{1}{\alpha}\left|\int_0^\alpha T_t(f_n)(x)\,\mathrm{d}t\right|\right)_{n\in\mathbb{N}}, \tag{2.4}$$

for every $x \in \Omega$ except those in a m-null subset of Ω.

The function $x \mapsto \left\|\overline{M_T(f)(x)}\right\|_{l^q}$ is m-measurable, because $x \mapsto M_T f_n(x)$ is m-measurable for every $n \in \mathbb{N}$ (see Subsection 2.1) and it is now very natural to wonder whether the conclusions of the original scalar Hopf-Dunford-Schwartz lemma remain true for $\overline{M_T}$ when one replaces $L^p(\Omega)$ by $L^p(l^q)$, $1 \leqslant p, q \leqslant +\infty$. The main theorem of this paragraph is a partial positive answer to that question, namely for $1 < p \leqslant q < +\infty$:

Theorem 2.3. *Let $(T_t)_{t\geqslant 0}$ be a strongly continuous semi-group which satisfies the contraction property. Assume that $1 < p \leqslant q < +\infty$. Then, for any $f \in L^p(l^q)$ we have*

$$\left\|\overline{M_T}f\right\|_{L^p(l^q)} \leqslant C(p,q)\left\|f\right\|_{L^p(l^q)}. \tag{2.5}$$

Remark 2.4. The case $p = q$ is a straightforward consequence of the scalar case. Because Theorem 2.1 holds for $p = +\infty$, one can easily check that Inequality (2.5) also holds for $1 < p < +\infty$ and $q = +\infty$.

The first step consists in showing that we can reduce the problem to a semi-group of linear *positive* operators. We need some definitions.

Definition 2.5. *Let T and S be two operators (non necessarily linear) on $L^p(\Omega)$.*

1. *T is positive if $|T(f)| \leqslant T(g)$ whenever $|f| \leqslant g$ almost everywhere, $f, g \in L^p(\Omega)$. In particular, if T is linear, it is equivalent to say that $T(f) \geqslant 0$ whenever $f \geqslant 0$.*
2. *T is dominated by S if for every $f \in L^p(\Omega)$, $|Tf| \leqslant S|f|$.*

These definitions extend to a semi-group of operators; for example, $(T_t)_{t\geqslant 0}$ is dominated by $(S_t)_{t\geqslant 0}$ if T_t is dominated by S_t for every $t \geqslant 0$.

The following result is the main tool in [24] for the author to get his vector-valued version of Hopf-Dunford-Schwartz lemma. Actually, it is a direct combination of the proof of [24, Theorem 3.1] and [24, Lemma 3.5]:

Theorem 2.6. *Suppose that $(T_t)_{t\geqslant 0}$ is a strongly continuous semi-group which satisfies the contraction property. Then there exists a strongly continuous semi-group $(S_t)_{t\geqslant 0}$ satisfying the contraction property, such that S_t is linear positive for any $t \geqslant 0$, and which dominates $(T_t)_{t\geqslant 0}$.*

Let then $(T_t)_{t\geqslant 0}$ be as in Theorem 2.3 and let $(S_t)_{t\geqslant 0}$ be a positive semi-group dominating $(T_t)_{t\geqslant 0}$, as given by Theorem 2.6. As usual, we denote by M_S (resp. $\overline{M_S}$) the scalar (resp. vector-valued) maximal operator associated with $(S_t)_{t\geqslant 0}$. Remark that $x \mapsto \left\|\overline{M_S}(f)(x)\right\|_{l^q}$ is still m-measurable for every $f = (f_n)_{n\in\mathbb{N}} \in L^p(l^q)$.

Now, the second and last step consists in showing that (2.5) is satisfied by $\overline{M_S}$ instead of $\overline{M_T}$, which would imply Theorem 2.3. It is based on a result by J. L. Rubio de Francia which appeals to the following definition.

Definition 2.7 (Definition V.1.20 of [11])**.** *An operator T in $L^p(\Omega)$ is called* linearizable *if there exists a* linear *operator U in $L^p(\Omega)$, whose values are B-valued functions (for some Banach space B), such that*

$$Tf(x) = \left\|Uf(x)\right\|_B,$$

for every $f \in L^p(\Omega)$ and every $x \in \Omega$.

Let us define the operator U in $L^p(\Omega)$ which takes f to $\left(\alpha \mapsto \dfrac{1}{\alpha}\int_0^\alpha S_t(f)\,\mathrm{d}t\right)$. According to the scalar Hopf-Dunford-Schwartz lemma (Theorem 2.1), M_S is bounded in $L^p(\Omega)$ for any $p > 1$. In particular it follows that, when $p>1$, U is a linear operator which takes $L^p(\Omega)$ to $L^p(\Omega, L^\infty(\mathbb{R}))$. In addition, we have

$$M_S f(x) = \|Uf(x)\|_{L^\infty(\mathbb{R})},$$

for every $f \in L^p(\Omega)$ and every $x \in \Omega$, which means that M_S is linearizable.

Therefore, it turns out that Theorem 2.3 is a consequence of the following:

Theorem 2.8 (Corollary V.1.23 of [11])**.** *Let T be a linearizable operator which is bounded in $L^p(\Omega)$ for some $1 \leqslant p < +\infty$. If T is positive, then for any $q \geqslant p$, T has a bounded extension to $L^p(l^q)$, i.e.*

$$\left\| \left(\sum_{n=0}^{+\infty} |T(f_n)|^q \right)^{1/q} \right\|_{L^p(\Omega)} \leqslant C(p,q) \left\| \left(\sum_{n=0}^{+\infty} |f_n|^q \right)^{1/q} \right\|_{L^p(\Omega)},$$

for any $f = (f_n)_{n\in\mathbb{N}}$ in $L^p(l^q)$.

Remark 2.9. (1) A priori, it should have been more difficult to deal with $\overline{M_S}$ since it is "greater" than $\overline{M_T}$. But $\overline{M_S}$ enjoys the property that it is associated with a *positive* operator.
(2) Theorem 2.3 allows one to recover the Fefferman-Stein inequalities of Theorem 1.1 when $1 < p \leqslant q$. Indeed, let

$$T_t f(x) = \frac{1}{(2\pi t)^{d/2}} \int e^{-\|x-y\|^2/2t} f(y)\,\mathrm{d}y.$$

We define the Euclidean heat maximal operator by

$$M_H f(x) = \sup_{\alpha>0} \frac{1}{\alpha} \left| \int_0^\alpha T_t f(x)\,\mathrm{d}t \right|.$$

Theorem 1.1 for $1 < p \leqslant q$ is now a simple consequence of Theorem 2.3 and the pointwise inequality

$$Mf(x) \lesssim M_H f(x), \quad x \in \mathbb{R}^d.$$

(3) Let us observe that Theorem 2.3 can still be extended to a more general context. Let B be a Banach space and let $\widetilde{T_t}$ be the linear extension of T_t to $L^p(\Omega, B)$. For a sequence $(f_n)_{n\in\mathbb{N}}$ of B-valued functions, we can define the maximal operator

$$\overline{M^B_{\widetilde{T}}} f := \left(M^B_{\widetilde{T}} f_n \right)_{n\in\mathbb{N}} = \left(\sup_{\alpha>0} \frac{1}{\alpha} \left\| \int_0^\alpha \widetilde{T}_t(f_n)\,\mathrm{d}t \right\|_B \right)_{n\in\mathbb{N}}. \tag{2.6}$$

We can check that $\overline{M^B_{\widetilde{T}}}$ satisfies the conclusions of Theorem 2.3 in this context, by following the proof of Theorem 2.3, using Taggart's result [24, Corollary 4.2] and noticing that, if $(S_t)_{t\geqslant 0}$ is a positive semi-group dominating $(T_t)_{t\geqslant 0}$, then $M^{B,+}_{\widetilde{T}} := M^B_{\widetilde{S}}$ is a linearizable positive operator, that is for $f \in L^p(\Omega, B)$,

$$M^{B,+}_{\widetilde{T}} f = \left\| \left(\alpha \mapsto \frac{1}{\alpha} \int_0^\alpha \widetilde{S}_t(f)\,\mathrm{d}t \right) \right\|_{L^\infty(\mathbb{R},B)}, \tag{2.7}$$

which takes $L^p(\Omega, B)$ into $L^p(\Omega)$. In particular, such a further extension of Theorem 2.3 is a (self-) improvement of Taggart's result [24, Corollary 4.2].

Since Theorem 2.8 does not hold for arbitrary $1 \leqslant p, q \leqslant +\infty$ (see [11]), our method does not give a complete vector-valued version of Hopf-Dunford-Schwartz lemma. Yet, we make the following conjecture, a confirmation of which would be a powerful result.

Conjecture. *Let $(T_t)_{t\geqslant 0}$ be a strongly continuous semi-group which satisfies the contraction property. Let $1 < q < +\infty$.*

(1) *If $f \in L^1(l^q)$, then for every $\lambda > 0$ we have*

$$m\left(\left\{x \in \Omega : \left\|\overline{M_T} f(x)\right\|_{l^q} > \lambda\right\}\right) \leqslant C(q) \frac{1}{\lambda} \|f\|_{L^1(l^q)}.$$

(2) *If $f \in L^p(l^q)$, with $1 < p < +\infty$, then we have*

$$\left\|\overline{M_T} f\right\|_{L^p(l^q)} \leqslant C(p,q) \|f\|_{L^p(l^q)}.$$

3. Some Fefferman-Stein inequalities for the Dunkl maximal operator

In this section, we present an application of our vector-valued Hopf-Dunford-Schwartz lemma (Theorem 2.3) in the setting of Dunkl analysis, which extends Fourier analysis on Euclidean spaces and analysis on Riemannian symmetric spaces of Euclidean type. Before stating our Fefferman-Stein type inequalities for the operator $\overline{M_\kappa^W} f = \left(M_\kappa^W f_n\right)_{n\in\mathbb{N}}$ (where M_κ^W is given by (1.1)), we give a brief account on the Dunkl theory for the reader's convenience.

3.1. **Background on Dunkl analysis.** For a large panorama of this theory, we refer to [18] and the references therein.

Let $W \subset \mathcal{O}(\mathbb{R}^d)$ be a finite reflection group associated with a reduced root system $\mathcal{R}$ (not necessarily crystallographic) and let $\kappa : \mathcal{R} \to \mathbb{R}_+$ be a multiplicity function, that is a W-invariant function.

The (rational) Dunkl operators $T_\xi^{\mathcal{R}}$ on $\mathbb{R}^d$, which were introduced in [8], are the following κ-deformations of directional derivatives ∂_ξ by reflections

$$T_\xi^{\mathcal{R}} f(x) = \partial_\xi f(x) + \frac{1}{2}\sum_{\alpha\in\mathcal{R}} \kappa(\alpha)\frac{f(x) - f(\sigma_\alpha(x))}{\langle x,\alpha\rangle}\langle \xi,\alpha\rangle, \quad x\in\mathbb{R}^d,$$

where σ_α denotes the reflection with respect to the hyperplane orthogonal to α. The most important property of these operators is their commutativity ([8]). Therefore, we are naturally led to consider the eigenfunction problem

$$T_\xi^{\mathcal{R}} f = \langle y,\xi\rangle f, \quad \forall \xi\in\mathbb{R}^d, \tag{3.1}$$

with $y \in \mathbb{C}^d$ a fixed parameter. Opdam has completely solved this problem ([16]).

Theorem 3.1. *Let $y \in \mathbb{C}^d$. There exists a unique $f = E_\kappa^W(\cdot, y)$ solution of* (3.1) *which is real-analytic on $\mathbb{R}^d$ and satisfies $f(0) = 1$. Moreover E_κ^W, called the Dunkl kernel, extends to a holomorphic function on $\mathbb{C}^d\times\mathbb{C}^d$.*

Unfortunately, the Dunkl kernel is explicitly known only in very few cases; nevertheless we know that this kernel has many properties in common with the classical exponential to which it reduces when $\kappa = 0$. The Dunkl kernel is of particular interest as it gives rise to an integral transform, which is taken with respect to a weighted Lebesgue measure invariant under the action of W and which generalizes the Euclidean Fourier transform.

More precisely, we introduce the following weighted Lebesgue measure

$$\mathrm{d}\mu_\kappa^W(x) = \prod_{\alpha\in\mathcal{R}} |\langle x,\alpha\rangle|^{\kappa(\alpha)}\,\mathrm{d}x = h(x)\,\mathrm{d}x,$$

where the weight h is W-invariant and homogeneous of degree $\gamma = \sum_{\alpha\in\mathcal{R}}\kappa(\alpha)$ (that is for every $\lambda > 0, h(\lambda x) = \lambda^\gamma h(x)$). We denote by L_κ^p the space $L^p(\mathbb{R}^d;\mu_\kappa^W)$ (for $1 \leqslant p \leqslant +\infty$) and we use the shorter notation $\|\cdot\|_{\kappa,p}$ instead of $\|\cdot\|_{L_\kappa^p}$. Then for every $f \in L_\kappa^1$, the Dunkl transform of f, denoted by $\mathcal{F}_\kappa^W(f)$, is defined by

$$\mathcal{F}_\kappa^W(f)(x) = c_\kappa^W \int_{\mathbb{R}^d} E_\kappa^W(-ix,y) f(y)\,\mathrm{d}\mu_\kappa^W(y), \quad x\in\mathbb{R}^d,$$

where c_κ^W is the following Mehta-type constant

$$c_\kappa^W = \left(\int_{\mathbb{R}^d} \mathrm{e}^{-\|x\|^2/2}\,\mathrm{d}\mu_\kappa^W(x)\right)^{-1}.$$

Let us point out that the Dunkl transform coincides with the Euclidean Fourier transform when $\kappa = 0$ and that it is more or less a Hankel transform when $d = 1$. The two main properties of the Dunkl transform are given in the following theorem ([3, 9]).

Theorem 3.2. (1) ***Inversion formula.*** *Let $f \in L_\kappa^1$. If $\mathcal{F}_\kappa^W(f)$ is in L_κ^1, then we have the following inversion formula*

$$f(x) = c_\kappa^W \int_{\mathbb{R}^d} E_\kappa^W(ix,y)\mathcal{F}_\kappa^W(f)(y)\,\mathrm{d}\mu_\kappa^W(y).$$

(2) ***Plancherel theorem.*** *The Dunkl transform has a unique extension to an isometric isomorphism of* L^2_κ.

The Dunkl transform shares many other properties with the Fourier transform. Therefore, it is natural to associate a generalized translation operator with this transform.

There are many ways to define the Dunkl translation but we use the definition which most underlines the analogy with the Fourier transform. It is the definition given in [25] with a different convention. Let $x \in \mathbb{R}^d$. The Dunkl translation $f \mapsto \tau_x^W f$ is defined on L^2_κ by the equation

$$\mathcal{F}_\kappa^W(\tau_x^W f)(y) = E_\kappa^W(ix, y)\mathcal{F}_\kappa^W(f)(y), \quad y \in \mathbb{R}^d.$$

In Fourier analysis, the translation operator $f \mapsto f(\cdot + x)$ (to which the Dunkl translation reduces when $\kappa = 0$) is positive and L^p-bounded. In the Dunkl setting, τ_x^W is not a positive operator ([17, 25]) and the L^p_κ-boundedness is still a challenging problem, apart from the trivial case where $p = 2$ (thanks to the Plancherel theorem and the fact that $|E_\kappa^W(ix, y)| \leqslant 1$). Moreover, its structure prevents from using covering methods.

3.2. **Fefferman-Stein type inequalities for the Dunkl maximal operator.** With all these definitions in mind, we can now state our main result.

Theorem 3.3. *Let* $1 < p \leqslant q < +\infty$. *Then, for any* $f \in L^p_\kappa(l^q)$ *we have*

$$\left\| \overline{M_\kappa^W} f \right\|_{L^p_\kappa(l^q)} \leqslant C(p,q) \, \|f\|_{L^p_\kappa(l^q)} . \tag{3.2}$$

Remark 3.4. In [4], complete Fefferman-Stein inequalites were given in the particular case where $W \simeq \mathbb{Z}_2^d$ but, as already claimed, the proof relies on some explicit formulas for $\tau_x^{\mathbb{Z}_2^d}$ which allow to construct a $\mathbb{Z}_2^d$-invariant maximal operator, which controls pointwise $M_\kappa^{\mathbb{Z}_2^d}$ and for which we can use covering argument together with interpolation.

We come to the proof of Theorem 3.3. Let us introduce the so-called Dunkl-type heat semi-group $(H_t^W)_{t\geqslant 0}$ which is associated with the Dunkl Laplacian $\Delta_\kappa^W = \sum_{j=1}^d (T_{e_j}^{\mathcal{R}})^2$ (see [19]). More precisely for every $f \in L^p_\kappa$, with $1 \leqslant p \leqslant +\infty$, and for every $t \geqslant 0$, it is given by

$$H_t^W f = \begin{cases} \int_{\mathbb{R}^d} f(y) Q_\kappa^W(\cdot, y, t) \, \mathrm{d}\mu_\kappa^W(y) & \text{if } t > 0 \\ f & \text{if } t = 0, \end{cases}$$

where

$$Q_\kappa^W(x, y, t) = \frac{c_\kappa^W}{(2t)^{\frac{d}{2}+\gamma}} \mathrm{e}^{-\frac{(\|x\|^2+\|y\|^2)}{4t}} E_\kappa^W\Big(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\Big) > 0, \quad x, y \in \mathbb{R}^d, t > 0.$$

According to [5] (see also [25]), we have the following pointwise inequality

$$M_\kappa^W f(x) \lesssim \sup_{\alpha > 0} \frac{1}{\alpha} \int_0^\alpha H_t^W |f(x)| \, \mathrm{d}t := M_{\kappa,H}^W f(x), \quad x \in \mathbb{R}^d.$$

Thanks to this inequality, it suffices to show that Inequality (3.2) is true by replacing $\overline{M_\kappa^W} f = \left(M_\kappa^W f_n\right)_{n\in\mathbb{N}}$ by $\overline{M_{\kappa,H}^W} f = \left(M_{\kappa,H}^W f_n\right)_{n\in\mathbb{N}}$, for any $f = (f_n)_{n\in\mathbb{N}}$ in $L^p_\kappa(l^q)$ positive. Now $(H_t)_{t\geqslant 0}$ is a symmetric diffusion semi-group on L^p_κ for $1 \leqslant p \leqslant +\infty$ (Theorem 2.6. in [5]). Thus, by our vector-valued Hopf-Dunford-Schwartz lemma (Theorem 2.3), we get the desired estimate (3.2).

Remark 3.5. Theorem 3.3 can be extended to a larger class of Dunkl-type convolution operators as presented in [4]. Therefore, this generalizes Theorem 4.1 in [4] (only for the range $1 < p \leqslant q < +\infty$ but for all reflection group W instead of the very particular case $\mathbb{Z}_2^d$).

Acknowledgment. We would like to thank the referee for carefully reading this paper and providing valuable improvements and references.

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