# ON THE BOUNDEDNESS OF THE DUNKL SPHERICAL MAXIMAL OPERATOR

LUC DELEAVAL

Abstract. In this article, we introduce a Dunkl-type spherical maximal operator associated with a finite reflection group and study its boundedness. In view of this study, we give some results on suitable square functions and maximal multiplier operators in the Dunkl setting.

## 1. Introduction

Introduced in [14] in order to study some properties of a generalized Laplacian associated with finite reflection groups, the Dunkl-type spherical mean operator turned out to be particularly important in harmonic analysis and special functions associated with root systems (see especially [2, 19, 25]). Quite remarkably, its positivity-preserving property, shown by Rösler in [19], implies a radial product formula for the non-symmetric counterpart of a generalized Bessel function (the Dunkl kernel), a result which strenghten a well-known conjecture on a product formula for multivariable Bessel functions (with non-negative multiplicities) associated with Weyl chambers.

In this article, we introduce and study a Dunkl-type spherical maximal operator $M_\kappa$, where $\kappa$ is a multiplicity function, that is, a function invariant under the action of a finite reflection group $W$ associated with a root system $\mathcal{R}$ (see Section 2 for more details). This operator is initially defined on the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ by means of the Dunkl-type spherical mean operators $(A_r^\kappa)_{r>0}$, namely

$$\begin{aligned} M_\kappa f(x) &= \sup_{r>0} |A_r^\kappa f(x)| \\ &:= \sup_{r>0} d_\kappa \left| \int_{S^{d-1}} \tau_x^\kappa f(-ry) \prod_{\alpha\in\mathcal{R}} |\langle y,\alpha\rangle|^{\kappa(\alpha)} d\sigma(y) \right|, \quad x\in\mathbb{R}^d, \end{aligned}$$

where $d_\kappa$ is a normalizing constant, $d\sigma$ is the normalized Lebesgue surface measure on the sphere $S^{d-1}$ and $\tau_x^\kappa$ is the Dunkl translation (see Section 2 for more details). This generalized translation, which reduces to the usual translation when the multiplicity function $\kappa$ is equal to 0, is not a positive operator and its $L^p(\mathbb{R}^d;\mu_\kappa)$-boundedness, with $d\mu_\kappa(x) = \prod_{\alpha\in\mathcal{R}} |\langle x,\alpha\rangle|^{\kappa(\alpha)} dx$, is not established for any finite reflection group. In fact, even an explicit formula for $\tau_x^\kappa$ is unknown in general.

Nevertheless, we succeed in proving, for any finite reflection group, the following theorem on the $L^p(\mathbb{R}^d;\mu_\kappa)$-boundedness of $M_\kappa$. This is the main result of the paper.

**Theorem 1.1.** *Let $d \geqslant 3$ and let $2\gamma_\kappa$ be the non-negative degree of homogeneity of the measure $\mu_\kappa$.*

2000 *Mathematics Subject Classification.* 42B15, 42B25.

*Key words and phrases.* Dunkl operator, spherical maximal operator, square function, multiplier.

The author is pleased to express his thanks to Olivier Guédon for his valuable comments.

(1) *If $p$ is a real number which satisfies*

$$\frac{d+2\gamma_\kappa}{d+2\gamma_\kappa-1} < p \leqslant +\infty,$$

*then, for every $f \in L^p(\mathbb{R}^d;\mu_\kappa)$, we have the estimate*

$$\|M_\kappa f\|_{L^p(\mathbb{R}^d;\mu_\kappa)} \leqslant C\|f\|_{L^p(\mathbb{R}^d;\mu_\kappa)}, \tag{1.1}$$

*where $C = C(d,\kappa,p)$ is a constant independent of $f$.*

(2) *The condition $p > \frac{d+2\gamma_\kappa}{d+2\gamma_\kappa-1}$ is necessary, that is, the inequality* (1.1) *is not valid if $p \leqslant \frac{d+2\gamma_\kappa}{d+2\gamma_\kappa-1}$.*

As we will see in this paper, the range of $p$ in Theorem 1.1 depends on the decay at infinite of an underlying multiplier (and some of its derivatives) associated with $M_\kappa$.

Of course, the previous theorem implies the following result (which is also available for any finite reflection group) on differentiation of functions.

**Corollary 1.1.** *Under the hypothesis of Theorem 1.1, if $p$ is a real number which satisfies*

$$\frac{d+2\gamma_\kappa}{d+2\gamma_\kappa-1} < p \leqslant +\infty,$$

*then, for every $f \in L^p(\mathbb{R}^d;\mu_\kappa)$,*

$$\lim_{r\to 0} A_r^\kappa f(x) = f(x)$$

*for almost all $x \in \mathbb{R}^d$.*

In the particular case where $\gamma_\kappa = 0$ (that is, $\kappa$ is equal to 0), Theorem 1.1 coincides with a theorem due to Stein for the classical spherical maximal operator (see [24]). Stein's result was particularly important since, together with an argument based on the Calderón-Zygmund method of rotations, it implies that the $L^p$-norm of the centered Hardy-Littlewood maximal operator is bounded above by some dimension-free constant (for $p > 1$), see [21] (and see [22] for full details of the proof). Theorem 1.1 might be a first step to prove that the Dunkl maximal operator (see [5, 25]) is bounded on $L^p(\mathbb{R}^d;\mu_\kappa)$ (for p>1) not only uniformly in $d$, but also in $\gamma_\kappa$. For the moment, the best upper bound known is $\sqrt{d+2\gamma_\kappa}$ (see [5]).

In order to prove Theorem 1.1, we shall introduce and study suitable square functions and maximal multiplier operators in the Dunkl setting, and we shall use some classical technics of the Littlewood-Paley theory, in particular some decompositions of multipliers into dyadic radial pieces with localized frequencies. More precisely, we shall dominate $M_\kappa$ by a series of Dunkl-type maximal multiplier operators $M_\kappa \leqslant \sum_{n=0}^{+\infty} M_\kappa^{m_n^\kappa}$ (where $m_n^\kappa$ is a radial multiplier), and we shall establish that each $M_\kappa^{m_n^\kappa}$ is of strong-type when $p=2$ and of weak-type when $p=1$. Then, we shall proceed by interpolation, and the range of $p$ in Theorem 1.1 is then relevant for series convergence. For the case $p=2$, we mainly use both the decay at infinite and a support property for $m_n^\kappa$, a precise upper bound for the $L^2(\mathbb{R}^d;\mu_\kappa)$-norm of a related Dunkl-type square function, and a rather classical method used by several authors in the standard case (see especially [3, 20]). For the case $p=1$, we need, together with some ideas presented in [11] in the standard case, two key arguments to bypass some difficulties related to the Dunkl translation: a formula for the Dunkl transform of $m_n^\kappa$ in terms of an operator (without explicit formula for it) which intertwines the usual directional derivatives and the Dunkl operators (see

Section 2 for more details), and a Funk-Hecke type formula for reflection invariant weights.

The paper is organized as follows. In the next section, we collect some definitions and results related to Dunkl theory. We then introduce in Section 3 the Dunkl-type spherical maximal operator and present the right setting for its study. Therefore, we also introduce and study in this section some square functions and maximal multiplier operators related to the Dunkl transform and the Dunkl convolution. Section 4 is devoted to the proof of Theorem 1.1.

Throughout the paper and if it is not necessary, a constant may vary from line to line without changing its notation.

## 2. Preliminaries

This section is devoted to the preliminaries and background. We only focus on the aspects of Dunkl theory which will be relevant in what follows. For a large survey about this theory, the reader may especially consult [10, 18] and the references therein. Concerning root systems and reflection groups, see [12].

2.1. **The Dunkl transform.** Let $W \subset \mathcal{O}(\mathbb{R}^d)$ be a finite reflection group associated with a reduced root system $\mathcal{R}$ (not necessarily crystallographic) and let $\kappa : \mathcal{R} \to \mathbb{C}$ be a multiplicity function, that is, a $W$-invariant function. We assume in this article that $\kappa$ takes value in $[0, +\infty[$.

The (rational) Dunkl operators $T_\xi^\kappa$ on $\mathbb{R}^d$, introduced in [6], are the following $\kappa$-deformations of directional derivatives $\partial_\xi$ by reflections

$$T_\xi^\kappa f(x) = \partial_\xi f(x) + \sum_{\alpha\in\mathcal{R}_+} \kappa(\alpha) \frac{f(x) - f(\sigma_\alpha(x))}{\langle x, \alpha\rangle} \langle \xi, \alpha\rangle, \quad x \in \mathbb{R}^d,$$

where $\sigma_\alpha$ denotes the reflection with respect to the hyperplane orthogonal to $\alpha$ and $\mathcal{R}_+$ denotes a positive subsystem of $\mathcal{R}$. The definition is of course independent of the choice of the positive subsystem since $\kappa$ is $W$-invariant. These operators map $\mathcal{P}_n^d$ to $\mathcal{P}_{n-1}^d$, where $\mathcal{P}_n^d$ is the space of homogeneous polynomials of degree $n$ in $d$ variables, and they mutually commute, that is, $T_\xi^\kappa T_{\xi'}^\kappa = T_{\xi'}^\kappa T_\xi^\kappa$ (see [6]). Moreover, there exists a unique linear isomorphism $V_\kappa$ (called intertwining operator) on $\mathcal{P} = \bigoplus_{n\geqslant 0} \mathcal{P}_n^d$ such that

$$V_\kappa(\mathcal{P}_n^d) = \mathcal{P}_n^d, \qquad V_\kappa|_{\mathcal{P}_0^d} = \mathrm{Id}|_{\mathcal{P}_0^d}, \qquad T_\xi^\kappa V_\kappa = V_\kappa \partial_\xi \quad \forall \xi \in \mathbb{R}^d.$$

Unfortunately and even if we have significant results for it (a Laplace-type representation which allows to extend it to various larger function spaces or its positivity for instance, see [17]), the intertwining operator is explicitly known only in some special cases: when the root system is orthogonal (see [13, 26]), of $A_2$-type (see [8]) and of $B_2$-type (see [9]). For $y \in \mathbb{C}^d$, let

$$E_\kappa(x, y) = V_\kappa\big(\mathrm{e}^{\langle \cdot, y\rangle}\big)(x), \quad x \in \mathbb{R}^d,$$

where $\langle \cdot, \cdot\rangle$ denotes the bilinear extension of the Euclidean inner product to $\mathbb{C}^d \times \mathbb{C}^d$. Then $E_\kappa(\cdot, y)$ is the unique real-analytic solution of the spectral problem

$$T_\xi^\kappa f = \langle \xi, y\rangle f \quad \forall \xi \in \mathbb{R}^d, \quad f(0) = 1,$$

and moreover, $E_\kappa$ extends to a holomorphic function on $\mathbb{C}^d \times \mathbb{C}^d$, see [15]. The kernel $E_\kappa$ is symmetric in its arguments and satisfies for all $z, z' \in \mathbb{C}^d$, $\lambda \in \mathbb{C}$ and $w \in W$

$$E_\kappa(\lambda z, z') = E_\kappa(z, \lambda z'), \quad E_\kappa(w(z), w(z')) = E_\kappa(z, z'). \tag{2.1}$$

The Dunkl kernel is of particular interest as it gives rise to an integral transform which is taken with respect to a weighted Lebesgue measure invariant under the action of $W$ and which generalizes the Euclidean Fourier transform.

More precisely, let us introduce the measure $d\mu_\kappa(x) = h_\kappa^2(x)dx$ where the weight defined on $\mathbb{R}^d$ by

$$h_\kappa^2(x) = \prod_{\alpha\in\mathcal{R}_+} |\langle x,\alpha\rangle|^{2\kappa(\alpha)} = \prod_{\alpha\in\mathcal{R}} |\langle x,\alpha\rangle|^{\kappa(\alpha)}$$

is homogeneous of degree $2\gamma_\kappa$ with

$$\gamma_\kappa = \sum_{\alpha\in\mathcal{R}_+} \kappa(\alpha),$$

and let us denote by $L^p_\kappa$ the space $L^p(\mathbb{R}^d;\mu_\kappa)$ (for $1 \leqslant p \leqslant +\infty$) and use the shorter notation $\|\cdot\|_{\kappa,p}$ instead of $\|\cdot\|_{L^p_\kappa}$.
Then for every $f \in L^1_\kappa$, the Dunkl transform of $f$, denoted by $\mathcal{F}_\kappa(f)$, is defined by

$$\mathcal{F}_\kappa(f)(x) = c_\kappa \int_{\mathbb{R}^d} E_\kappa(-ix,y)f(y)d\mu_\kappa(y), \quad x\in\mathbb{R}^d,$$

where $c_\kappa$ is the Mehta-type constant

$$c_\kappa = \left(\int_{\mathbb{R}^d} \mathrm{e}^{-\frac{\|x\|^2}{2}} d\mu_\kappa(x)\right)^{-1}.$$

The inverse Dunkl transform is defined by $\mathcal{F}^\kappa(f)(x) = \mathcal{F}_\kappa(f)(-x)$. Let us point out that the Dunkl transform coincides with the Euclidean Fourier transform when $\kappa = 0$ (since $T^0_\xi = \partial_\xi$ and $V_0 = \mathrm{Id}$) and that it is more or less a Hankel transform when $d = 1$ (and then $W \simeq \mathbb{Z}_2$) since we have (see [7])

$$E_\kappa(ix,y) = \frac{\Gamma\big(\kappa+\frac12\big)}{\Gamma(\kappa)\Gamma\big(\frac12\big)} \int_{-1}^{1} \mathrm{e}^{itxy}(1+t)(1-t^2)^{\kappa-1}dt = j_{\kappa-\frac12}(xy) + \frac{ixy}{2\kappa+1} j_{\kappa+\frac12}(xy),$$

where $\kappa$ is the only value taken by the multiplicity function and where $j_\alpha$ is the normalized spherical Bessel function of order $\alpha$, that is,

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^{+\infty} \frac{(-1)^n z^{2n}}{n!2^{2n}\Gamma(n+\alpha+1)}.$$

The Dunkl transform shares many properties with the Euclidean Fourier transform and some of these properties (see [4, 7]) are collected below.

**Theorem 2.1.**

(1) *The Dunkl transform of a radial function is radial.*
(2) ***Inversion formula.*** *Let $f \in L^1_\kappa$. If $\mathcal{F}_\kappa(f)$ is in $L^1_\kappa$, then we have the inversion formula $f = \mathcal{F}^\kappa(\mathcal{F}_\kappa(f))$, that is,*

$$f(x) = c_\kappa \int_{\mathbb{R}^d} E_\kappa(ix,y)\mathcal{F}_\kappa(f)(y)d\mu_\kappa(y).$$

(3) *The Dunkl transform is a homeomorphism of $\mathcal{S}(\mathbb{R}^d)$.*
(4) ***Plancherel theorem.*** *The Dunkl transform has a unique extension to an isometric isomorphism of $L^2_\kappa$.*

2.2. **The Dunkl translation and convolution.** Since many properties of the Euclidean Fourier transform carry over to the Dunkl transform, it is natural to associate a generalized translation operator with this transform.

There are many ways to define the Dunkl translation but we use the definition which most underlines the analogy with the Euclidean Fourier transform. It is the definition given in [25] with a different convention. Let $x \in \mathbb{R}^d$. The Dunkl translation $f \mapsto \tau_x^\kappa f$ is defined on $L^2_\kappa$ by the equation

$$\mathcal{F}_\kappa(\tau_x^\kappa f)(y) = E_\kappa(ix, y)\mathcal{F}_\kappa(f)(y), \quad y \in \mathbb{R}^d.$$

If $f$ is a Schwartz function, the Dunkl translation can be defined pointwise. Indeed, the inversion formula allows us to write

$$\tau_x^\kappa f(y) = c_\kappa \int_{\mathbb{R}^d} E_\kappa(ix, z)E_\kappa(iy, z)\mathcal{F}_\kappa(f)(z)d\mu_\kappa(z). \tag{2.2}$$

In Fourier analysis, the translation operator $f \mapsto f(\cdot + x)$ (to which the Dunkl translation reduces when $\kappa = 0$) is positive and $L^p$-bounded. In the Dunkl setting, $\tau_x^\kappa$ is not a positive operator in full generality (see [16, 25]) and the $L^p_\kappa$-boundedness is still a challenging problem, apart from the trivial case where $p = 2$ (thanks to Plancherel's theorem and the estimate $|E_\kappa(ix, y)| \leqslant 1$ proved in [17]). An important partial answer to this problem and a substitute result for the positivity of $\tau_x^\kappa$ are given in the following theorem (see [25]), where $L^p_{\kappa,\mathrm{rad}}$ is the subspace of radial functions in $L^p_\kappa$.

**Theorem 2.2.**

(1) *For every $p$ satisfying $1 \leqslant p \leqslant 2$ and for every $x \in \mathbb{R}^d$, the Dunkl translation $\tau_x^\kappa : L^p_{\kappa,\mathrm{rad}} \to L^p_\kappa$ is a bounded operator.*
(2) *Let $f \in L^1_{\kappa,\mathrm{rad}}$ be a bounded and positive function. Then, $\tau_x^\kappa f \geqslant 0$ for every $x \in \mathbb{R}^d$.*

A decisive argument in the proof of the previous theorem is the following formula, which will be relevant in the sequel. If $f$ is a radial function in $\mathcal{S}(\mathbb{R}^d)$ with $f(\xi) = \overline{f}(\|\xi\|)$, then (see [19])

$$\tau_x^\kappa f(y) = V_\kappa\Big(\overline{f}\Big(\sqrt{\|x\|^2 + \|y\|^2 + 2\langle \cdot, y\rangle}\Big)\Big)(x). \tag{2.3}$$

The Dunkl convolution is classically defined by means of the Dunkl translation, that is, for every $f, g \in L^2_\kappa$,

$$(f *_\kappa g)(x) = c_\kappa \int_{\mathbb{R}^d} f(y)\tau_x^\kappa g(-y)d\mu_\kappa(y), \quad x \in \mathbb{R}^d.$$

It satisfies the basic property

$$\mathcal{F}_\kappa(f *_\kappa g) = \mathcal{F}_\kappa(f)\mathcal{F}_\kappa(g).$$

If the Dunkl translation can be extended as a bounded operator on $L^p_\kappa$, then the Dunkl convolution will satisfy the usual Young's inequality. At present and if we consider a reflection group in full generality, we can only say something about convolution with radial functions (see [25]).

**Theorem 2.3.** *Let $g$ be a bounded radial function in $L^1_\kappa$. Then, the operator $f \mapsto f *_\kappa g$ initially defined on $L^1_\kappa \cap L^2_\kappa$ extends to $L^p_\kappa$, for every $1 \leqslant p \leqslant +\infty$, as a bounded operator.*

## 3. Dunkl spherical maximal operator

In this section, we define the Dunkl spherical maximal operator $M_\kappa$ and give a more convenient expression for it. Moreover, we introduce in the Dunkl setting two objects closely related to $M_\kappa$: a maximal multiplier operator and an appropriate square function. We prove for these objects some results which will be relevant in the proof of Theorem 1.1.

3.1. **Definition and a more convenient expression.** Following Mejjaoli and Trimèche [14] (see also [19]), we may define the Dunkl-type spherical mean operators $(A_r^\kappa)_{r>0}$ on $\mathcal{S}(\mathbb{R}^d)$ by

$$A_r^\kappa f(x) = d_\kappa \int_{S^{d-1}} \tau_x^\kappa f(-ry) h_\kappa^2(y) d\sigma(y), \quad x \in \mathbb{R}^d,\ r > 0,$$

with

$$d_\kappa = \left( \int_{S^{d-1}} h_\kappa^2(y) d\sigma(y) \right)^{-1}.$$

The Dunkl spherical maximal operator is classically defined by means of $(A_r^\kappa)_{r>0}$ by

$$M_\kappa f(x) = \sup_{r>0} |A_r^\kappa f(x)|, \quad x \in \mathbb{R}^d. \tag{3.1}$$

Let us give a more convenient expression for $M_\kappa$. For $f \in \mathcal{S}(\mathbb{R}^d)$ and by use of the equality (2.2), we can write for every $x \in \mathbb{R}^d$

$$\begin{aligned} A_r^\kappa f(x) &= d_\kappa \int_{S^{d-1}} \left( c_\kappa \int_{\mathbb{R}^d} E_\kappa(ix,z) E_\kappa(-iry,z) \mathcal{F}_\kappa(f)(z) d\mu_\kappa(z) \right) h_\kappa^2(y) d\sigma(y) \\ &= c_\kappa \int_{\mathbb{R}^d} E_\kappa(ix,z) \mathcal{F}_\kappa(f)(z) \left( d_\kappa \int_{S^{d-1}} E_\kappa(-irz,y) h_\kappa^2(y) d\sigma(y) \right) d\mu_\kappa(z), \end{aligned}$$

where we have used Fubini's theorem and some properties of the Dunkl kernel (the first equality of (2.1) together with the fact that $E_\kappa$ is symmetric in its arguments). Since (see [19, page 2425])

$$d_\kappa \int_{S^{d-1}} E_\kappa(-irz,y) h_\kappa^2(y) d\sigma(y) = j_{d/2+\gamma_\kappa-1}(r\|z\|), \tag{3.2}$$

we have

$$A_r^\kappa f(x) = c_\kappa \int_{\mathbb{R}^d} E_\kappa(ix,z) \mathcal{F}_\kappa(f)(z) j_{d/2+\gamma_\kappa-1}(r\|z\|) d\mu_\kappa(z),$$

and (3.1) is therefore equivalent to

$$M_\kappa f(x) = \sup_{r>0} c_\kappa \left| \int_{\mathbb{R}^d} E_\kappa(ix,y) \mathcal{F}_\kappa(f)(y) j_{d/2+\gamma_\kappa-1}(r\|y\|) d\mu_\kappa(y) \right|.$$

By setting

$$\widetilde{j_\kappa}(\xi) := j_{d/2+\gamma_\kappa-1}(\|\xi\|), \quad \xi \in \mathbb{R}^d, \tag{3.3}$$

we finally write

$$M_\kappa f(x) = \sup_{r>0} c_\kappa \left| \int_{\mathbb{R}^d} E_\kappa(ix,y) \mathcal{F}_\kappa(f)(y) \widetilde{j_\kappa}(ry) d\mu_\kappa(y) \right|. \tag{3.4}$$

With the previous expression in mind, it is obvious that the right setting for the study of $M_\kappa$ is the Littlewood-Paley theory. More precisely, let $\varphi_0$ be a radial and smooth function on $\mathbb{R}^d$ satisfying

$$\varphi_0(x) = \begin{cases} 1 & \text{if } \|x\| \leqslant 1 \\ 0 & \text{if } \|x\| \geqslant 2. \end{cases}$$

Then, we set for every positive integer $n$

$$\varphi_n(x) = \varphi_0(2^{-n}x) - \varphi_0(2^{1-n}x), \tag{3.5}$$

and we define the following dyadic radial pieces associated with the multiplier $\widetilde{j_\kappa}$

$$\forall n \geqslant 0, \quad m_n^\kappa = \varphi_n \widetilde{j_\kappa}. \tag{3.6}$$

Since we have obviously

$$\sum_{n=0}^{+\infty} \varphi_n = 1,$$

then we can write

$$\widetilde{j_\kappa} = \sum_{n=0}^{+\infty} m_n^\kappa.$$

Therefore, we have the following pointwise inequality

$$M_\kappa f(x) \leqslant \sum_{n=0}^{+\infty} M_\kappa^{m_n^\kappa} f(x), \quad x \in \mathbb{R}^d, \tag{3.7}$$

where we have set

$$\begin{aligned} M_\kappa^{m_n^\kappa} f(x) &= \sup_{r>0} c_\kappa \left| \int_{\mathbb{R}^d} E_\kappa(ix,y)\mathcal{F}_\kappa(f)(y) m_n^\kappa(ry) d\mu_\kappa(y) \right| \\ &= \sup_{r>0} \left| \mathcal{F}^\kappa\big(\mathcal{F}_\kappa(f)(\cdot) m_n^\kappa(r\,\cdot)\big)(x) \right|. \end{aligned}$$

We now turn to the study of such Dunkl-type maximal multiplier operator and to a closely related object, a suitable square function.

3.2. **Dunkl-type maximal multiplier operator and square function.** Let $m \in \mathcal{S}(\mathbb{R}^d)$ be a radial function. We denote by $M_\kappa^m$ the maximal operator associated with the multiplier $m$ and initially defined for $f \in \mathcal{S}(\mathbb{R}^d)$ by

$$M_\kappa^m f(x) = \sup_{r>0} \left| \mathcal{F}^\kappa\big(\mathcal{F}_\kappa(f)(\cdot) m(r\,\cdot)\big)(x) \right|, \quad x \in \mathbb{R}^d.$$

The following proposition, which has an interest in its own, will be useful in the proof of our main result. It deals with the $L_\kappa^p$-boundedness of such a Dunkl-type maximal operator.

**Proposition 3.1.** *$M_\kappa^m$ extends to $L_\kappa^p$, for every $1 < p < +\infty$, as a bounded operator.*

In order to prove this proposition, let us recall some facts on the Dunkl convolution with a radial kernel. For $\phi \in L_\kappa^1$ and $r > 0$, the dilation $\phi_r$ is defined by

$$\phi_r(x) = \frac{1}{r^{d+2\gamma_\kappa}} \phi\Big(\frac{x}{r}\Big), \quad x \in \mathbb{R}^d.$$

*Remark.* By a change of variables, we have obviously $\mathcal{F}_\kappa(\phi_t)(\cdot) = \mathcal{F}_\kappa(\phi)(t\,\cdot)$.

If $\phi \in L^1_\kappa$ is a radial function which satisfies for every $x \in \mathbb{R}^d$

$$|\phi(x)| \leqslant \frac{C}{(1+\|x\|)^{d+2\gamma_\kappa+1}}$$

and such that $\mathcal{F}_\kappa(\phi) \in L^1_\kappa$, then we claim (see [25, Theorem 6.2]) that for every $1 < p < +\infty$ and every $f \in L^p_\kappa$

$$\sup_{r>0} \big|(f *_\kappa \phi_r)(x)\big| \leqslant C\mathcal{M}_\kappa f(x), \quad x \in \mathbb{R}^d,$$

where $\mathcal{M}_\kappa$ is the well-known Dunkl-type maximal operator (see [5] or [25]) and $C$ is a constant independent of $f$.

With this recall in mind, we are now in a position to prove Proposition 3.1.

*Proof.* Since $m$ is a Schwartz radial function, then $\mathcal{F}^\kappa(m)$ is a Schwartz radial function and since we can write for every $x \in \mathbb{R}^d$

$$\mathcal{F}^\kappa\big(\mathcal{F}_\kappa(f)(\cdot)m(r\,\cdot)\big)(x) = \Big(f *_\kappa (\mathcal{F}^\kappa(m))_r\Big)(x)$$

with

$$|\mathcal{F}^\kappa(m)(x)| \leqslant \frac{C}{(1+\|x\|)^{d+2\gamma_\kappa+1}},$$

then

$$M^m_\kappa f(x) \leqslant C\mathcal{M}_\kappa f(x),$$

where $C$ is a constant independent of $f$. By applying the Hardy-Littlewood type theorem for $\mathcal{M}_\kappa$ (see [5, Theorem 1.3] or [25, Theorem 6.1]) which asserts in particular that $\mathcal{M}_\kappa$ maps $L^p_\kappa$ to itself for $1 < p < +\infty$, the proposition is proved. □

Closely related to that Dunkl-type maximal multiplier operator is the following square function. For a radial function $m \in \mathcal{S}(\mathbb{R}^d)$, we denote by $g^m_\kappa$ the square function associated with the multiplier $m$ and initially defined for every $f \in \mathcal{S}(\mathbb{R}^d)$ by

$$g^m_\kappa(f)(x) = \left(\int_0^{+\infty} \Big|\mathcal{F}^\kappa\big(\mathcal{F}_\kappa(f)(\cdot)m(t\,\cdot)\big)(x)\Big|^2 \frac{dt}{t}\right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^d.$$

If the multiplier $m$ is supported in an annulus, then we can give a precise upper bound for the $L^2_\kappa$-norm of $g^m_\kappa(f)$. Indeed, we have the following proposition.

**Proposition 3.2.** *Let $r$ be a positive real number. Suppose that $m$ is supported in the annulus $\{x \in \mathbb{R}^d : r \leqslant \|x\| \leqslant \rho r\}$, $\rho > 1$, and is bounded by $C$. Then, for every $f \in L^2_\kappa$, we have*

$$\|g^m_\kappa(f)\|_{\kappa,2} \leqslant C\sqrt{\ln(\rho)}\|f\|_{\kappa,2},$$

*where $C$ is the same constant in both the hypothesis and conclusion of the proposition.*

*Proof.* We can write

$$\int_{\mathbb{R}^d} |g^m_\kappa(f)(x)|^2 d\mu_\kappa(x) = \int_0^{+\infty} \left(\int_{\mathbb{R}^d} \Big|\big(\mathcal{F}_\kappa(f)(\cdot)m(t\,\cdot)\big)(x)\Big|^2 d\mu_\kappa(x)\right)\frac{dt}{t}$$

where we have used Fubini's theorem and Plancherel's theorem. By using again Fubini's theorem we get

$$\int_{\mathbb{R}^d} |g^m_\kappa(f)(x)|^2 d\mu_\kappa(x) = \int_{\mathbb{R}^d} \left(\int_0^{+\infty} \Big|\big(\mathcal{F}_\kappa(f)(\cdot)m(t\,\cdot)\big)(x)\Big|^2 \frac{dt}{t}\right) d\mu_\kappa(x).$$

The assumptions on $m$ allow us to write for $x \in \mathbb{R}^d \setminus \{0\}$

$$\int_0^{+\infty} \Big|\big(\mathcal{F}_\kappa(f)(\cdot)m(t\,\cdot)\big)(x)\Big|^2 \frac{dt}{t} = |\mathcal{F}_\kappa(f)(x)|^2 \int_{\frac{r}{\|x\|}}^{\frac{\rho r}{\|x\|}} |m(tx)|^2 \frac{dt}{t} \leqslant C^2 \ln(\rho) |\mathcal{F}_\kappa(f)(x)|^2.$$

Then we have

$$\int_{\mathbb{R}^d} |g_\kappa^m(f)(x)|^2 d\mu_\kappa(x) \leqslant C^2 \ln(\rho) \int_{\mathbb{R}^d} |\mathcal{F}_\kappa(f)(x)|^2 d\mu_\kappa(x)$$

and the desired conclusion is deduced by Plancherel's theorem. □

## 4. Proof of the main result

This section is devoted to the proof of our main result. Let us start with the first point of Theorem 1.1.

### 4.1. The case where $\frac{d+2\gamma_\kappa}{d+2\gamma_\kappa-1} < p \leqslant +\infty$.

This case is mainly based on two intermediate results on the $L^p_\kappa$-boundedness of each $M_\kappa^{m_n^\kappa}$(with $n \geqslant 1$): a weak-type result when $p = 1$ and a strong-type result when $p = 2$. Let us begin with the case $p = 2$.

**Proposition 4.1.** *Let $f \in L^2_\kappa$ and $n \geqslant 1$. Then, we have the estimate*

$$\big\|M_\kappa^{m_n^\kappa} f\big\|_{\kappa,2} \leqslant \frac{C}{2^{\frac{n(d+2\gamma_\kappa-2)}{2}}} \|f\|_{\kappa,2},$$

*where $C = C(d,\kappa)$ is a constant independent of $f$ and $n$.*

*Proof.* Let $n \geqslant 1$. Since $m_n^\kappa$ is a Schwartz radial function, then $\mathcal{F}^\kappa(m_n^\kappa)$ is a Schwartz radial function and we claim that for almost $x \in \mathbb{R}^d$ (see [25, Theorem 6.2] and the first remark at the end of this proof)

$$\Big(f *_\kappa (\mathcal{F}^\kappa(m_n^\kappa))_r\Big)(x) \to \Big(c_\kappa \int_{\mathbb{R}^d} \mathcal{F}^\kappa(m_n^\kappa)(y) d\mu_\kappa(y)\Big) f(x)$$

as $r$ goes to 0. Since we have

$$c_\kappa \int_{\mathbb{R}^d} \mathcal{F}^\kappa(m_n^\kappa)(y) d\mu_\kappa(y) = m_n^\kappa(0) = 0,$$

we deduce that for almost $x \in \mathbb{R}^d$

$$\Big(f *_\kappa (\mathcal{F}^\kappa(m_n^\kappa))_r\Big)(x) \to 0$$

as $r$ goes to 0. Therefore, we can write for almost $x \in \mathbb{R}^d$

$$\begin{aligned}\Big(f *_\kappa (\mathcal{F}^\kappa(m_n^\kappa))_r\Big)^2(x) &= \int_0^r \frac{d}{dt}\Big(\Big(f *_\kappa (\mathcal{F}^\kappa(m_n^\kappa))_t\Big)^2(x)\Big) dt \\ &= 2\int_0^r \Big(f *_\kappa (\mathcal{F}^\kappa(m_n^\kappa))_t\Big)(x) \Big(f *_\kappa (\mathcal{F}^\kappa(\tilde{m}_n^\kappa))_t\Big)(x) \frac{dt}{t},\end{aligned}$$

where

$$\tilde{m}_n^\kappa(x) = \langle x, \nabla m_n^\kappa(x)\rangle.$$

Thus, for almost all $x \in \mathbb{R}^d$, we get by enlarging the domain of the integral

$$\Big|f *_\kappa (\mathcal{F}^\kappa(m_n^\kappa))_r(x)\Big|^2 \leqslant 2\int_0^{+\infty} \Big|f *_\kappa (\mathcal{F}^\kappa(m_n^\kappa))_t(x)\Big| \Big|f *_\kappa (\mathcal{F}^\kappa(\tilde{m}_n^\kappa))_t(x)\Big| \frac{dt}{t},$$

and this is equivalent to

$$\Big|\mathcal{F}^\kappa\big(\mathcal{F}_\kappa(f)(\cdot)m_n^\kappa(r\,\cdot)\big)(x)\Big|^2 \leqslant 2\int_0^{+\infty}\Big|\mathcal{F}^\kappa\big(\mathcal{F}_\kappa(f)(\cdot)m_n^\kappa(t\,\cdot)\big)(x)\Big|\,\Big|\mathcal{F}^\kappa\big(\mathcal{F}_\kappa(f)(\cdot)\tilde{m}_n^\kappa(t\,\cdot)\big)(x)\Big|\frac{dt}{t}.$$

We deduce by taking the supremum over every $r>0$ that

$$\big(M_\kappa^{m_n^\kappa}f(x)\big)^2 \leqslant 2\int_0^{+\infty}\Big|\mathcal{F}^\kappa\big(\mathcal{F}_\kappa(f)(\cdot)m_n^\kappa(t\,\cdot)\big)(x)\Big|\,\Big|\mathcal{F}^\kappa\big(\mathcal{F}_\kappa(f)(\cdot)\tilde{m}_n^\kappa(t\,\cdot)\big)(x)\Big|\frac{dt}{t}$$

for almost all $x\in\mathbb{R}^d$. By using the Cauchy-Schwarz inequality we obtain for almost all $x\in\mathbb{R}^d$

$$\big(M_\kappa^{m_n^\kappa}f(x)\big)^2 \leqslant 2\bigg(\int_0^{+\infty}\Big|\mathcal{F}^\kappa\big(\mathcal{F}_\kappa(f)(\cdot)m_n^\kappa(t\,\cdot)\big)(x)\Big|^2\frac{dt}{t}\bigg)^{\frac{1}{2}}\bigg(\int_0^{+\infty}\Big|\mathcal{F}^\kappa\big(\mathcal{F}_\kappa(f)(\cdot)\tilde{m}_n^\kappa(t\,\cdot)\big)(x)\Big|^2\frac{dt}{t}\bigg)^{\frac{1}{2}},$$

that is to say

$$\big(M_\kappa^{m_n^\kappa}f(x)\big)^2 \leqslant 2\,g_\kappa^{m_n^\kappa}(f)(x)g_\kappa^{\tilde{m}_n^\kappa}(f)(x).$$

By integrating over $\mathbb{R}^d$ and by using again the Cauchy-Schwarz inequality we are led to

$$\big\|M_\kappa^{m_n^\kappa}f\big\|_{\kappa,2}^2 \leqslant 2\big\|g_\kappa^{m_n^\kappa}(f)\big\|_{\kappa,2}\big\|g_\kappa^{\tilde{m}_n^\kappa}(f)\big\|_{\kappa,2}. \tag{4.1}$$

On one hand, we obviously notice that

$$\operatorname{supp}(m_n^\kappa)\subset\{x\in\mathbb{R}^d : 2^{n-1}\leqslant\|x\|\leqslant 2^{n+1}\},\ \operatorname{supp}(\tilde{m}_n^\kappa)\subset\{x\in\mathbb{R}^d : 2^{n-1}\leqslant\|x\|\leqslant 2^{n+1}\}. \tag{4.2}$$

On the other hand, we claim that

$$\|m_n^\kappa\|_\infty \leqslant \frac{C_1(d,\kappa)}{2^{\frac{n(d+2\gamma_\kappa-1)}{2}}},\quad \|\tilde{m}_n^\kappa\|_\infty \leqslant \frac{C_2(d,\kappa)}{2^{\frac{n(d+2\gamma_\kappa-3)}{2}}}, \tag{4.3}$$

where both $C_1(d,\kappa)$ and $C_2(d,\kappa)$ are independent of $n$. Indeed, thanks to the following well-known estimate for the usual Bessel function (see for instance [1, page 238])

$$\sup_{x\geqslant 0} x^{1/2}|J_\alpha(x)| < +\infty,\quad \forall\alpha>-\frac{1}{2},$$

and since

$$j_\alpha(z) = 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha},$$

we can write for every $x\in\mathbb{R}^d\setminus\{0\}$

$$|m_n^\kappa(x)| \leqslant \frac{C_1(d,\kappa)}{\|x\|^{\frac{d-1}{2}+\gamma_\kappa}},\quad |\tilde{m}_n^\kappa(x)| \leqslant \frac{C_2(d,\kappa)}{\|x\|^{\frac{d-3}{2}+\gamma_\kappa}},$$

where we have used for $\tilde{m}_n^\kappa$ the formula

$$\frac{d}{dt}j_\alpha(t) = -\frac{t}{2(\alpha+2)}j_{\alpha+1}(t),\quad \forall\alpha>-\frac{1}{2}.$$

Since both $m_n^\kappa$ and $\tilde{m}_n^\kappa$ are localized near $\|x\|\simeq 2^n$ in virtue of (4.2), we can assert that (4.3) is proved. Invoking (4.2) and (4.3), we are led thanks to Proposition 3.2

to

$$\begin{aligned}\left\|g_\kappa^{m_n^\kappa}(f)\right\|_{\kappa,2}\left\|g_\kappa^{\tilde{m}_n^\kappa}(f)\right\|_{\kappa,2} &\leqslant \left(\frac{C_1(d,\kappa)\sqrt{\ln 4}}{2^{\frac{n(d+2\gamma_\kappa-1)}{2}}}\|f\|_{\kappa,2}\right)\left(\frac{C_2(d,\kappa)\sqrt{\ln 4}}{2^{\frac{n(d+2\gamma_\kappa-3)}{2}}}\|f\|_{\kappa,2}\right)\\ &\leqslant \frac{C(d,\kappa)}{2^{n(d+2\gamma_\kappa-2)}}\|f\|_{\kappa,2}^2,\end{aligned}$$

where $C(d,\kappa)$ is a constant independent of $n$. If we use this estimate in (4.1) we get

$$\left\|M_\kappa^{m_n^\kappa}f\right\|_{\kappa,2}^2 \leqslant \frac{C(d,\kappa)}{2^{n(d+2\gamma_\kappa-2)}}\|f\|_{\kappa,2}^2,$$

which is exactly what we wanted to prove. □

*Remarks.*

- In [25, Theorem 6.2], Thangavelu and Xu have proved that, for suitable $\phi$ satisfying $c_\kappa \int_{\mathbb{R}^d}\phi(x)d\mu_\kappa(x) = 1$, we have $f *_\kappa \phi_r \to f$ almost everywhere as $r$ goes to 0. In fact, it is straightforward to adapt their proof (and the proof of Theorem 5.2) in order to remove the assumption $c_\kappa \int_{\mathbb{R}^d}\phi(x)d\mu_\kappa(x) = 1$ and to prove that for suitable $\phi$
$$f *_\kappa \phi_r \to \left(c_\kappa \int_{\mathbb{R}^d}\phi(x)d\mu_\kappa(x)\right)f$$
almost everywhere as $r$ goes to 0.
- We can easily use the idea of the previous proof in order to establish the following generalization of Proposition 4.1. Let $m \in L_\kappa^1$ be a radial function such that $\mathcal{F}_\kappa(m) \in L_\kappa^1$ is bounded. We denote by $M_\kappa^m$ the maximal operator associated with the multiplier $m$ and initially defined for $f \in \mathcal{S}(\mathbb{R}^d)$ by
$$M_\kappa^m f(x) = \sup_{r>0}\left|\mathcal{F}^\kappa\big(\mathcal{F}_\kappa(f)(\cdot)m(r\,\cdot)\big)(x)\right|, \quad x \in \mathbb{R}^d.$$
If $m$ satisfies
$$|m(x)| \leqslant \frac{1}{\|x\|^a}, \quad \left|\langle x, \nabla m(x)\rangle\right| \leqslant \frac{1}{\|x\|^b}, \quad x \in \mathbb{R}^d \setminus \{0\},$$
with $a+b>0$, then $M_\kappa^m$ maps $L_\kappa^2$ to itself. Indeed, it suffices to decompose the multiplier $m$ into the radial pieces $m_n := \varphi_n m$ (with $\varphi_n$ defined as in (3.5)) in order to prove that for every $n \geqslant 1$
$$\left\|M_\kappa^{m_n}f\right\|_{\kappa,2} \leqslant \frac{C}{2^{\frac{n(a+b)}{2}}}\|f\|_{\kappa,2};$$
then, since $M_\kappa^{m_0}$ maps $L_\kappa^2$ to itself, it remains to invoke the inequality $M_\kappa^m \leqslant \sum_{n=0}^{+\infty} M_\kappa^{m_n}$.

We now turn to the second proposition we need for the proof of the first point of Theorem 1.1. This is a weak-type result for each maximal multiplier operator $M_\kappa^{m_n^\kappa}$ (with $n \geqslant 1$).

**Proposition 4.2.** *Let $f \in L_\kappa^1$ and $n \geqslant 1$. Then, for every $\lambda > 0$, we have the estimate*

$$\mu_\kappa\Big(\big\{x \in \mathbb{R}^d : M_\kappa^{m_n^\kappa}f(x) > \lambda\big\}\Big) \leqslant C\,\frac{2^n}{\lambda}\|f\|_{\kappa,1},$$

*where $C = C(d,\kappa)$ is a constant independent of $f$, $n$ and $\lambda$.*

In order to prove this proposition, we need the following lemma. For a radial function $f$ on $\mathbb{R}^d$, recall the notation $f(\xi) = \overline{f}(\|\xi\|)$.

**Lemma 4.1.** *Let $n \geqslant 1$. Then, for every $x \in \mathbb{R}^d$, we have*
$$\mathcal{F}^\kappa(m_n^\kappa)(x) = d_\kappa \int_{S^{d-1}} V_\kappa\left(\overline{\mathcal{F}^\kappa(\varphi_n)}\left(\sqrt{\|x\|^2+1-2\langle x,\cdot\rangle}\right)\right)(y)h_\kappa^2(y)d\sigma(y).$$

*Proof.* Let $x \in \mathbb{R}^d$. We can write for every $n \geqslant 1$
$$\begin{aligned}\mathcal{F}^\kappa(m_n^\kappa)(x) &= c_\kappa \int_{\mathbb{R}^d} E_\kappa(ix,y)\varphi_n(y)\tilde{j}_\kappa(y)d\mu_\kappa(y)\\ &= c_\kappa \int_{\mathbb{R}^d} E_\kappa(ix,y)\varphi_n(y)j_{d/2+\gamma_\kappa-1}(\|y\|)d\mu_\kappa(y),\end{aligned}$$
where we have used (3.6) in the first step and the definition of $\tilde{j}_\kappa$ (see (3.3)) in the second step. Thanks to the equation (3.2) we have
$$\mathcal{F}^\kappa(m_n^\kappa)(x) = c_\kappa \int_{\mathbb{R}^d} E_\kappa(ix,y)\varphi_n(y)\left(d_\kappa \int_{S^{d-1}} E_\kappa(-iy,z)h_\kappa^2(z)d\sigma(z)\right)d\mu_\kappa(y).$$
Thanks to the first equality of (2.1) and the fact that $E_\kappa$ is symmetric in its argument, we have $E_\kappa(-iy,z) = E_\kappa(-iz,y)$. Then
$$\mathcal{F}^\kappa(m_n^\kappa)(x) = c_\kappa \int_{\mathbb{R}^d} E_\kappa(ix,y)\varphi_n(y)\left(d_\kappa \int_{S^{d-1}} E_\kappa(-iz,y)h_\kappa^2(z)d\sigma(z)\right)d\mu_\kappa(y),$$
from which we can deduce by the Fubini theorem that
$$\mathcal{F}^\kappa(m_n^\kappa)(x) = d_\kappa \int_{S^{d-1}} \left(c_\kappa \int_{\mathbb{R}^d} E_\kappa(ix,y)E_\kappa(-iz,y)\varphi_n(y)d\mu_\kappa(y)\right)h_\kappa^2(z)d\sigma(z).$$
For every $y \in \mathbb{R}^d$, we have $\varphi_n(-y) = \varphi_n(y)$, then a change of variables gives us
$$\begin{aligned}\mathcal{F}^\kappa(m_n^\kappa)(x) &= d_\kappa \int_{S^{d-1}} \left(c_\kappa \int_{\mathbb{R}^d} E_\kappa(ix,-y)E_\kappa(-iz,-y)\varphi_n(y)d\mu_\kappa(y)\right)h_\kappa^2(z)d\sigma(z)\\ &= d_\kappa \int_{S^{d-1}} \left(c_\kappa \int_{\mathbb{R}^d} E_\kappa(-ix,y)E_\kappa(iz,y)\varphi_n(y)d\mu_\kappa(y)\right)h_\kappa^2(z)d\sigma(z)\end{aligned}$$
where we have used the first equality of (2.1) in the second step. Since $\mathcal{F}^\kappa(\varphi_n)$ is a Schwartz function, we can write by using the equality (2.2)
$$\mathcal{F}^\kappa(m_n^\kappa)(x) = d_\kappa \int_{S^{d-1}} \tau_z^\kappa\left(\mathcal{F}^\kappa(\varphi_n)\right)(-x)h_\kappa^2(z)d\sigma(z).$$
Invoking (2.3) ($\mathcal{F}^\kappa(\varphi_n)$ is a radial function since $\varphi_n$ is radial), we are led to
$$\mathcal{F}^\kappa(m_n^\kappa)(x) = d_\kappa \int_{S^{d-1}} V_\kappa\left(\overline{\mathcal{F}^\kappa(\varphi_n)}\left(\sqrt{\|x\|^2+1-2\langle x,\cdot\rangle}\right)\right)(z)h_\kappa^2(z)d\sigma(z)$$
and the lemma is proved. □

Now, we can prove Proposition 4.2.

*Proof.* Let $n \geqslant 1$. In view of [25, Theorem 6.2] and since $\mathcal{F}^\kappa(m_n^\kappa)$ is a Schwartz radial function, we claim that it is enough to prove that for every $x \in \mathbb{R}^d$
$$\left|\mathcal{F}^\kappa(m_n^\kappa)(x)\right| \leqslant \frac{C(d,\kappa)2^n}{\left(1+\|x\|\right)^{d+2\gamma_\kappa+1}}. \tag{4.4}$$
Indeed, if (4.4) is proved, then we have
$$M_\kappa^{m_n^\kappa}f(x) = \sup_{r>0}\left|\left(f *_\kappa (\mathcal{F}^\kappa(m_n^\kappa))_r\right)(x)\right| \leqslant C(d,\kappa)\tilde{C}(d,\kappa)2^n\mathcal{M}_\kappa f(x),$$
and we can conclude by using the Hardy-Littlewood type theorem for the Dunkl maximal operator $\mathcal{M}_\kappa$ (see [5, Theorem 1.3] or [25, Theorem 6.1]). Therefore, we

are left with the task of establishing (4.4). Let $x \in \mathbb{R}^d$. Thanks to the previous lemma, we have

$$\mathcal{F}^\kappa(m_n^\kappa)(x) = d_\kappa \int_{S^{d-1}} V_\kappa\Big(\overline{\mathcal{F}^\kappa(\varphi_n)}\Big(\sqrt{\|x\|^2+1-2\langle x,\cdot\rangle}\Big)\Big)(y)h_\kappa^2(y)d\sigma(y).$$

By the Funk-Hecke type formula for reflection invariant weight functions (see [10, page 195]) we get

$$\mathcal{F}^\kappa(m_n^\kappa)(x) = C(d,\kappa)\int_{-1}^{1}\overline{\mathcal{F}^\kappa(\varphi_n)}\Big(\sqrt{\|x\|^2+1-2\|x\|t}\Big)\sqrt{1-t^2}^{\,d+2\gamma_\kappa-3}dt.$$

Since the definition of $\varphi_n$ allows us to write $\mathcal{F}^\kappa(\varphi_n) = \Psi_{2^{-n}}$ with $\Psi \in \mathcal{S}(\mathbb{R}^d)$, then we can write

$$\Big|\mathcal{F}^\kappa(m_n^\kappa)(x)\Big| \leqslant C(d,\kappa)\int_{-1}^{1}\frac{2^{n(d+2\gamma_\kappa)}}{\Big(1+2^n\sqrt{\|x\|^2+1-2\|x\|t}\Big)^{d+2\gamma_\kappa+2}}\sqrt{1-t^2}^{\,d+2\gamma_\kappa-3}dt.$$

Therefore

$$\Big|\mathcal{F}^\kappa(m_n^\kappa)(x)\Big| \leqslant C(d,\kappa)\sum_{l=-1}^{+\infty}\int_{I_l(x)}\frac{2^{n(d+2\gamma_\kappa)}}{\Big(1+2^n\sqrt{\|x\|^2+1-2\|x\|t}\Big)^{d+2\gamma_\kappa+2}}\sqrt{1-t^2}^{\,d+2\gamma_\kappa-3}dt,$$

where we have set

$$I_{-1}(x) = [-1,1]\cap\Big\{t\in\mathbb{R} : \sqrt{\|x\|^2+1-2\|x\|t} \leqslant 2^{-n}\Big\}$$
$$I_l(x) = [-1,1]\cap\Big\{t\in\mathbb{R} : 2^{l-n} < \sqrt{\|x\|^2+1-2\|x\|t} \leqslant 2^{l+1-n}\Big\}, \quad \forall l \geqslant 0.$$

We split the sum in order to write

$$\Big|\mathcal{F}^\kappa(m_n^\kappa)(x)\Big| \leqslant C(d,\kappa)(\Sigma_1+\Sigma_2),$$

with

$$\Sigma_1 = \sum_{l=-1}^{n}\int_{I_l(x)}\frac{2^{n(d+2\gamma_\kappa)}}{\Big(1+2^n\sqrt{\|x\|^2+1-2\|x\|t}\Big)^{d+2\gamma_\kappa+2}}\sqrt{1-t^2}^{\,d+2\gamma_\kappa-3}dt$$
$$\Sigma_2 = \sum_{l=n+1}^{+\infty}\int_{I_l(x)}\frac{2^{n(d+2\gamma_\kappa)}}{\Big(1+2^n\sqrt{\|x\|^2+1-2\|x\|t}\Big)^{d+2\gamma_\kappa+2}}\sqrt{1-t^2}^{\,d+2\gamma_\kappa-3}dt.$$

Then, in order to prove (4.4), it is enough to prove that

$$\Sigma_1 \leqslant \frac{C(d,\kappa)2^n}{\big(1+\|x\|\big)^{d+2\gamma_\kappa+1}}, \quad \Sigma_2 \leqslant \frac{C(d,\kappa)2^n}{\big(1+\|x\|\big)^{d+2\gamma_\kappa+1}}.$$

Let us begin with an observation which will be relevant for both $\Sigma_1$ and $\Sigma_2$. For $t \in I_l(x)$, $l \geqslant -1$, we have

$$\|x\| \leqslant 2^{l+1-n}+1. \tag{4.5}$$

Indeed, for $x \in \mathbb{R}^d$ and $t \in I_l(x)$ fixed, we can write

$$\|x\|-t \leqslant \sqrt{(\|x\|-t)^2} \leqslant \sqrt{(\|x\|-t)^2+(1-t^2)} = \sqrt{\|x\|^2+1-2\|x\|t},$$

that is to say

$$\|x\| \leqslant \sqrt{\|x\|^2 + 1 - 2\|x\|t} + t.$$

Since $t \in I_l(x)$, then $t \leqslant 1$ and (4.5) is proved.

We now turn to the desired estimate for $\Sigma_1$. Thanks to (4.5) together with the following trivial observation

$$-1 \leqslant l \leqslant n \Longrightarrow 2^{l+1-n} + 1 \leqslant 3,$$

we get

$$\Sigma_1 \leqslant 2^{n(d+2\gamma_\kappa)} \chi_{B_3(0)}(x) \sum_{l=-1}^{n} \int_{I_l(x)} \frac{\sqrt{1-t^2}^{d+2\gamma_\kappa-3}}{\left(1 + 2^n \sqrt{\|x\|^2 + 1 - 2\|x\|t}\right)^{d+2\gamma_\kappa+2}} dt,$$

where $\chi_X$ is the characteristic function of the set $X$ and $B_R(\xi)$ is the Euclidean ball of radius $R$ centered at $\xi$. For every $t \in I_l(x)$, $0 \leqslant l \leqslant n$, we have

$$\frac{1}{\left(1 + 2^n \sqrt{\|x\|^2 + 1 - 2\|x\|t}\right)^{d+2\gamma_\kappa+2}} \leqslant \frac{1}{2^{l(d+2\gamma_\kappa+2)}},$$

and the above inequality is obviously true for $l = -1$. Thus

$$\Sigma_1 \leqslant 2^{n(d+2\gamma_\kappa)} \chi_{B_3(0)}(x) \sum_{l=-1}^{n} \left( \frac{1}{2^{l(d+2\gamma_\kappa+2)}} \int_{I_l(x)} \sqrt{1-t^2}^{d+2\gamma_\kappa-3} dt \right).$$

Now we claim that, for every $-1 \leqslant l \leqslant n$,

$$\int_{I_l(x)} \sqrt{1-t^2}^{d+2\gamma_\kappa-3} dt \leqslant 2^{(l+1-n)(d+2\gamma_\kappa-1)+1}.$$

Indeed, we have

$$\begin{aligned}
\int_{I_l(x)} \sqrt{1-t^2}^{d+2\gamma_\kappa-3} dt &\leqslant \int_{[-1,1]\cap\left\{t\in\mathbb{R}:\sqrt{1-t^2}\leqslant 2^{l+1-n}\right\}} \sqrt{1-t^2}^{d+2\gamma_\kappa-3} dt \\
&\leqslant 2^{(l+1-n)(d+2\gamma_\kappa-3)} \int_{[-1,1]\cap\left\{t\in\mathbb{R}:\sqrt{1-t^2}\leqslant 2^{l+1-n}\right\}} dt \\
&\leqslant 2^{(l+1-n)(d+2\gamma_\kappa-3)} \int_{[-1,1]\cap\left\{t\in\mathbb{R}:\sqrt{1-|t|}\leqslant 2^{l+1-n}\right\}} dt,
\end{aligned}$$

and the following obvious observation

$$t \in [-1,1] \cap \left\{t \in \mathbb{R} : \sqrt{1-|t|} \leqslant 2^{l+1-n}\right\} \Longrightarrow 1 - 2^{2(l+1-n)} \leqslant |t| \leqslant 1$$

then implies that

$$\begin{aligned}
\int_{I_l(x)} \sqrt{1-t^2}^{d+2\gamma_\kappa-3} dt &\leqslant 2^{(l+1-n)(d+2\gamma_\kappa-3)+1} \int_{1-2^{2(l+1-n)}}^{1} dt \\
&= 2^{(l+1-n)(d+2\gamma_\kappa-1)+1}.
\end{aligned}$$

Therefore

$$\Sigma_1 \leqslant 2 \sum_{l=-1}^{n} \frac{2^{n(d+2\gamma_\kappa)} 2^{(l+1-n)(d+2\gamma_\kappa-1)}}{2^{l(d+2\gamma_\kappa+2)}} \chi_{B_3(0)}(x).$$

Since

$$\begin{aligned}
\sum_{l=-1}^{n} \frac{2^{n(d+2\gamma_\kappa)} 2^{(l+1-n)(d+2\gamma_\kappa-1)}}{2^{l(d+2\gamma_\kappa+2)}} &= 2^n 2^{d+2\gamma_\kappa-1} \sum_{l=-1}^{n} \frac{1}{2^{3l}} \\
&\leqslant C(d,\kappa) 2^n,
\end{aligned}$$

we can deduce that

$$\Sigma_1 \leqslant C(d,\kappa)2^n \chi_{B_3(0)}(x) \leqslant \frac{C(d,\kappa)2^n}{\big(1+\|x\|\big)^{d+2\gamma_\kappa+1}},$$

and it remains to prove the same estimate for $\Sigma_2$.

We can write thanks to (4.5) and the definition of $I_l(x)$

$$\begin{aligned}\Sigma_2 &= \sum_{l=n+1}^{+\infty}\int_{I_l(x)} \frac{2^{n(d+2\gamma_\kappa)}}{\Big(1+2^n\sqrt{\|x\|^2+1-2\|x\|t}\Big)^{d+2\gamma_\kappa+2}}\sqrt{1-t^2}^{\,d+2\gamma_\kappa-3}dt\\ &\leqslant 2^{n(d+2\gamma_\kappa)}\sum_{l=n+1}^{+\infty}\frac{\chi_{B_{2^{l+1-n}+1}(0)}(x)}{2^{l(d+2\gamma_\kappa+2)}}\int_{I_l(x)}\sqrt{1-t^2}^{\,d+2\gamma_\kappa-3}dt.\end{aligned}$$

Since for every $l \geqslant n+1$ we have the inclusion $B_{2^{l+1-n}+1}(0) \subset B_{2^{l+2-n}}(0)$ and since we can write

$$\int_{I_l(x)}\sqrt{1-t^2}^{\,d+2\gamma_\kappa-3}dt \leqslant \int_{-1}^{1}\sqrt{1-t^2}^{\,d+2\gamma_\kappa-3}dt \leqslant C(d,\kappa),$$

then

$$\begin{aligned}\Sigma_2 &\leqslant C(d,\kappa)2^{n(d+2\gamma_\kappa)}\sum_{l=n+1}^{+\infty}\frac{\chi_{B_{2^{l+2-n}}(0)}(x)}{2^{l(d+2\gamma_\kappa+2)}}\\ &\leqslant \frac{C(d,\kappa)2^{n(d+2\gamma_\kappa)}}{\big(1+\|x\|\big)^{d+2\gamma_\kappa+1}}\sum_{l=n+1}^{+\infty}\frac{\big(1+2^{l+2-n}\big)^{d+2\gamma_\kappa+1}}{2^{l(d+2\gamma_\kappa+2)}}\\ &\leqslant \frac{C(d,\kappa)2^n}{\big(1+\|x\|\big)^{d+2\gamma_\kappa+1}}\sum_{l=n+1}^{+\infty}\frac{2^{(l-n)\big((d+2\gamma_\kappa+1)-(d+2\gamma_\kappa+2)\big)}}{2^{n\big((d+2\gamma_\kappa+2)+1-(d+2\gamma_\kappa)\big)}}\\ &\leqslant \frac{C(d,\kappa)2^n}{\big(1+\|x\|\big)^{d+2\gamma_\kappa+1}}\sum_{l=n+1}^{+\infty}\frac{2^{-(l-n)}}{2^{3n}}\\ &\leqslant \frac{C(d,\kappa)2^n}{\big(1+\|x\|\big)^{d+2\gamma_\kappa+1}}\end{aligned}$$

and the proposition is completely proved.

□

With the results of Proposition 4.1 and Proposition 4.2 in mind, the proof of the first part of Theorem 1.1 is nearly obvious.

*Proof of the first part of Theorem 1.1.* By the Marcinkiewicz interpolation theorem (see for instance [23]), we can deduce from Proposition 4.1 and Proposition 4.2 that for every $n \geqslant 1$ and every $f \in L^p_\kappa$, $1 < p \leqslant 2$,

$$\big\|M_\kappa^{m_n^\kappa}f\big\|_{\kappa,p} \leqslant C(d,\kappa,p)\big(2^n\big)^{(\frac{2}{p}-1)}\big(2^{-(\frac{d+2\gamma_\kappa-2}{2})n}\big)^{(2-\frac{2}{p})}\|f\|_{\kappa,p},$$

that is to say

$$\big\|M_\kappa^{m_n^\kappa}f\big\|_{\kappa,p} \leqslant C(d,\kappa,p)2^{(\frac{d+2\gamma_\kappa}{p}-(d+2\gamma_\kappa-1))n}\|f\|_{\kappa,p}.$$

For $p > (d+2\gamma_\kappa)/(d+2\gamma_\kappa-1)$, the series

$$\sum_{n\geqslant 1} 2^{(\frac{d+2\gamma_\kappa}{p}-(d+2\gamma_\kappa-1))n}$$

converges. Moreover, we have thanks to Proposition 3.1 that $M_\kappa^{m_0^\kappa}$ maps $L^p_\kappa$ to itself for every $1 < p < +\infty$. Therefore, in view of (3.7), we obtain that $M_\kappa$ is bounded on $L^p_\kappa$ for every $(d + 2\gamma_\kappa)/(d + 2\gamma_\kappa - 1) < p \leqslant 2$. For $p > 2$, we of course proceed by interpolation between the $L^2_\kappa$ case and the $L^\infty$ case (which is a consequence of the fact that the Dunkl spherical mean operator is a contraction on $L^p_\kappa$, $1 \leqslant p \leqslant +\infty$, see [19] and [25]). □

4.2. **The case where** $p \leqslant \frac{d+2\gamma_\kappa}{d+2\gamma_\kappa-1}$**.** We now prove the second point of Theorem 1.1.

In view of this, let $f$ be the radial and smooth function with compact support given on $\mathbb{R}^d$ by

$$f(x) = \begin{cases} \mathrm{e}^{\frac{-1}{1-\|x\|^2}} & \text{if } \|x\| < 1 \\ 0 & \text{if } \|x\| \geqslant 1. \end{cases}$$

Of course, $f$ belongs to $L^p_\kappa$. However, $M_\kappa f \notin L^p_\kappa$. Indeed, we can write for every $x \in \mathbb{R}^d$

$$M_\kappa f(x) \geqslant d_\kappa \left| \int_{S^{d-1}} \tau_x^\kappa f(-\|x\|y) h_\kappa^2(y) d\sigma(y) \right|,$$

and since $f$ is a Schwartz radial function, we can use (2.3) to get

$$M_\kappa f(x) \geqslant d_\kappa \left| \int_{S^{d-1}} V_\kappa \Big( \overline{f} \Big( \sqrt{2\|x\|^2 - 2\|x\|\langle \cdot, y\rangle} \Big) \Big)(x) h_\kappa^2(y) d\sigma(y) \right|.$$

By the Funk-Hecke type formula for reflection invariant weight functions (see [10, page 195]) we have

$$\begin{aligned} \int_{S^{d-1}} V_\kappa \Big( \overline{f} \Big( \sqrt{2\|x\|^2 - 2\|x\|\langle x, y\rangle} \Big) \Big) h_\kappa^2(y) d\sigma(y) \\ = C(d,\kappa) \int_{-1}^{1} \overline{f} \Big( \sqrt{2\|x\|^2 - 2\|x\|^2 t} \Big) \sqrt{1-t^2}^{\,d+2\gamma_\kappa - 3} dt, \end{aligned}$$

and thus

$$M_\kappa f(x) \geqslant C(d,\kappa) \int_{-1}^{1} \overline{f} \Big( \sqrt{2\|x\|^2(1-t)} \Big) \sqrt{1-t^2}^{\,d+2\gamma_\kappa - 3} dt.$$

For $\|x\| \geqslant N$ with $N$ large enough, we can write

$$\begin{aligned} M_\kappa f(x) &\geqslant C(d,\kappa) \int_{1-\frac{1}{8\|x\|^2}}^{1} \overline{f} \Big( \sqrt{2\|x\|^2(1-t)} \Big) \sqrt{1-t^2}^{\,d+2\gamma_\kappa - 3} dt \\ &\geqslant C(d,\kappa) \int_{1-\frac{1}{8\|x\|^2}}^{1} \overline{f} \Big( \frac{1}{2} \Big) \sqrt{1-t^2}^{\,d+2\gamma_\kappa - 3} dt \\ &\geqslant C(d,\kappa) \mathrm{e}^{-\frac{4}{3}} \int_{1-\frac{1}{8\|x\|^2}}^{1} \sqrt{1-t^2}^{\,d+2\gamma_\kappa - 3} dt \\ &\geqslant C(d,\kappa) \int_{1-\frac{1}{8\|x\|^2}}^{1} t \sqrt{1-t^2}^{\,d+2\gamma_\kappa - 3} dt. \end{aligned}$$

Since

$$\begin{aligned}\int_{1-\frac{1}{8\|x\|^2}}^{1} t\sqrt{1-t^2}^{d+2\gamma_\kappa-3}dt &= \frac{1}{d+2\gamma_\kappa-1}\Big(1-\Big(1-\frac{1}{8\|x\|^2}\Big)^2\Big)^{\frac{d+2\gamma_\kappa-1}{2}}\\ &= C(d,\kappa)\Big(2-\frac{1}{8\|x\|^2}\Big)^{\frac{d+2\gamma_\kappa-1}{2}}\frac{1}{\big(2\sqrt{2}\|x\|\big)^{d+2\gamma_\kappa-1}},\end{aligned}$$

we conclude that for $\|x\|\geqslant N$

$$M_\kappa f(x)\geqslant\frac{C(d,\kappa)}{\|x\|^{d+2\gamma_\kappa-1}}.$$

Since $p\leqslant\frac{d+2\gamma_\kappa}{d+2\gamma_\kappa-1}$, we have

$$\begin{aligned}\int_{\{x\in\mathbb{R}^d:\|x\|\geqslant N\}}\frac{1}{\|x\|^{p(d+2\gamma_\kappa-1)}}d\mu_\kappa(x) &= C(d,\kappa)\int_N^{+\infty}\frac{1}{r^{p(d+2\gamma_\kappa-1)}}r^{d+2\gamma_\kappa-1}dr\\ &= +\infty,\end{aligned}$$

and then $M_\kappa f\notin L^p_\kappa$.

## References


[1] George E. Andrews, Richard Askey, and Ranjan Roy. *Special functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1999.

[2] Salem Ben Saïd and Bent Ørsted. The wave equation for Dunkl operators. *Indag. Math. (N.S.)*, 16(3-4):351–391, 2005.

[3] J. Bourgain. On high-dimensional maximal functions associated to convex bodies. *Amer. J. Math.*, 108(6):1467–1476, 1986.

[4] Marcel de Jeu. The Dunkl transform. *Invent. Math.*, 113(1):147–162, 1993.

[5] Luc Deleaval. Two results on the Dunkl maximal operator. *Studia Math.*, 203(1):47–68, 2011.

[6] Charles F. Dunkl. Differential-difference operators associated to reflection groups. *Trans. Amer. Math. Soc.*, 311(1):167–183, 1989.

[7] Charles F. Dunkl. Hankel transforms associated to finite reflection groups. In *Hypergeometric functions on domains of positivity, Jack polynomials, and applications (Tampa, FL, 1991)*, volume 138 of *Contemp. Math.*, pages 123–138. Amer. Math. Soc., Providence, RI, 1992.

[8] Charles F. Dunkl. Intertwining operators associated to the group $S_3$. *Trans. Amer. Math. Soc.*, 347(9):3347–3374, 1995.

[9] Charles F. Dunkl. An intertwining operator for the group $B_2$. *Glasg. Math. J.*, 49(2):291–319, 2007.

[10] Charles F. Dunkl and Yuan Xu. *Orthogonal polynomials of several variables*, volume 81 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2001.

[11] Loukas Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2008.

[12] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[13] M. Maslouhi and E. H. Youssfi. The Dunkl intertwining operator. *J. Funct. Anal.*, 256(8):2697–2709, 2009.

[14] H. Mejjaoli and K. Trimèche. On a mean value property associated with the Dunkl Laplacian operator and applications. *Integral Transform. Spec. Funct.*, 12(3):279–302, 2001.

[15] E. M. Opdam. Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group. *Compositio Math.*, 85(3):333–373, 1993.

[16] Margit Rösler. Bessel-type signed hypergroups on **R**. In *Probability measures on groups and related structures, XI (Oberwolfach, 1994)*, pages 292–304. World Sci. Publ., River Edge, NJ, 1995.

[17] Margit Rösler. Positivity of Dunkl's intertwining operator. *Duke Math. J.*, 98(3):445–463, 1999.

[18] Margit Rösler. Dunkl operators: theory and applications. In *Orthogonal polynomials and special functions (Leuven, 2002)*, volume 1817 of *Lecture Notes in Math.*, pages 93–135. Springer, Berlin, 2003.

[19] Margit Rösler. A positive radial product formula for the Dunkl kernel. *Trans. Amer. Math. Soc.*, 355(6):2413–2438 (electronic), 2003.

[20] José L. Rubio de Francia. Maximal functions and Fourier transforms. *Duke Math. J.*, 53(2):395–404, 1986.

[21] E. M. Stein. The development of square functions in the work of A. Zygmund. *Bull. Amer. Math. Soc. (N.S.)*, 7(2):359–376, 1982.

[22] E. M. Stein and J.-O. Strömberg. Behavior of maximal functions in $\mathbf{R}^n$ for large $n$. *Ark. Mat.*, 21(2):259–269, 1983.

[23] Elias M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[24] Elias M. Stein. Maximal functions. I. Spherical means. *Proc. Nat. Acad. Sci. U.S.A.*, 73(7):2174–2175, 1976.

[25] Sundaram Thangavelu and Yuan Xu. Convolution operator and maximal function for the Dunkl transform. *J. Anal. Math.*, 97:25–55, 2005.

[26] Yuan Xu. Orthogonal polynomials for a family of product weight functions on the spheres. *Canad. J. Math.*, 49(1):175–192, 1997.

Laboratoire d'Analyse et de Mathématiques appliquées, Université Paris-Est Marne la Vallée, 5 boulevard Descartes, Champs sur Marne, 77454 Marne la Vallée, Cédex 2, France

*E-mail address*: luc.deleaval@u-pem.fr