DUNKL KERNEL AND DUNKL TRANSLATION FOR A POSITIVE SUBSYSTEM OF ORTHOGONAL ROOTS

LUC DELEAVAL

Abstract. In this article, we give an explicit formula for the Dunkl kernel associated with a positive subsystem of orthogonal roots. We then prove a product formula for this kernel which allows us to show that the Dunkl translation is a bounded operator in this setting. We finally study the Dunkl maximal operator for such subsystems.

1. Introduction

Dunkl operators provide an essential tool to extend Euclidean Fourier analysis and analysis on Riemannian symmetric spaces of Euclidean type. Since their invention in 1989, these operators have largely contributed, in the setting of root systems and associated reflection groups, to the development of harmonic analysis and to the theory of multivariable hypergeometric functions. In particular, an integral transform which generalizes the Euclidean Fourier transform and which is called the Dunkl transform plays an important role in this development. One of the most challenging problem in the Dunkl analysis is to find explicit formulae for the kernel of this transform, the well-known Dunkl kernel. Some significant results exist (such as asymptotic behavior for instance, see [23]) but an explicit formula is still lacking except in the cases where the root system is of A_2-type ([12]), of B_2-type ([13]) and where the reflection group is $\mathbb{Z}_2^d$ ([11, 26]). Another key tool associated with Dunkl operators is the generalized translation or Dunkl translation. This operator which plays the role of $f \mapsto f(\cdot + x)$ in the Dunkl setting is not positive in general and its boundedness is still a challenging problem. It is the main reason that we cannot apply the tools of real analysis (covering lemma, Calderón-Zygmund decomposition...) in order to study the Dunkl maximal operator for instance. In fact, the Dunkl translation is rather well understood only in two cases: when it acts on radial functions and when the reflection group is $\mathbb{Z}_2^d$.

In this article, we prove in particular that the Dunkl translation is bounded in the case of a positive subsystem of orthogonal roots. For that purpose, we establish a product formula for the Dunkl kernel for which we first give an explicit formula thanks to a recent work of Maslouhi and Youssfi (see [16] and also [17]). These results generalize the $\mathbb{Z}_2^d$ case. The paper is organized as follows. In the next section we collect some definitions and results related to Dunkl operators which will be relevant for the sequel. In section 3 we give an explicit formula for the Dunkl kernel in the case of a positive subsytem of orthogonal roots and we prove a product formula for it. The interesting case of a root system of A_1-type is also studied in order to show an equality between special functions. Section 4 is devoted to the

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2000 *Mathematics Subject Classification.* 33C52, 42B10, 42B25.

Key words and phrases. Dunkl operators, Dunkl kernel, Special function, Maximal operator.

Dunkl translation. Due to our product formula for the Dunkl kernel, we will prove that this translation is a bounded operator for such subsystems and a study of the Dunkl maximal operator in this setting is given in the last section.

2. Preliminaries

This section is devoted to the preliminaries and background. We only focus on the aspects of the Dunkl theory we need. For a large survey about this theory, the reader may especially consult [14, 21] and the references therein.

Let $W \subset \mathcal{O}(\mathbb{R}^d)$ be a finite reflection group associated with a reduced root system $\mathcal{R}$ (not necessarily crystallographic) and let $\kappa : \mathcal{R} \to \mathbb{C}$ be a multiplicity function, that is a W-invariant function. We assume in this article that κ takes value in $[0, +\infty[$.

The (rational) Dunkl operators $T_\xi^{\mathcal{R}}$ on $\mathbb{R}^d$, which were introduced in [9], are the following κ-deformations of directional derivatives ∂_ξ by reflections

$$T_\xi^{\mathcal{R}} f(x) = \partial_\xi f(x) + \sum_{\alpha \in \mathcal{R}_+} \kappa(\alpha) \frac{f(x) - f(\sigma_\alpha(x))}{\langle x, \alpha \rangle} \langle \xi, \alpha \rangle, \quad x \in \mathbb{R}^d,$$

where σ_α denotes the reflection with respect to the hyperplane orthogonal to α and $\mathcal{R}_+$ denotes a positive subsystem of $\mathcal{R}$. The definition is of course independent of the choice of a positive subsystem since κ is W-invariant. The most important property of these operators is their commutativity, that is to say $T_\xi^{\mathcal{R}} T_{\xi'}^{\mathcal{R}} = T_{\xi'}^{\mathcal{R}} T_\xi^{\mathcal{R}}$ ([9]). Therefore, we are naturally led to consider the eigenfunction problem

$$T_\xi^{\mathcal{R}} f = \langle y, \xi \rangle f \quad \forall \xi \in \mathbb{R}^d \tag{2.1}$$

with $y \in \mathbb{C}^d$ a fixed parameter. This problem has been completely solved by Opdam ([18]).

Theorem 2.1. *Let $y \in \mathbb{C}^d$. There exists a unique $f = E_\kappa^W(\cdot, y)$ solution of*

$$T_\xi^{\mathcal{R}} f = \langle y, \xi \rangle f \quad \forall \xi \in \mathbb{R}^d$$

which is real-analytic on $\mathbb{R}^d$ and satisfies $f(0) = 1$. Moreover the Dunkl kernel E_κ^W extends to a holomorphic function on $\mathbb{C}^d \times \mathbb{C}^d$.

In fact, the existence of a solution has been already proved by Dunkl ([10]). Indeed, he noticed the existence of an intertwining operator V_κ^W which satisfies

$$V_\kappa^W(\mathcal{P}_n^d) = \mathcal{P}_n^d, \qquad V_\kappa^W{}_{|_{\mathcal{P}_0^d}} = \mathrm{Id}_{|_{\mathcal{P}_0^d}}, \qquad T_\xi^{\mathcal{R}} V_\kappa^W = V_\kappa^W \partial_\xi \quad \forall \xi \in \mathbb{R}^d,$$

where $\mathcal{P}_n$ denotes the space of homogeneous polynomials of degree n in d variables. Since the exponential function is solution of (2.1) when $\kappa = 0$ (that is to say when $T_\xi^{\mathcal{R}} = \partial_\xi$), he naturally set $E_\kappa(\cdot, y) = V_\kappa^W(\mathrm{e}^{\langle \cdot, y \rangle})$. Unfortunately, the Dunkl kernel is explicitly known only in some special cases; when the root system is of A_2-type ([12]), of B_2-type ([13]) and when the reflection group is $\mathbb{Z}_2^d$ ([11, 26]). Nevertheless we know that this kernel has many properties in common with the classical exponential to which it reduces when $\kappa = 0$. For significant results on this kernel and the intertwining operator, the reader may especially consult [6, 10, 18, 19, 20, 22]. An object closely related with this kernel is the generalized Bessel function (or Dunkl-type Bessel function) defined for $x \in \mathbb{C}^d$ and $y \in \mathbb{C}^d$ by

$$J_\kappa^W(x, y) = \frac{1}{\# W} \sum_{w \in W} E_\kappa^W(w(x), y)$$

where $\#W$ denotes the order of W (for further details about J_κ^W, see subsection 3.2). The Dunkl kernel is of particular interest as it gives rise to an integral transform which is taken with respect to a weighted Lebesgue measure invariant under the action of W and which generalizes the Euclidean Fourier transform.

More precisely, let us introduce the measure $\mathrm{d}\mu_\kappa^W(x) = h_\kappa^2(x)\,\mathrm{d}x$ where the weight given by

$$h_\kappa^2(x) = \prod_{\alpha\in\mathcal{R}_+} |\langle x,\alpha\rangle|^{2\kappa(\alpha)}$$

is homogeneous of degree $2\gamma_\kappa$ with

$$\gamma_\kappa = \sum_{\alpha\in\mathcal{R}_+} \kappa(\alpha),$$

and let us denote by $L^p(\mu_\kappa^W)$ the space $L^p(\mathbb{R}^d;\mu_\kappa^W)$ (for $1 \leqslant p \leqslant +\infty$) and use the shorter notation $\|\cdot\|_{W,\kappa,p}$ instead of $\|\cdot\|_{L^p(\mu_\kappa^W)}$.
Then for every $f \in L^1(\mu_\kappa^W)$, the Dunkl transform of f denoted by $\mathcal{F}_\kappa^W(f)$ is defined by

$$\mathcal{F}_\kappa^W(f)(x) = c_\kappa^W \int_{\mathbb{R}^d} E_\kappa^W(-ix,y)f(y)\,\mathrm{d}\mu_\kappa^W(y), \quad x\in\mathbb{R}^d,$$

where c_κ^W is the Mehta-type constant

$$c_\kappa^W = \left(\int_{\mathbb{R}^d} \mathrm{e}^{-\frac{\|x\|^2}{2}}\,\mathrm{d}\mu_\kappa^W(x)\right)^{-1}.$$

Let us point out that the Dunkl transform coincides with the Euclidean Fourier transform when $\kappa = 0$. The two main properties of the Dunkl transform are given in the following theorem ([6, 11]).

Theorem 2.2.

(1) ***Inversion formula.*** *Let $f \in L^1(\mu_\kappa^W)$. If $\mathcal{F}_\kappa^W(f)$ is in $L^1(\mu_\kappa^W)$, then we have the following inversion formula*

$$f(x) = c_\kappa^W \int_{\mathbb{R}^d} E_\kappa^W(ix,y)\mathcal{F}_\kappa^W(f)(y)\,\mathrm{d}\mu_\kappa^W(y).$$

(2) ***Plancherel theorem.*** *The Dunkl transform has a unique extension to an isometric isomorphism of $L^2(\mu_\kappa^W)$.*

The Dunkl transform shares many other properties with the Fourier transform. Therefore, it is natural to associate a generalized translation operator with this transform.

There are many ways to define the Dunkl translation but we use the definition which most underlines the analogy with the Fourier transform. It is the definition given in [25] with a different convention. Let $x \in \mathbb{R}^d$. The Dunkl translation $f \mapsto \tau_x^{W,\kappa} f$ is defined on $L^2(\mu_\kappa^W)$ by the equation

$$\mathcal{F}_\kappa^W(\tau_x^{W,\kappa} f)(y) = E_\kappa^W(ix,y)\mathcal{F}_\kappa^W(f)(y), \quad y\in\mathbb{R}^d.$$

In Fourier analysis, the translation operator $f \mapsto f(\cdot + x)$ (to which the Dunkl translation reduces when $\kappa = 0$) is positive and L^p-bounded. In the Dunkl setting, $\tau_x^{W,\kappa}$ is not a positive operator ([19, 25]) and the $L^p(\mu_\kappa^W)$-boundedness is still a challenging problem, apart from the trivial case where $p = 2$ (thanks to the Plancherel theorem and the fact that $|E_\kappa^W(ix,y)| \leqslant 1$). The most general result we have is given in the following theorem ([25]), where we denote by $L^p_{\mathrm{rad}}(\mu_\kappa^W)$ the space of radial functions of $L^p(\mu_\kappa^W)$.

Theorem 2.3.

(1) *For every p satisfying $1 \leqslant p \leqslant 2$ and for every $x \in \mathbb{R}^d$, the Dunkl translation $\tau_x^{W,\kappa} : L^p_{\mathrm{rad}}(\mu_\kappa^W) \to L^p(\mu_\kappa^W)$ is a bounded operator.*

(2) *Let $f \in L^1(\mu_\kappa^W)$ be a bounded, radial and positive function. Then, for every $x \in \mathbb{R}^d$ we have $\tau_x^{W,\kappa} f \geqslant 0$.*

At the moment, an explicit formula for $\tau_x^{W,\kappa}$ is only known in two cases: when f is radial and when the reflection group is $\mathbb{Z}_2$ (from which also follows a formula in the $\mathbb{Z}_2^d$ case), both due to Rösler (see respectively [22] and [19]).
Let us finish this section by recalling some facts concerning the one-dimensional case. The reflection group is $W = \{\mathrm{Id}, \sigma\} \simeq \mathbb{Z}_2$ with $\sigma(x) = -x$ and the corresponding Dunkl kernel is given for every $x \in \mathbb{R}$ and for every $y \in \mathbb{R}$ by

(2.2)
$$E_\kappa^{\mathbb{Z}_2}(ix, y) = \frac{\Gamma\big(\kappa + \frac{1}{2}\big)}{\Gamma\big(\kappa\big)\Gamma\big(\frac{1}{2}\big)} \int_{-1}^{1} \mathrm{e}^{itxy}(1+t)(1-t^2)^{\kappa-1}\,\mathrm{d}t = j_{\kappa-\frac{1}{2}}(xy) + \frac{ixy}{2\kappa+1} j_{\kappa+\frac{1}{2}}(xy),$$

where κ is the only value taken by the multiplicity function and where j_κ is the normalized Bessel function of order κ, that is to say

$$j_\kappa(z) = \Gamma(\kappa+1) \sum_{n=0}^{+\infty} \frac{(-1)^n z^{2n}}{n! 2^{2n} \Gamma(n+\kappa+1)}.$$

A product formula for this kernel has been established by Rösler (see [19]). Before stating this formula, we have to introduce some notations.

Notations.

(1) For $x, y, z \in \mathbb{R}$, we set

$$\rho_{x,y,z} = \begin{cases} \frac{1}{2xy}(x^2 + y^2 - z^2) & \text{if } x, y \neq 0 \\ 0 & \text{if } x = 0 \text{ or } y = 0, \end{cases}$$

and we define

$$\varrho(x,y,z) = \frac{1}{2}(1 - \rho_{x,y,z} + \rho_{z,x,y} + \rho_{z,y,x}).$$

(2) For $x, y, z > 0$, we put

$$K_\kappa(x,y,z) = 2^{2\kappa-2} \frac{\Gamma\big(\kappa+\frac{1}{2}\big)}{\Gamma\big(\kappa\big)\Gamma\big(\frac{1}{2}\big)} \frac{\Delta(x,y,z)^{2\kappa-2}}{(xyz)^{2\kappa-1}} \chi_{[|x-y|, x+y]}(z),$$

where $\Delta(x,y,z)$ is the area of the triangle with sides x, y, z, that is to say

$$\Delta(x,y,z) = \frac{1}{4}\sqrt{(x+y+z)(x+y-z)(x-y+z)(y+z-x)}.$$

(3) For $x, y, z \in \mathbb{R} \setminus \{0\}$ we finally set

$$\mathcal{K}_\kappa(x,y,z) = K_\kappa\big(|x|, |y|, |z|\big)\varrho(x,y,z).$$

Remark. Let us notice the symmetries

$$\mathcal{K}_\kappa(x,y,z) = \mathcal{K}_\kappa(y,x,z) = \mathcal{K}_\kappa(-x,z,y) = \mathcal{K}_\kappa(-z,y,-x).$$

With these notations in mind we can now give the product formula for the one-dimensional Dunkl kernel.

Theorem 2.4. *Let $x, y \in \mathbb{R}$.*

(1) *For every $z \in \mathbb{R}$ we have*

$$E_\kappa^{\mathbb{Z}_2}(ix,z)E_\kappa^{\mathbb{Z}_2}(iy,z) = \int_{\mathbb{R}} E_\kappa^{\mathbb{Z}_2}(iz,z')\,\mathrm{d}\nu_{x,y}^{\mathbb{Z}_2,\kappa}(z'),$$

where the measure $\nu_{x,y}^{\mathbb{Z}_2,\kappa}$ is given by

$$\mathrm{d}\nu_{x,y}^{\mathbb{Z}_2,\kappa}(z') = \begin{cases} \mathcal{K}_\kappa(x,y,z')\,\mathrm{d}\mu_\kappa^{\mathbb{Z}_2}(z') & \text{if } x,y \neq 0 \\ \mathrm{d}\delta_x(z') & \text{if } y = 0 \\ \mathrm{d}\delta_y(z') & \text{if } x = 0. \end{cases}$$

(2) *The measure $\nu_{x,y}^{\mathbb{Z}_2,\kappa}$ satisfies*

(a) $\operatorname{supp}\nu_{x,y}^{\mathbb{Z}_2,\kappa} = \Big[-|x|-|y|, -\big||x|-|y|\big|\Big] \bigcup \Big[\big||x|-|y|\big|, |x|+|y|\Big]$ *for $x, y \neq 0$.*

(b) $\nu_{x,y}^{\mathbb{Z}_2,\kappa}(\mathbb{R}) = 1$ *and* $\|\nu_{x,y}^{\mathbb{Z}_2,\kappa}\| = \int_{\mathbb{R}} |\mathrm{d}\nu_{x,y}^{\mathbb{Z}_2}| \leqslant 4$, *for $x, y \in \mathbb{R}$.*

As an important consequence of this product formula we have the following result for the Dunkl translation (see [19]).

Theorem 2.5. *Let $x \in \mathbb{R}$ and let p satisfy $1 \leqslant p \leqslant +\infty$. For every $f \in L^p(\mu_\kappa^{\mathbb{Z}_2})$ we have*

$$\|\tau_x^{\mathbb{Z}_2,\kappa} f\|_{\mathbb{Z}_2,\kappa,p} \leqslant 4\|f\|_{\mathbb{Z}_2,\kappa,p}.$$

One of the key argument in the proof of the above theorem is the integral representation

$$\begin{aligned}\tau_x^{\mathbb{Z}_2,\kappa} f(y) &= \frac{1}{2}\int_{-1}^{1} f\Big(\sqrt{x^2+y^2+2xyt}\Big)\Big(1 + \frac{x+y}{\sqrt{x^2+y^2+2xyt}}\Big)\psi_\kappa(t)\,\mathrm{d}t \\ &\quad + \frac{1}{2}\int_{-1}^{1} f\Big(-\sqrt{x^2+y^2+2xyt}\Big)\Big(1 - \frac{x+y}{\sqrt{x^2+y^2+2xyt}}\Big)\psi_\kappa(t)\,\mathrm{d}t\end{aligned}$$

where $\psi_\kappa(t) = \frac{\Gamma(\kappa+\frac{1}{2})}{\Gamma(\kappa)\Gamma(\frac{1}{2})}(1+t)(1-t^2)^{\kappa-1}$, and which can be reformulated (using a change of variables) as follows

$$\tau_x^{\mathbb{Z}_2,\kappa} f(y) = \int_{\mathbb{R}} f(z)\,\mathrm{d}\nu_{x,y}^{\mathbb{Z}_2,\kappa}(z). \tag{2.3}$$

Remark. Recently, Amri, Anker and Sifi have proved in [2] the following optimal estimate

$$\|\tau_x^{\mathbb{Z}_2,\kappa} f\|_{\mathbb{Z}_2,\kappa,p} \leqslant \left(\sqrt{2}\,\frac{\Gamma\big(\kappa+\frac{1}{2}\big)^2}{\Gamma\big(\kappa+\frac{1}{4}\big)\Gamma\big(\kappa+\frac{3}{4}\big)}\right)^{2|\frac{1}{p}-\frac{1}{2}|}\|f\|_{\mathbb{Z}_2,\kappa,p}.$$

In this article, we will generalize in particular these two previous theorems in the setting of a positive subsystem of orthogonal roots.

3. Dunkl kernel and product formula

This section is devoted to the study of the Dunkl kernel associated with a positive subsystem of orthogonal roots. More precisely, we fix a positive subsystem $\mathcal{R}_+ = \{\alpha^1, \ldots, \alpha^m\}$ composed by m vectors of $\mathbb{R}^d$ which are pairwise orthogonal (with $1 \leqslant m \leqslant d$) and we fix a multiplicity function κ on the root system $\mathcal{R} = \{\pm\alpha^j : 1 \leqslant j \leqslant m\}$. Since there are m conjugacy classes of reflections, this multiplicity function takes m values $\kappa_1, \dots, \kappa_m$ respectively associated with $\alpha^1, \dots, \alpha^m$. We assume that these values are positive. From now on, W will always denote the reflection group associated with such a root system $\mathcal{R}$ unless otherwise specified. The measure μ_κ^W is given by

$$\mathrm{d}\mu_\kappa^W(x) = \prod_{j=1}^m |\langle x, \alpha^j\rangle|^{2\kappa_j}\,\mathrm{d}x.$$

3.1. **Explicit formula and product formula.** We will establish in this subsection a product formula for the Dunkl kernel which will allow us to prove in the next section that the Dunkl translation $\tau_x^{W,\kappa}$ is a bounded operator. In order to do that, we first give an explicit formula for the Dunkl kernel which is an easy consequence of the following integral representation of the intertwining operator V_κ^W recently proved by Maslouhi and Youssfi ([16, 17]).

Theorem 3.1. *Let $\mathcal{R}_+ = \{\alpha^1, \dots, \alpha^m\}$ be a positive subsystem composed by m vectors of $\mathbb{R}^d$ pairwise orthogonal (with $1 \leqslant m \leqslant d$) and let W be the associated reflection group. Let us denote by κ_j the value of the multiplicity function at α^j and let $h : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^d$ be the function defined by*

$$h(t,x) = x + \sum_{j=1}^m (t_j - 1)\frac{\langle x, \alpha^j\rangle}{\langle \alpha^j, \alpha^j\rangle}\alpha^j.$$

Then the intertwining operator V_κ^W is given by the following integral representation

$$V_\kappa^W f(x) = \int_{[-1,1]^m} f\big(h(t,x)\big)\prod_{j=1}^m \frac{\Gamma\big(\kappa_j + \frac{1}{2}\big)}{\Gamma\big(\kappa_j\big)\Gamma\big(\frac{1}{2}\big)}(1+t_j)(1-t_j^2)^{\kappa_j - 1}\,\mathrm{d}t. \tag{3.1}$$

Remark. This formula generalizes the one given in the $\mathbb{Z}_2^d$ case by Xu (see [26]).

Of course we will use (3.1) to give a suitable expression of the Dunkl kernel. Before doing this, we introduce some notations.

Notations. We complete the family $\{\alpha^j : 1 \leqslant j \leqslant m\}$ with $d-m$ vectors denoted by $\beta^1, \dots, \beta^{d-m}$ such that the family $\{\alpha^j, \beta^l : 1 \leqslant j \leqslant m,\, 1 \leqslant l \leqslant d-m\}$ is an orthogonal basis of $\mathbb{R}^d$.
For $x \in \mathbb{R}^d$, we will respectively denote by x_{α^j} and x_{β^l} (for $1 \leqslant j \leqslant m$ and $1 \leqslant l \leqslant d-m$) the following real numbers

$$x_{\alpha^j} = \frac{\langle x, \alpha^j\rangle}{\|\alpha^j\|}, \quad x_{\beta^l} = \frac{\langle x, \beta^l\rangle}{\|\beta^l\|}.$$

We also set $x_{\alpha,\beta}$ for the following vector of $\mathbb{R}^d$

$$x_{\alpha,\beta} = (x_{\alpha^1}, \dots, x_{\alpha^m}, x_{\beta^1}, \dots, x_{\beta^{d-m}}).$$

Finally, we will denote by $\overline{\kappa}^m$ the multiplicity function associated with the root system $\mathcal{R} = \{\pm e_j : 1 \leqslant j \leqslant m\}$ of $\mathbb{R}^m$ and such that for every $1 \leqslant j \leqslant m$

$$\overline{\kappa}^m(e_j) = \kappa_j = \kappa(\alpha^j),$$

and we will denote by $\overline{\kappa}^d$ the multiplicity function associated with the root system $\mathcal{R} = \{\pm e_j : 1 \leqslant j \leqslant d\}$ of $\mathbb{R}^d$ and such that

$$\overline{\kappa}^d(e_j) = \begin{cases} \kappa_j = \kappa(\alpha^j) & \text{if } 1 \leqslant j \leqslant m \\ 0 & \text{if } m < j \leqslant d. \end{cases}$$

We can now give the explicit formula of the Dunkl kernel.

Proposition 3.1. *Let x, y be two elements of $\mathbb{R}^d$. We have*

$$E_\kappa^W(ix,y) = E_{\kappa^d}^{\mathbb{Z}_2^d}(ix_{\alpha,\beta}, y_{\alpha,\beta}). \tag{3.2}$$

Proof. By applying the previous theorem to the function $y \mapsto \mathrm{e}^{i\langle x,y\rangle}$ we obtain

$$\begin{aligned} E_k^W(ix,.)(y) &= V_\kappa^W(\mathrm{e}^{i\langle x,y\rangle}) \\ &= \int_{[-1,1]^m} \mathrm{e}^{i\langle x,h(t,y)\rangle} \prod_{j=1}^m \frac{\Gamma\big(\kappa_j+\frac{1}{2}\big)}{\Gamma\big(\kappa_j\big)\Gamma\big(\frac{1}{2}\big)}(1+t_j)(1-t_j^2)^{\kappa_j-1}\,\mathrm{d}t. \end{aligned}$$

Since we can write

$$h(t,y) = \sum_{j=1}^m t_j \frac{y_{\alpha^j}}{\|\alpha^j\|}\alpha^j + \sum_{l=1}^{d-m}\frac{y_{\beta^l}}{\|\beta^l\|}\beta^l,$$

we are led to

$$E_\kappa^W(ix,y) = \prod_{l=1}^{d-m} \mathrm{e}^{ix_{\beta^l}y_{\beta^l}} \prod_{j=1}^m \int_{-1}^1 \mathrm{e}^{it_j x_{\alpha^j}y_{\alpha^j}} \frac{\Gamma\big(\kappa_j+\frac{1}{2}\big)}{\Gamma\big(\kappa_j\big)\Gamma\big(\frac{1}{2}\big)}(1+t_j)(1-t_j^2)^{\kappa_j-1}\,\mathrm{d}t_j.$$

If we now use (2.2) we get

$$E_\kappa^W(ix,y) = \prod_{l=1}^{d-m} \mathrm{e}^{ix_{\beta^l}y_{\beta^l}} \prod_{j=1}^m E_{\kappa_j}^{\mathbb{Z}_2}(ix_{\alpha^j}, y_{\alpha^j}),$$

from which we deduce the desired result. □

Remark. The formula of Proposition 3.1 is of course independent of the choice of the vectors β_l. This fact follows from the equality

$$\prod_{l=1}^{d-m} \mathrm{e}^{ix_{\beta^l}y_{\beta^l}} = \mathrm{e}^{i\langle x,y\rangle}\mathrm{e}^{-i\sum_{j=1}^m x_{\alpha^j}y_{\alpha^j}}.$$

Then the kernel can be reformulated as follows

$$E_\kappa^W(ix,y) = \mathrm{e}^{i\langle x,y\rangle}\mathrm{e}^{-i\sum_{j=1}^m \frac{\langle x,\alpha^j\rangle\langle y,\alpha^j\rangle}{\|\alpha^j\|^2}} \prod_{j=1}^m E_{\kappa_j}^{\mathbb{Z}_2}\Big(i\frac{\langle x,\alpha^j\rangle}{\|\alpha^j\|}, \frac{\langle y,\alpha^j\rangle}{\|\alpha^j\|}\Big).$$

We now prove a product formula for the Dunkl kernel E_κ^W.

Theorem 3.2. *Let $x, y, z \in \mathbb{R}^d$. We have the following product formula*

$$\begin{aligned} &E_\kappa^W(ix,z)E_\kappa^W(iy,z) \\ &\quad = \int_{\mathbb{R}^m\times\mathbb{R}^{d-m}} E_\kappa^W(iz, A^{-1}z')\Big(\bigotimes_{j=1}^m \mathrm{d}\nu_{x_{\alpha^j},y_{\alpha^j}}^{\mathbb{Z}_2,\kappa_j} \otimes \bigotimes_{l=1}^{d-m}\mathrm{d}\delta_{(x+y)_{\beta^l}}\Big)(z'), \end{aligned}$$

where A is the matrix given by

$$A = \begin{pmatrix} \frac{\alpha^1_1}{\|\alpha^1\|} & \dots & \frac{\alpha^1_d}{\|\alpha^1\|} \\ \vdots & & \vdots \\ \frac{\alpha^m_1}{\|\alpha^m\|} & \dots & \frac{\alpha^m_d}{\|\alpha^m\|} \\ \frac{\beta^1_1}{\|\beta^1\|} & \dots & \frac{\beta^1_d}{\|\beta^1\|} \\ \vdots & & \vdots \\ \frac{\beta^{d-m}_1}{\|\beta^{d-m}\|} & \dots & \frac{\beta^{d-m}_d}{\|\beta^{d-m}\|} \end{pmatrix}.$$

Remark. We then have the equality $x_{\alpha,\beta} = Ax$. Depending on the context we will use the notation we consider the most suitable.

Proof. Let $x, y, z \in \mathbb{R}^d$. Thanks to Proposition 3.1 we can write

$$E^W_\kappa(ix,z)E^W_\kappa(iy,z) = E^{\mathbb{Z}_2^d}_{\overline{\kappa}^d}(ix_{\alpha,\beta}, z_{\alpha,\beta})E^{\mathbb{Z}_2^d}_{\overline{\kappa}^d}(iy_{\alpha,\beta}, z_{\alpha,\beta}),$$

that is to say

$$E^W_\kappa(ix,z)E^W_\kappa(iy,z) = \prod_{l=1}^{d-m} \mathrm{e}^{i(x+y)_{\beta^l} z_{\beta^l}} \prod_{j=1}^{m} E^{\mathbb{Z}_2}_{\kappa_j}(ix_{\alpha^j}, z_{\alpha^j})E^{\mathbb{Z}_2}_{\kappa_j}(iy_{\alpha^j}, z_{\alpha^j}).$$

Using the one-dimensional product formula (see Theorem 2.4) we are led to

$$E^W_\kappa(ix,z)E^W_\kappa(iy,z) = \prod_{j=1}^{m} \int_{\mathbb{R}} \prod_{l=1}^{d-m} \mathrm{e}^{i(x+y)_{\beta^l} z_{\beta^l}} E^{\mathbb{Z}_2}_{\kappa_j}(iz_{\alpha^j}, z'_j)\, \mathrm{d}\nu^{\mathbb{Z}_2,\kappa_j}_{x_{\alpha^j}, y_{\alpha^j}}(z'_j),$$

or equivalently

$$\begin{aligned} &E^W_\kappa(ix,z)E^W_\kappa(iy,z) \\ &\quad = \int_{\mathbb{R}^m} \prod_{l=1}^{d-m} \mathrm{e}^{i(x+y)_{\beta^l} z_{\beta^l}} E^{\mathbb{Z}_2^m}_{\overline{\kappa}^m}\big(i(z_{\alpha^1}, \dots, z_{\alpha^m}), (z'_1, \dots, z'_m)\big) \bigotimes_{j=1}^{m} \mathrm{d}\nu^{\mathbb{Z}_2,\kappa_j}_{x_{\alpha^j}, y_{\alpha^j}}(z'_j). \end{aligned} \tag{3.3}$$

Let us note that we have for every j satisfying $1 \leqslant j \leqslant m$

$$z'_j = \big(A^{-1}\big(z'_1, \dots, z'_m, (x+y)_{\beta^1}, \dots, (x+y)_{\beta^{d-m}}\big)\big)_{\alpha^j}. \tag{3.4}$$

Indeed we have by definition

$$\begin{aligned} &\big(A^{-1}\big(z'_1, \dots, z'_m, (x+y)_{\beta^1}, \dots, (x+y)_{\beta^{d-m}}\big)\big)_{\alpha^j} \\ &\qquad = \frac{\langle A^{-1}\big(z'_1, \dots, z'_m, (x+y)_{\beta^1}, \dots, (x+y)_{\beta^{d-m}}\big), \alpha^j\rangle}{\|\alpha^j\|}, \end{aligned}$$

and then

$$\begin{aligned} &\big(A^{-1}\big(z'_1, \dots, z'_m, (x+y)_{\beta^1}, \dots, (x+y)_{\beta^{d-m}}\big)\big)_{\alpha^j} \\ &\qquad = \frac{\langle\big(z'_1, \dots, z'_m, (x+y)_{\beta^1}, \dots, (x+y)_{\beta^{d-m}}\big), A\,\alpha^j\rangle}{\|\alpha^j\|} = z'_j, \end{aligned}$$

where we have used the fact that A is an orthogonal matrix. We can also remark that we have for every l satisfying $1 \leqslant l \leqslant d-m$

$$(x+y)_{\beta^l} = \big(A^{-1}\big(z'_1, \dots, z'_m, (x+y)_{\beta^1}, \dots, (x+y)_{\beta^{d-m}}\big)\big)_{\beta^l}. \tag{3.5}$$

Using (3.4) and (3.5), (3.3) can be reformulated as follows

$$E_\kappa^W(ix,z)E_\kappa^W(iy,z) = \int_{\mathbb{R}^m} E_{\overline{\kappa}^d}^{\mathbb{Z}_2^d}(iz_{\alpha,\beta}, (A^{-1}(z_1',\dots,z_m',(x+y)_{\beta^1},\dots,(x+y)_{\beta^{d-m}}))_{\alpha,\beta}) \bigotimes_{j=1}^m \mathrm{d}\nu_{x_{\alpha^j},y_{\alpha^j}}^{\mathbb{Z}_2,\kappa_j}(z_j'),$$

and due to Proposition 3.1 we then obtain

$$E_\kappa^W(ix,z)E_\kappa^W(iy,z) = \int_{\mathbb{R}^m} E_\kappa^W\big(iz, A^{-1}\big(z_1',\dots,z_m',(x+y)_{\beta^1},\dots,(x+y)_{\beta^{d-m}}\big)\big) \bigotimes_{j=1}^m \mathrm{d}\nu_{x_{\alpha^j},y_{\alpha^j}}^{\mathbb{Z}_2,\kappa_j}(z_j'),$$

and the product formula is proved. □

Remark. This product formula does not depend of the choice of the vectors β^l. This is a consequence of (3.3), that is to say

$$E_\kappa^W(ix,z)E_\kappa^W(iy,z) = \int_{\mathbb{R}^m} \prod_{l=1}^{d-m} \mathrm{e}^{i(x+y)_{\beta^l} z_{\beta^l}} E_{\overline{\kappa}^m}^{\mathbb{Z}_2^m}\big(i(z_{\alpha^1},\dots,z_{\alpha^m}),(z_1',\dots,z_m')\big) \bigotimes_{j=1}^m \mathrm{d}\nu_{x_{\alpha^j},y_{\alpha^j}}^{\mathbb{Z}_2,\kappa_j}(z_j'),$$

together with the fact that

$$\prod_{l=1}^{d-m} \mathrm{e}^{i(x+y)_{\beta^l} z_{\beta^l}} = \mathrm{e}^{i\langle x+y,z\rangle}\mathrm{e}^{-i\sum_{j=1}^m (x+y)_{\alpha^j} z_{\alpha^j}}.$$

We now turn to an interesting particular case.

3.2. **An interesting particular case.** We consider in this subsection the particular case of a root system of A_1-type, that is to say we fix in $\mathbb{R}^2$ the root system $\mathcal{R} = \{\pm(e_1 - e_2)\}$ so that $W \simeq \mathfrak{S}_2$, and we choose $\mathcal{R}_+ = \{e_1 - e_2\}$.
This orthogonal system is of interest for the following reason. Due to some works of Baker and Forrester (see [4, 5]) we explicitly know the generalized Bessel function $J_\kappa^{\mathfrak{S}_2}$: it is a multivariate hypergeometric function ${}_0F_0^{(\alpha)}$ of two arguments which is expressed in terms of Gegenbauer polynomials (see [15]). Thus, by use of the formula of the Dunkl kernel in the A_1-case (obtained as a particular case of Proposition 3.1), we will be in a position to give an equality which links normalized Bessel functions and Gegenbauer polynomials. Even if we are only interested in the two-dimensional case and in the function ${}_0F_0^{(\alpha)}$, we introduce Jack polynomials and multivariate hypergeometric function in full generality for the reader convenience. For more details, one may especially consult the Stanley's article [24] for Jack polynomials and the Yan's article [27] for multivariate hypergeometric functions.

We call partition (with no more than d parts) every sequence $\lambda = (\lambda_1,\dots,\lambda_d)$ of nonnegative integers in decreasing order

$$\lambda_1 \geqslant \cdots \geqslant \lambda_d \geqslant 0.$$

The set of all such partitions is denoted by $\mathbb{N}^{d,P}$. The number of elements $\lambda_j \neq 0$ (the nonzero elements are the parts of λ) is the length $l(\lambda)$ of the partition and the weight $|\lambda|$ of the partition is defined by

$$|\lambda| = \lambda_1 + \cdots + \lambda_d.$$

We identify a partition $\lambda \in \mathbb{N}^{d,P}$ with its diagram $\{(j,k) : 1 \leqslant j \leqslant l(\lambda), 1 \leqslant k \leqslant \lambda_j\}$ and the conjugate (or dual) partition λ' to $\lambda \in \mathbb{N}^{d,P}$ is the sequence $\lambda' = (\lambda'_1, \dots, \lambda'_m)$ defined by $\lambda'_j = \#\{k : \lambda_k \geqslant j\}$, that is to say λ'_j is the number of nodes in the jth column of λ. The upper hook-length h^*_λ at $(j,k) \in \lambda$ and the lower hook-length h^λ_* at $(j,k) \in \lambda$ are respectively given by

$$h^*_\lambda(j,k) = \lambda'_k - j + \alpha(\lambda_j - k + 1)$$
$$h^\lambda_*(j,k) = \lambda'_k - j + 1 + \alpha(\lambda_j - k)$$

with α a positive real number.
For $\lambda \in \mathbb{N}^{d,P}$ and α a positive real number, the Jack polynomial of Jack parameter α is the unique (up to a normalization) symmetric homogeneous eigenfunction $C^{(\alpha)}_\lambda$ of the operator

$$\sum_{j=1}^d x_j^2 \frac{\partial^2}{\partial x_j^2} + \frac{2}{\alpha} \sum_{\substack{j,k=1 \\ j\neq k}}^d \frac{x_j^2}{x_j - x_k} \frac{\partial}{\partial x_j}$$

corresponding to the eigenvalue

$$v_\lambda = \sum_{j=1}^d \Big(\lambda_j \Big(\lambda_j - 1 - \frac{2}{\alpha}(j-1) \Big) \Big) + \frac{2}{\alpha} |\lambda| (d-1).$$

We adopt in this note the normalization

$$(x_1 + \cdots + x_d)^n = \sum_{\substack{\lambda \in \mathbb{N}^{d,P} \\ |\lambda| = n}} C^{(\alpha)}_\lambda(x).$$

The other classical normalization is the following one (see [24]), where the Jack polynomials $\mathcal{J}^{(\alpha)}_\lambda$ are chosen so that

$$(x_1 + \cdots + x_d)^n = \sum_{\substack{\lambda \in \mathbb{N}^{d,P} \\ |\lambda| = n}} \frac{\alpha^{|\lambda|}\, |\lambda|!}{h(\lambda)} \mathcal{J}^{(\alpha)}_\lambda(x),$$

with $h(\lambda)$ given by

$$h(\lambda) = \prod_{(j,k)\in\lambda} h^*_\lambda(j,k) h^\lambda_*(j,k).$$

Thus, the relation between $C^{(\alpha)}_\lambda$ and $\mathcal{J}^{(\alpha)}_\lambda$ is given for $\lambda \in \mathbb{N}^{d,P}$ of weight n by

$$C^{(\alpha)}_\lambda(x) = \frac{\alpha^n\, n!}{h(\lambda)} \mathcal{J}^{(\alpha)}_\lambda. \tag{3.6}$$

Before introducing the definition of the multivariate hypergeometric function, we recall that the generalized Pochhammer symbol is defined by

$$(z)^{(\alpha)}_\lambda = \prod_{j=1}^d \Big(z - \frac{1}{\alpha}(j-1) \Big)_{\lambda_j},$$

where z is a complex number, α is a positive real number, λ is an element of $\mathbb{N}^{d,P}$ and $(z)_{\lambda_j}$ is the usual Pochhammer symbol.

Definition. Let p and q be two integers, λ be an element of $\mathbb{N}^{d,P}$ and α be a positive real number. Let $a_1,\dots,a_p,b_1,\dots,b_q$ be complex numbers such that $(b_j)_\lambda^{(\alpha)} \neq 0$ for each j satisfying $1 \leqslant j \leqslant q$. The multivariate hypergeometric function (with two arguments) ${}_pF_q^{(\alpha)}$ is defined by the serie

$$ {}_pF_q^{(\alpha)}\big(a_1,\dots,a_p;b_1,\dots,b_q;x;y\big) = \sum_{n=0}^{+\infty} \sum_{\substack{\lambda\in\mathbb{N}^{d,P}\\ |\lambda|=n}} \frac{(a_1)_\lambda^{(\alpha)}\cdots(a_p)_\lambda^{(\alpha)}}{(b_1)_\lambda^{(\alpha)}\cdots(b_q)_\lambda^{(\alpha)}} \frac{C_\lambda^{(\alpha)}(x)\,C_\lambda^{(\alpha)}(y)}{|\lambda|!\,C_\lambda^{(\alpha)}(\mathbf{1})}, $$

where $\mathbf{1}$ is the vector $(1,\dots,1)$ of $\mathbb{R}^d$.

Thanks to Baker and Forrester ([4, 5]), we know that the generalized Bessel function is given in the case where the root system is of A_{d-1}-type (and so $W \simeq \mathfrak{S}_d$) by

$$ J_\kappa^{\mathfrak{S}_d}(x,y) = {}_0F_0^{(1/\kappa)}(x,y) = \sum_{n=0}^{+\infty} \sum_{\substack{\lambda\in\mathbb{N}^{d,P}\\ |\lambda|=n}} \frac{C_\lambda^{(1/\kappa)}(x)\,C_\lambda^{(1/\kappa)}(y)}{|\lambda|!\,C_\lambda^{(1/\kappa)}(\mathbf{1})}, \tag{3.7} $$

and, in the case where the root system is of B_d-type (and so $W \simeq \mathfrak{S}_d \ltimes \mathbb{Z}_2^d$) by

$$ \begin{aligned} J_\kappa^{\mathfrak{S}_d\ltimes\mathbb{Z}_2^d}(x,y) &= {}_0F_1^{(1/\kappa_2)}\left(b;\Big(\frac{x_1^2}{2},\dots,\frac{x_d^2}{2}\Big),\Big(\frac{y_1^2}{2},\dots,\frac{y_d^2}{2}\Big)\right) \\ &= \sum_{n=0}^{+\infty} \sum_{\substack{\lambda\in\mathbb{N}^{d,P}\\ |\lambda|=n}} \frac{C_\lambda^{(1/\kappa_2)}\Big(\frac{x_1^2}{2},\dots,\frac{x_d^2}{2}\Big)\,C_\lambda^{(1/\kappa_2)}\Big(\frac{y_1^2}{2},\dots,\frac{y_d^2}{2}\Big)}{(b)_\lambda^{(1/\kappa_2)}|\lambda|!\,C_\lambda^{(1/\kappa_2)}(\mathbf{1})}, \end{aligned} $$

with κ_1 and κ_2 the two values of the multiplicity function (respectively on the roots $\pm e_j$, $\pm e_j \pm e_k$) and

$$ b = \kappa_1 + (d-1)\kappa_2 + \frac{1}{2}. $$

Let us point out that Demni has recently proved in [8] an explicit formula for the generalized Bessel function in terms of multivariate hypergeometric function in the case where the root system is of D_d-type.
Thanks to the expression of the generalized Bessel function $J_\kappa^{\mathfrak{S}_2}$ and to the expression of the Dunkl kernel $E_\kappa^{\mathfrak{S}_2}$, we will prove a formula which links normalized Bessel functions and Gegenbauer polynomials. In order to become more precise, let us recall that the Gegenbauer polynomial U_n^α is given for a real number $\alpha > -\frac{1}{2}$ and for every integer n by

$$ U_n^\alpha(x) = \frac{(2\alpha)_n}{\big(\alpha+\frac{1}{2}\big)_n} P_n^{(\alpha-1/2,\alpha-1/2)}(x), $$

where $P_n^{(\alpha,\beta)}$ is the usual Jacobi polynomial (see the book [3] of Andrews, Askey and Roy for more details on special functions).

Remark. The Gegenbauer polynomials are usually denoted by C_n^α. We choose a different notation in this note in order to avoid confusions with the Jack polynomials.

We now give the formula that we want to prove.

Theorem 3.3. *Let κ be the positive value of the multiplicity function associated with $\mathcal{R} = \{\pm(e_1 - e_2)\}$. Let $x = (x_1, x_2) \in \mathbb{R}^2, y = (y_1, y_2) \in \mathbb{R}^2$ such that $x_1x_2 > 0, y_1y_2 > 0$. We write for $\lambda \in \mathbb{N}^{2,P}$*

$$c_\kappa(\lambda) = \prod_{(j,k)\in\lambda} \frac{\big(2-(j-1)+\frac{1}{\kappa}(k-1)\big)}{h(\lambda)} = \prod_{(j,k)\in\lambda} \left(\frac{\big(2-(j-1)+\frac{1}{\kappa}(k-1)\big)}{h^*_\lambda(j,k)h^\lambda_*(j,k)}\right),$$

and we write

$$G(\lambda,\kappa,x,y) = (x_1x_2y_1y_2)^{\frac{\lambda_1+\lambda_2}{2}}\, U^\kappa_{\lambda_1-\lambda_2}\Big(\frac{x_1+x_2}{2\sqrt{x_1x_2}}\Big) U^\kappa_{\lambda_1-\lambda_2}\Big(\frac{y_1+y_2}{2\sqrt{y_1y_2}}\Big).$$

Then we have the following equality

$$j_{\kappa-\frac{1}{2}}\Big(\frac{i}{2}(x_1-x_2)(y_1-y_2)\Big) = \mathrm{e}^{\frac{-(x_1+x_2)(y_1+y_2)}{2}} \sum_{n=0}^{+\infty} \sum_{\substack{\lambda\in\mathbb{N}^{2,P}\\ |\lambda|=n}} \frac{c_\kappa(\lambda)\big((\lambda_1-\lambda_2)!\big)^2}{\kappa^{|\lambda|}(2\kappa)^2_{\lambda_1-\lambda_2}} G(\lambda,\kappa,x,y).$$

In order to prove this theorem we need the following lemma which gives us an explicit formula for the two-dimensional Jack polynomials.

Lemma 3.1. *Let α be a positive real number and let λ be an element of $\mathbb{N}^{2,P}$. The Jack polynomial $C^{(\alpha)}_\lambda$ is given for every $x = (x_1, x_2) \in \mathbb{R}^2$ satisfying $x_1x_2 > 0$ by*

$$C^{(\alpha)}_\lambda(x) = \frac{c_{\alpha^{-1}}(\lambda)\alpha^{|\lambda|}\,|\lambda|!\,(\lambda_1-\lambda_2)!}{(\frac{2}{\alpha})_{\lambda_1-\lambda_2}} (x_1x_2)^{\frac{\lambda_1+\lambda_2}{2}}\, U^{\frac{1}{\alpha}}_{\lambda_1-\lambda_2}\Big(\frac{x_1+x_2}{2\sqrt{x_1x_2}}\Big).$$

Proof. For every $\lambda \in \mathbb{N}^{2,P}$ we know thanks to a result of Lassalle (see [15]) that

$$\mathcal{J}^{(\alpha)}_\lambda(x) = \mathcal{J}^{(\alpha)}_\lambda(\mathbf{1}) \frac{(\lambda_1-\lambda_2)!}{(\frac{2}{\alpha})_{\lambda_1-\lambda_2}} (x_1x_2)^{\frac{\lambda_1+\lambda_2}{2}}\, U^{\frac{1}{\alpha}}_{\lambda_1-\lambda_2}\Big(\frac{x_1+x_2}{2\sqrt{x_1x_2}}\Big).$$

Using the following formula due to Stanley (see Theorem 5.4 in [24])

$$\mathcal{J}^{(\alpha)}_\lambda(\mathbf{1}) = \prod_{(j,k)\in\lambda} \big(2-(j-1)+\alpha(k-1)\big),$$

we get that

$$\mathcal{J}^{(\alpha)}_\lambda(x) = \prod_{(j,k)\in\lambda} \big(2-(j-1)+\alpha(k-1)\big) \frac{(\lambda_1-\lambda_2)!}{(\frac{2}{\alpha})_{\lambda_1-\lambda_2}} (x_1x_2)^{\frac{\lambda_1+\lambda_2}{2}}\, U^{\frac{1}{\alpha}}_{\lambda_1-\lambda_2}\Big(\frac{x_1+x_2}{2\sqrt{x_1x_2}}\Big).$$

The link between $\mathcal{J}^{(\alpha)}_\lambda$ and $C^{(\alpha)}_\lambda$ (equality (3.6)) leads us to

$$\frac{h(\lambda)}{\alpha^{|\lambda|}\,|\lambda|!} C^{(\alpha)}_\lambda(x) = \prod_{(j,k)\in\lambda} \big(2-(j-1)+\alpha(k-1)\big) \frac{(\lambda_1-\lambda_2)!}{(\frac{2}{\alpha})_{\lambda_1-\lambda_2}} (x_1x_2)^{\frac{\lambda_1+\lambda_2}{2}}\, U^{\frac{1}{\alpha}}_{\lambda_1-\lambda_2}\Big(\frac{x_1+x_2}{2\sqrt{x_1x_2}}\Big),$$

and the result easily follows. □

Remark. We have in particular

$$C_\lambda^{(\alpha)}(\mathbf{1}) = c_{\alpha^{-1}}(\lambda)\alpha^{|\lambda|}\,|\lambda|!, \tag{3.8}$$

where we have used the well-known equality (see [3, page 302] for instance)

$$U_n^\alpha(1) = \frac{(2\alpha)_n}{n!}.$$

Thanks to the definition of $c(\lambda)$ and according to the following equality given in the previous proof

$$\prod_{(j,k)\in\lambda}\big(2-(j-1)+\alpha(k-1)\big) = \mathcal{J}_\lambda^{(\alpha)}(\mathbf{1}),$$

we obtain

$$C_\lambda^{(\alpha)}(\mathbf{1}) = \frac{\alpha^{|\lambda|}\,|\lambda|!}{h(\lambda)}\,\mathcal{J}_\lambda^{(\alpha)}(\mathbf{1}),$$

and it is in agreement with (3.6).

We can now prove Theorem 3.3.

Proof. According to the definition of the generalized Bessel function we have for every $x=(x_1,x_2)\in\mathbb{R}^2$ and every $y=(y_1,y_2)\in\mathbb{R}^2$ satisfying $x_1x_2>0, y_1y_2>0$

$$J_\kappa^{\mathfrak{S}_2}(x,y) = \frac{1}{2}\Big(E_\kappa^{\mathfrak{S}_2}(x,y) + E_\kappa^{\mathfrak{S}_2}\big((x_2,x_1),y\big)\Big).$$

Using the explicit formula of the Dunkl kernel $E_\kappa^{\mathfrak{S}_2}$ (particular case of Proposition 3.1) we get

$$J_\kappa^{\mathfrak{S}_2}(x,y) = \frac{1}{2}\,\mathrm{e}^{\frac{(x_1+x_2)(y_1+y_2)}{2}}\left(E_\kappa^{\mathbb{Z}_2}\Big(\frac{x_1-x_2}{\sqrt{2}},\frac{y_1-y_2}{\sqrt{2}}\Big) + E_\kappa^{\mathbb{Z}_2}\Big(\frac{x_2-x_1}{\sqrt{2}},\frac{y_1-y_2}{\sqrt{2}}\Big)\right).$$

Since we have thanks to (2.2)

$$E_\kappa^{\mathbb{Z}_2}\Big(\frac{x_1-x_2}{\sqrt{2}},\frac{y_1-y_2}{\sqrt{2}}\Big) + E_\kappa^{\mathbb{Z}_2}\Big(\frac{x_2-x_1}{\sqrt{2}},\frac{y_1-y_2}{\sqrt{2}}\Big) = 2j_{\kappa-\frac{1}{2}}\Big(\frac{i}{2}(x_1-x_2)(y_1-y_2)\Big),$$

we claim that

$$J_\kappa^{\mathfrak{S}_2}(x,y) = \mathrm{e}^{\frac{(x_1+x_2)(y_1+y_2)}{2}}\,j_{\kappa-\frac{1}{2}}\Big(\frac{i}{2}(x_1-x_2)(y_1-y_2)\Big).$$

Due to the explicit formula (3.7) we can also write

$$J_\kappa^{\mathfrak{S}_2}(x,y) = {}_0F_0^{(1/\kappa)}(x,y) = \sum_{n=0}^{+\infty}\sum_{\substack{\lambda\in\mathbb{N}^{2,P}\\|\lambda|=n}}\frac{C_\lambda^{(1/\kappa)}(x)\,C_\lambda^{(1/\kappa)}(y)}{|\lambda|!\,C_\lambda^{(1/\kappa)}(\mathbf{1})}.$$

Therefore we have the following equality

$$\mathrm{e}^{\frac{(x_1+x_2)(y_1+y_2)}{2}}\,j_{\kappa-\frac{1}{2}}\Big(\frac{i}{2}(x_1-x_2)(y_1-y_2)\Big) = \sum_{n=0}^{+\infty}\sum_{\substack{\lambda\in\mathbb{N}^{2,P}\\|\lambda|=n}}\frac{C_\lambda^{(1/\kappa)}(x)\,C_\lambda^{(1/\kappa)}(y)}{|\lambda|!\,C_\lambda^{(1/\kappa)}(\mathbf{1})}. \tag{3.9}$$

According to Lemma 3.1 and to (3.8) given in the previous remark we are led after simplifications to

$$\frac{\kappa^{|\lambda|} C_\lambda^{(1/\kappa)}(x)\, C_\lambda^{(1/\kappa)}(y)}{c_\kappa(\lambda)|\lambda|!\, C_\lambda^{(1/\kappa)}(\mathbf{1})} = \left(\frac{(\lambda_1-\lambda_2)!}{(2\kappa)_{\lambda_1-\lambda_2}}\right)^2 (x_1x_2y_1y_2)^{\frac{\lambda_1+\lambda_2}{2}}\, U^\kappa_{\lambda_1-\lambda_2}\Big(\frac{x_1+x_2}{2\sqrt{x_1x_2}}\Big) U^\kappa_{\lambda_1-\lambda_2}\Big(\frac{y_1+y_2}{2\sqrt{y_1y_2}}\Big),$$

that is to say

$$\frac{C_\lambda^{(1/\kappa)}(x)\, C_\lambda^{(1/\kappa)}(y)}{|\lambda|!\, C_\lambda^{(1/\kappa)}(\mathbf{1})} = \frac{c_\kappa(\lambda)\big((\lambda_1-\lambda_2)!\big)^2}{\kappa^{|\lambda|}(2\kappa)^2_{\lambda_1-\lambda_2}} G(\lambda,\kappa,x,y).$$

We now use this equality in (3.9) in order to obtain

$$\mathrm{e}^{\frac{(x_1+x_2)(y_1+y_2)}{2}} j_{\kappa-\frac{1}{2}}\Big(\frac{i}{2}(x_1-x_2)(y_1-y_2)\Big) = \sum_{n=0}^{+\infty} \sum_{\substack{\lambda\in\mathbb{N}^{2,P}\\ |\lambda|=n}} \frac{c_\kappa(\lambda)\big((\lambda_1-\lambda_2)!\big)^2}{\kappa^{|\lambda|}(2\kappa)^2_{\lambda_1-\lambda_2}} G(\lambda,\kappa,x,y),$$

and the theorem is proved. □

We now return to the general case, that is to say we consider a positive subsystem of orthogonal roots. We will see in the following section that the product formula of Theorem 3.2 allows us to prove that the Dunkl translation is a bounded operator.

4. Dunkl translation

Let us begin with an integral representation of the Dunkl translation.

Proposition 4.1. *Let $x\in\mathbb{R}^d$. For every f in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ we have the following integral representation*

$$\tau_x^{W,\kappa}(f)(y) = \int_{\mathbb{R}^m\times\mathbb{R}^{d-m}} f(A^{-1}z)\Big(\bigotimes_{j=1}^{m} \mathrm{d}\nu^{\mathbb{Z}_2,\kappa_j}_{x_{\alpha^j},y_{\alpha^j}} \otimes \bigotimes_{l=1}^{d-m} \mathrm{d}\delta_{(x+y)_{\beta^l}}\Big)(z), \quad y\in\mathbb{R}^d. \tag{4.1}$$

Proof. Since $f\in\mathcal{S}(\mathbb{R}^d)$ we can apply the inversion formula of Theorem 2.2 to get

$$\tau_x^{W,\kappa}(f)(y) = c_\kappa^W \int_{\mathbb{R}^d} E_\kappa^W(ix,z)E_\kappa^W(iy,z)\mathcal{F}_\kappa^W(f)(z)\,\mathrm{d}\mu_\kappa^W(z).$$

Using the product formula of Theorem 3.2 we obtain after an interversion

$$\tau_x^{W,\kappa}(f)(y) = \int_{\mathbb{R}^m\times\mathbb{R}^{d-m}} \Big(c_\kappa^W \int_{\mathbb{R}^d} E_\kappa^W(iz,A^{-1}z')\mathcal{F}_\kappa^W(f)(z)\,\mathrm{d}\mu_\kappa^W(z)\Big)\,\mathrm{d}\varrho_{x,y}(z')$$

where we have set for the sake of readability

$$\mathrm{d}\varrho_{x,y}(z') = \Big(\bigotimes_{j=1}^{m} \mathrm{d}\nu^{\mathbb{Z}_2,\kappa_j}_{x_{\alpha^j},y_{\alpha^j}} \otimes \bigotimes_{l=1}^{d-m} \mathrm{d}\delta_{(x+y)_{\beta^l}}\Big)(z').$$

We deduce the desired result by using again the inversion formula of Theorem 2.2. □

The formula is of course independent of the choice of the vectors β^l since the product formula of the Dunkl kernel E_κ^W is independent of these vectors. The previous proposition implies that the Dunkl translation is a bounded operator. More precisely we have the following result.

Theorem 4.1. *Let $x \in \mathbb{R}^d$ and let p be a real number which satisfies $1 \leqslant p \leqslant +\infty$. The Dunkl translation $\tau_x^{W,\kappa}$ extends to $L^p(\mu_\kappa^W)$ as a bounded operator. In particular we have*

$$\|\tau_x^{W,\kappa} f\|_{W,\kappa,p} \leqslant 4^m \|f\|_{W,\kappa,p}.$$

Proof. Let $x \in \mathbb{R}^d$ and let p be a real number satisfying $1 \leqslant p \leqslant +\infty$. We will prove that for every $f \in L^p(\mu_\kappa^W)$

$$y \mapsto \int_{\mathbb{R}^m} f\big(A^{-1}\big(z_1, \dots, z_m, (x+y)_{\beta^1}, \dots, (x+y)_{\beta^{d-m}}\big)\big) \bigotimes_{j=1}^m \mathrm{d}\nu_{x_{\alpha^j}, y_{\alpha^j}}^{\mathbb{Z}_2,\kappa_j}(z_j)$$

is an element of $L^p(\mu_\kappa^W)$. Assume that $1 < p < +\infty$ (the case $p = +\infty$ is trivial and the case $p = 1$ use similar arguments to those we will present). Let q be a real number such that $\frac{1}{p} + \frac{1}{q} = 1$.
In order to simplify the notation we set for this proof

$$v_{z,x,y} = \big(z_1, \dots, z_m, (x+y)_{\beta^1}, \dots, (x+y)_{\beta^{d-m}}\big).$$

Since we can write

$$\Big| f\big(A^{-1} v_{z,x,y}\big) \prod_{j=1}^m \mathcal{K}_{\kappa_j}(x_{\alpha^j}, y_{\alpha^j}, z_j) \Big| \\ = \big| f\big(A^{-1} v_{z,x,y}\big) \big| \prod_{j=1}^m |\mathcal{K}_{\kappa_j}(x_{\alpha^j}, y_{\alpha^j}, z_j)|^{\frac{1}{p}} \prod_{j=1}^m |\mathcal{K}_{\kappa_j}(x_{\alpha^j}, y_{\alpha^j}, z_j)|^{\frac{1}{q}},$$

we obtain by applying the Hölder inequality

$$\Big| \int_{\mathbb{R}^m} f\big(A^{-1} v_{z,x,y}\big) \bigotimes_{j=1}^m \mathrm{d}\nu_{x_{\alpha^j}, y_{\alpha^j}}^{\mathbb{Z}_2,\kappa_j}(z_j) \Big|^p \\ \leqslant 4^{\frac{mp}{q}} \Big(\int_{\mathbb{R}^m} \big| f\big(A^{-1} v_{z,x,y}\big) \big|^p \prod_{j=1}^m |\mathcal{K}_{\kappa_j}(x_{\alpha^j}, y_{\alpha^j}, z_j)| \, \mathrm{d}\mu_{\overline{\kappa}^m}^{\mathbb{Z}_2^m}(z) \Big),$$

where we have used the fact that (according to (2)(b) of Theorem 2.4)

$$\Big(\int_{\mathbb{R}^m} \prod_{j=1}^m |\mathcal{K}_{\kappa_j}(x_{\alpha^j}, y_{\alpha^j}, z_j)| \, \mathrm{d}\mu_{\overline{\kappa}^m}^{\mathbb{Z}_2^m}(z) \Big)^{\frac{p}{q}} \leqslant 4^{\frac{mp}{q}}.$$

We integrate and then use the Fubini theorem to write

$$\int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^m} f\big(A^{-1} v_{z,x,y}\big) \bigotimes_{j=1}^m \mathrm{d}\nu_{x_{\alpha^j}, y_{\alpha^j}}^{\mathbb{Z}_2,\kappa_j}(z_j) \Big|^p \mathrm{d}\mu_\kappa^W(y) \tag{4.2}$$

$$\leqslant 4^{\frac{mp}{q}} \int_{\mathbb{R}^m} \Big(\int_{\mathbb{R}^d} \big| f\big(A^{-1} v_{z,x,y}\big) \big|^p \prod_{j=1}^m \big| \mathcal{K}_{\kappa_j}(x_{\alpha^j}, y_{\alpha^j}, z_j) \langle y, \alpha^j \rangle^{2\kappa_j} \big| \, \mathrm{d}y \Big) \mathrm{d}\mu_{\overline{\kappa}^m}^{\mathbb{Z}_2^m}(z).$$

The change of variables $y' = Ay$ allows us to write

$$\int_{\mathbb{R}^d} \big|f\big(A^{-1}v_{z,x,y}\big)\big|^p \prod_{j=1}^m \big|\mathcal{K}_{\kappa_j}(x_{\alpha^j}, y_{\alpha^j}, z_j)\langle y, \alpha^j\rangle^{2\kappa_j}\big|\,\mathrm{d}y = \prod_{j=1}^m \|\alpha^j\|^{2\kappa_j} \times \int_{\mathbb{R}^d} \big|f\big(A^{-1}\big(z_1, \dots, z_m, y'_{m+1} + x_{\beta^1}, \dots, y'_d + x_{\beta^{d-m}}\big)\big)\big|^p k(x, y'_{1,m}, z)\,\mathrm{d}y',$$

where we have set $k(x, y'_{1,m}, z) = \prod_{j=1}^m \big|\mathcal{K}_{\kappa_j}(x_{\alpha^j}, y'_j, z_j){y'_j}^{2\kappa_j}\big|$. Thus we can assert that

$$\int_{\mathbb{R}^d} \big|f\big(A^{-1}v_{z,x,y}\big)\big|^p \prod_{j=1}^m \big|\mathcal{K}_{\kappa_j}(x_{\alpha^j}, y_{\alpha^j}, z_j)\langle y, \alpha^j\rangle^{2\kappa_j}\big|\,\mathrm{d}y = \prod_{j=1}^m \|\alpha^j\|^{2\kappa_j} \int_{\mathbb{R}^d} \big|f\big(A^{-1}\big(z_1, \dots, z_m, y'_{m+1}, \dots, y'_d\big)\big)\big|^p k(x, y'_{1,m}, z)\,\mathrm{d}y'$$

and therefore (4.2) is equivalent to

$$(4.3)\quad 4^{\frac{-mp}{q}} \prod_{j=1}^m \|\alpha^j\|^{-2\kappa_j} \int_{\mathbb{R}^d} \bigg|\int_{\mathbb{R}^m} f\big(A^{-1}v_{z,x,y}\big) \bigotimes_{j=1}^m \mathrm{d}\nu^{\mathbb{Z}_2,\kappa_j}_{x_{\alpha^j}, y_{\alpha^j}}(z_j)\bigg|^p \mathrm{d}\mu^W_\kappa(y) \leqslant \int_{\mathbb{R}^m} \bigg(\int_{\mathbb{R}^d} \big|f\big(A^{-1}\big(z_1, \dots, z_m, y'_{m+1}, \dots, y'_d\big)\big)\big|^p k(x, y'_{1,m}, z)\,\mathrm{d}y'\bigg)\,\mathrm{d}\mu^{\mathbb{Z}_2^m}_{\kappa^m}(z).$$

In order to simplify the notations, let us temporarily set

$$\big(z_1, \dots, z_m, y'_{m+1}, \dots, y'_d\big) = w(z, y'_{m+1,d}).$$

Since we have from one hand

$$\int_{\mathbb{R}^d} \big|f\big(A^{-1}\big(z_1, \dots, z_m, y'_{m+1}, \dots, y'_d\big)\big)\big|^p k(x, y'_{1,m}, z)\,\mathrm{d}y' = \int_{\mathbb{R}^{d-m}} \big|f\big(A^{-1}w(z, y'_{m+1,d})\big)\big|^p \bigg(\int_{\mathbb{R}^m} k(x, y'_{1,m}, z)\,\mathrm{d}y'_1 \dots \mathrm{d}y'_m\bigg)\,\mathrm{d}y'_{m+1} \dots \mathrm{d}y'_d,$$

and from the other hand (due to the symmetric properties of each $\mathcal{K}_{\kappa_j}$ and due to (2)(b) of Theorem 2.4)

$$\int_{\mathbb{R}^m} k(x, y'_{1,m}, z)\,\mathrm{d}y'_1 \dots \mathrm{d}y'_m = \int_{\mathbb{R}^m} \prod_{j=1}^m \big|\mathcal{K}_{\kappa_j}(x_{\alpha^j}, y'_j, z_j){y'_j}^{2\kappa_j}\big|\,\mathrm{d}y'_1 \dots \mathrm{d}y'_m \leqslant 4^m,$$

we claim that we can deduce from (4.3) that

$$4^{\frac{-mp}{q}} \prod_{j=1}^m \|\alpha^j\|^{-2\kappa_j} \int_{\mathbb{R}^d} \bigg|\int_{\mathbb{R}^m} f\big(A^{-1}v_{z,x,y}\big) \bigotimes_{j=1}^m \mathrm{d}\nu^{\mathbb{Z}_2,\kappa_j}_{x_{\alpha^j}, y_{\alpha^j}}(z_j)\bigg|^p \mathrm{d}\mu^W_\kappa(y) \leqslant 4^m \int_{\mathbb{R}^m \times \mathbb{R}^{d-m}} \big|f\big(A^{-1}w(z, y'_{m+1,d})\big)\big|^p \prod_{j=1}^m |z_j|^{2\kappa_j}\,\mathrm{d}z_1 \dots \mathrm{d}z_m \mathrm{d}y'_{m+1} \dots \mathrm{d}y'_d.$$

Thanks to the change of variables

$$z' = A^{-1}w(z, y'_{m+1,d}) = A^{-1}\big(z_1, \dots, z_m, y'_{m+1}, \dots, y'_d\big),$$

we are led to

$$4^{\frac{-mp}{q}}\prod_{j=1}^m\|\alpha^j\|^{-2\kappa_j}\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^m}f\big(A^{-1}v_{z,x,y}\big)\bigotimes_{j=1}^m\mathrm{d}\nu^{\mathbb{Z}_2,\kappa_j}_{x_{\alpha^j},y_{\alpha^j}}(z_j)\Big|^p\,\mathrm{d}\mu^W_\kappa(y)$$
$$\leqslant 4^m\int_{\mathbb{R}^d}\big|f(z')\big|^p\prod_{j=1}^m\Big(\frac{|\langle z',\alpha^j\rangle|}{\|\alpha^j\|}\Big)^{2\kappa_j}\mathrm{d}z',$$

that is to say

$$\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^m}f\big(A^{-1}v_{z,x,y}\big)\bigotimes_{j=1}^m\mathrm{d}\nu^{\mathbb{Z}_2,\kappa_j}_{x_{\alpha^j},y_{\alpha^j}}(z_j)\Big|^p\,\mathrm{d}\mu^W_\kappa(y)$$
$$\leqslant 4^{m+\frac{mp}{q}}\int_{\mathbb{R}^d}\big|f(z')\big|^p\,\mathrm{d}\mu^W_\kappa(z').$$

The theorem is proved. □

Of course the boundedness of the Dunkl translation gives us the following immediate consequence for the Dunkl convolution which is naturally defined by means of $\tau^{W,\kappa}_x$, that is to say

$$(f\underset{W,\kappa}{*}g)(x)=c^W_\kappa\int_{\mathbb{R}^d}f(y)\tau^{W,\kappa}_x g(-y)\,\mathrm{d}\mu^W_\kappa(y),\quad x\in\mathbb{R}^d.$$

Corollary 4.1. *Let $p,q,r\in[1,+\infty]$ satisfying*

$$\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}.$$

Then the map $(f,g)\mapsto f\underset{W,\kappa}{*}g$ initially defined on $L^2(\mu^W_\kappa)\times L^2(\mu^W_\kappa)$ extends to a bounded map from $L^p(\mu^W_\kappa)\times L^q(\mu^W_\kappa)$ to $L^r(\mu^W_\kappa)$ and we have in particular

$$\|f\underset{W,\kappa}{*}g\|_{W,\kappa,r}\leqslant c^W_\kappa 4^m\|f\|_{W,\kappa,p}\|g\|_{W,\kappa,q}.$$

Therefore we have for Dunkl analysis associated with a positive subsystem of orthogonal roots the same powerful tools that in the $\mathbb{Z}_2^d$ case, and the analysis in this setting is closely related to the one associated with $\mathbb{Z}_2^d$. We now turn to the study of the Dunkl maximal operator in this setting.

5. Dunkl maximal operator

Let us begin with a sharp estimate for the Dunkl translation of χ_{B_r} (B_r is the Euclidean ball of radius r centered at the origin) which will be a key result in our study of the Dunkl maximal operator. We set for every $x\in\mathbb{R}$ and every $r>0$

$$I(x,r)=]\max\{0;|x|-r\},|x|+r[.$$

Theorem 5.1. *For every $x\in\mathbb{R}^d$, every $y\in\mathbb{R}^d$ and every $r>0$ we have*

$$|\tau^{W,\kappa}_x(\chi_{B_r})(y)|\leqslant C\prod_{j=1}^m\frac{\mu^{\mathbb{Z}_2}_{\kappa_j}(]-r,r[)}{\mu^{\mathbb{Z}_2}_{\kappa_j}(I(x_{\alpha^j},r))},$$

where $C=C(m,\kappa)$ is a constant independent of x,y,r.

In order to prove this theorem, we first introduce a notation.

Notation. For $x, y \in \mathbb{R}$ we denote by $\nu_{x,y}^{\mathbb{Z}_2,\kappa,+}$ the measure given by

$$\mathrm{d}\nu_{x,y}^{\mathbb{Z}_2,\kappa,+}(z) = \begin{cases} \frac{1}{2}K_\kappa(|x|,|y|,|z|)(1-\rho_{x,y,z})\,\mathrm{d}\mu_\kappa^{\mathbb{Z}_2}(z) & \text{if } x, y \neq 0 \\ \mathrm{d}\delta_x(z) & \text{if } y = 0 \\ \mathrm{d}\delta_y(z) & \text{if } x = 0. \end{cases}$$

Let us point out that this measure is positive. Indeed, it is a simple consequence of the following observation

$$|z| \in \Big[\big||x|-|y|\big|, |x|+|y|\Big] \Longrightarrow |\rho_{x,y,z}| \leqslant 1.$$

We also need the following result which is due to Abdelkefi and Sifi (see [1]).

Lemma 5.1. *For every $x \in \mathbb{R}$, every $y \in \mathbb{R}$ and every $r > 0$ we have*

$$|\tau_x^{\mathbb{Z}_2,\kappa}(\chi_{]-r,r[})(y)| \leqslant C\frac{\mu_\kappa^{\mathbb{Z}_2}(]-r,r[)}{\mu_\kappa^{\mathbb{Z}_2}(I(x,r))},$$

where $C = C(\kappa)$ is a positive constant independent of x, y, r.

We can now give the proof of Theorem 5.1.

Proof. For every $x \in \mathbb{R}^d$, every $y \in \mathbb{R}^d$ and every $r > 0$ we write

$$\tau_x^{W,\kappa}(\chi_{B_r})(y) = \int_{\mathbb{R}^m} \big(\chi_{B_r} \circ A^{-1}\big)\big(z_1,\dots,z_m,(x+y)_{\beta^1},\dots,(x+y)_{\beta^{d-m}}\big) \bigotimes_{j=1}^m \mathrm{d}\nu_{x_{\alpha^j},y_{\alpha^j}}^{\mathbb{Z}_2,\kappa_j}(z_j).$$

Since the function χ_{B_r} is radial we can deduce that

$$\tau_x^{W,\kappa}(\chi_{B_r})(y) = \int_{\mathbb{R}^m} \chi_{B_r}\big(z_1,\dots,z_m,(x+y)_{\beta^1},\dots,(x+y)_{\beta^{d-m}}\big) \bigotimes_{j=1}^m \mathrm{d}\nu_{x_{\alpha^j},y_{\alpha^j}}^{\mathbb{Z}_2,\kappa_j}(z_j).$$

Thanks to (2)(b) of Theorem 2.4 we can apply the Fubini theorem to obtain

$$(5.1) \qquad \tau_x^{W,\kappa}(\chi_{B_r})(y) = \int_{\mathbb{R}^{m-1}} \left(\int_{\mathbb{R}} \chi_{B_r}(v_{z,x,y})\,\mathrm{d}\nu_{x_{\alpha^1},y_{\alpha^1}}^{\mathbb{Z}_2,\kappa_1}(z_1)\right) \bigotimes_{j=2}^m \mathrm{d}\nu_{x_{\alpha^j},y_{\alpha^j}}^{\mathbb{Z}_2,\kappa_j}(z_j),$$

where we have set as in the proof of Theorem 4.1

$$v_{z,x,y} = (z_1,\dots,z_m,(x+y)_{\beta^1},\dots,(x+y)_{\beta^{d-m}}).$$

If $x_{\alpha^1} = 0$ or if $y_{\alpha^1} = 0$ we obviously have

$$\tau_x^{W,\kappa}(\chi_{B_r})(y) = \int_{\mathbb{R}^{m-1}} \left(\int_{\mathbb{R}} \chi_{B_r}(v_{z,x,y})\,\mathrm{d}\nu_{x_{\alpha^1},y_{\alpha^1}}^{\mathbb{Z}_2,\kappa_1,+}(z_1)\right) \bigotimes_{j=2}^m \mathrm{d}\nu_{x_{\alpha^j},y_{\alpha^j}}^{\mathbb{Z}_2,\kappa_j}(z_j).$$

Otherwise, (5.1) can be reformulated as follows

$$(5.2) \quad \tau_x^{W,\kappa}(\chi_{B_r})(y) = \int_{\mathbb{R}^{m-1}} \left(\int_{\mathbb{R}} \chi_{B_r}(v_{z,x,y}) K_{\kappa_1}(|x_{\alpha^1}|,|y_{\alpha^1}|,|z_1|)\varrho(x_{\alpha^1},y_{\alpha^1},z_1)\,\mathrm{d}\mu_{\kappa_1}^{\mathbb{Z}_2}(z_1)\right) \bigotimes_{j=2}^m \mathrm{d}\nu_{x_{\alpha^j},y_{\alpha^j}}^{\mathbb{Z}_2,\kappa_j}(z_j).$$

Let us note that

$$z_1 \mapsto \rho_{z_1,x_{\alpha^1},y_{\alpha^1}} \quad \text{and} \quad z_1 \mapsto \rho_{z_1,y_{\alpha^1},x_{\alpha^1}}$$

are odd. Since the function $z_1 \mapsto \chi_{B_r}(v_{z,x,y})$ is even, we can deduce from (5.2) that

$$\tau_x^{W,\kappa}(\chi_{B_r})(y) = \int_{\mathbb{R}^{m-1}} \left(\int_{\mathbb{R}} \frac{1}{2} \chi_{B_r}(v_{z,x,y}) K_\kappa(|x_{\alpha^1}|, |y_{\alpha^1}|, |z_1|)(1-\rho_{x_{\alpha^1}, y_{\alpha^1}, z_1}) \, \mathrm{d}\mu_{\kappa_1}^{\mathbb{Z}_2}(z_1) \right) \bigotimes_{j=2}^{m} \mathrm{d}\nu_{x_{\alpha^j}, y_{\alpha^j}}^{\mathbb{Z}_2, \kappa_j}(z_j),$$

that is to say

$$\tau_x^{W,\kappa}(\chi_{B_r})(y) = \int_{\mathbb{R}^{m-1}} \left(\int_{\mathbb{R}} \chi_{B_r}(v_{z,x,y}) \, \mathrm{d}\nu_{x_{\alpha^1}, y_{\alpha^1}}^{\mathbb{Z}_2, \kappa_1, +}(z_1) \right) \bigotimes_{j=2}^{m} \mathrm{d}\nu_{x_{\alpha^j}, y_{\alpha^j}}^{\mathbb{Z}_2, \kappa_j}(z_j).$$

By similar arguments we are led to

$$\tau_x^{W,\kappa}(\chi_{B_r})(y) = \int_{\mathbb{R}^m} \chi_{B_r}(v_{z,x,y}) \bigotimes_{j=1}^{m} \mathrm{d}\nu_{x_{\alpha^j}, y_{\alpha^j}}^{\mathbb{Z}_2, \kappa_j, +}(z_j).$$

Since the measure is positive we claim that

$$|\tau_x^{W,\kappa}(\chi_{B_r})(y)| \leqslant \int_{\mathbb{R}^m} \chi_{Q_r}(v_{z,x,y}) \bigotimes_{j=1}^{m} \mathrm{d}\nu_{x_{\alpha^j}, y_{\alpha^j}}^{\mathbb{Z}_2, \kappa_j, +}(z_j),$$

where Q_r is the following cube $Q_r = \left\{ x \in \mathbb{R}^d : |x_j| < r, \ 1 \leqslant j \leqslant d \right\}$. Since we can separate the variables we get

$$|\tau_x^{W,\kappa}(\chi_{B_r})(y)| \leqslant \prod_{j=1}^{m} \int_{\mathbb{R}} \chi_{]-r,r[}(z_j) \, \mathrm{d}\nu_{x_{\alpha^j}, y_{\alpha^j}}^{\mathbb{Z}_2, \kappa_j, +}(z_j).$$

As the function $z_j \mapsto \chi_{]-r,r[}(z_j)$ is even (for every $1 \leqslant j \leqslant m$) we have

$$\int_{\mathbb{R}} \chi_{]-r,r[}(z_j) \, \mathrm{d}\nu_{x_{\alpha^j}, y_{\alpha^j}}^{\mathbb{Z}_2, \kappa_j, +}(z_j) = \int_{\mathbb{R}} \chi_{]-r,r[}(z_j) \, \mathrm{d}\nu_{x_{\alpha^j}, y_{\alpha^j}}^{\mathbb{Z}_2, \kappa_j}(z_j),$$

and so

$$\int_{\mathbb{R}} \chi_{]-r,r[}(z_j) \, \mathrm{d}\nu_{x_{\alpha^j}, y_{\alpha^j}}^{\mathbb{Z}_2, \kappa_j, +}(z_j) = \tau_{x_{\alpha^j}}^{\mathbb{Z}_2, \kappa_j}(\chi_{]-r,r[})(y_{\alpha^j}),$$

where we have used (2.3). Therefore we can assert that

$$|\tau_x^{W,\kappa}(\chi_{B_r})(y)| \leqslant \prod_{j=1}^{m} \tau_{x_{\alpha^j}}^{\mathbb{Z}_2, \kappa_j}(\chi_{]-r,r[})(y_{\alpha^j}).$$

We obtain the desired result by applying the one-dimensional inequality of Lemma 5.1. □

The estimate of Theorem 5.1 is of interest in order to study the Dunkl maximal operator M_κ^W which has been introduced in [25] for a general reflection group and which is defined for $f \in L^2(\mu_\kappa^W)$ by

$$M_\kappa^W f(x) = \sup_{r>0} \frac{1}{\mu_\kappa^W(B_r)} \left| \int_{\mathbb{R}^d} f(y) \tau_x^{W,\kappa}(\chi_{B_r})(-y) \, \mathrm{d}\mu_\kappa^W(y) \right|, \quad x \in \mathbb{R}^d.$$

The two main results for this operator are a scalar maximal theorem for a general reflection group (see [25]) and its extension to the vector-valued case for $W \simeq \mathbb{Z}_2^d$ (see [7]). We will prove a vector-valued theorem for M_κ^W when W is the reflection group associated with a positive subsystem of orthogonal roots, assuming that $m = d$. More precisely, the theorem we want to prove is the following one, where we denote by $\mathcal{M}_\kappa^W$ the Dunkl-type Fefferman-Stein operator given for a sequence $f = (f_n)_{n\geqslant 1}$ of measurable functions by

$$\mathcal{M}_\kappa^W f = \left(M_\kappa^W f_n\right)_{n\geqslant 1}.$$

Theorem 5.2. *Let W be the reflection group associated with a positive subsystem of orthogonal roots $\mathcal{R}_+ = \{\alpha^j; 1 \leqslant j \leqslant d\}$ and let $\kappa_1, \dots, \kappa_d$ be the positive values of the multiplicity function κ respectively associated with $\alpha^1, \dots, \alpha^d$. Let $f = (f_n)_{n\geqslant 1}$ be a sequence of measurable functions defined on $\mathbb{R}^d$.*

(1) *Let $1 < r < +\infty$. If $\|f\|_{\ell^r} \in L^1(\mu_\kappa^W)$, then for every $\lambda > 0$ we have*

$$\mu_\kappa^W\left(\left\{x \in \mathbb{R}^d : \|\mathcal{M}_\kappa^W f(x)\|_{\ell^r} > \lambda\right\}\right) \leqslant \frac{C}{\lambda}\big\| \|f\|_{\ell^r}\big\|_{W,\kappa,1},$$

where $C = C(d, \kappa, \alpha^1, \dots, \alpha^d, r)$ is a positive constant independent of $(f_n)_{n\geqslant 1}$ and λ.

(2) *Let $1 < r < +\infty$ and let $1 < p < +\infty$. If $\|f\|_{\ell^r} \in L^p(\mu_\kappa^W)$, then we have*

$$\big\| \|\mathcal{M}_\kappa^W f\|_{\ell^r}\big\|_{W,\kappa,p} \leqslant C\big\| \|f\|_{\ell^r}\big\|_{W,\kappa,p},$$

where $C = C(d, \kappa, \alpha^1, \dots, \alpha^d, p, r)$ is a positive constant independent of $(f_n)_{n\geqslant 1}$.

In order to prove this theorem, we will construct a more convenient operator which controls M_κ^W pointwise. The idea for the construction is to use the inequality of Theorem 5.1 (in order to bypass the lack of information concerning the structure of the Dunkl translation) which justifies the following definition.

Definition. Let M_κ^{W,R_α} be the weighted maximal operator defined by

$$M_\kappa^{W,R_\alpha} f(x) = \sup_{r>0} \frac{1}{\mu_{\overline{\kappa}^d}^{\mathbb{Z}_2^d}(R_\alpha(x,r))} \int_{\{y : \tilde{y}_\alpha \in R_\alpha(x,r)\}} |f(y)|\, \mathrm{d}\mu_\kappa^W(y),$$

where we have set $\tilde{y}_\alpha = (|y_{\alpha^1}|, \dots, |y_{\alpha^d}|)$ and

$$R_\alpha(x,r) = I(x_{\alpha^1}, r) \times \cdots \times I(x_{\alpha^d}, r).$$

With this definition in mind we can now state the following proposition.

Proposition 5.1. *For every $x \in \mathbb{R}^d$ we have the inequality*

$$M_\kappa^W f(x) \leqslant C M_\kappa^{W,R_\alpha} f(x), \tag{5.3}$$

where $C = C(d, \kappa, \alpha^1, \dots, \alpha^d)$ is a positive constant independent of f and x.

Proof. Let $x \in \mathbb{R}^d$ and let $r > 0$. In the proof of Theorem 5.1, we have proved that for every $y \in \mathbb{R}^d$

$$|\tau_x^{W,\kappa}(\chi_{B_r})(y)| \leqslant \prod_{j=1}^d \tau_{x_{\alpha^j}}^{\mathbb{Z}_2,\kappa_j}(\chi_{]-r,r[})(y_{\alpha^j}).$$

Due to (2)(a) of Theorem 2.4, we have the following observation

$$|y_{\alpha^j}| \notin I(x_{\alpha^j}, r) \Longrightarrow \tau_{x_{\alpha^j}}^{\mathbb{Z}_2,\kappa_j}(\chi_{]-r,r[})(y_{\alpha^j}) = 0.$$

Thus we claim that

$$\tau_x^{W,\kappa}(\chi_{B_r})(y) = 0 \quad \text{if} \quad \tilde{y}_\alpha \notin R_\alpha(x,r).$$

The above property and the result of Theorem 5.1 imply that

$$\left|\int_{\mathbb{R}^d} f(y)\tau_x^{W,\kappa}(\chi_{B_r})(-y)\,\mathrm{d}\mu_\kappa^W(y)\right| \leqslant C\int_{\{y:\tilde{y}_\alpha\in R_\alpha(x,r)\}}|f(y)|\prod_{j=1}^d \frac{\mu_{\kappa_j}^{\mathbb{Z}_2}(]-r,r[)}{\mu_{\kappa_j}^{\mathbb{Z}_2}(I(x_{\alpha^j},r))}\,\mathrm{d}\mu_\kappa^W(y),$$

that is to say

(5.4)

$$\left|\int_{\mathbb{R}^d} f(y)\tau_x^{W,\kappa}(\chi_{B_r})(-y)\,\mathrm{d}\mu_\kappa^W(y)\right| \leqslant \frac{C\mu_{\overline{\kappa}^d}^{\mathbb{Z}_2^d}(Q_r)}{\mu_{\overline{\kappa}^d}^{\mathbb{Z}_2^d}(R_\alpha(x,r))}\int_{\{y:\tilde{y}_\alpha\in R_\alpha(x,r)\}}|f(y)|\,\mathrm{d}\mu_\kappa^W(y).$$

Let us notice that $\mu_{\overline{\kappa}^d}^{\mathbb{Z}_2^d}(Q_r) = C\mu_\kappa^W(B_r)$ with

$$C = \frac{2^d(2\gamma_\kappa+d)}{\prod_{j=1}^d(2\kappa_j+1)}\left(\int_{S^{d-1}} h_\kappa^2(y)\,\mathrm{d}y\right)^{-1}.$$

Indeed, we have on one hand

$$\mu_{\overline{\kappa}^d}^{\mathbb{Z}_2^d}(Q_r) = \prod_{j=1}^d \mu_{\kappa_j}^{\mathbb{Z}_2}(]-r,r[) = 2^d\prod_{j=1}^d\Big(\frac{1}{2\kappa_j+1}\Big)r^{2\gamma_\kappa+d},$$

and on the other hand, changing to polar coordinates gives

$$\mu_\kappa^W(B_r) = \int_0^r u^{2\gamma_\kappa+d-1}\,\mathrm{d}u\int_{S^{d-1}} h_\kappa^2(y)\,\mathrm{d}y = \frac{1}{2\gamma_\kappa+d}\left(\int_{S^{d-1}} h_\kappa^2(y)\,\mathrm{d}y\right)r^{2\gamma_\kappa+d}.$$

Therefore we can deduce from (5.4) that

$$\frac{1}{\mu_\kappa^W(B_r)}\left|\int_{\mathbb{R}^d} f(y)\tau_x^{W,\kappa}(\chi_{B_r})(-y)\,\mathrm{d}\mu_\kappa^W(y)\right| \leqslant \frac{C}{\mu_{\overline{\kappa}^d}^{\mathbb{Z}_2^d}(R_\alpha(x,r))}\int_{\{y:\tilde{y}_\alpha\in R_\alpha(x,r)\}}|f(y)|\,\mathrm{d}\mu_\kappa^W(y)$$

from which the desired conclusion follows. □

Thanks to the previous proposition, it is sufficient to show the following vector-valued theorem for the operator M_κ^{W,R_α} in order to prove Theorem 5.2.

Theorem 5.3. *Let $f=(f_n)_{n\geqslant 1}$ be a sequence of measurable functions defined on $\mathbb{R}^d$ and let $\mathcal{M}_\kappa^{W,R_\alpha}$ be the operator given by $\mathcal{M}_\kappa^{W,R_\alpha}f = \big(M_\kappa^{W,R_\alpha}f_n\big)_{n\geqslant 1}$.*

(1) *Let $1<r<+\infty$. If $\|f\|_{\ell^r}\in L^1(\mu_\kappa^W)$, then for every $\lambda>0$ we have*

$$\mu_\kappa^W\Big(\Big\{x\in\mathbb{R}^d : \|\mathcal{M}_\kappa^{W,R_\alpha}f(x)\|_{\ell^r}>\lambda\Big\}\Big) \leqslant \frac{C}{\lambda}\big\|\,\|f\|_{\ell^r}\big\|_{W,\kappa,1},$$

where $C=C(d,\kappa,\alpha^1,\dots,\alpha^d,r)$ is a positive constant independent of $(f_n)_{n\geqslant 1}$ and λ.

(2) *Let $1<r<+\infty$ and let $1<p<+\infty$. If $\|f\|_{\ell^r}\in L^p(\mu_\kappa^W)$, then we have*

$$\big\|\,\|\mathcal{M}_\kappa^{W,R_\alpha}f\|_{\ell^r}\big\|_{W,\kappa,p} \leqslant C\big\|\,\|f\|_{\ell^r}\big\|_{W,\kappa,p},$$

where $C=C(d,\kappa,\alpha^1,\dots,\alpha^d,p,r)$ is a positive constant independent of $(f_n)_{n\geqslant 1}$.

Proof. Let us first notice that we can write

$$M_\kappa^{W,R_\alpha} f(x) = \prod_{j=1}^{d} \|\alpha^j\|^{2\kappa_j} M_{\overline{\kappa}^d}^{\mathbb{Z}_2^d,R}\big(f \circ A^{-1}\big)(Ax), \quad x \in \mathbb{R}^d, \tag{5.5}$$

where $M_{\overline{\kappa}^d}^{\mathbb{Z}_2^d,R}$ is the operator which has been defined in [7] (with a different notation) by

$$M_{\overline{\kappa}^d}^{\mathbb{Z}_2^d,R} f(x) = \sup_{r>0} \frac{1}{\mu_{\overline{\kappa}^d}^{\mathbb{Z}_2^d}(R(x,r))} \int_{\{y:\tilde{y}\in R(x,r)\}} |f(y)| \, \mathrm{d}\mu_{\overline{\kappa}^d}^{\mathbb{Z}_2^d}(y),$$

where we have set $\tilde{y} = (|y_1|, \ldots, |y_d|)$ and

$$R(x,r) = I(x_1,r) \times \cdots \times I(x_d,r).$$

Moreover, we point out that we have for every $g \in L^p(\mu_\kappa^W)$

$$\|g\|_{W,\kappa,p} = \Big(\prod_{j=1}^{d} \|\alpha^j\|^{2\kappa_j}\Big)^{\frac{1}{p}} \|g \circ A^{-1}\|_{\mathbb{Z}_2^d,\overline{\kappa}^d,p}. \tag{5.6}$$

Then, combining the arguments (5.5) and (5.6) together with the vector-valued theorem for $M_{\overline{\kappa}^d}^{\mathbb{Z}_2^d,R}$ (see Theorem 3.2 in [7]), we claim that the theorem is proved. □

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Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie, 4 place Jussieu, 75005 Paris, France

E-mail address: deleaval@math.jussieu.fr