The Ginzburg-Landau Functional with a Discontinuous and Rapidly Oscillating Pinning Term

Part II: the Non-Zero Degree Case

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ABSTRACT. We consider minimizers of a 2D Ginzburg-Landau energy with a discontinuous and rapidly oscillating pinning term, subject to a Dirichlet boundary condition of degree $d > 0$. The pinning term models an unbounded number of small impurities in a superconductor. We prove that for a strongly Type-II superconductor with impurities, the minimizers have exactly $d$ isolated zeros (vortices). These vortices are of degree 1 and pinned by the impurities. As in the standard case studied by Bethuel, Brezis, and Hélein in [BBH94], the macroscopic location of vortices is governed by vortex/vortex and vortex/boundary repelling effects. In some special cases, we prove that their macroscopic location tends to minimize the renormalized energy of Bethuel-Brezis-Hélein. In addition, impurities affect the microscopic location of vortices. Our techniques allow us to work with impurities having different sizes. In this situation, we prove that vortices are pinned by the largest impurities.

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1. Introduction

In this article, we let $\Omega \subset \mathbb{R}^2$ be a smooth, simply connected domain, and we let $a_\varepsilon : \Omega \to \{b, 1\}$, $b \in (0, 1)$ be a measurable function. We associate to $a_\varepsilon$ the pinned Ginzburg-Landau energy

\[ E_\varepsilon(u) = \frac{1}{2} \int_\Omega \left\{ |\nabla u(x)|^2 + \frac{1}{2\varepsilon^2} \left( a_\varepsilon(x)^2 - |u(x)|^2 \right)^2 \right\} \, dx. \]

Here, $u : \Omega \to \mathbb{C}$ is in the Sobolev space $H^1(\Omega, \mathbb{C})$, and $\varepsilon > 0$ is the inverse of the Ginzburg-Landau parameter. Our study takes place in the asymptotic $\varepsilon \to 0$.

Our goal is to consider a discontinuous and rapidly oscillating pinning term (the pinning term is $a_\varepsilon : \Omega \to \{b, 1\}$). Our pinning term is periodic with respect to a $\delta \times \delta$-grid with $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ (in some cases, we drop the periodic hypothesis).

We are interested in the minimization of (1.1) in $H^1(\Omega, \mathbb{C})$ subject to a Dirichlet boundary condition: we fix $g \in C^\infty(\partial \Omega, S^1)$, and thus the set of the test functions is $H^1_g := \{ u \in H^1(\Omega, \mathbb{C}) \mid \text{tr}_{\partial \Omega} u = g \}$.

The situation where $d = \text{deg}_{\partial \Omega}(g) = 0$ was studied in detail in [DMM11]. The non-zero degree case ($d = \text{deg}_{\partial \Omega}(g) > 0$) is the purpose of the present article. Recall that for $\Gamma \subset \mathbb{R}^2$ a Jordan curve and $g \in H^{1/2}(\Gamma, S^1)$, the degree (winding number) of $g$ is defined as

\[ \text{deg}_\Gamma(g) := \frac{1}{2\pi} \int_\Gamma g \wedge \nu \, d\tau. \]

Here "$\wedge$" stands for the vectorial product in $\mathbb{C}$; that is, $z_1 \wedge z_2 = \text{Im}(\overline{z_1} z_2)$, $z_1, z_2 \in \mathbb{C}$, $\nu$ is the direct unit tangent vector of $\Gamma$ ($\nu = \nu^\perp$ where $\nu$ is the outward normal unit vector of $\text{int}(\Gamma)$, the bounded open set whose boundary is $\Gamma$) and $\nu^\perp$ is the tangential derivative on $\Gamma$.

This energy is a simplification of the full Ginzburg-Landau energy (see equation (1.2) below) whose minimizers model the state of a superconductor. A superconductor is a material which, when cooled to near absolute zero, loses all electrical resistance. In addition, a superconductor exhibits the Meissner effect: when a magnetic field is applied, an opposed magnetic field appears in the material in order to avoid the penetration of the applied field in the bulk of the superconductor (the material is perfectly diamagnetic).

If the applied magnetic field is too large, then the superconductivity is destroyed in some areas of the material. According to the behavior of the superconductor submitted to a large magnetic field, two kinds of superconductors are defined: the Type-I and the Type-II superconductors. Type-I superconducting materials are those in which the transition between the superconducting state (absence of resistance and Meissner effect everywhere) and the normal state (an electrical resistance exists in the material and the magnetic field penetrates everywhere)
is abrupt. Type-II superconductors have an intermediate state (it is called the mixed state) where areas in normal state coexist with superconducting states (see [Ser99a, Ser99b, and SS07]). For a Type-II superconductor, the areas where the superconductivity is destroyed are called the vorticity defects.

The parameter $\varepsilon$ corresponds to a material parameter, which is small for Type-II superconductors (see [Tin96, SS07]). Because we treat the asymptotic $\varepsilon \to 0$, we are considering (strongly) Type-II superconductors.

The pinning term $a_\varepsilon : \Omega \to \{b, 1\}$ allows us to model a heterogenous superconductor (see [Kac09] or Introduction of [Dos10]). The domain $\Omega$ can be viewed as a cross-section of multi-filamentary wires with a large number of thin superconducting rods. While using a superconductor, one may observe an energy dissipation, which is a consequence of the motion of the vorticity defects (see [BS65]). Heterogenous superconductors allow us to decrease the energy dissipation by pinning the vorticity defects (see [Dev02]).

Physical informations which can be obtained with the simplification of the full Ginzburg-Landau energy are quantization and location of zeros of minimizers. Their zeros represent the centers of the vorticity defects. If $u_\varepsilon$ is a minimizer, then $\{ |u_\varepsilon| \leq b/2 \}$ are the vorticity defects. Here, the superconductor is a cylinder whose cross section is $\Omega$, and the vorticity defects (under some special conditions) take the form of small wires parallel to the superconductor ([Tin96, SS07]).

Before going further, let us summarize two previous works in related directions: [LM99] and [ASS01]. In these works, the role of the pinning term is identified: namely, its points of minimum attract the vorticity defects.

In [LM99], Lassoued and Mironescu consider the case where $a_\varepsilon \equiv a$. Here, the pinning term is

$$a = \begin{cases} 
    b & \text{in } \omega, \\
    1 & \text{in } \Omega \setminus \omega,
\end{cases}$$

where $0 < b < 1$, and $\omega$ is a smooth inner domain of $\Omega$. These authors prove that the vorticity defects are quantified by $\text{deg}_{\partial \Omega}(\varrho)$, localized in $\omega$, and that their position is governed by a renormalized energy (in the spirit of [BBH94]).

In [ASS01], Aftalion, Sandier and Serfaty consider a smooth and $\varepsilon$-dependent pinning term $a_\varepsilon$. Their study allows us to consider the case where the pinning term has fast oscillations: it is a (rapidly oscillating) perturbation of a fixed smooth function $\tilde{b} : \Omega \to [b, 1]$ such that $a_\varepsilon \geq \tilde{b}$.

In contrast with [LM99] (and with this article), [ASS01] is dedicated to the study of a full Ginzburg-Landau energy $GL_\varepsilon$ with the pinning term $a_\varepsilon$:

$$(1.2) \quad GL_\varepsilon(u, A) = \frac{1}{2} \int_{\Omega} \left\{ |(\nabla - iA)u|^2 + |\text{curl } A - h_{\text{ex}}|^2 + \frac{1}{2\varepsilon^2} (a_\varepsilon^2 - |u|^2)^2 \right\}.$$

Here, we denoted by $A : \Omega \to \mathbb{R}^2$ the electromagnetic vector potential of the induced field, and by $h_{\text{ex}} = h_{\text{ex}}(\varepsilon) \gg 1$ the intensity of the applied magnetic field (see [SS07] for more details).
The authors of [ASS01] considered the following hypotheses on \( a_\varepsilon, b \):

- \( |\nabla a_\varepsilon| \leq Ch_\varepsilon \)
- There is \( \sigma_\varepsilon \in \mathbb{R} \) such that \( \sigma_\varepsilon = o((\ln |\ln \varepsilon|)^{-1/2}) \); and for all \( x \in \Omega \), we have \( \min_{B(x, \sigma_\varepsilon)} \{a_\varepsilon - b\} = 0 \).

Here, \( B(x, r) \) is the open ball centered at \( x \in \mathbb{R}^2 \) with radius \( r > 0 \).

In the study of the full Ginzburg-Landau functional without pinning term \( GL_0^\varepsilon \) (\( GL_0^\varepsilon \) is obtained from (1.2) by taking \( a_\varepsilon \equiv 1 \)), the vorticity defects appear for a large applied magnetic field. They are characterized by two facts: (1) the presence of isolated zeros \( x_i \) of a map \( u \) with a non-zero degree on small circles centered in \( x_i \); and (2) the existence of a magnetic field inside the domain (\( \text{curl}(A) \approx h_\varepsilon \) inside small discs). The nature of the superconductivity ensures that both facts appear together. Assume that the intensity of the applied field \( h_\varepsilon \) depends on \( 0 < \varepsilon < 1 \), and that \( h_\varepsilon/|\ln \varepsilon| \to \Lambda \in \mathbb{R}^+ \). For the full Ginzburg-Landau energy without pinning term \( GL_0^\varepsilon \), it is well known (see, e.g., [SS07]) that there is an inner domain \( \omega_\Lambda \) (non-decreasing with respect to \( \Lambda \)) such that, when \( \varepsilon \to 0 \), the vorticity defects are “uniformly located” by \( \omega_\Lambda \) (in this situation, the number of vortices is unbounded).

In [ASS01] (a study of a full Ginzburg-Landau functional with a pinning term), the authors prove the existence of \( \omega_\Lambda \), an inner set of \( \Omega \), where the penetration of the magnetic field is located. In contrast with the situation where there is no pinning term, the presence of \( a_\varepsilon \) ensures that, in general, the vortices are not uniformly located in \( \omega_\Lambda \). Although in the proofs of the main results of [ASS01] the minimum points of \( \tilde{b} \) seem to play the role of a pinning site, this fact is not proved. The authors expect that the most favorable pinning sites should be close to the minima of \( \tilde{b} \); thus \( \omega_\Lambda \) should be close to the points of minimum of \( \tilde{b} \).

One of our goals is to prove that the minimum points of a rapidly oscillating and discontinuous pinning term attract the vorticity defects. Below, we state a simplified version of our main result. A more precise statement is in Theorem 1.5.

**Theorem 1.1.** Assume that \( \omega_\varepsilon = \{a_\varepsilon = b\} \) is a periodic set whose connected components (the inclusions) are diluted (i.e., the inclusions are small and their inter-distance is much larger than their size) as represented in Figure 1.1.

Assume also that \( \varepsilon \) is much smaller than the size of the inclusions (Hypothesis (1.3)) and that \( \deg_{\Omega}(g) = d > 0 \).

Then, for small \( \varepsilon \):

- A minimizer \( u_\varepsilon \) of \( E_\varepsilon \) has exactly \( d \) zeros, and all of these zeros are of degree 1; thus \( u_\varepsilon \) has exactly \( d \) vortices of degree 1. [Quantization]
- The vorticity defects are trapped by the inclusions: \( \{|u_\varepsilon| \leq b/2\} \subset \omega_\varepsilon \). [Pinning by the Inclusions]
- When \( \varepsilon \to 0 \), the vortices tend to minimize the renormalized energy of Bethuel-Brezis-Hélein. [Macroscopic Location]
- When \( \varepsilon \to 0 \), the location of a vortex inside an inclusion tends to depend only on \( b \) and on the geometry of the inclusion. [Microscopic Location]
G-L Energy with a Discontinuous and Rapidly Oscillating Pinning Term

We now construct our (periodic) pinning term \( a_\varepsilon \).

**Construction 1.2 (The periodic pinning term).** Let
- \( \delta = \delta(\varepsilon) \in (0, 1) \), \( \lambda = \lambda(\varepsilon) \in (0, 1) \);
- \( Y = (-\frac{1}{2}, \frac{1}{2})^2 \) be the unit cell;
- \( \omega \) be a smooth, bounded, and simply connected open set such that \((0, 0) \in \omega \) and \( \bar{\omega} \subset Y \).

For \( k, \ell \in \mathbb{Z} \), we denote the following:

\[
Y_{k,\ell}^\delta := (\delta k, \delta \ell) + \delta \cdot Y, \quad \Omega_{\delta}^{\text{incl}} = \bigcup_{Y_{k,\ell}^\delta \subset \Omega} Y_{k,\ell}^\delta, \quad \omega^{\lambda} = \lambda \cdot \omega, \\
\omega_{\text{per}}^{\lambda} = \bigcup_{(k, \ell) \in \mathbb{Z}^2} \{(k, \ell) + \omega^{\lambda}\}, \quad \omega_\varepsilon = \bigcup_{Y_{k,\ell}^\delta \subset \Omega} \{(\delta k, \delta \ell) + \delta \cdot \omega^{\lambda}\}.
\]

For \( b \in (0, 1) \), we define

\[
a^{\lambda} : \mathbb{R}^2 \to \{b, 1\}, \quad a_\varepsilon : \mathbb{R}^2 \to \{b, 1\}, \\
x \to \begin{cases} b & \text{if } x \in \omega_{\text{per}}^{\lambda} \\ 1 & \text{otherwise} \end{cases}, \quad x \to \begin{cases} b & \text{if } x \in \omega_\varepsilon \\ 1 & \text{otherwise} \end{cases}.
\]

The function \( a^{\lambda} \) is a dilatation by \( \delta^{-1} \) of \( a_\varepsilon \) in \( \delta^{-1} \cdot \Omega_{\delta}^{\text{incl}} \), and extended by \( Y \) periodicity outside \( \delta^{-1} \cdot \Omega_{\delta}^{\text{incl}} \).

The values of the periodic pinning term are represented in Figure 1.1. The connected components of \( \{a_\varepsilon = b\} = \omega_\varepsilon \) are called inclusions or impurities.

In the rest of this article, \( \lambda = \lambda(\varepsilon) \) and \( \delta = \delta(\varepsilon) \) are functions of \( \varepsilon \). We assume that \( \delta \to 0 \) as \( \varepsilon \to 0 \). In addition, we assume that either \( \lambda \equiv 1 \) or \( \lambda \to 0 \) as \( \varepsilon \to 0 \). The case \( \lambda \to 0 \) is the diluted case.

We make the (technical) assumption on the size of inclusions:

\[
\lim_{\varepsilon} \frac{|\ln(\lambda \delta)|^3}{|\ln \varepsilon|} = 0. 
\]

**Remark 1.3.**
- Hypothesis (1.3) is slightly more restrictive than asking that \( \lambda \delta / \varepsilon^\alpha \to +\infty \) for all \( \alpha \in (0, 1) \). It is technical; a more natural hypothesis should be \( \lambda \delta / \varepsilon \to +\infty \) or \( \lambda \delta / \varepsilon^\alpha \to +\infty \) for some \( \alpha \in (0, 1) \), or \( \ln(\lambda \delta) / \ln \varepsilon \to 0 \).
- In [ASS01] and in the situation where we have a bounded number of zeros (the applied magnetic field is not too large), the smooth pinning term \( a_\varepsilon^0 \) satisfies the condition \( |\nabla a_\varepsilon^0| \leq C |\ln \varepsilon| \). In order to compare this assumption with (1.3), we may consider a regularization of our pinning term by a mollifier \( \rho_t(x) = t^{-2} \rho(x/t) \). A suitable scale \( t \) to have a complete view of the variations of \( a_\varepsilon^0 \) is \( t = \lambda \delta \). Thus, \( |\nabla(\rho_\lambda \delta * a_\varepsilon)| \) is of order of \( 1 / (\lambda \delta) \). Consequently, the condition (1.3) allows us to
(a) The pinning term is periodic on a $\delta \times \delta$-grid.

(b) The parameter $\lambda$ controls the size of an inclusion in the cell.

\textbf{Figure 1.1.} The periodic pinning term
consider a more rapidly oscillating pinning term than in [ASS01]. Indeed, we have \( \ln |\nabla a_0^\varepsilon| \leq \ln |\ln \varepsilon| \); on the other hand, (1.3) is equivalent to \( \ln (\rho \lambda \delta * a_\varepsilon) \leq \ln (\lambda \delta) \leq o(\ln \varepsilon^{1/3}) \).

The goal of this article is to study the minimizers of

\[
E_\varepsilon(u) = \frac{1}{2} \int_{\Omega} \left\{ |\nabla u|^2 + \frac{1}{2\varepsilon^2} (a_\varepsilon^2 - |u|^2)^2 \right\}, \quad u \in H^1_\varepsilon
\]

in the asymptotic \( \varepsilon \to 0 \). A standard method (initiated in [LM99]) consists in decoupling \( E_\varepsilon \) into a sum of two functionals. The key tool in this method is \( U_\varepsilon \), the unique global minimizer of \( E_\varepsilon \) in \( H^1_\varepsilon \) (see [LM99]). Clearly, \( U_\varepsilon \) satisfies

\[
\begin{aligned}
-\Delta U_\varepsilon &= \frac{1}{\varepsilon^2} U_\varepsilon (a_\varepsilon^2 - U_\varepsilon^2) \quad \text{in } \Omega, \\
U_\varepsilon &= 1 \quad \text{on } \partial \Omega.
\end{aligned}
\] (1.4)

From the uniqueness of \( U_\varepsilon \), by construction of a test function, it is easy to get that \( b \leq U_\varepsilon \leq 1 \).

This special solution may be seen as a regularization of \( a_\varepsilon \). For example, one may easily prove that \( U_\varepsilon \) is exponentially close to \( a_\varepsilon \) far away from \( \partial \omega_\varepsilon \) (a more complete description of \( U_\varepsilon \) is done in Appendix D.1). Namely, we have the following result.

**Proposition 1.4.** There are \( C, \alpha > 0 \) independent of \( \varepsilon, R > 0 \) such that

\[
|a_\varepsilon - U_\varepsilon| \leq Ce^{-\alpha R/\varepsilon} \quad \text{in } V_R := \{ x \in \Omega \mid \text{dist}(x, \partial \omega_\varepsilon) \geq R \},
\] (1.5)

\[
|\nabla U_\varepsilon| \leq \frac{Ce^{-\alpha R/\varepsilon}}{\varepsilon} \quad \text{in } W_R := \{ x \in \Omega \mid \text{dist}(x, \partial \omega_\varepsilon), \text{dist}(x, \partial \Omega) \geq R \}.
\] (1.6)

A similar result was proved in [DM11, Proposition 2]. The above proposition follows by the same arguments.

As in [LM99], we define

\[
F_\varepsilon(v) = \frac{1}{2} \int_{\Omega} \left\{ U_\varepsilon^2 |\nabla v|^2 + \frac{1}{2\varepsilon^2} U_\varepsilon^4 (1 - |v|^2)^2 \right\}.
\] (1.7)

Then, we have for all \( v \in H^1_\varepsilon \) (see [LM99]) that \( E_\varepsilon(U_\varepsilon v) = E_\varepsilon(U_\varepsilon) + F_\varepsilon(v) \).

Therefore, \( u_\varepsilon \) is a minimizer of \( E_\varepsilon \) in \( H^1_\varepsilon \) if and only if \( v_\varepsilon = u_\varepsilon / U_\varepsilon \) is a minimizer of \( F_\varepsilon \) in \( H^1_\varepsilon \). Because \( 1 \geq U_\varepsilon \geq b \), the study of a minimizer \( u_\varepsilon = U_\varepsilon v_\varepsilon \) of \( E_\varepsilon \) in \( H^1_\varepsilon \) (location of zeros and asymptotics) can be performed by combining the asymptotic of \( U_\varepsilon \) with the one of \( v_\varepsilon \).

Our main result is the following.
**Theorem 1.5.** Assume that \(\lambda, \delta\) satisfy (1.3) and that \(\lambda \to 0\).

Quantization. There are \(\varepsilon_0 > 0, c > 0,\) and \(\eta_0 > 0\) such that for \(0 < \varepsilon < \varepsilon_0\):

1. \(v_\varepsilon\) has exactly \(d\) zeros \(x_1^\varepsilon, \ldots, x_d^\varepsilon\);
2. \(B(x_i^\varepsilon, c\lambda\delta) \subset \omega_\varepsilon\) for \(i = 1, \ldots, d\) [pinning by inclusions];
3. For \(\rho = \rho(\varepsilon) \downarrow 0\) such that \(\frac{|\ln \rho|}{|\ln \varepsilon|} \to 0\), there is \(C > 0\) independent of \(\varepsilon\) satisfying
   \[|v_\varepsilon| \geq 1 - C \sqrt{\frac{|\ln \rho|}{|\ln \varepsilon|}} \quad \text{in } \Omega \setminus \bigcup B(x_i^\varepsilon, \rho);\]
4. For \(\varepsilon < \varepsilon_0\),
   - there are two repulsive effects: \(|x_i^\varepsilon - x_j^\varepsilon| \geq \eta_0\) for \(i \neq j\), and, also,
   - \(\deg_{B(x_i^\varepsilon, \delta)}(v_\varepsilon) = 1\).

**Location:**
- The macroscopic location of the zeros tends to minimize the renormalized energy of Bethuel-Brezis-Helein, that is,
  \[W_{\theta} : \{\{x_1, \ldots, x_d\} \subset \Omega \mid x_i \neq x_j \text{ for } i \neq j\} \to \mathbb{R},\]
  defined in [BBH94, Chapter I, equation (47)]:
  \[\limsup_{\varepsilon \to 0} W_{\theta}(x_1^\varepsilon, \ldots, x_d^\varepsilon) = \min_{a_1, \ldots, a_d \in \Omega} W_{\theta}(a_1, \ldots, a_d).\]
- The microscopic location of the zeros inside \(\omega_\varepsilon\) tends to depend only on \(\omega\) and \(b\):
  - Since \(x_i^\varepsilon \in \omega_\varepsilon\), we have \(x_i^\varepsilon = (k_i^\varepsilon \delta, \ell_i^\varepsilon \delta) + \lambda \delta y_i^\varepsilon\) with \(k_i^\varepsilon, \ell_i^\varepsilon \in \mathbb{Z}\) and \(y_i^\varepsilon \in \omega;\)
  - For \(\varepsilon_n \downarrow 0\) such that \(y_i^{\varepsilon_n} \to \hat{a}_i\), we have \(\hat{a}_i \in \omega\) which minimizes a renormalized energy \(\tilde{W}_{\theta} : \omega \to \mathbb{R}\) (given in [DM11, equation (90)]) which depends only on \(\omega\) and \(b \in (0, 1)\).

**Remark 1.6.**
1. The renormalized energy defined in [BBH94]
   \[W_{\theta} : \{\{x_1, \ldots, x_d\} \subset \Omega \mid x_i \neq x_j \text{ for } i \neq j\} \to \mathbb{R}\]
   governs the location of the zeros in the situation where \(a_\varepsilon \equiv 1\) (homogeneous case): the zeros tend to minimize \(W_{\theta}\). In [BBH94, Chapter I], the authors define a renormalized energy in a more general setting:
   \[W_{\theta}^{BBH} : \left\{\{(x_1, d_1), \ldots, (x_N, d_N)\} \mid x_i \in \Omega, x_i \neq x_j \text{ for } i \neq j, \quad d_i \in \mathbb{Z} \text{ is such that } \sum_{i=1}^{N} d_i = d \right\} \to \mathbb{R}\]
   Here \(W_{\theta}(x_1, \ldots, x_d) = W_{\theta}^{BBH}(\{(x_1, 1), \ldots, (x_d, 1)\})\); that is, in this article we will consider only the renormalized energy with the degrees \(d_i\) equal to 1, and thus we do not specify the degrees in its notation.
(2) From smoothness of $W_g$ (see [BBH94] and [CM96]), the Location section of Theorem 1.5 implies that, up to passing to a subsequence, the zeros converge to a minimizer of $W_g$.

(3) This macroscopic location is strongly correlated with the Dirichlet boundary condition $g \in C^\infty(\partial\Omega, S^1)$ (because $W_g$ depends on $g$).

(4) The result about the macroscopic position of the zeros in the case of the periodic and diluted pinning term may be summed up as follows: the macroscopic position of the zeros tends to be the same as in the homogenous case ($\alpha_\varepsilon \equiv 1$).

(5) The asymptotic microscopic location of the zeros (positioned inside an inclusion) is independent of the boundary condition. For example, in the situation $\omega = B(0, r_0)$ (i.e., where the inclusions are discs), this asymptotic location should be the center of the inclusion. This fact is not proved yet.

(6) In Assertion (4) of the Quantization section,

$$\deg_{\partial B(x_\varepsilon^i, \delta)}(v_\varepsilon) \overset{\text{def}}{=} \deg_{\partial B(x_\varepsilon^i, \delta)}(\frac{v_\varepsilon}{|v_\varepsilon|}).$$

(7) It is obvious that Theorem 1.5 implies Theorem 1.1.

2. Main Results

We present in this section several extensions of the above result, dropping either the dilution of the inclusion ($\lambda \equiv 1$ instead of $\lambda \to 0$) or the periodic structure. The main results of this section are obtained under the condition that $\lambda \delta$ satisfies (1.3).

Our main results are expressed in four theorems:

- Theorem 2.2 gives information on the zeros of minimizers $u_\varepsilon, v_\varepsilon$ (quantization and location).
- Theorem 2.5 establishes the asymptotic behavior of $v_\varepsilon$.
- Theorem 2.6 establishes, under the additional hypothesis $\lambda \to 0$, that the microscopic position of the zeros is independent of the boundary condition $g$.
- Theorem 2.7 gives an expansion of $F_\varepsilon(v_\varepsilon)$.

The techniques developed in this paper allow us to consider either the case $\lambda \to 0$ or the case $\lambda \equiv 1$. The results in the diluted case ($\lambda \to 0$) are more precise. One may drop the periodic structure for the pinning term and consider impurities (the connected components of $\omega_\varepsilon = \{a_\varepsilon = b\}$) with different sizes (adding the hypothesis $\lambda \to 0$).

More precisely, we may consider the pinning term defined as follows.
Construction 2.1 (The general diluted pinning term).

- Fix $P \in \mathbb{N}^*$, $j \in \{1, \ldots, P\}$, and $1 > \varepsilon > 0$. We consider $M^\varepsilon_j \in \mathbb{N}$ and
  
  $$M^\varepsilon_j = \begin{cases} \emptyset & \text{if } M^\varepsilon_j = 0, \\ \{1, \ldots, M^\varepsilon_j\} & \text{if } M^\varepsilon_j \in \mathbb{N}. \end{cases}$$

- The sets $M^\varepsilon_j$ are such that, for sufficiently small $\varepsilon$, one may fix $y^\varepsilon_{i,j} \in \Omega$ such that for $(i,j) \neq (i',j')$, $i \in M^\varepsilon_j$, $i' \in M^\varepsilon_{j'}$ we have
  
  \begin{equation}
  |y^\varepsilon_{i,j} - y^\varepsilon_{i',j'}| \geq \delta^j + \delta^j' \text{ and } \text{dist}(y^\varepsilon_{i,j}, \partial \Omega) \geq \delta^j.
  \end{equation}

  We denote $\hat{M}^\varepsilon_j := \{y^\varepsilon_{i,j} | i \in M^\varepsilon_j\}$.

  For the sake of simplicity, we assume that there is $\eta > 0$ such that for small $\varepsilon$, we have $M^\varepsilon_1 \geq d = \text{deg}(\partial \Omega(g))$ and

  \begin{equation}
  \min \{ \min_{i=1,\ldots,d} \text{dist}(y^\varepsilon_{i,1}, \partial \Omega), \min_{i \neq i'} \{ \min_{j=1,\ldots,d} |y^\varepsilon_{i,j} - y^\varepsilon_{i',j}| \} \} \geq \eta.
  \end{equation}

- We now define the domain which models the impurities:

  $$\omega^\varepsilon = \bigcup_{j=1}^P \bigcup_{i \in \hat{M}^\varepsilon_j} \{y^\varepsilon_{i,j} + \delta^j \cdot \omega^\lambda\}, \quad \omega^\lambda = \lambda \cdot \omega.$$

The general diluted pinning term is

$$a^\varepsilon : \mathbb{R}^2 \to \{b, 1\}, \quad x \mapsto \begin{cases} 1 & \text{if } x \notin \omega^\varepsilon, \\ b & \text{if } x \in \omega^\varepsilon. \end{cases}$$

The values of the pinning term are represented in Figure 2.1.

Our main results are the following.

Theorem 2.2. Assume that $\lambda, \delta$ satisfy (1.3); also, if the pinning term is not periodic (represented in Figure 2.1), we assume that $\lambda \to 0$.

Then there is $\varepsilon_0 > 0$ such that:

1. For $0 < \varepsilon < \varepsilon_0$, $v^\varepsilon$ has exactly $d$ zeros $x^\varepsilon_1, \ldots, x^\varepsilon_d$;
2. There are $c > 0$ and $\eta_0 > 0$ such that for $\varepsilon < \varepsilon_0$, $B(x^\varepsilon_i, c \lambda \delta) \subset \omega^\varepsilon$ and

   $$\min_{i \neq i'} \{ \min_{j=1,\ldots,d} |x^\varepsilon_j - x^\varepsilon_{i'}|, \text{dist}(x^\varepsilon_j, \partial \Omega) \} \geq \eta_0.$$

   In particular, if the pinning term is not periodic, then the zeros are trapped by the largest inclusions (those of size $\lambda \delta$).

3. For $\rho = \rho(\varepsilon) \downarrow 0$ such that $|\ln \rho| / |\ln \varepsilon| \to 0$, we have for $\varepsilon < \varepsilon_0$ that

   $$|v^\varepsilon| \geq 1 - C \sqrt{\frac{|\ln \rho|}{|\ln \varepsilon|}} \text{ in } \Omega \setminus \bigcup B(x^\varepsilon_i, \rho).$$

   Here, $C$ is independent of $\varepsilon$. 

For $\varepsilon < \varepsilon_0$, $\deg_{\partial B(0, \lambda \delta)}(v_\varepsilon) = 1$.

**Remark 2.3.** Hypothesis (2.2) is used to simplify the statements. Without this hypothesis, some of the results are subject to technical considerations on $\delta, \lambda, b, P, \ldots$. For example, if we consider the pinning term $a_\varepsilon$ defined in $\Omega = B(0, 2)$ by

$$a_\varepsilon : B(0, 2) \rightarrow \{b, 1\},$$

$$x \mapsto \begin{cases} b & \text{if } x \in B(0, \lambda \delta) \cup B(1, \lambda \delta^P), \\ 1 & \text{otherwise,} \end{cases}$$

and $g \in C^\infty(\partial \Omega, S^1)$ such that $\deg_{\partial \Omega}(g) = 2$, then Hypothesis (2.2) is not satisfied. In this situation, we may prove that, for sufficiently small $\varepsilon$, the minimizer $v_\varepsilon$ has exactly two zeros; and if $2|\ln \lambda| + [3 - P + b^2(P - 1)]|\ln \delta| \rightarrow -\infty$ (respectively, $+\infty$), then the zeros are in $B(0, \lambda \delta)$ (respectively, there is one zero inside $B(0, \lambda \delta)$ and one zero inside $B(1, \lambda \delta^P)$).

Roughly speaking, the previous condition expresses that if $\lambda \delta$ is sufficiently large in comparison with $\lambda \delta^P$ (depending on $b$), then it is cheaper (in terms of energy) to have two vortices in the disc $B(0, \lambda \delta)$. Otherwise, the repulsion cost between both vortices is too large, and then the configuration with one vortex per inclusion is energetically favorable.

**Remark 2.4.** The fact that the vortices are trapped by the largest inclusions is a direct consequence of the dilution of the inclusions. The dilution of the inclusion is related to two properties: namely, small inclusions and inter-distance between two inclusions much larger than the sizes of each inclusion.
In order to illustrate this remark, we may consider the following pinning term defined in $\Omega = B(0, 1)$ (represented in Figure 2.2):

$$a_\varepsilon \begin{cases} a_\varepsilon \text{ is periodic in } \{ (x_1, x_2) \in \Omega \mid x_1 > 0 \} \text{ with respect to a } \\
\delta^2 \times \delta^2\text{-grid without dilution; that is, } \lambda = 1, \\
a_\varepsilon(x_1, x_2) = \begin{cases} b & \text{if } (x_1, x_2) \in B((-\frac{\varepsilon}{2}, 0), \delta), \\
1 & \text{if } (x_1, x_2) \notin B((-\frac{\varepsilon}{2}, 0), \delta) \text{ and } x_1 \leq 0. \end{cases} \end{cases}$$

In $\{(x_1, x_2) \in \Omega \mid x_1 > 0\}$, the inclusions are discs with a sufficiently large radius to occupy a large part of the $\delta^2 \times \delta^2\text{-periodic cells.}$

We fix $g \in C^\infty(\partial \Omega, S^1)$ such that $\text{deg}_{\partial \Omega}(g) = 1$. By adapting the arguments of this article, we may prove that for small $\varepsilon$, a minimizer $u_\varepsilon$ of $E_\varepsilon$ (with the pinning term represented in Figure 2.2) has a unique zero. Moreover, if the radius of the discs in the $\delta^2 \times \delta^2\text{-periodic array is sufficiently large (in order for the discs to occupy a large part of the periodic cells), then for small } \varepsilon, \text{ this unique zero is trapped by the smallest inclusions.}$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The dilution is necessary; the unique zero is in the smallest inclusions.}
\end{figure}

**Theorem 2.5.** Assume that $\lambda, \delta$ satisfy (1.3); moreover, if the pinning term is not periodic (represented in Figure 2.1), then we assume also that $\lambda \to 0$. 
Let $\varepsilon_n \to 0$, up to a subsequence; we therefore have the existence of $a_1, \ldots, a_d \in \Omega$, $d$ distinct points such that $x_i^{\varepsilon_n} \to a_i$ and

$$|v_{\varepsilon_{n}}| - 1 \text{ and } v_{\varepsilon_{n}} - v_{*} \text{ in } H^{1}_{\text{loc}}(\Omega \setminus \{a_1, \ldots, a_d\}, \mathbb{S}^1),$$

where $v_{*}$ solves

$$\begin{cases}
-\text{div}(A\nabla v_{*}) = (A\nabla v_{*} \cdot \nabla v_{*})v_{*} & \text{ in } \Omega \setminus \{a_1, \ldots, a_d\}, \\
v_{*} = g & \text{ on } \partial \Omega.
\end{cases}$$

Here, $A$ is the homogenized matrix of $a^2(\cdot/\delta) \text{Id}_{\mathbb{R}^2}$ if $\lambda = 1$ and $A = \text{Id}_{\mathbb{R}^2}$ if $\lambda = 0$.

In addition, for each $M > 0$, we have that $v_{\varepsilon,i}(\cdot) = v_{\varepsilon}(x_i^\varepsilon + (\varepsilon/\delta)\cdot)$ converges, up to a subsequence, in $C^1(B(0,M))$ to $f(|x|)(x/|x|)g_0$, where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is the universal function defined in [Mir96b], and $g_0 \in \mathbb{R}$.

**Theorem 2.6.** Assume, in addition to the hypotheses of Theorem 2.5, that $\lambda \to 0$.

Let $|x| = [(x_1, x_2)] = [(x_1), (x_2)] \in \mathbb{Z}^2$ be the vectorial integer part of the point $x \in \mathbb{R}^2$.

For $x_i^\varepsilon$ a zero of $v_{\varepsilon}$, let $y_i^\varepsilon = x_i^\varepsilon/\delta - [x_i^\varepsilon/\delta] \in \omega$. Then, as $\varepsilon \to 0$, up to passing to a subsequence, we have $y_i^\varepsilon \to \hat{a}_i \in \omega$. Here, $\hat{a}_i$ minimizes a renormalized energy $\hat{W}_1 : \omega \to \mathbb{R}$ (given in [DM11, equation (90)]) which depends only on $\omega$ and $b$. In particular, $\hat{a}_i$ is independent of the boundary condition $g$.

**Theorem 2.7.** Assume that $\lambda, \delta$ satisfy (1.3); moreover, if the pinning term is not periodic (represented in Figure 2.1), we assume that $\lambda \to 0$.

We then have the expansion $F_\varepsilon(v_{\varepsilon}) = J_{\varepsilon} + db^2(\pi \ln b + \gamma) + o_{\varepsilon}(1)$, where $J_{\varepsilon}$ is defined in (3.6), and $\gamma > 0$ is the universal constant defined in [BBH94, Lemma IX.1].

**Remark 2.8.** All these results hold if we replace the discontinuous pinning term by a regularization with a mollifier. A suitable scale should be much smaller than the size of the inclusions; for example, we may consider the scale $(\lambda \delta)^p+2$.

The rest of this article is divided into two parts. In Section 3, we consider two auxiliary minimization problems for weighted Dirichlet functionals associated to $\mathbb{S}^1$-valued maps. Then, in Section 4, we devote attention to the proofs of Theorems 1.5, 2.2, 2.5, 2.6, and 2.7. The main tool here is an $\eta$-ellipticity result (Lemma 4.2). This lemma reduces (under the assumption that $\lambda, \delta$ satisfy (1.3)) the study of $F_\varepsilon$ to the one of the auxiliary problems considered in Section 3.

3. **Shrinking Holes for Weighted Dirichlet Functionals**

This section is devoted to the study of two minimization problems, and is divided into three subsections.

The first and second subsections relate to minimizations of weighted Dirichlet functionals among $\mathbb{S}^1$-valued maps. In both subsections, the considered weights
are the more general ones: \( \alpha \in L^\infty(\mathbb{R}^2, [b^2, 1]) \). The third subsection deals with the weight \( \alpha = U'_\varepsilon \) in the situation where \( U_\varepsilon \) is the minimizer of \( E_\varepsilon \) in \( H^1_1 \), with \( a_\varepsilon \) represented in Figure 1.1 (the periodic case with or without dilution) or in Figure 3.1 (the general diluted case).

**Notation 3.1.** In Section 3, we fix the following:

- a smooth, simply connected bounded open set \( \Omega \subset \mathbb{R}^2 \);
- a boundary condition \( g \in C^\infty(\partial \Omega, S^1) \) such that \( d := \deg\Omega(g) > 0 \);
- a smooth, bounded, and simply connected open set \( \Omega' \subset \mathbb{R}^2 \) such that \( \Omega \subset \Omega' \);
- an extension of \( g \) which is in \( C^\infty(\Omega' \setminus \Omega, S^1) \) (this extension is also denoted by \( g \)).

We also consider (uniformly bounded) families \( \{x, d\} = \{(x_1, d_1), \ldots, (x_N, d_N)\} \) of points/degrees such that

- \( x_i \in \Omega, x_i \neq x_{i'} \) for \( i \neq i' \);
- the values \( d_i \) are such that \( d_i \in \mathbb{N}^* \) and \( \sum_i d_i = d \) (thus \( N \leq d \)).

According to the considered problems, for \( 0 < \rho \leq 8^{-1} \min_{i \neq i'} |x_i - x_{i'}| \), we will use the following perforated domains:

- \( \Omega_\rho := \Omega_\rho(x) = \Omega \setminus \bigcup_i B(x_i, \rho) \);
- \( \Omega'_\rho := \Omega'_\rho(x) = \Omega' \setminus \bigcup_i B(x_i, \rho) \).

### 3.1. Existence results.

In this subsection, we prove the existence of solutions of two minimization problems whose studies will be the purpose of the rest of Section 3 (Subsections 3.2 and 3.3).

More precisely, in Subsection 3.1.1 we prove the existence of minimal maps, and in Subsection 3.1.2 we prove the existence of minimal perforations.

#### 3.1.1. Existence of minimal maps defined in a fixed perforated domain.

Let \( x = (x_1, \ldots, x_N) \) be \( 1 \leq N \leq d \) distinct points of \( \Omega \), and \( d = (d_1, \ldots, d_N) \in (\mathbb{N}^*)^N \) be such that \( \sum_i d_i = d \).

For \( 0 < \rho < 8^{-1} \min_{i \neq j} |x_i - x_j| \), we denote \( \Omega_\rho = \Omega \setminus \bigcup_i B(x_i, \rho) \). We define

\[ I_\rho(x, d) = \{ w \in H^1(\Omega_\rho, S^1) \mid w = g \text{ on } \partial \Omega \text{ and } \deg_{B(x_i, \rho)}(w) = d_i \}, \]

and for \( 0 < \rho < 8^{-1} \min \{\min_{i \neq j} |x_i - x_j|, \min_{i} \text{dist}(x_i, \partial \Omega)\} \), we define

\[ J_\rho(x, d) = \{ w \in H^1(\Omega_\rho, S^1) \mid w = g \text{ on } \partial \Omega \} \]

and \( u(x_i + \rho e^{i\theta}) = e^{i(d_i \theta + \theta_i)}, \theta_i \in \mathbb{R} \). From the compatibility condition \( \deg_{\partial \Omega}(g) = d = \sum d_i \), we have \( I_\rho(x, d), J_\rho(x, d) \neq \emptyset \) and it is clear that \( J_\rho(x, d) \subset I_\rho(x, d) \).

In Subsection 3.2, we shall compare the minimal energies corresponding to a weighted Dirichlet functional in the above sets. Here, we just state existence results.
Proposition 3.2. Let $\alpha \in L^\infty(\Omega)$ be such that $b^2 \leq \alpha \leq 1$. Consider the minimization problems

$$\hat{J}_{p,\alpha}(x, d) = \inf_{w \in J_p} \frac{1}{2} \int_{\Omega_p} \alpha |\nabla w|^2$$

and

$$\hat{J}_{p,\alpha}(x, d) = \inf_{w \in J_p} \frac{1}{2} \int_{\Omega_p} \alpha |\nabla w|^2.$$  

In both minimization problems, the infima are attained. Moreover, if $\alpha \in W^{1,\infty}(\Omega)$, then, denoting $w_{p,\alpha}^{\text{deg}}$ (respectively $w_{p,\alpha}^{\text{Dir}}$) a global minimizer of $\frac{1}{2} \int_{\Omega_p} \alpha |\nabla w|^2$ in $J_p(x, d)$ (respectively in $J_p(x, d)$), we have $w_{p,\alpha}^{\text{deg}} \in H^2(\Omega, S^1)$ (respectively $w_{p,\alpha}^{\text{Dir}} \in H^2(\Omega, S^1)$) and

$$\begin{align}
\begin{cases}
- \text{div}(\alpha \nabla w_{p,\alpha}^{\text{deg}}) = \alpha |\nabla w_{p,\alpha}^{\text{deg}}|^2 w_{p,\alpha}^{\text{deg}} & \text{in } \Omega_p, \\
w_{p,\alpha}^{\text{deg}} \in J_p \text{ and } w_{p,\alpha}^{\text{deg}} \Delta w_{p,\alpha}^{\text{deg}} = 0 & \text{on } \partial B(x_i, \rho), i = 1, \ldots, N, \end{cases}
\end{align}
$$

(3.1)

$$\begin{align}
\begin{cases}
- \text{div}(\alpha \nabla w_{p,\alpha}^{\text{Dir}}) = \alpha |\nabla w_{p,\alpha}^{\text{Dir}}|^2 w_{p,\alpha}^{\text{Dir}} & \text{in } \Omega_p, \\
w_{p,\alpha}^{\text{Dir}} \in J_p \text{ and } \int_{\partial B(x_i, \rho)} \alpha w_{p,\alpha}^{\text{Dir}} \Delta w_{p,\alpha}^{\text{Dir}} = 0 & \text{in } \Omega_p.
\end{cases}
\end{align}
$$

(3.2)

(The proof of this standard result is postponed to Appendix A.) In the special case $\alpha = U_p^2$, we denote

$$\hat{J}_{p,\alpha}(x, d) = \inf_{w \in J_p} \frac{1}{2} \int_{\Omega_p} U_p^2 |\nabla w|^2$$

and

$$\hat{J}_{p,\alpha}(x, d) = \inf_{w \in J_p} \frac{1}{2} \int_{\Omega_p} U_p^2 |\nabla w|^2.$$  

3.1.2. Existence of optimal perforated domains. For $\alpha \in L^\infty(\mathbb{R}^2, [b^2, 1])$, we define

$$\begin{align}
I_{p,\alpha} := & \inf_{x_1, \ldots, x_N \in \Omega} \inf_{w \in H^1_0(\Omega, S^1)} \frac{1}{2} \int_{\Omega_p} \alpha |\nabla w|^2 \\
d_{1, \ldots, d_p > 0, \sum d_i = d} \text{degs}(x_i, \theta) \text{w}(w) = d_i
\end{align}
$$

(3.3)

and

$$\begin{align}
J_{p,\alpha} := & \inf_{x_1, \ldots, x_N \in \Omega} \inf_{w \in H^1_0(\Omega, S^1)} \frac{1}{2} \int_{\Omega_p} \alpha |\nabla w|^2 \\
d_{1, \ldots, d_p > 0, \sum d_i = d} \text{degs}(x_i, \theta) \text{w}(w) = d_i
\end{align}
$$

(3.4)

Here, $\Omega_p = \Omega' \setminus \bigcup B(x_i, \rho)$. In the special case $\alpha = U_p^2$, we denote

$$\begin{align}
I_{p,\alpha} := & \inf_{x_1, \ldots, x_N \in \Omega} \inf_{w \in H^1_0(\Omega, S^1)} \frac{1}{2} \int_{\Omega_p} U_p^2 |\nabla w|^2 \\
d_{1, \ldots, d_p > 0, \sum d_i = d} \text{degs}(x_i, \theta) \text{w}(w) = d_i
\end{align}
$$

(3.5)
and

\[ J_{p,\epsilon} := \inf_{x_1, \ldots, x_d \in \Omega \atop \|x_i - x_j\| \geq 8\rho} \inf_{\|w\|_{H^1(\Omega, \mathcal{L})}} \frac{1}{2} \int_{\Omega} |\nabla w|^2. \]

We have the following result.

**Proposition 3.3.** For \( \alpha \in L^\infty(\mathbb{R}^2, [b^2, 1]) \), there are points \( x^{\text{deg}}_{p,\alpha} \in \Omega^N \) (with \( 1 \leq N \leq d \)), \( x^{\text{dir}}_{p,\alpha} \in \Omega^d \), and degrees \( d_{p,\alpha} \in (\mathbb{N}^*)^N \) (with \( d_{p,\alpha} = (d_1, \ldots, d_N) \), \( \sum d_i = d \)) such that \( \{x^{\text{deg}}_{p,\alpha}, d_{p,\alpha}\} \) minimizes \( I_{p,\alpha} \) and \( x^{\text{dir}}_{p,\alpha} \) minimizes \( J_{p,\alpha} \).

The proof of this result is in Appendix B.

3.2. Dirichlet vs. degree conditions in a fixed perforated domain. Let \( \eta_{\text{stop}} > 0 \) be such that \( \eta_{\text{stop}} < 10^{-5} \cdot 9^{-d^2} \text{diam}(\Omega) \), and let \( N \in \{1, \ldots, d\} \).

Consider, then, \( x_1, \ldots, x_N \in \Omega \), \( N \) distinct points of \( \Omega \) such that \( \eta_{\text{stop}} < 10^{-3} \cdot 9^{-d^2} \min_i \text{dist}(x_i, \partial \Omega) \), and let \( \rho > 0 \) be such that

\[ \min \left\{ \eta_{\text{stop}}, \min i \neq j |x_i - x_j| \right\} \geq 8\rho. \]

Roughly speaking, \( \eta_{\text{stop}} \) controls the distance between the points and \( \partial \Omega \).

The main result of this section is the following.

**Proposition 3.4.** There is \( C_0 > 0 \) depending only on \( g \), \( \Omega \), \( \eta_{\text{stop}} \), and \( b \), such that for \( \alpha \in L^\infty(\Omega, [b^2, 1]) \), we have

\[ \hat{J}_{p,\alpha}(x, d) \leq \hat{J}_{p,\alpha}(x, d) \leq \hat{J}_{p,\alpha}(x, d) + C_0. \]

Here, \( \hat{J}_{p,\alpha} \) and \( \hat{J}_{p,\alpha} \) are defined in Proposition 3.2.

The rigorous proof of Proposition 3.4 is presented in Appendix C. Here, we simply present the main lines of the proof.

Two situations are possible:

1. \( N = 1 \) or points \( x_1, \ldots, x_N \) are well separated: \( \frac{1}{2} \min_i \neq j |x_i - x_j| > \eta_{\text{stop}}. \)
2. The points \( x_1, \ldots, x_N \) are not well separated: \( \frac{1}{2} \min_i \neq j |x_i - x_j| \leq \eta_{\text{stop}}. \)

If the points are well separated (or \( N = 1 \)), Proposition 3.4 can be easily proved: it is a direct consequence of Proposition C.4 and Lemma C.3 in Appendix C. These results, whose statements and proofs are postponed to Appendix C, essentially show the existence of test functions in two kinds of domains:

- the thin domain \( \Omega_{10^{-1}\eta_{\text{stop}}} = \Omega \setminus \bigcup_i B(x_i, 10^{-1}\eta_{\text{stop}}) \) obtained by perforating \( \Omega \) by “large”, “well-separated” and “far from \( \partial \Omega \)” discs;
- the thick annulars \( B(x_i, 10^{-1}\eta_{\text{stop}}) \setminus B(x_i, \rho). \)

The proof proceeds in three steps:
Step 1: Using Lemma C.3, we get a constant $C_1$, depending only on $\rho, \Omega, \eta$, such that $J^{(1)}_{\rho, \eta, \alpha}(x, d) \leq C_1$.

Step 2: With the help of Proposition C.4, we obtain the existence of a constant $C_0$ (depending only on $b$) such that for $d \in \mathbb{N}$, denoting $A^c_\rho = B(x_i, 10^{-1} \eta) \setminus B(x_i, \rho)$, we have

$$
\inf_{w \in H^1(A^c_\rho, \mathbb{S}^1)} \int_{A^c_\rho} \frac{1}{2} |\nabla w|^2 \leq \inf_{\deg_{\partial B(x_i, \rho)}(w) = d} \frac{1}{2} \int_{A^c_\rho} \alpha |\nabla w|^2 + C_b d^2.
$$

Step 3: By extending a minimizer of $J^{(1)}_{\rho, \eta, \alpha}(x, d)$ by those of $\frac{1}{2} \int_{A^c_\rho} \alpha |\nabla \cdot |^2$ with Dirichlet conditions, we can construct a map which proves the result, taking $C_0 = C_1 + d^3 C_b$.

### 3.3. “Optimal” perforated domains for the degree conditions.

Recall that we fixed $\Omega' \supset \Omega$, a smooth bounded domain such that $\text{dist}(\partial \Omega', \Omega) > 0$, and that we fixed a smooth $\mathbb{S}^1$-valued extension of $g$ to $\Omega' \setminus \Omega$ (still denoted by $g$).

In this subsection, we focus on the case $\alpha = U^2$, where $U$ is the global minimizer of $E_\rho$ in $H^1$. Here, $E_\rho$ has either the periodic pinning term $\alpha_\rho$ (defined by Construction 1.2) or the general diluted pinning term (defined in Construction 2.1).

We extend $U$ with the value 1 outside $\Omega$, to study the minimization problem

$$
I_{\rho, \epsilon} := \inf_{X_1, \ldots, X_N \in \Omega} \inf_{d_1, \ldots, d_N > 0, \sum d_i = d} \int_{\Omega' \setminus B(x_i, \rho)} U^2 |\nabla w|^2,
$$

where $\Omega' = \Omega' \setminus \bigcup \{B(x_i, \rho)\}$ and $H^1(\Omega', \mathbb{S}^1) = \{w \in H^1(\Omega', \mathbb{S}^1) \mid w = g \text{ in } \Omega' \setminus \Omega \cup B(x_i, \rho)\}$.

We assume that Hypothesis (1.3) holds, and that $|\ln (\lambda \delta)|^{3/2} \ln \epsilon \to 0$. This is not optimal for the statements, but it makes the proofs simpler (this hypothesis may be relaxed, but it appears as a crucial and technical hypothesis for the methods developed in Section 4; e.g., see Remark 4.11).

A first purpose of this section is the study of the behavior of $I_{\rho, \epsilon}$ when $\rho = \rho(\epsilon) \to 0$ as $\epsilon \to 0$. In view of the application we have in mind, we suppose that $\lambda \delta^{p+1} \gg \rho(\epsilon) \geq \epsilon$, but this is not crucial for our arguments (here, $p = 1$ if $U$ is associated with the periodic pinning term).

A second objective of our study is to exhibit the behavior of almost minimal configurations $\{x_i^n, \ldots, x_n^n, (d_1^n, \ldots, d_N^n)\}$.

For fixed $\rho, \epsilon$, the existence of a minimal configuration of points $x_{\rho, \epsilon}$ is the purpose of Proposition 3.3. In this section, we consider only almost minimal configurations.
Notation 3.5. For $\varepsilon_n \downarrow 0$, we say that $\{(x_1^n, \ldots, x_N^n), (d_1^n, \ldots, d_N^n)\}$ is an almost minimal configuration for $I_{\rho, \varepsilon_n}$ ($\rho = \rho(\varepsilon_n) \downarrow 0$) when $x_1^n, \ldots, x_N^n \in \Omega$, $|x_i^n - x_j^n| \geq 8\rho$, $d_1^n, \ldots, d_N^n > 0$, $\sum d_i^n = d$, and there is $C > 0$ (independent of $n$) such that

$$\inf_{w \in H^1_0(\Omega^n, S^1)} \left\{ \frac{1}{2} \int_{\Omega^n} U_{\varepsilon_n}^2 |
abla w|^2 - I_{\rho, \varepsilon_n} \right\} = C.$$  

Roughly speaking, we establish in this section two repelling effects for the points: a Coulomb repulsion between the points and a confinement effect, and an attractive effect for the points by the inclusions $\omega_\varepsilon$.

3.3.1. The case of the periodic pinning term. The main result of this section establishes that when $\varepsilon_n, \rho \downarrow 0$, for sufficiently large $n$, an almost minimal configuration $\{(x_1^n, \ldots, x_N^n), (d_1^n, \ldots, d_N^n)\}$ is such that

- The points $x_i^n$ cannot be mutually close (repulsion);
- The degrees $d_i^n$ are necessarily all equal to 1 (quantization);
- The points $x_i^n$ cannot approach $\partial \Omega$ (confinement);
- There is $c > 0$ such that $B(x_i^n, c\lambda\delta) \subset \omega_\varepsilon$ for all $i$ (pinning).

These facts are expressed in the following proposition (whose proof is postponed to Appendix D).

Proposition 3.6 (The case of a periodic pinning term). Assume that $\lambda, \delta$ satisfy (1.3), and let $\omega_\varepsilon$ be the periodic pinning term (represented in Figure 1.1).

Let $\varepsilon_n \downarrow 0$, $\rho = \rho(\varepsilon_n) \downarrow 0$, and $x_1^n, \ldots, x_N^n \in \Omega$ be such that $|x_i^n - x_j^n| \geq 8\rho$, $\rho \geq \varepsilon_n$, and let $d_1^n, \ldots, d_N^n \in \mathbb{N}^n$ be such that $\sum d_i^n = d$.

1. Assume that there is $i_0 \in \{1, \ldots, N\}$ such that $d_{i_0}^n \neq 1$ or that there are $i_0 \neq i_1$ such that $|x_{i_0}^n - x_{i_1}^n| \to 0$. Then

$$\inf_{w \in H^1_0(\Omega^n, S^1)} \left\{ \frac{1}{2} \int_{\Omega^n} U_{\varepsilon_n}^2 |
abla w|^2 - I_{\rho, \varepsilon_n} \right\} \to \infty.$$  

2. Assume that there is $i_0 \in \{1, \ldots, N\}$ such that $\text{dist}(x_{i_0}^n, \partial \Omega) \to 0$. Then

$$\inf_{w \in H^1_0(\Omega^n, S^1)} \left\{ \frac{1}{2} \int_{\Omega^n} U_{\varepsilon_n}^2 |
abla w|^2 - I_{\rho, \varepsilon_n} \right\} \to \infty.$$  

3. Assume that $\rho/(\lambda \delta) \to 0$ and that there is $i_0 \in \{1, \ldots, N\}$ such that $x_{i_0}^n \notin \omega_{\varepsilon_n}$, or such that $x_{i_0}^n \in \omega_{\varepsilon_n}$ and $\text{dist}(x_{i_0}^n, \partial \omega_{\varepsilon_n})/(\lambda \delta) \to 0$. Then

$$\inf_{w \in H^1_0(\Omega^n, S^1)} \left\{ \frac{1}{2} \int_{\Omega^n} U_{\varepsilon_n}^2 |
abla w|^2 - I_{\rho, \varepsilon_n} \right\} \to \infty.$$
A straightforward consequence of Proposition 3.6 is the following result.

**Corollary 3.7.**

(1) Consider an almost minimal configuration \( \{x_{\rho,\varepsilon}, d_{\rho,\varepsilon}\} \in \Omega^N \times \mathbb{N}^N \); that is, assume there is \( w \in H^1_0(\Omega^\prime \setminus \bigcup B(x_{\rho,\varepsilon}^i, \rho), \mathbb{S}^1) \) verifying

\[
\deg_{\partial B(x_{\rho,\varepsilon}^i, \rho)}(w) = d_{i,\rho,\varepsilon} \quad \text{and} \quad \frac{1}{2} \int_{\Omega \setminus \bigcup B(x_{\rho,\varepsilon}^i, \rho)} U_\varepsilon^2 |\nabla w|^2 \leq I_{\rho,\varepsilon} + C.
\]

(Here, \( C \) is independent of \( \varepsilon \).)

Then, \( d_i = 1 \) for all \( i \), and there is some \( \eta_0 \) independent of \( \varepsilon \) such that, for small \( \varepsilon \), we have \( |x_{\rho,\varepsilon}^i - x_{\rho,\varepsilon}^j|, \text{dist}(x_{\rho,\varepsilon}^i, \partial \Omega) \geq \eta_0 \) for all \( i \neq j, \ i, j \in \{1, \ldots, N\} \). In particular, we have \( N = d \).

(2) If, in addition, \( \rho = \rho(\varepsilon) \) is such that \( \rho \geq \varepsilon \) and \( \rho/(\lambda \delta) \to 0 \), then there is \( C > 0 \), independent of \( \varepsilon \) such that, for small \( \varepsilon \), we have \( B(x_{\rho,\varepsilon}^i, c\lambda \delta) \subset \omega_{\varepsilon} \).

**Proof of Corollary 3.7.** We prove the first part. Let \( C > 0 \). We argue by contradiction and assume that for all \( n \in \mathbb{N}^N \), there are \( 0 < \varepsilon_n \leq \rho = \rho(\varepsilon_n) \leq 1/n \), \( x_n = x_{\rho,\varepsilon_n} \) and \( d_n = (d_1, \ldots, d_N) \) satisfying the hypotheses of Corollary 3.7, and such that \( \min\{|x_n^i - x_n^j|, \text{dist}(x_n^i, \partial \Omega)\} \to 0 \) or such that there is \( i \in \{1, \ldots, N\} \) for which we have \( d_i \neq 1 \).

By construction, we have that \( \{x_{\rho,\varepsilon_n}, d_n\} \) is an almost minimal configuration for \( I_{\rho,\varepsilon_n} \) with \( \rho = \rho(\varepsilon_n) \geq \varepsilon_n \). Clearly, from Proposition 3.6 we find a contradiction.

The proof of the second part is similar. \( \square \)

We end this subsection by the following direct consequence of Corollary 3.7.

**Corollary 3.8.** For sufficiently small \( \varepsilon, \rho \), an almost minimal configuration \( (x_1, \ldots, x_d) \) for \( J_{\rho,\varepsilon} \) is an almost minimal configuration for \( I_{\rho,\varepsilon} \).

Moreover, there is \( C_0 > 0 \) such that \( J_{\rho,\varepsilon} \leq I_{\rho,\varepsilon} + C_0 \), with \( C_0 \) being independent of small \( \varepsilon, \rho \).

**Proof.** Let \( C > 0 \) and let \( (x_1, \ldots, x_d), (x_1', \ldots, x_d') \in \Omega^d \) be such that

\[
\hat{J}_{\rho,\varepsilon}(x_1, \ldots, x_d) \leq J_{\rho,\varepsilon} + C \quad \text{and} \quad \hat{J}_{\rho,\varepsilon}(x_1', \ldots, x_d') \leq I_{\rho,\varepsilon} + C.
\]

From Corollary 3.7, there is \( \eta_0 = \eta_0(C) > 0 \) such that for \( \varepsilon \leq \rho \leq \eta_0 \), we have \( \min \text{dist}(x_i', \partial \Omega) \geq \eta_0 \). Using Proposition 3.4, there is \( C_0 > 0 \) such that

\[
\hat{J}_{\rho,\varepsilon}(x_1, \ldots, x_d) \leq \hat{J}_{\rho,\varepsilon}(x_1, \ldots, x_d) \leq J_{\rho,\varepsilon} + C \leq \hat{J}_{\rho,\varepsilon}(x_1', \ldots, x_d') + C \leq \hat{J}_{\rho,\varepsilon}(x_1', \ldots, x_d') + C + C_0 \leq I_{\rho,\varepsilon} + 2C + C_0. \quad \square
\]
3.3.2. A more precise result for the case of the periodic pinning term with dilution. In this section, we focus on the periodic pinning term (represented in Figure 1.1) with dilution: $\lambda \to 0$.

**Notation 3.9.** We define two kinds of configuration of distinct points of $\Omega$:
- We say that for $\varepsilon_n \downarrow 0$ and $\rho = \rho(\varepsilon_n) \to 0$, $d$ distinct points of $\Omega$, $x_n = (x_1^n, \ldots, x_d^n)$ form a quasi-minimizer of $I_{\rho,\varepsilon_n}$ when $J_{\rho,\varepsilon_n}(x_n) - J_{\rho,\varepsilon_n} \to 0$.
- We say that for $\varepsilon_n \downarrow 0$ and $\rho = \rho(\varepsilon_n) \to 0$, $d$ distinct points of $\Omega$, $x_n = (x_1^n, \ldots, x_d^n)$ form a quasi-minimizer of $W_{\rho}$, the renormalized energy of Bethuel-Brezis-Hélein (see [BBH94] or Remark 1.6 (1)), when $W_\rho(x_n) \to \min W_\rho$.

**Proposition 3.10 (Asymptotic location of optimal perforations).** Assume that $\lambda, \delta$ satisfy (1.3) and that $\lambda \to 0$.

Let $\varepsilon_n \downarrow 0$, $\rho = \rho(\varepsilon_n) \to 0$, $\rho \geq \varepsilon_n$, and $x_n = (x_1^n, \ldots, x_d^n)$ be $d$ distinct points of $\Omega$.

If the points $x_n$ form a quasi-minimizer of $I_{\rho,\varepsilon_n}$, then $x_n = (x_1^n, \ldots, x_d^n)$ form a quasi-minimizer of $W_{\rho}$.

This proposition is proved in Appendix E.

**Remark 3.11.** Moreover, by using results of Appendices D and E, we may also prove that if $x_n$ is a quasi minimizer, then, up to translating each point of $x_n$ with a vector $u_i^\varepsilon \in B(0, \lambda \delta)$, we have that $x_n$ is a quasi minimizer of $I_{\rho,\varepsilon_n}$.

3.3.3. The case of a general pinning term with variable sizes of inclusions. We assume that $\alpha$, $\beta$ is the general pinning term represented in Figure 2.1 with the hypothesis on the dilution: $\lambda \to 0$.

**Proposition 3.12 (The case of a non-periodic pinning term).** Assume that $\lambda, \delta$ satisfy (1.3) and $\lambda \to 0$.

Let $\rho = \rho(\varepsilon)$ such that $\rho \geq \varepsilon$ and $\rho/(\lambda \delta^{3/2}) \to 0$. If $\{x_{\rho,\varepsilon}, d_{\rho,\varepsilon}\}$ is an almost minimal configuration for $I_{\rho,\varepsilon}$, then $N = d$ (thus $d_i = 1$ for all $i$), and there are $c, \eta_0 > 0$ (independent of $\varepsilon$) such that for sufficiently small $\varepsilon$:

1. $|x_i^{\rho,\varepsilon} - x_j^{\rho,\varepsilon}|, \text{dist}(x_i^{\rho,\varepsilon}, \partial \Omega) \geq \eta_0$ for all $i \neq j$, $i, j \in \{1, \ldots, d\}$.

2. $B(x_i^{\rho,\varepsilon}, c \lambda \delta) \subset \omega_\varepsilon$ (the points $x_i^{\rho,\varepsilon}$ are trapped by the largest inclusions).

Moreover, there is $C_0 > 0$ such that $I_{\rho,\varepsilon} \leq I_{\rho,\varepsilon} + C_0$, $C_0$, being independent of small $\varepsilon$, $\rho$. Thus, an almost minimal configuration $x_{\rho,\varepsilon}$ for $I_{\rho,\varepsilon}$ is an almost minimal configuration for $I_{\rho,\varepsilon}$.

This proposition is proved in Appendix E (Subsection E.3).

4. The Pinned Ginzburg-Landau Functional

In this section, we turn to the main purpose of this article: the study of minimizers of $F_\varepsilon$ (defined in (1.7)) in $H_0^1$. The pinning term is the periodic one (represented in Figure 1.1) or the general diluted pinning term (represented in Figure 2.1).
Recall that we fix \( \delta = \delta(\varepsilon), \delta \to 0, \lambda = \lambda(\varepsilon) \), and \( \lambda \equiv 1 \) or \( \lambda \to 0 \) satisfying (1.3). If the pinning term is not periodic, then we add the hypothesis \( \lambda \to 0 \).

4.1. Sharp upper bound, \( \eta \)-ellipticity and uniform convergence. In this subsection, we first (Subsection 4.1.1) get a sharp upper bound for the minimal energy, and state a fundamental tool (the \( \eta \)-ellipticity lemma). Then (Subsection 4.1.2), we prove the first pinning effect: \(|v_\varepsilon| \approx 1 \) “far away” from \( \omega_\varepsilon \).

4.1.1. Sharp upper bound and an \( \eta \)-ellipticity result. We may easily prove the following upper bound.

**Lemma 4.1.** Assume that \( \rho / (\lambda \delta) \to 0 \) (or \( \rho / (\lambda \delta^{3/2}) \to 0 \) if the pinning term is not periodic); we then have

\[
\inf_{v \in H^1_{0}(\Omega, C)} F_\varepsilon(v) \leq \beta b^2 \left( \pi \ln \frac{b \rho}{\varepsilon} + \gamma \right) + J_{\rho, \varepsilon} + o_\varepsilon(1),
\]

where \( \gamma > 0 \) is a universal constant defined in [BBH94, Lemma IX.1].

**Proof.** We construct a suitable test function \( \tilde{w}_\varepsilon \in H^1_{0}(\Omega, C) \) (for sufficiently small \( \varepsilon \)). From Proposition 3.3, one may consider \((x_1^\varepsilon, \ldots, x_d^\varepsilon) = x^\varepsilon \in \Omega^d\), a minimal configuration for \( J_{\rho, \varepsilon} \).

Note that since \( \rho / (\lambda \delta) \to 0 \) (or \( \rho / (\lambda \delta^{3/2}) \to 0 \) if the pinning term is not periodic), from Corollaries 3.7 and 3.8 (or Proposition 3.12 if the pinning term is not periodic) there exist \( \eta > 0 \) and \( c > 0 \) such that for small \( \varepsilon \), we have \( B(x_i^\varepsilon, c \lambda \delta) \subset \omega_\varepsilon \) and \( \min_i \min_{i \neq j} \{|x_i - x_j|, \text{dist}(x_i, \partial \Omega)\} \geq \eta \).

We consider the test function

\[
\tilde{w}_\varepsilon(x) = \begin{cases} 
\omega_\varepsilon & \text{in } \Omega^\rho, \\
\alpha^\varepsilon \omega_\varepsilon/(\beta \rho)(x - x_i^\varepsilon) & \text{in } B(x_i^\varepsilon, \rho).
\end{cases}
\]

Here, the constants \( \alpha^\varepsilon \in S^1 \) are such that \( \omega_\varepsilon(x_i^\varepsilon + \rho e^{i\theta}) = \alpha^\varepsilon e^{i\theta} \).

Estimate (4.1) is obtained by using Proposition 1.4, as well as the fact that \( E^0_\varepsilon(u_\varepsilon) = \pi |\ln \varepsilon| + \gamma + o_\varepsilon(1) \) as \( \varepsilon \to 0 \) (see [BBH94, Lemma IX.1]).

Note that

\[
I_{\rho, \varepsilon} \leq J_{\rho, \varepsilon} \leq \pi d |\ln \rho| + C.
\]

We now turn to the \( \eta \)-ellipticity.
We denote by $v_\varepsilon$ a global minimizer of $F_\varepsilon$ in $H^1_\varepsilon$. We extend $|v_\varepsilon|$ with the value 1 outside $\Omega$.

One of the main ingredients in this work is the following result.

**Lemma 4.2 (η-ellipticity lemma).** Let $0 < \alpha < \frac{1}{2}$. Then the following results hold:

1. For $\varepsilon < \varepsilon_0$, if $F_{\varepsilon}(v_\varepsilon, B(x, \varepsilon^{\alpha}) \cap \Omega) \leq \varepsilon^2 |\ln \varepsilon| - C_1$, then we have
   
   $$|v_\varepsilon| \geq 1 - C \varepsilon^{2\alpha} \text{ in } B(x, \varepsilon^{2\alpha}).$$

   Here, $\chi_\varepsilon \in (0, 1)$ is such that $\chi_\varepsilon \to 0$ and $\varepsilon_0 > 0$, $C > 0$, $C_1 > 0$ depend only on $b, \alpha, \chi, \Omega, \|g\|_{C^1}$.

2. For $\varepsilon < \varepsilon_0$, if $F_{\varepsilon}(v_\varepsilon, B(x, \varepsilon^{\alpha}) \cap \Omega) \leq C_0 |\ln \varepsilon|$, then we have
   
   $$|v_\varepsilon| \geq \mu \text{ in } B(x, \varepsilon^{2\alpha}).$$

   Here, $\mu \in (0, 1)$ and $C_0 > 0$ depend only on $b, \alpha, \mu, \Omega, \|g\|_{C^1}$.

This result is a direct consequence of Lemma 1 in [DM11].

**4.1.2. Uniform convergence of $|v_\varepsilon|$ outside $\omega_\varepsilon$.** With the help of Lemma 4.2, we are now in a position to establish the uniform convergence of $|v_\varepsilon|$ to 1 far away from $\overline{\omega_\varepsilon}$.

**Proposition 4.3.** Let $\mu > 0$ be such that

$$
\begin{cases}
10^{-2} \cdot \text{dist}(\omega, \partial Y) > \mu > 0 & \text{if } a_\varepsilon \text{ is periodic,} \\
10^{-2} > \mu > 0 & \text{otherwise,}
\end{cases}
$$

and let $K^\mu_\varepsilon = \{x \in \Omega | \text{dist}(x, \omega_\varepsilon) \geq \mu \lambda \delta\}$.

Then, for sufficiently small $\varepsilon$, we have $|v_\varepsilon| \geq 1 - C \sqrt{\frac{|\ln(\lambda \delta)|}{|\ln \varepsilon|}}$ in $K^\mu_\varepsilon$. Here, $C$ is independent of $\varepsilon$ and $\mu$.

Furthermore, if we have $|v_\varepsilon(x)| < 1 - C \sqrt{\frac{|\ln(\lambda \delta)|}{|\ln \varepsilon|}}$ for some small $\varepsilon$, then

$$F_\varepsilon(v_\varepsilon, B(x, \varepsilon^{1/4})) \geq \frac{2(\pi d + 1)}{b^2(1 - b^2)} |\ln(\lambda \delta)|.$$
then we have \(|v_\varepsilon| \geq 1 - C\chi|B(x, \varepsilon^{1/2})|\). In order to prove Proposition 4.3, we argue by contradiction. There are \(\varepsilon_n \downarrow 0, \mu > 0, \) and \(x_n \in K^\mu_{\Omega}\) such that \(|v_{\varepsilon_n}(x_n)| < 1 - C\chi|B(x, \varepsilon^{1/2})|\). From (1.5), we find

\begin{equation}
|U_{\varepsilon_n} - 1| \leq C_0 e^{-\alpha \mu / (2\varepsilon)} \quad \text{in } K^\mu_{\Omega}, \quad \xi = \frac{\varepsilon_n}{2\mu}.
\end{equation}

Consequently, Lemma 4.2, the definition of \(C\), and (4.3) imply that, for large \(n\),

\begin{equation}
\frac{1}{2} \int_{B(x_n, \varepsilon_n)^4} \left\{ |\nabla v_{\varepsilon_n}|^2 + \frac{1}{2\varepsilon_n^2} (1 - |v_{\varepsilon_n}|^2)^2 \right\} \geq \frac{2(\pi d + 1)}{b^2(1 - b^2)} \ln (\lambda \delta) + o_\varepsilon(1).
\end{equation}

We extend \(v_\varepsilon\) to \(\Omega' := \Omega + B(0,1)\) with the help of an \(\varepsilon\)-independent smooth \(S^1\)-valued map \(v\) such that \(v = g\) on \(\partial\Omega\). We also extend \(U_\varepsilon\) and \(a_\varepsilon\) with the value 1 outside \(\Omega\).

For \(n\) sufficiently large, we have

\begin{equation}
\frac{1}{2} \int_{\Omega'} \left\{ |\nabla v_{\varepsilon_n}|^2 + \frac{1}{2\varepsilon_n^2} (1 - |v_{\varepsilon_n}|^2)^2 \right\} \leq C |\ln \varepsilon_n|.
\end{equation}

Theorem 4.1 in [SS07], applied with \(r = 10^{-2} \cdot \lambda \delta \mu\) and for large \(n\), implies the existence of \(B^n = \{B^n_j\}\), a finite disjoint covering by balls of

\begin{equation}
\left\{ x \in \Omega' \mid \text{dist}(x, \partial \Omega') > \frac{\varepsilon_n}{b} \text{ and } 1 - |v_{\varepsilon_n}(x)| \geq \left(\frac{\varepsilon_n}{b}\right)^{1/8} \right\}
\end{equation}

such that \(\text{rad}(B^n) \leq 10^{-2} \cdot \lambda \delta \mu\), satisfying

\begin{equation}
\frac{1}{2} \sum_j d^n_j \left\{ |\nabla v_{\varepsilon_n}|^2 + \frac{b^2}{2\varepsilon_n^2} (1 - |v_{\varepsilon_n}|^2)^2 \right\} \\
geq \pi \delta \sum_j d^n_j (|\ln \varepsilon_n| - |\ln (\lambda \delta)|) - C = \pi \delta \sum_j d^n_j |\ln \xi| - C.
\end{equation}

Here, \(\text{rad}(B^n) = \sum_j \text{rad}(B^n_j)\), \(\text{rad}(B)\) stands for the radius of the ball \(B\), \(\xi = \varepsilon_n / (\lambda \delta)\), and the integers \(d^n_j\) are defined by

\[d^n_j = \begin{cases} |	ext{deg}_{\partial B^n_j}(v_{\varepsilon_n})| & \text{if } B^n_j \subset \{ x \in \Omega' \mid \text{dist}(x, \partial \Omega') > \varepsilon_n / b \}, \\ 0 & \text{otherwise}.\end{cases}\]

Since \(B_j \subset \Omega + B_{1/2} \subset \{ x \in \Omega' \mid \text{dist}(x, \partial \Omega') > \varepsilon_n / b \}\), we obtain

\begin{equation}
\frac{1}{2} \sum_j d^n_j \left\{ |\nabla v_{\varepsilon_n}|^2 + \frac{b^2}{2\varepsilon_n^2} (1 - |v_{\varepsilon_n}|^2)^2 \right\} \geq \pi \delta |\ln \xi| - C.
\end{equation}
From (4.3) and (1.3), we have

\[ F_\xi(v_{\varepsilon_n}, \bigcup_j B_j \cup B(x_n, \varepsilon_1^{1/4})) \geq b^2 \left( 1 - \varepsilon_n^2 \right) \left\{ |\nabla v_{\varepsilon_n}|^2 + \frac{1}{2\varepsilon_n^2} (1 - |v_{\varepsilon_n}|^2)^2 \right\} + o_n(1). \]

By combining (4.1) (with \( \rho = \lambda^2 \delta^2 \)), (4.2), (4.4), (4.5), and (4.6), we find that

\[ \pi db^2 \ln \left( \frac{(\lambda \delta)}{\xi} \right) + \pi d |\ln (\lambda \delta^2)| \]

\[ \geq F_{\varepsilon_n}(v_{\varepsilon_n}, \Omega') - O(1) \]

\[ \geq F_{\varepsilon_n}(v_{\varepsilon_n}, \bigcup_j B_j \cup B(x_n, \varepsilon_1^{1/4})) - O(1) \]

\[ \geq \pi db^2 |\ln \xi| + 2(\pi d + 1) |\ln (\lambda \delta)| - O(1), \]

which is a contradiction. This completes the proof of Proposition 4.3.

\[ \square \]

4.2. Bad discs.

4.2.1. Construction and first properties of bad discs. A fundamental argument in this article is for the use of ad-hoc coverings of \( \{|v| \leq \frac{7}{8}\} \) by small discs. The best radius for a covering of \( \{|v| \leq \frac{7}{8}\} \) should be of the order of \( \varepsilon \). But the construction of such a covering needs some preliminary results.

Roughly speaking, the way to get a “sharp” covering is to consider a trivial covering and to “clean” it by dropping some discs with the help of an “energetical test” (\( \eta \)-ellipticity result).

Here, we use two kinds of energetical tests: Lemma 4.2 and Theorem III.3 in [BBH94]. Theorem III.3 in [BBH94] gives the most precise results (it allows dealing with discs with radius \( O(\varepsilon) \)), but it needs a bound on the potential part \( \varepsilon^{-2} \int_\Omega (1 - |v|)^2 \). In the case without pinning term \( (a_\varepsilon \equiv 1) \), this estimate is obtained with a Pohozaev identity. In our case, the use of a Pohozaev identity is delicate because it forces us to handle \( \nabla U_\varepsilon \), which has a wild behavior close to \( \partial \omega_\varepsilon \). Thus, in order to bound \( \varepsilon^{-2} \int_\Omega (1 - |v_\varepsilon|)^2 \), we must do a preliminary analysis; once this is done, we will be in a position to prove this bound (this is the purpose of Proposition 4.13).

This preliminary analysis uses larger discs (discs with radius \( \rho, \varepsilon \ll \rho \ll \lambda \delta^{p+1} \)). The construction of intermediate coverings is done via Lemma 4.2.

We first consider the following covering.
Notation 4.4 (A trivial covering of $\Omega$ by discs). For $\varepsilon > 0$, we fix a family of discs $(B(x_i, \varepsilon^{1/4}))_{i \in I}$ such that
\[
x_i \in \Omega, \quad \forall \ i \in I,
\]
\[
B \left(x_i, \frac{\varepsilon^{1/4}}{4} \right) \cap B \left(x_j, \frac{\varepsilon^{1/4}}{4} \right) = \emptyset \quad \text{if } i \neq j,
\]
\[
\bigcup_{i \in I} B(x_i, \varepsilon^{1/4}) \supset \Omega.
\]

Then, we select discs (using Lemma 4.2) and define the following sets.

Notation 4.5 (The initial good/bad discs).

- Let $C_0 = C_0(\frac{1}{4}, \frac{7}{8})$, $\varepsilon_0 = \varepsilon_0(\frac{1}{4}, \frac{7}{8})$ be defined by Lemma 4.2.2 with $\alpha = \frac{1}{4}$ and $\mu = \frac{7}{8}$.

  For $\varepsilon < \varepsilon_0$, we say that $B(x_i, \varepsilon^{1/4})$ is an initial good disc if
  \[
  F_\varepsilon(v_\varepsilon, B(x_i, \varepsilon^{1/4}) \cap \Omega) \leq C_0 |\ln \varepsilon|,
  \]
  and that $B(x_i, \varepsilon^{1/4})$ is an initial bad disc if
  \[
  F_\varepsilon(v_\varepsilon, B(x_i, \varepsilon^{1/4}) \cap \Omega) > C_0 |\ln \varepsilon|.
  \]

- We let $J = J(\varepsilon) := \{i \in I \mid B(x_i, \varepsilon^{1/4}) \text{ is an initial bad disc} \}$.

An easy consequence of Lemma 4.1 is the following result.

Lemma 4.6. The number of initial bad discs is bounded. Moreover, there is an integer $N$ which depends only on $g$ and $\Omega$ such that $\text{Card} J \leq N$.

Proof. Since each point of $\Omega$ is covered by at most $C > 0$ (universal constant) discs $B(x_i, \varepsilon^{1/4})$, we have $\sum_{i \in I} F_\varepsilon(v_\varepsilon, B(x_i, \varepsilon^{1/4}) \cap \Omega) \leq CF_\varepsilon(v_\varepsilon, \Omega)$. The previous assertion implies that $\text{Card} J \leq C\pi d/C_0 + 1$. \qed

Let $\rho(\varepsilon) = \rho \downarrow 0$ be such that
\[
\frac{\rho}{\lambda \delta^{P+1}} \to 0 \quad \text{and} \quad \frac{|\ln \rho|^3}{|\ln \varepsilon|} \to 0.
\]

Note that, from Assumption (1.3), such a $\rho$ exists; for instance, $\rho = (\lambda \delta)^{P+2}$ (recall that if the pinning term is periodic, then $P = 1$).

The following result is a straightforward variant of Theorem IV.1 in [BBH94].

Lemma 4.7 (Separation of the initial bad discs). Let $\varepsilon \downarrow 0$. Then (possibly after passing to a subsequence and relabeling the indices), we may choose $J' \subset J$ and a constant $\kappa$ independent of $n$ such that
\[
J' = \{1, \ldots, N'\}, \quad N' = \text{Cst},
\]
\[
|x_i - x_j| \geq 16\kappa \rho \quad \text{for } i, j \in J', \ i \neq j,
\]
and
\[
\bigcup_{i \in I} B(x_i, \varepsilon_i^{1/4}) \subset \bigcup_{i \in J'} B(x_i, \kappa \rho).
\]
**Notation 4.8 (The $\rho$-bad disc).** For $i \in J'$, we say that $B(x_i, 2\kappa \rho)$ is a $\rho$-bad disc.

**Proposition 4.9.** Let $B(x_i, 2\kappa \rho)$ be a $\rho$-bad disc; then, we have

1. $\rho / \text{dist}(B(x_i, 2\kappa \rho), \partial \Omega) \to 0$,
2. $\deg_{\partial B(x_i, 2\kappa \rho)}(v_{\epsilon_n}) > 0$,
3. $F_{\epsilon_n}(v_{\epsilon_n}, B(x_i, 2\kappa \rho)) \geq 2b^2 \deg_{\partial B(x_i, 2\kappa \rho)}(v_{\epsilon_n}) \ln(\rho / \epsilon_n) - O(1)$,
4. $|v_{\epsilon_n}| \geq 1 - C \sqrt{|\ln \rho|/|\ln \epsilon_n|} \ln \Omega \setminus \bigcup_{i \in J'} B(x_i, 2\kappa \rho)$.

**Proof.** We prove Assertions (1), (2), and (3). Set

$$J'_0 := \{i \in J' \mid \deg_{\partial (B(x_i, 2\kappa \rho) \cap \Omega)}(v_{\epsilon_n}) > 0\}.$$ 

Since $|v_{\epsilon_n}| \geq \frac{7}{8}$ in $\Omega \setminus \bigcup_{i \in J'} B(x_i, 2\kappa \rho)$, we have

$$0 < d = \sum_{i \in J'} \deg_{\partial (B(x_i, 2\kappa \rho) \cap \Omega)}(v_{\epsilon_n}) \leq \sum_{i \in J'_0} \deg_{\partial (B(x_i, 2\kappa \rho) \cap \Omega)}(v_{\epsilon_n}).$$

Consequently, $J'_0 \neq \emptyset$.

Up to a subsequence, we may assume that $J'_0$ is independent of $n$.

From Proposition 4.3, for all $i \in J'_0$, we have $\text{dist}(B(x_i, \epsilon^{1/4}), \partial \Omega) \gtrsim \delta$ (or $\delta^p$ if the pinning term is not periodic). Consequently, for $i \in J'_0$, we find

$$\frac{\text{dist}(B(x_i, 2\kappa \rho), \partial \Omega)}{\rho} \to +\infty,$$

since $\rho / (\lambda \delta^{p+1}) \to 0$.

Assertions (1), (2), and (3) will follow from the estimate

$$F_{\epsilon_n}(v_{\epsilon_n}, B(x_i, 2\kappa \rho)) \geq b^2 \pi \deg_{\partial B(x_i, 2\kappa \rho)}(v_{\epsilon_n}) \ln \frac{\rho}{\epsilon_n} - O(1),$$

valid for $i \in J'_0$. Indeed, assume for the moment that (4.11) holds for $i \in J'_0$. Then, by combining (4.1), (4.2), (4.7), (4.8), (4.9), and (4.11), we find that $J'_0 = J'$, that is, Assertion (2) holds. Consequently, Assertion (2) combined with (4.10) yield Assertion (1). And, from Assertion (2) and (4.11), Assertion (3) holds.

We now turn to the proof of (4.11), which relies on Proposition 4.1 in [SS07].

We apply this proposition in the domain $B = B(0, 2\kappa)$, to the function $v'(x) = v_{\epsilon_n}(\rho (x - x_i))$, and with the rescaled parameter $\xi_{\text{meso}} = \epsilon / \rho$.

Note that $\epsilon \ll \xi_{\text{meso}} \ll \rho \ll \lambda \delta^{p+1}$ and $|\ln \epsilon| \sim |\ln \xi_{\text{meso}}| \gg |\ln(\lambda \delta)|$, from (4.8).

Clearly, $v'$ satisfies

$$\int_{B} \left\{ |\nabla v'|^2 + \frac{1}{\xi_{\text{meso}}^2} (1 - |v'|^2)^2 \right\} = \int_{B(x_i, 2\kappa \rho)} \left\{ |\nabla v_{\epsilon_n}|^2 + \frac{1}{\xi^2} (1 - |v_{\epsilon_n}|^2)^2 \right\} = O(|\ln \epsilon|) = O(|\ln \xi_{\text{meso}}|).$$
Hence, one may apply the following result of Sandier and Serfaty: there is \((B_j)_{j \in I},\) a finite covering of
\[
\left\{ x \in B \left( 0, 2\kappa - \frac{\xi_{\text{meso}}}{b} \right) : |v'(x)| \leq 1 - \left( \frac{\xi_{\text{meso}}}{b} \right)^{1/8} \right\}
\]
with disjoint balls \(B_j\) of radius \(r_j < 10^{-3}\) such that
\[
\frac{1}{2} \int_{B \cap \bigcup B_j} \left( |\nabla v'|^2 + \frac{b^2}{\xi_{\text{meso}}^2} (1 - |v'|^2)^2 \right) \geq \pi \sum_j d_j \ln \xi_{\text{meso}} - O(1).
\]

Here,
\[
d_j = \begin{cases} \deg_{\partial B_j}(v') & \text{if } B_j \subset B \left( 0, 2\kappa - \frac{\xi_{\text{meso}}}{b} \right), \\ 0 & \text{otherwise.} \end{cases}
\]

Note that \(\{|v_{\epsilon_n}| \leq \frac{7}{3} \} \subset \bigcup_j B(x_i, \epsilon_{\text{meso}}^{1/4}) \subset \bigcup_j B(x_i, \kappa \rho),\) from construction. Consequently, if \(\deg_{\partial (B_j \cap B(0,2\kappa - \xi_{\text{meso}}/b))}(v') \neq 0,\) then we have \(B_j \subset B \left( 0, \frac{3}{2}\kappa \right).\)

Therefore,
\[
\sum d_j = \deg_{\partial B(0,2\kappa)}(v') = \deg_{\partial B(x_i,2\kappa \rho)}(v_{\epsilon_n})
\]
and
\[
\frac{1}{2} \int_{B(x_i,2\kappa \rho)} \left( |\nabla v_{\epsilon_n}|^2 + \frac{1}{2\xi_{\text{meso}}^2} (1 - |v_{\epsilon_n}|^2)^2 \right) \geq \pi \deg_{\partial B(x_i,2\kappa \rho)}(v_{\epsilon_n}) \ln \xi_{\text{meso}} - O(1)
\]
\[
= \pi \deg_{\partial B(x_i,2\kappa \rho)}(v_{\epsilon_n}) \ln \frac{\rho}{\epsilon_n} - O(1).
\]

Thus \((4.11)\) holds.

The last assertion is obtained using Lemmas 4.1 and 4.2. Indeed, note that the proof of \((4.11)\) gives a more precise result:
\[
F_{\epsilon_n} \left( v_{\epsilon_n}, B \left( x_i, \frac{3}{2}\kappa \rho \right) \right) \geq b^2 \pi \deg_{\partial B(x_i,2\kappa \rho)}(v_{\epsilon_n}) \ln \frac{\rho}{\epsilon_n} - O(1).
\]

Let \(x \in \Omega \setminus \bigcup_j B(x_i, 2\kappa \rho);\) then \(B(x, \epsilon_{\text{meso}}^{1/4}) \cap B(x_i, \frac{3}{2}\kappa \rho) = \emptyset.\) Consequently, using Lemma 4.1 and the previous lower bound, we obtain
\[
F_{\epsilon_n}(v_{\epsilon_n}, B(x, \epsilon_{\text{meso}}^{1/4}) \leq J_{2\kappa \rho, \epsilon_n} + C_0 \leq \pi d |\ln \rho| + \tilde{C}_0.
\]

Thus, from Lemma 4.2, there is \(C > 0,\) independent of \(x,\) such that \(|v_{\epsilon_n}(x)| \geq 1 - C \sqrt{|\ln \rho|/|\ln \epsilon_n|} \).

4.2.2. Location and degree of bad discs. Let
\[
w_n = \frac{v_{\epsilon_n}}{|v_{\epsilon_n}|} \in H^1 \left( \Omega \setminus \bigcup_{j} B(x_i, 2\kappa \rho), \mathbb{R}^1 \right).
\]
**Proposition 4.10.** The map \( w_n \) is an almost minimal function for \( I_{2\kappa,\varepsilon} \) in the sense that
\[
\frac{1}{2} \int_{\Omega \setminus \bigcup_{j'} B(x_i, 2\kappa \rho)} U_{\varepsilon}^2 \left| \nabla w_n \right|^2 \leq I_{2\kappa,\varepsilon} + O(1).
\]

**Proof.** Denote \( K_n = \frac{1}{2} \int_{\Omega \setminus \bigcup_{j'} B(x_i, 2\kappa \rho)} U_{\varepsilon}^2 \left| \nabla w_n \right|^2; \) then we have
\[
K_n \leq F_{\varepsilon,n} \left( v_{\varepsilon,n}, \Omega \setminus \bigcup_{j'} B(x_i, 2\kappa \rho) \right) + \int_{\Omega \setminus \bigcup_{j'} B(x_i, 2\kappa \rho)} U_{\varepsilon}^2 (1 - |v_{\varepsilon,n}|^2) \left| \nabla w_n \right|^2
\]
\[
= F_{\varepsilon,n} \left( v_{\varepsilon,n}, \bigcup_{j'} B(x_i, 2\kappa \rho) \right) + \int_{\Omega \setminus \bigcup_{j'} B(x_i, 2\kappa \rho)} U_{\varepsilon}^2 (1 - |v_{\varepsilon,n}|^2) \left| \nabla w_n \right|^2
\]
\[
\leq I_{2\kappa,\varepsilon} + C \sqrt{\frac{\ln \rho}{\ln \varepsilon}} \int_{\Omega \setminus \bigcup_{j'} B(x_i, 2\kappa \rho)} U_{\varepsilon}^2 \left| \nabla w_n \right|^2 + O(1)
\]
by (4.1), Corollary 3.8, Proposition 4.9
\[
\leq I_{2\kappa,\varepsilon} + C \sqrt{\frac{\ln \rho}{\ln \varepsilon n}} F_{\varepsilon,n} \left( v_{\varepsilon,n}, \bigcup_{j'} B(x_i, 2\kappa \rho) \right) + O(1)
\]
by (4.1), Proposition 4.9
\[
\leq I_{2\kappa,\varepsilon} + C \sqrt{\frac{\ln \rho}{\ln \varepsilon n}} + O(1)
\]
by (4.1), (4.2), Proposition 4.9
\[
\leq I_{2\kappa,\varepsilon} + O(1)
\]
by (4.8). \( \square \)

**Remark 4.11.** Note that the penultimate line in the proof of Proposition 4.10 is the main use of (1.3) (which is expressed in (4.8)).

By combining Proposition 3.6 with Proposition 4.10 in the periodic case, or with Proposition 3.12 if the pinning term is not periodic, we obtain the following result.

**Corollary 4.12.** The configuration
\[
\{(x_1, \ldots, x_N), (\deg_{\partial B(x_i, 2\kappa \rho)} (v_{\varepsilon,n}), \ldots, \deg_{\partial B(x_{N'}, 2\kappa \rho)} (v_{\varepsilon,n}))\}
\]
is an almost minimal configuration of \( I_{2\kappa,\varepsilon} \); consequently, \( \deg_{\partial B(x_i, 2\kappa \rho)} (v_{\varepsilon,n}) = 1 \) for all \( i, N' = d \), and there is \( \eta_0 > 0 \) independent of large \( n \) such that
\[
\min_{i \neq j} \min_{i} |x_i - x_j|, \min_{i} \text{dist}(x_i, \partial \Omega) > 2\eta_0, \quad B(x_i, 2\eta_0 \lambda \delta) \subset \omega_{\varepsilon}.
\]

**4.3.** \( H^1_{\text{loc}} \)-weak convergence. In order to keep notation simple, from now on we replace \( 2\kappa \rho \) by \( \rho/2 \).
Using Corollary 4.12, there is \( \{a_1, \ldots, a_d\} \subset \Omega \) such that (possibly after passing to a subsequence), we have \( x_i^n = x_i \to a_i \). Let \( \rho_0 > 0 \) be defined as
\[
\rho_0 = 10^{-2} \cdot \min_{k \neq \ell} \{ \text{dist}(a_k, \partial \Omega), |a_k - a_\ell| \}.
\]

4.3.1. The contribution of the modulus is bounded in the whole domain.
We are now in position to bound the potential part of \( F_\varepsilon(v_\varepsilon) \). More precisely, we have the following result.

**Proposition 4.13.** We have
\[
\int_\Omega \left\{ |\nabla| v_{\varepsilon n} | |^2 + \frac{1}{\varepsilon_n^2} (1 - |v_{\varepsilon n}|^2)^2 \right\} = O(1).
\]

**Proof.** From \((4.1)\), Proposition 4.9 (Assertions (1), (2) and (3)), and Proposition 4.10, we infer that
\[
\int_{\Omega \cup B(x_i, \rho/2)} \left\{ |\nabla| v_{\varepsilon n} | |^2 + \frac{1}{\varepsilon_n^2} (1 - |v_{\varepsilon n}|^2)^2 \right\} = O(1).
\]
Consequently, it suffices to obtain a similar estimate in \( B(x_i, \rho) \). Note that \( B(x_i, \rho) \subset \omega_{\varepsilon n} \). Thus, if we set
\[
u'(x) = \frac{u_{\varepsilon n}(x_i + \rho x)}{b} : B(0, 1) \to \mathbb{C},
\]
then \( u' \) solves
\[
-\Delta u' = \frac{1}{[\varepsilon_n/(b \rho)]^2} u'(1 - |u'|^2) \quad \text{in} \quad B(0, 1).
\]

From [BOS05], we obtain
\[
\frac{1}{2} \int_{B(0,1/2)} \left\{ |\nabla| u'| |^2 + \frac{b^2 \rho^2}{2 \varepsilon_n^2} (1 - |u'|^2)^2 \right\} = O(1).
\]
This estimate is the subject of Theorem 1 (for the potential part) and Proposition 1 in [BOS05] (for the gradient of the modulus); see also Corollary 1 in [BOS05].

Set \( K_n = \frac{1}{2} \int_{B(0,1/2)} \left\{ |\nabla| u'| |^2 + (b^2 \rho^2/(2\varepsilon_n^2))(1 - |u'|^2)^2 \right\}. \) Using Proposition 1.4, we obtain
\[
K_n = O(1) = \frac{1}{2} b^2 \int_{B(x_i, \rho/2)} \left\{ |\nabla| U_{\varepsilon n} v_{\varepsilon n} | |^2 + \frac{b^4}{2 \varepsilon_n^2} \left( 1 - \frac{|U_{\varepsilon n} v_{\varepsilon n}|^2}{b^2} \right)^2 \right\}
\]
\[
= \frac{1}{2} b^2 \int_{B(x_i, \rho/2)} \left\{ |\nabla| v_{\varepsilon n} | |^2 + \frac{b^2}{2 \varepsilon_n^2} (1 - |v_{\varepsilon n}|^2)^2 \right\} + o_n(1).
\]
Consequently, Proposition 4.13 holds. \( \square \)
4.3.2. We bound the energy in a fixed perforated domain.

**Proposition 4.14.** For $0 < \eta \leq \rho_0$, there is $C(\eta) > 0$, independent of $n$, such that we have

\[
\frac{1}{2} \int_{\Omega \cup B(a_i, \eta)} \left| \nabla u_{\epsilon_n} \right|^2 \leq C(\eta).
\]

**Proof.** We argue by contradiction and assume that there is $\eta > 0$ such that, up to passing to a subsequence, we have \( \int_{\Omega \cup B(a_i, \eta)} \left| \nabla u_{\epsilon_n} \right|^2 \to \infty \). Then, because

\[
\int_{\Omega \cup B(a_i, \eta)} \left| \nabla u_{\epsilon_n} \right|^2 = \int_{\Omega \cup B(a_i, \eta)} \left| u_{\epsilon_n} \right|^2 \left| \nabla w_n \right|^2 + \left| \nabla \left( \left| u_{\epsilon_n} \right| \right) \right|^2,
\]

from Propositions 4.9 and 4.13 we get \( \int_{\Omega \cup B(a_i, \eta)} \left| \nabla w_n \right|^2 \to \infty \). Thus, we have

\[
\int_{\Omega \cup B(x_i, 10^{-1} \eta)} \left| \nabla w_n \right|^2 \to \infty.
\]

It is clear that we may get a map \( \tilde{w}_n \in J_{10^{-1}\eta}(x_{\epsilon_n}, 1) \) such that \( \int_{\Omega \cup B(x_i, 10^{-1} \eta)} \left| \nabla \tilde{w}_n \right|^2 \leq C(\eta) \).

For \( i = 1, \ldots, d \), using Proposition C.4 (Appendix C, Section C.3, p. 55), we get the existence of a map \( \tilde{w}_{i,n} \in H^1(B(x_i, 10^{-1} \eta) \setminus B(x_i, \rho/2), \mathbb{R}^1) \) such that

\[
\tilde{w}_{i,n}(x_i + 10^{-1} \eta e^{i\theta}) = \tilde{w}_n(x_i + 10^{-1} \eta e^{i\theta}) \text{ and}
\]

\[
\int_{B(x_i, 10^{-1} \eta) \setminus B(x_i, \rho/2) \setminus B(x_i, 10^{-1} \eta)} \frac{U_{\epsilon_n}^2}{\left| \nabla \tilde{w}_{i,n} \right|^2} \left| \nabla \tilde{w}_{i,n} \right|^2 \leq \int_{B(x_i, 10^{-1} \eta) \setminus B(x_i, \rho/2) \setminus B(x_i, 10^{-1} \eta)} \frac{U_{\epsilon_n}^2}{\left| \nabla w_n \right|^2} + O(1),
\]

Therefore by extending \( \tilde{w}_n \) with \( \tilde{w}_{i,n} \) in \( B(x_i, 10^{-1} \eta) \setminus B(x_i, \rho/2) \), we get a map still denoted \( \tilde{w}_n \in H^1_0(\Omega \setminus \bigcup B(x_i, \rho/2), \mathbb{R}^1) \) such that

\[
\frac{1}{2} \int_{\Omega \cup B(x_i, \rho/2)} \left| \nabla w_n \right|^2 - \frac{1}{2} \int_{\Omega \cup B(x_i, \rho/2)} \left| \nabla \tilde{w}_n \right|^2 \to \infty,
\]

which is in contradiction with Proposition 4.10.

Consequently, there is \( u_* \in H^1_{\text{loc}}(\hat{\Omega} \setminus \{a_1, \ldots, a_d\}, \mathbb{R}^1) \) such that, up to passing to a subsequence, \( u_{\epsilon_n} \to u_* \) in \( H^1_{\text{loc}}(\Omega \setminus \{a_1, \ldots, a_d\}) \). The next section is dedicated to the limiting equation of \( u_* \).

4.3.3. We establish the limiting equation. In order to obtain the expression of the homogenized problem, we use the unfolding operator (see [CDG08, Definition 2.1]).

In the case of a periodic pinning term, the use of the unfolding operator needs a slightly modification of the cell period. More precisely, instead of considering the \( \delta \times \delta \)-grid whose vertices are the points \( \{\delta(k, \ell) + (\frac{1}{2}, \frac{1}{2}) \mid k, \ell \in \mathbb{Z}\} \), we consider the one whose vertices are \( \{\delta(k, \ell) \mid k, \ell \in \mathbb{Z}\} \).

Thus instead of having cells which contain one inclusion at their center, we have cells with quarters of inclusion at their vertices (see Figure 4.1).
More specifically, we define, for $\Omega_0 \subset \mathbb{R}^2$, an open set and $\delta > 0$:

$$
T_\delta : L^2(\Omega_0) \to L^2(\Omega_0 \times \tilde{Y})
$$

$$
\phi \to T_\delta(\phi)(x, y) = \begin{cases} 
\phi\left(\delta\left[\frac{x}{\delta}\right] + \delta y\right) & \text{for } (x, y) \in \tilde{\Omega}_\delta^\text{incl} \times \tilde{Y}, \\
0 & \text{for } (x, y) \in \Lambda_\delta \times \tilde{Y}.
\end{cases}
$$

Here, $\tilde{Y} = (0, 1)^2$, $[s]$ is the integer part of $s \in \mathbb{R}$, and

$$
\tilde{\Omega}_\delta^\text{incl} := \bigcup_{\tilde{Y}_0^\delta \subset \Omega_0, K \in \mathbb{Z}^2} \tilde{Y}_0^\delta, \Lambda_\delta := \Omega_0 \setminus \tilde{\Omega}_\delta^\text{incl} \quad \text{and} \quad \left[\frac{x}{\delta}\right] := \left([\frac{x_1}{\delta}], [\frac{x_2}{\delta}]\right).
$$

An adaptation of a result of Sauvageot ([Sau09, Theorem 4]) gives the following result.

**Proposition 4.15.** Let $\Omega_0 \subset \mathbb{R}^2$ be a smooth bounded open set. Let $v_n \in H^2(\Omega_0, \mathbb{C})$ be such that

1. $|v_n| \leq 1$ and $\int_{\Omega_0} (1 - |v_n|^2)^2 \to 0$;
2. $v_n - v_* \text{ in } H^1(\Omega_0)$ for some $v_* \in H^1(\Omega_0, S^1)$;
3. there are $H_n \in W^{1,\infty}(\Omega_0, [b^2, 1])$ and $\delta = \delta_n \downarrow 0$ such that
   $$
   T_\delta(H_n)(x, y) \to H_0(y) \quad \text{in } L^2(\Omega_0 \times \tilde{Y});
   $$
4. $-\text{div}(H_n \nabla v_n) = v_n f_n, f_n \in L^\infty(\Omega_0, \mathbb{R}).$

Then, $v_*$ is a solution of

$$
-\text{div}(\mathcal{A} \nabla v_*) = (\mathcal{A} \nabla v_* \cdot \nabla v_*) v_*
$$

where $\mathcal{A}$ is the homogenized matrix of $H_0(\cdot/\delta) \text{Id}_{\mathbb{R}^2}$ (see Appendix F for more details about $\mathcal{A}$).

The proof of Proposition 4.15 is postponed to Appendix F.

We apply the above proposition with $\Omega_0 = \Omega \cup \bigcup B(\alpha_i, \eta)$, and $\delta = \delta_n \downarrow 0$ as the sequence which defines $a_{\epsilon_n}$ and $H_n = U_{\epsilon_n}^2$. By application of Proposition 1.4, we obtain

$$
T_\delta(U_{\epsilon_n}^2)(x, y) \to L^2(\Omega_0 \times \tilde{Y}) \begin{cases}
\tilde{a}^2(\gamma) & \text{if } \lambda \equiv 1, \\
1 & \text{if } \lambda \to 0.
\end{cases}
$$

Note that, in the case of the periodic pinning term, the $\tilde{Y}$-periodic extension of $\tilde{a}$ in $\mathbb{R}^2$ is equal to the $Y$-periodic extension of $1 - (1 - b^2)1_{\Omega_0}$ which is $a^1$ (defined as Construction 1.2).

We find that $v_*$ solves

$$
-\text{div}(\mathcal{A} \nabla v_*) = (\mathcal{A} \nabla v_* \cdot \nabla v_*) v_* \quad \text{if } \lambda \equiv 1,
$$

$$
-\Delta v_* = |\nabla v_*|^2 v_* \quad \text{if } \lambda \to 0.
$$

Here, $\mathcal{A}$ is the homogenized matrix of $[a^1(\cdot/\delta)]^2 \text{Id}_{\mathbb{R}^2}$. 
(a) Four period-cells which are obtained from \( Y = (-1/2, 1/2)^2 \), and the new period-cell (in dashes) obtained from \( \tilde{Y} = (0, 1)^2 \).

(b) The new unit-cell \( \tilde{Y} \) with four quarters of an inclusion and the values of \( \tilde{a} = a_1^\lambda \).

**Figure 4.1.** The modification of the unit cell
4.4. The small bad discs.

4.4.1. Definitions. Now, we use the bound on the potential part of the energy of the minimizers $\varepsilon^{-2} \int_{\Omega} (1 - |v_\varepsilon|^2)^2 \leq C$, Proposition 4.13.

With the help of this bound, in the spirit of [BBH94, Theorem III.3], we may detect the vorticity defects (the connected components of $\{|v_\varepsilon| \geq \frac{7}{8}\}$) by smaller discs (discs with radius of order of $\varepsilon$) than the $\rho$-bad discs (Notation 4.8).

Notation 4.16 (The small bad discs). The construction is done as follows:

- We consider a covering of $\Omega$ as in Notation 4.4 (page 25). We fix $\rho = \rho(\varepsilon) \downarrow 0$ such that Assumption (4.8) holds. For sufficiently small $\varepsilon$, we denote $S_\rho(\varepsilon) = \left\{ B \left( x, \frac{R}{2} \right) \mid B(x, R) \text{ given by Notation 4.8, page 26} \right\}$.

- Since we have that $|\nabla u_\varepsilon|, |\nabla U_\varepsilon| \leq C/\varepsilon$ (exactly as in [BBH93]), we get that $|\nabla v_\varepsilon| \leq C/\varepsilon$. Following [BBH94, Theorem III.3], for $\ell \geq 2$, there are $\kappa_\ell, \mu_\ell > 0$ (depending only on $\Omega, g$ and $\ell$) such that for $x \in \Omega$, if

$$\frac{1}{\varepsilon^2} \int_{B(x, \frac{\kappa_\ell \varepsilon}{4})} \left( 1 - |v_\varepsilon|^2 \right)^2 \leq \mu_\ell,$$

then $|v_\varepsilon| \geq 1 - \frac{1}{\ell^2}$ in $B(x, \kappa_\ell \varepsilon)$.

- We fix $\ell \geq 2$, and drop the subscript $\ell$. We now consider a covering of $\bigcup_{B \in S_\rho(\varepsilon)} B$ by discs $(B(x_i^\varepsilon, \kappa_\varepsilon))_{i \in I}$ such that

$$x_i^\varepsilon \in \bigcup_{B \in S_\rho(\varepsilon)} B, \quad \forall \ i \in I,$$

$$B \left( x_i^\varepsilon, \frac{\kappa_\varepsilon}{4} \right) \cap B \left( x_j^\varepsilon, \frac{\kappa_\varepsilon}{4} \right) = \emptyset \quad \text{if } i \neq j,$$

$$\bigcup_{i \in I} B(x_i^\varepsilon, \kappa_\varepsilon) \supset \bigcup_{B \in S_\rho(\varepsilon)} B.$$

We say that $B(x_i^\varepsilon, \kappa_\varepsilon)$ is a small good disc if $\frac{1}{\varepsilon^2} \int_{B(x_i^\varepsilon, 2\kappa_\varepsilon)} \left( 1 - |v_\varepsilon|^2 \right)^2 < \mu$.

- If $B(x_i^\varepsilon, \kappa_\varepsilon)$ is not a small good disc, then we call it a small bad disc. We denote $J \subset I$ the set of indices of small bad discs.

Following [BBH94], using Proposition 4.13, there is $N_\ell = N > 0$ (depending only on $\Omega, g$ and $\ell$) such that Card($J$) $\leq N$.

Using a standard separation process (Lemma C.1), for $\varepsilon_n \downarrow 0$ (possibly after passing to a subsequence and relabeling the discs), there exist $J^* \subset J$ and
\[ \kappa' \in \{ \kappa, \ldots, 9^{N-1}\kappa \} \text{ such that} \]
\[
\left\{ |v_{\varepsilon_n}| < 1 - \frac{1}{\ell^2} \right\} \subset \bigcup_{i \in J} B(x_{i}^{\varepsilon_n}, \kappa \varepsilon_n) \subset \bigcup_{i \in J'} B(x_{i}^{\varepsilon_n}, \kappa' \varepsilon_n)
\]

and
\[
\frac{|x_{i}^{\varepsilon_n} - x_{j}^{\varepsilon_n}|}{\varepsilon_n} \geq 8\kappa' \text{ if } i, j \in J', i \neq j.
\]

By a standard iterative procedure, we may assume that the small bad discs are mutually far away in the \( \varepsilon \)-scale.

**Proposition 4.17.** Possibly after passing to a subsequence, we have, for large \( R \) and \( J'' \subset J' \),
\[
\left\{ |v_{\varepsilon_n}| < 1 - \frac{1}{\ell^2} \right\} \subset \bigcup_{i \in J''} B(x_{i}^{\varepsilon_n}, R\varepsilon_n),
\]
where, for \( i \neq j \),
\[
\frac{|x_{i}^{\varepsilon_n} - x_{j}^{\varepsilon_n}|}{\varepsilon_n} \to \infty \text{ as } n \to \infty.
\]

**Notation 4.18 (The small and separated bad discs).** The elements of the set \( \{ B(x_{i}^{\varepsilon_n}, R\varepsilon_n) \mid i \in J'' \} \) obtained in Proposition 4.17 are the small and separated bad discs.

**4.4.2. Each \( \rho \)-bad disc contains exactly one small and separated bad disc.**

By construction, we know that the small and separated bad discs (defined in Notation 4.18) are covered by the \( \rho \)-bad discs defined in Notation 4.8 (page 26).

We next prove that there are exactly \( d \) small bad discs, and consequently, there is exactly one small and separated bad disc per \( \rho \)-bad disc.

**Proposition 4.19.** For large \( n \) and for all \( i \in J'' \), we have
\[
\deg_{\partial B(x_{i}^{\varepsilon_n}, R\varepsilon_n)}(v_{\varepsilon_n}) = 1.
\]

**Proof.** First we prove that, for large \( n \) and for all \( i \), we have
\[
\deg_{\partial B(x_{i}^{\varepsilon_n}, R\varepsilon_n)}(v_{\varepsilon_n}) \neq 0.
\]

We argue by contradiction and assume that, up to a subsequence, there is \( i \) such that we have \( \deg_{\partial B(x_{i}^{\varepsilon_n}, R\varepsilon_n)}(v_{\varepsilon_n}) = 0. \)

Set
\[
M_n = \min \left\{ b \min_{i \neq j} \frac{|x_{i}^{\varepsilon_n} - x_{j}^{\varepsilon_n}|}{8R\varepsilon_n}, \delta^{-1} \right\}
\]
and set
\[
u_{\varepsilon_n} : B(0, M_n) \to \mathbb{C}, \quad x \mapsto \frac{u_{\varepsilon_n}((\varepsilon_n/b)x + x_{i}^{\varepsilon_n})}{b}.
\]
Note that \( B(x_i^k, M_n \epsilon_n) \subset \omega_{\epsilon_n} \), and by Proposition 4.17, we have \( M_n \to \infty \).

It is easy to see that \( u_n' \) solves \( -\Delta u_n' = u_n'(1 - |u_n'|^2) \). Following [BMR94], up to a subsequence, we have

\[
(4.14) \quad u_n' \to u_0 \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^2);
\]

here, \( u_0 : \mathbb{R}^2 \to \mathbb{C} \) solves \( -\Delta u_0 = u_0(1 - |u_0|^2) \) in \( \mathbb{R}^2 \).

Two cases then occur: \( \int_{\mathbb{R}^2} (1 - |u_0|^2)^2 < \infty \) or \( \int_{\mathbb{R}^2} (1 - |u_0|^2)^2 = \infty \). Assume first that \( \int_{\mathbb{R}^2} (1 - |u_0|^2)^2 < \infty \). From [BMR94], noting that the degree of \( u_0 \) on large circles centered in 0 is 0, we obtain that \( u_0 = Cyt \in \mathbb{S}^1 \) and consequently \( \int_{\mathbb{R}^2} (1 - |u_0|^2)^2 = 0 \).

Since \( u_n' \to u_0 \) in \( L^4(B(0, 2bR)) \) \( (R \geq \kappa) \), we find that

\[
\int_{B(0, 2bR)} (1 - |u_n'|^2)^2 = \frac{b^2}{\epsilon_n^2} \int_{B(x_i^k, 2R \epsilon_n)} (1 - \frac{|u_n|}{b})^2 \\
= \frac{b^2}{\epsilon_n^2} \int_{B(x_i^k, 2R \epsilon_n)} (1 - |u_{\epsilon_n}|^2)^2 + o_n(1) \to 0.
\]

Since \( B(x_i^k, \kappa \epsilon_n) \) is a small bad disc and \( B(x_i^k, 2 \kappa \epsilon_n) \subset B(x_i^k, 2R \epsilon_n) \), we have a contradiction.

Therefore \( \int_{\mathbb{R}^2} (1 - |u_0|^2)^2 = \infty \). Consequently, there is \( M_0 > 0 \) such that

\[
\int_{B(0, bM_0)} (1 - |u_0|^2)^2 \geq \sup_n \left\{ \frac{4b^2}{\epsilon_n^2} \int_{\Omega} (1 - |v_{\epsilon_n}|^2)^2 \right\}.
\]

Thus, for large \( n \) we have

\[
\int_{B(0, bM_n)} (1 - |u_n'|^2)^2 = \frac{b^2}{\epsilon_n^2} \int_{B(x_i^k, M_n \epsilon_n)} (1 - \frac{|u_{\epsilon_n}|}{b})^2 \\
= \frac{b^2}{\epsilon_n^2} \int_{B(x_i^k, M_n \epsilon_n)} (1 - |v_{\epsilon_n}|^2)^2 + o_n(1) \\
\geq \sup_n \left\{ \frac{2b^2}{\epsilon_n^2} \int_{\Omega} (1 - |v_{\epsilon_n}|^2)^2 \right\},
\]

which is a contradiction with \( B(x_i^k, M_n \epsilon_n) \subset \Omega \).

Consequently, we obtain that for large \( n \), \( \deg_{\partial B(x_i^k, R \epsilon_n)} (v_{\epsilon_n}) \neq 0 \).

We now prove that

\[
(4.15) \quad \deg_{\partial B(x_i^k, R \epsilon_n)} (v_{\epsilon_n}) = 1 \quad \text{for all } i \text{ and large } n.
\]
Note that each small bad disc contains at least a zero of $v_{\varepsilon}$. Consequently, for $\rho$ satisfying (4.8), all small bad discs are included in a $\rho$-bad disc $B(y, \rho)$ defined in Notation 4.8 (page 26). (For the sake of simplicity, we wrote $B(y, \rho)$ instead of $B(y, 2\kappa \rho)$).

If $B(y, \rho)$ is a $\rho$-bad disc, we denote $\Lambda_y = \{ i \in J'' \mid x_i^{\varepsilon_n} \in B(y, \rho) \}$. Clearly, if $\text{Card}(\Lambda_y) = 1$, then (4.15) holds.

We define

$$a^y_n := \begin{cases} 10^{-2} \min_{i,j \in \Lambda_y, i \neq j} |x_i^{\varepsilon_n} - x_j^{\varepsilon_n}| & \text{if Card}(\Lambda_y) > 1, \\ R\varepsilon_n & \text{otherwise.} \end{cases}$$

From Proposition 4.17, if Card($\Lambda_y$) > 1, then $a^y_n / \varepsilon_n \to \infty$.

For simplicity, we assume that $y = 0$, and we let$
\tilde{B} = B(0, 8) \setminus \bigcup_{i \in \Lambda_0} B\left(x_i^{\varepsilon_n}, \frac{a^0_n}{\rho}, \frac{a^0_n}{\rho} \right)$. 

**Remark 4.20.** Note that from Corollary 4.12 we have $B(y, 16\rho) \subset \omega_\varepsilon$. Clearly, we are in position to apply Theorem 2 in [HS95] in the perforated domain $\tilde{B}$. After scaling, we find that

$$\frac{1}{2} \int_{B(y, 8\rho) \setminus \bigcup B(x_i^{\varepsilon_n}, a^y_n)} |\nabla v_{\varepsilon_n}|^2 \geq \pi \left| \sum_{i \in \Lambda_y} \deg_{\partial B(x_i^{\varepsilon_n}, R\varepsilon_n)}(v_{\varepsilon_n}) \right| \ln \frac{\rho}{a^y_n} - C.$$ 

In order to prove (4.15), we observe the case where there exists $y$ such that Card($\Lambda_y$) > 1. Recall that if for all $y$, center of a $\rho$-bad disc, we have Card($\Lambda_y$) = 1, then (4.15) holds. Since $\deg_{\partial B(x_i^{\varepsilon_n}, R\varepsilon_n)}(v_{\varepsilon_n}) \neq 0$, if Card($\Lambda_y$) > 1, then we have

$$\sum_{i \in \Lambda_y} \left| \deg_{\partial B(x_i^{\varepsilon_n}, R\varepsilon_n)}(v_{\varepsilon_n}) \right| > 1.$$ 

We obtain easily the following lower bound for $i \in \Lambda_y$:

$$\frac{1}{2} \int_{B(x_i^{\varepsilon_n}, a^y_n) \setminus B(x_i^{\varepsilon_n}, R\varepsilon_n)} |\nabla v_{\varepsilon_n}|^2 \geq \pi \left| \deg_{\partial B(x_i^{\varepsilon_n}, R\varepsilon_n)}(v_{\varepsilon_n}) \right| \frac{a^y_n}{R\varepsilon_n} - C.$$ 

Summing for $i \in \Lambda_y$, we obtain that

$$\sum_{i \in \Lambda_y} \frac{1}{2} \int_{B(x_i^{\varepsilon_n}, a^y_n) \setminus B(x_i^{\varepsilon_n}, R\varepsilon_n)} |\nabla v_{\varepsilon_n}|^2 \geq 2\pi \left| \frac{a^y_n}{R\varepsilon_n} - C. \right.$$
Consequently, we deduce that
\[
\sum_{y'} \frac{1}{2} \int_{B(y',\delta)} |\nabla v_{\varepsilon_n}|^2 \\
\geq \pi d \ln \frac{\delta}{R_{\varepsilon_n}} + \pi \sum_{y \text{ such that } \text{Card}(A_y) > 1} \ln \frac{\varepsilon_n}{R_{\varepsilon_n}} - O(1).
\]

From Lemma 4.1 and Propositions 4.9 and 4.10, we get
\[
\frac{1}{2} \int_{\bigcup B(y',\delta) \cup \bigcup B(x_{\varepsilon_n},R_{\varepsilon_n})} U_{\varepsilon_n}^2 |\nabla v_{\varepsilon_n}|^2 = \pi d b^2 \ln \frac{\delta}{R_{\varepsilon_n}} + O(1).
\]

Combining the previous estimates, we obtain that
\[
\{ y \text{ center of } \rho \text{-bad discs } | \text{Card}(A_y) > 1 \} = \emptyset,
\]
and thus \( \text{deg}_{B(x_{\varepsilon_n},R_{\varepsilon_n})}(v_{\varepsilon_n}) = 1 \) for large \( n \).

\[\Box\]

**Corollary 4.21.** For large \( n \), there is a unique zero inside each small and separated bad disc defined in Notation 4.18 (page 34).

**Proof.** From Proposition 4.19, one may assume that \( v_{\varepsilon_n}(x_{\varepsilon_n}) = 0 \).

Let \( i \in \{1, \ldots, d\} \). In view of (4.14), if we denote
\[
(4.16) \quad u_0 : B(0, M_n) \to \mathbb{C}, \quad x \mapsto \frac{u_{\varepsilon_n}((\varepsilon_n/b)x + x_{\varepsilon_n})}{b},
\]
then, up to a subsequence,
\[
(4.17) \quad u_n \to u_0 \text{ in } C^1(B(0,bR)).
\]

Here, \( M_n \) is defined in (4.13).

Using the main result of [Mir96b], we have the existence of a universal function \( f : \mathbb{R}^+ \to [0,1] \) such that
\[
(4.18) \quad u_0(x) = f(|x|) e^{i(\theta x + \theta_1)}
\]
where \( x = |x| e^{i\theta} \), \( \theta_1 \in \mathbb{R} \), and \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) is increasing.

Therefore, we may apply Theorem 2.3 in [BCP93] in order to obtain that, for large \( n \), \( u_n \) has a unique zero in \( B(0,bR) \). Consequently, for large \( n \), \( v_{\varepsilon_n} \) has a unique zero in \( B(x_{\varepsilon_n}^{\varepsilon_n},R_{\varepsilon_n}) \). \( \Box \)
Corollary 4.22. One may consider that $R$ depends only on $\ell$ (is independent of the extraction we consider); that is, for $\ell \geq 2$, there is $R_\ell > 0$ such that for small $\varepsilon$, denoting $\{x_\ell^i | i \in \{1, \ldots, d\}\}$ (the set of zeros of a minimizer $v_\ell$), we have

$$\left\{ |v_\varepsilon| < 1 - \frac{1}{\ell^2} \right\} \subset \bigcup_i B(x_\ell^i, R_\ell \varepsilon).$$

Proof. From Corollary 4.21, one may assume that $v_{\varepsilon_n}(x_{\ell_n}^i) = 0$.

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as in (4.18) and $u_n$ as in (4.16). For $\ell \geq 2$, consider $R_\ell > 0$ such that

$$\ell \mapsto R_\ell$$

is increasing and $f(b R_\ell) \geq 1 - \frac{1}{2 \ell^2}$.

Note that from [Sha94], one may consider $R_\ell \approx \sqrt{2 \ell/b}$.

The full sequence $|u_n|$ converges to $f$ in $L^\infty[B(0, b \max\{R, R_\ell\})]$, by uniqueness of $f$. Consequently, for $n$ sufficiently large, since $f$ is not decreasing,

$$\left\{ |v_{\varepsilon_n}| < 1 - \frac{1}{\ell^2} \right\} \subset \bigcup_i B(x_{\ell_n}^i, R_{\varepsilon_n}).$$

\[ \square \]

4.5. Asymptotic expansion of $F_\varepsilon(v_\varepsilon)$. This section is essentially devoted to prove Theorem 2.7. The key argument in this proof is Proposition 4.23.

4.5.1. Statement of the main result and a corollary. We state a technical and fundamental result and a direct corollary.

Proposition 4.23. For all $\varepsilon_n \downarrow 0$, up to a subsequence, there is $\rho = \rho(\varepsilon_n)$ such that $\varepsilon_n \ll \rho \ll \lambda \delta^{3/2}$ and such that, when $n \to \infty$, the following holds:

$$F_{\varepsilon_n}(v_{\varepsilon_n}) \geq J_{\rho, \varepsilon_n} + db^2 \left( \pi \ln \frac{b \rho}{\varepsilon_n} + \gamma \right) + o_n(1),$$

where $J_{\rho, \varepsilon}$ is defined in (3.6) and $\gamma$ is the universal constant defined in [BBH94, Lemma IX.1].

Proposition 4.23 is proved in Subsection 4.5.3.

Corollary 4.24. Let $\varepsilon_n \downarrow 0$, $\rho$ be as in Proposition 4.23. Then we have

$$J_{\varepsilon_n, \varepsilon_n} - J_{\rho, \varepsilon_n} = \pi db^2 \ln \frac{\rho}{\varepsilon_n} + o_n(1).$$

Proof. Using Proposition 3.3, we may consider $x_n = (x_1^n, \ldots, x_d^n) \in \Omega^d$ a minimal configuration of points for $J_{\rho, \varepsilon_n}$, that is, such that $J_{\rho, \varepsilon_n}(x_n, 1) = J_{\rho, \varepsilon_n}$. Combining Corollaries 3.7 and 3.8 (or Proposition 3.12 if the pinning term is not periodic), we have the existence of $c > 0$ such that $B(x_i^n, c \lambda \delta) \subset \omega_{\varepsilon_n}$. 
Therefore, for a minimal map $w_n$ of $\hat{J}_{\rho,\varepsilon_n}(x_n,1)$ (Proposition 3.2), we may easily construct a map $\tilde{w}_n \in H^1(\Omega \setminus \bigcup_i B(x_i,\varepsilon_n), S^1)$ such that $\tilde{w}_n \in J_{\varepsilon_n}(x_n,1)$ and

$$J_{\varepsilon_n,\varepsilon_n} \leq \frac{1}{2} \int_{\Omega \setminus \bigcup_i B(x_i,\varepsilon_n)} U_{\varepsilon_n}^2 |\nabla \tilde{w}_n|^2$$

$$= \frac{1}{2} \int_{\Omega \setminus \bigcup_i B(x_i,\rho)} U_{\varepsilon_n}^2 |\nabla w_n|^2 + \frac{1}{2} \int_{\bigcup_i B(x_i,\rho) \setminus \bigcup_i B(x_i,\varepsilon_n)} U_{\varepsilon_n}^2 |\nabla \tilde{w}_n|^2$$

$$= J_{\rho,\varepsilon_n} + d b^2 \pi \ln \frac{\rho}{\varepsilon_n} + o_n(1).$$

On the other hand, Lemma 4.1 combined with Proposition 4.23 yields

$$J_{\rho,\varepsilon_n} + d b^2 \left( \pi \ln \frac{b \rho}{\varepsilon_n} + \gamma \right) + o_n(1) \leq F_{\varepsilon_n}(v_{\varepsilon_n}) \leq J_{\varepsilon_n,\varepsilon_n} + d b^2 (\pi \ln b + \gamma).$$

We conclude with the help of (4.20) and (4.21). \qed

**4.5.2. Proof of Theorem 2.7.** We are now in a position to prove Theorem 2.7, that is, that

$$F_{\varepsilon}(v_{\varepsilon}) = J_{\varepsilon,\varepsilon} + d b^2 (\pi \ln b + \gamma) + o_{\varepsilon}(1).$$

Indeed, using Lemma 4.1, it suffices to prove that

$$F_{\varepsilon}(v_{\varepsilon}) \geq J_{\varepsilon,\varepsilon} + d b^2 (\pi \ln b + \gamma) + o_{\varepsilon}(1).$$

This estimate is equivalent to the statement that, for all $\varepsilon_n \downarrow 0$, up to a subsequence, we have

$$F_{\varepsilon_n}(v_{\varepsilon_n}) \geq J_{\varepsilon_n,\varepsilon_n} + d b^2 (\pi \ln b + \gamma) + o_{\varepsilon_n}(1).$$

Let $\varepsilon_n \downarrow 0$. Then, up to a subsequence, there is $\rho = \rho_n$ given by Proposition 4.23 such that

$$F_{\varepsilon_n}(v_{\varepsilon_n}) \geq J_{\rho,\varepsilon_n} + d b^2 \left( \pi \ln \frac{b \rho}{\varepsilon_n} + \gamma \right) + o_{\varepsilon_n}(1).$$

We deduce from Corollary 4.24 that

$$F_{\varepsilon_n}(v_{\varepsilon_n}) \geq J_{\varepsilon_n,\varepsilon_n} - d b^2 \ln \frac{\rho}{\varepsilon_n} + d b^2 \left( \pi \ln \frac{b \rho}{\varepsilon_n} + \gamma \right) + o_{\varepsilon_n}(1)$$

$$= J_{\varepsilon_n,\varepsilon_n} + d b^2 (\pi \ln b + \gamma) + o_{\varepsilon_n}(1),$$

which ends the proof of Theorem 2.7.
4.5.3. Proof of Proposition 4.23. In order to construct $\rho$, we first define a suitable extraction.

For $\ell \in \mathbb{N} \setminus \{0, 1\}$, consider $R_\ell$ given by Corollary 4.22. Using Proposition 4.19 and Corollary 4.21, for sufficiently large $n$, $v_{\varepsilon_n}$ has exactly $d$ zeros $x_{1}^{\varepsilon_n} = x_{1}, \ldots, x_{d}^{\varepsilon_n} = x_{d}$. Clearly, these zeros are well separated and far from $\partial \Omega$ (independently of $n$).

Fix $i \in \{1, \ldots, d\}$, and consider

$$u_n' : B \left( 0, \frac{\lambda^2 \delta^2}{\varepsilon_n} \right) \to \mathbb{C}, \quad x \mapsto \frac{u_{\varepsilon_n}((\varepsilon_n/b)x + x_i)}{b}. $$

For simplicity, assume $x_i = 0$.

Up to a subsequence, one has, as in (4.18),

$$u_n' \to u_0 \text{ in } C^2_{\text{loc}}(\mathbb{R}^2, \mathbb{C}), \quad u_0(x) = f(|x|)e^{i(\theta + \phi)}$$

where $x = |x|e^{i\theta}$, $\theta \in \mathbb{R}$, and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing.

Consequently, for $\ell \in \mathbb{N} \setminus \{0, 1\}$, one may construct an extraction $(n_\ell)_{\ell \geq 2}$ such that, denoting $u_n' = u_\ell = |u_\ell|e^{i(\theta + \phi)}$ and $v_{\varepsilon_n} = v_\ell$, we have

(4.22) \[ \left\{ |v_\ell| < 1 - \frac{1}{\ell^2} \right\} \subset \bigcup_i B(x_i, R_\ell \varepsilon_n), \]

$$\rho_\ell := R_\ell \varepsilon_n \leq \frac{\lambda^2 \delta^2}{\ell},$$

(4.23) \[ \left| \int_{B(0, \ell b R_\ell)} \left| \nabla u_\ell \right|^2 \right| + \frac{1}{2} (1 - |u_\ell|^2)^2 - \int_{B(0, \ell b R_\ell)} |\nabla u_0|^2 + \frac{1}{2} (1 - |u_0|^2)^2 \right| \leq \frac{1}{\ell}, \]

and

(4.24) \[ \| \phi_\ell - \phi \|_{L^\infty(B(0, 2\ell b R_\ell))} \leq \frac{1}{\ell}. \]

Here, $R_\ell \approx \sqrt{\ell}/b$ and is defined in Corollary 4.22.

Following the proof of Proposition 1, Step 2 in [BMR94], one has

(4.25) \[ \int_{B(0, \lambda^2 \delta^2 \varepsilon_n)} \left| \nabla \phi_\ell \right|^2 \leq C \quad \text{independently of } \ell. \]

In $B(0, \lambda^2 \delta^2 \varepsilon_n) \setminus B(0, \varepsilon_n)$, we denote $v_{n_\ell} = v_\ell = |v_\ell|e^{i(\theta + \phi)} (e^{i\theta} = x/|x|)$.

By conformal invariance, (4.24) implies that

(4.26) \[ \| \phi_\ell - \phi \|_{L^\infty(\partial B(0, 2\ell b R_\ell))} + |\phi_\ell|_{H^{1/2}(\partial B(0, 2\ell b R_\ell))} \leq \frac{C}{\ell}. \]
Denote $W_\ell = B(0, 2\rho_\ell) \setminus \overline{B(0, \rho_\ell)}$ and consider $\psi_\ell^i \in H^{1/2}(\partial W_\ell, \mathbb{R})$ such that

$$\psi_\ell^i = \begin{cases} \phi_\ell - \theta_i & \text{on } \partial B(0, \rho_\ell), \\ 0 & \text{on } \partial B(0, 2\rho_\ell). \end{cases}$$

Using (4.26), it is clear that $\|\psi_\ell^i\|_{L^\infty(\partial W_\ell)} + |\psi_\ell^i|_{H^{1/2}(\partial W_\ell)} = O(1/\ell)$. From this, it is straightforward that there exists a constant $C_0 > 0$ (independent of $\ell$) and $\Psi_\ell^i \in H^1(W_\ell, \mathbb{R})$ such that

$$\text{tr}_{\partial W_\ell} \psi_\ell^i = \psi_\ell^i \quad \text{and} \quad \frac{1}{2} \int_{W_\ell} |\nabla \psi_\ell^i|^2 \leq \frac{C_0}{\ell^2}.$$  

Finally, we define $\Psi_\ell \in H^1(\Omega \setminus \bigcup B(x_i, \rho_\ell), \mathbb{R})$ by

$$\Psi_\ell = \begin{cases} \Psi_\ell^i (\cdot - x_i) & \text{in } x_i + W_\ell, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\tilde{w}_\ell = \frac{v_\ell}{|v_\ell|} e^{-i\psi_\ell^i} \in J_\ell(x_i, 1) \quad \text{with } x = (x_1, \ldots, x_d).$$

Therefore, denoting $w_\ell = v_\ell/|v_\ell| = e^{i(\theta + \phi_\ell)}$, $U_\ell = U_{\ell\eta}$, and $\Omega_{\rho_\ell} = \Omega \setminus \overline{B(x_i, \rho_\ell)}$, we have

$$\hat{J}_{\rho_\ell, \eta}(x_i, 1) \leq \frac{1}{2} \int_{\Omega_{\rho_\ell}} U_\ell^2 |\nabla \tilde{w}_\ell|^2 - \frac{1}{2} \int_{\Omega_{\rho_\ell}} U_\ell^2 |\nabla w_\ell|^2 + 2U_\ell^2 \nabla (\theta + \phi_\ell) \cdot \nabla \Psi_\ell + o_\ell(1).$$

From (4.25), we obtain easily that

$$\left| \int_{\Omega_{\rho_\ell}} \nabla (\theta + \phi_\ell) \cdot \nabla \Psi_\ell \right| = \sum_i \left| \int_{x_i + W_\ell} \nabla (\theta + \phi_\ell) \cdot \nabla \Psi_\ell^i (\cdot - x_i) \right| = o_\ell(1)$$

and consequently

$$(4.27) \quad \hat{J}_{\rho_\ell, \eta}(x, 1) \leq \frac{1}{2} \int_{\Omega_{\rho_\ell}} U_\ell^2 |\nabla w_\ell|^2 + o_\ell(1).$$

On the other hand, from direct computations, one has

$$\frac{1}{2} \int_{\Omega_{\rho_\ell}} U_\ell^2 |\nabla v_\ell|^2 \geq \frac{1}{2} \int_{\Omega_{\rho_\ell}} U_\ell^2 |\nabla w_\ell|^2 + \frac{1}{2} \int_{\Omega_{\rho_\ell}} U_\ell^2 (|v_\ell|^2 - 1) |\nabla (\theta + \phi_\ell)|^2.$$
Using the same argument as Mironescu in [Mir96b], one may obtain that

\[(4.28) \quad \frac{1}{2} \int_{\Omega_{\rho_\ell}} (1 - |v_\ell|^2)^{1/2} |\nabla (\vartheta + \phi_\ell)|^2 \leq C, \quad \text{with } C \text{ independent of } \ell.\]

From (4.28) and (4.22), we obtain

\[
\frac{1}{2} \int_{\Omega_{\rho_\ell}} U_\ell^2 |\nabla v_\ell|^2 \geq \frac{1}{2} \int_{\Omega_{\rho_\ell}} U_\ell^2 |\nabla w_\ell|^2 - o_\ell(1).
\]

Therefore, with (4.27),

\[(4.29) \quad F_{\epsilon_{n_\ell}}(v_\ell, B(x_i, \rho_\ell)) + o_\ell(1) \geq \frac{1}{2} \int_{\Omega_{\rho_\ell}} U_\ell^2 |\nabla \vartheta|^2 + o_\ell(1) \geq \hat{J}_{\rho_\ell, \epsilon_{n_\ell}}(x, 1).
\]

In order to complete the proof of (4.19), it suffices to estimate the contribution of the discs $B(x_i, \rho_\ell)$.

Using (4.23), we have

\[
F_{\epsilon_{n_\ell}}(v_\ell, B(x_i, \rho_\ell)) = \frac{b^2}{2} \int_{B(0, R_\ell)} |u_\ell| \frac{1}{b} \left(1 - \frac{|u_\ell|^2}{b^2}\right)^2 + o_\ell(1)
\]

\[
= \frac{b^2}{2} \int_{B(0, R_\ell)} |\nabla u_\ell|^2 + \frac{1}{2} (1 - |u_\ell|^2)^2 + o_\ell(1)
\]

\[
= \frac{b^2}{2} \int_{B(0, R_\ell)} |\nabla u_0|^2 + \frac{1}{2} (1 - |u_0|^2)^2 + o_\ell(1).
\]

From Proposition 3.11 in [SS07], we have

\[
\frac{1}{2} \int_{B(0, R_\ell)} |\nabla u_0|^2 + \frac{1}{2} (1 - |u_0|^2)^2 = \pi \ln(b R_\ell) + y + o_\ell(1),
\]

and hence

\[(4.30) \quad F_{\epsilon_{n_\ell}}(v_\ell, B(x_i, \rho_\ell)) = b^2 \left[\pi \ln(b R_\ell) + y\right] + o_\ell(1).
\]

By combining (4.29) with (4.30), we obtain (4.19) with $\rho_\ell = R_\ell \epsilon_{n_\ell}$.

### 4.6. Proof of Theorems 1.5, 2.2, 2.5, and 2.6.

We prove the Quantization part of Theorems 1.5 and 2.2.

- The existence of exactly $d$ zeros is a direct consequence of Corollary 4.21.
- By Proposition 4.19 and Corollary 4.21, we obtain that the zeros are well separated, well included in $\omega_\ell$, and that $v_\ell$ has a degree equal to 1 on small circles around the zeros.
The lower bound for $|\nu_\varepsilon|$ is given by Proposition 4.9.4 in the case where $\rho$ satisfies (4.8). The general case is obtained by noting that

$$F_\varepsilon(v_\varepsilon, \Omega \setminus \bigcup \overline{B(x_i^\varepsilon, \rho/2)} ) \leq \frac{1}{2} \int_{\Omega \setminus \bigcup \overline{B(x_i^\varepsilon, \rho/2)} } U_{\varepsilon, x_i}^2 \left| \nabla \frac{v_\varepsilon}{|v_\varepsilon|} \right|^2 + O(1) \leq \pi d |\ln \rho| + O(1) \leq \pi(d + 1) |\ln \rho|.$$

Here, the estimate

$$\frac{1}{2} \int_{\Omega \setminus \bigcup \overline{B(x_i^\varepsilon, \rho/2)} } U_{\varepsilon, x_i}^2 \left| \nabla \frac{v_\varepsilon}{|v_\varepsilon|} \right|^2 \leq \pi d |\ln \rho| + O(1)$$

is easily obtained by arguing by contradiction. It then suffices to apply Lemma 4.2 with $\chi = \sqrt{\frac{\pi(d + 1) |\ln \rho|}{|\ln \varepsilon|}}$ in order to get the result. \( \square \)

We prove the Macroscopic Location section of Theorem 1.5. This part of Theorem 1.5 is a direct consequence of Theorem 2.7 (proved in Subsection 4.5.2), Proposition 4.9, (4.30), and Proposition 3.10.

Indeed, from Theorem 2.7, Proposition 4.9.4, and (4.30), we get that the zeros form a quasi-minimizer of $J_{\rho, \varepsilon}$ (defined in Notation 3.9, page 20). By using Proposition 3.10, we deduce that they are a quasi-minimizer of the renormalized energy $W_\theta$ (defined in Notation 3.9). Thus, by smoothness of $W_\theta$, the zeros tend to a minimal configuration of $W_\theta$. \( \square \)

We prove the Microscopic Location section of Theorem 1.5 and 2.6. For the case where $\lambda \to 0$, the fact that we may localize the zeros inside the inclusions (Microscopic Location section of Theorem 1.5 and Theorem 2.6) is obtained via Theorem 4 in [DM11].

Indeed, we take $f_n(x) = \text{tr}_{\partial \overline{B((k_n, \ell_n), \delta/2)}} v_{\varepsilon_n}((x_i^\varepsilon, \rho_{\varepsilon_n}) + \delta x)$ with $(k_n, \ell_n)$ a center of a cell containing a zero of $v_{\varepsilon_n}$. Using the main result of [Mir96a], one may easily prove that $f_n$ satisfies the conditions (A1) and (A2) in [DM11], thus we can apply Theorem 4 in [DM11] and infer that the location of the zero inside the inclusion is governed by a renormalized energy which is independent of the boundary condition. \( \square \)

Theorem 2.5 is obtained by combining the following facts:

- The weak $H^1$-convergence of $v_{\varepsilon_n}$ to $v_*$ is a direct consequence of Proposition 4.14. The limiting equation for $v_*$ is a direct consequence of Proposition 4.15 (this is explained immediately after Proposition 4.15).
- The behavior in an $\varepsilon$-neighborhood of the zeros of $v_{\varepsilon_n}$ is given by (4.16), (4.17), and (4.18) (noting that in (4.17), we have $R = R_\ell \to +\infty$ as $\ell \to \infty$). \( \square \)
5. PERSPECTIVES

We present in this section some natural questions that remain open.

(1) The first natural problem is the study of a full Ginzburg-Landau energy obtained by replacing the Dirichlet boundary condition by a more physical condition, namely, the application of an external magnetic field. This problem is related to the work of Aftalion-Sandier-Serfaty [ASS01]. In particular, a very interesting problem should be to consider the same energy as in [ASS01] with the pinning terms of this article, and to prove the pinning effects of the inclusion.

(2) What is the macroscopic location of the vortices in the non-diluted case? It is proved that in the diluted case ($\lambda \to 0$), vortices have the same asymptotical position as in the homogenous case. A natural question is: what is this location in the non-diluted case ($\lambda \equiv 1$)? For example, it should be natural to expect that their limiting location is a limit (when $\rho \to 0$) of a minimal family of points $(x_1^\rho, \ldots, x_d^\rho)$, either for

$$(x_1, \ldots, x_d) \mapsto \inf_{w \in H^1_0(\Omega, S^1)} \frac{1}{2} \int_{\Omega_\rho} A \nabla w \cdot \nabla w,$$

with $x_1, \ldots, x_d \in \Omega$, $\text{dist}(x_i, \partial \Omega) \geq 8\rho$, $|x_i - x_j| \geq 8\rho$, and $\Omega_\rho = \Omega \setminus \bigcup B(x_i, \rho)$; or for

$$(x_1, \ldots, x_d) \mapsto \inf_{\text{deg}_{\partial B(x_i, \rho)}(w) = 1} \frac{1}{2} \int_{\Omega_\rho'} A \nabla w \cdot \nabla w,$$

with $x_1, \ldots, x_d \in \Omega$, $|x_i - x_j| \geq 8\rho$ and $\Omega_\rho' = \Omega' \setminus \bigcup B(x_i, \rho)$.

Here, $A$ is the homogenized matrix of $a^2(\cdot/\delta) \text{Id}_{\mathbb{R}^2}$.

(3) In the non-diluted case, does the microscopic location of a vortex depend on the boundary condition $g$? It is proved that when $\lambda \to 0$, this limiting location depends only on $\omega$ and $b$. Is this still the case when $\lambda \equiv 1$?

(4) In the diluted case and in the special situation where the inclusions are discs, it is natural to expect that the limiting microscopic location of a vortex is the center of the inclusion. Prove or disprove this fact.

(5) A natural improvement of the results of this article would be to replace the technical hypothesis on the size of the inclusions (Hypothesis (1.3)) by either $\lambda \delta/\varepsilon \to +\infty$ or $\lambda \delta/\varepsilon^{\alpha} \to +\infty$ for some $\alpha \in (0, 1)$ or $\ln(\lambda \delta)/\ln \varepsilon \to 0$.

APPENDIX A. PROOF OF PROPOSITION 3.2

We prove the existence of minimal map in $I_\rho$ and in $J_\rho$. The main ingredient is the fact that these sets are closed under $H^1$-weak convergence (see [Las96] or below). Thus, considering a minimizing sequence for $\frac{1}{2} \int_{\Omega_\rho} \alpha |\nabla \cdot |^2$ in the above sets, we obtained the result.
We consider the following:

- We let \( \theta_i : \Omega_\rho \to \mathbb{R} \) be the main argument of \( x - x_i \), that is, \( e^{i\theta_i} = (x - x_i)/|x - x_i| \). Note that \( \theta_i \) represents multivalued functions with smooth (single valued) gradients.

- For \( d_i \in \mathbb{N}^* \) (given by the definition of \( I_\rho \) or \( J_\rho \), we let \( \theta_0 = \sum d_i \theta_i \), and thus \( e^{i\theta_0} = \prod (x - x_i)/|x - x_i|^{d_i} \).

From Lemma 11 in [Bre], there is \( \phi_0 \in C^\infty(\partial\Omega, \mathbb{R}) \) such that \( \partial e^{i\theta_0} = e^{i\phi_0} \).

Note that

\[
\begin{align*}
(A.1) \quad w \in I_\rho & \iff \begin{cases} 
    w = e^{i(\theta_0 + \phi)} & \text{with } \phi \in H^1(\Omega_\rho, \mathbb{R}), \\
    \text{tr}_{\partial\Omega} \phi = \phi_0,
  \end{cases} \\
(A.2) \quad w \in J_\rho & \iff \begin{cases} 
    w = e^{i(\theta_0 + \phi)} & \text{with } \phi \in H^1(\Omega_\rho, \mathbb{R}), \\
    \sum_{j=1} \partial_j \theta_j + \phi = \text{Cst}_i & \text{on } \partial B(x_i, \rho), \\
    \text{tr}_{\partial\Omega} \phi = \phi_0.
  \end{cases}
\end{align*}
\]

Clearly, from (A.1) and (A.2), \( I_\rho \) and \( J_\rho \) are \( H^1 \)-weakly closed.

We now prove the second part of Proposition 3.2.

One may easily obtain that, for some \( \lambda : \Omega_\rho \to \mathbb{R} \), denoting \( w = e^{i(\theta_0 + \phi)} \), \( \phi \in H^1(\Omega_\rho, \mathbb{R}) \) (and thus \( w \in I_\rho \)), we have

\[
(A.3) \quad -\text{div}(\alpha \nabla w) = \lambda w \iff \{- \text{div}(\alpha \nabla (\theta_0 + \phi)) = 0 \text{ and } \lambda = \alpha |\nabla w|^2 \}.
\]

This observation is a direct consequence of the following identity:

\[
-\text{div}(\alpha \nabla e^{i(\theta_0 + \phi)}) = -\text{div}(\alpha \nabla (\theta_0 + \phi)) e^{i(\theta_0 + \phi)} + \alpha |\nabla (\theta_0 + \phi)|^2 e^{i(\theta_0 + \phi)}.
\]

Note that, under this notation, one has \( |\nabla w| = |\nabla (\theta_0 + \phi)| \). Thus \( w \) is a minimizer in \( I_\rho \) or \( J_\rho \) if and only if \( \theta_0 + \phi \) minimizes the weighted Dirichlet functional under the condition fixed by the right-hand side of (A.1) or (A.2).

Consequently, we find that \( \theta_0 + \phi \) minimizes the weighted Dirichlet functional under its Dirichlet boundary condition; therefore, we obtain easily that

\[
-\text{div}(\alpha \nabla (\theta_0 + \phi)) = 0.
\]

The identity \( \nabla (\theta_0 + \phi) = w \wedge \nabla w \) yields \( -\text{div}(\alpha \nabla w) = \lambda w \). Here, for \( v \in \mathbb{C} \) and \( V \in \mathbb{C}^2 \), we denoted

\[
v \wedge V = -V \wedge v = \begin{pmatrix} v \wedge V_1 \\ v \wedge V_2 \end{pmatrix} \in \mathbb{R}^2.
\]

Hence, the Euler-Lagrange equations in (3.1) and (3.2) are direct consequences of (A.3).

The condition on the boundary of the holes for \( w^\text{deg}_{p,\alpha} \) (respectively, \( w^\text{Dir}_{p,\alpha} \)) follows from multiplying the equation satisfied by \( \theta_0 + \phi^\text{deg}_{p,\alpha} \), \( w^\text{deg}_{p,\alpha} = e^{i(\theta_0 + \phi^\text{deg}_{p,\alpha})} \) (respectively, \( \theta_0 + \phi^\text{Dir}_{p,\alpha} \), \( w^\text{Dir}_{p,\alpha} = e^{i(\theta_0 + \phi^\text{Dir}_{p,\alpha})} \) by \( \psi \in D(\Omega, \mathbb{R}) \) (respectively, \( \psi \in D(\Omega, \mathbb{R}) \) such that \( \psi \equiv \text{Cst}_i \) in \( B(x_i, \rho) \)).
Since $\alpha$ is sufficiently smooth, we can rewrite the Euler-Lagrange equation as
\[
-\Delta \phi = \frac{\nabla \alpha \cdot \nabla (\phi + \theta_0)}{\alpha} \quad \text{with} \quad \frac{\nabla \alpha \cdot \nabla (\phi + \theta_0)}{\alpha} \in L^2(\Omega_\rho).
\]

So, by elliptic regularity, $\phi^{\text{deg}}_p,\phi^{\text{Dir}}_p, \phi^{\text{Dir}}_{p,\alpha} \in H^2(\Omega_\rho, \mathbb{R})$, and consequently, $w^{\text{deg}}_p, w^{\text{Dir}}_p, w^{\text{Dir}}_{p,\alpha} \in H^2(\Omega_\rho, S^1)$.  

**APPENDIX B. PROOF OF PROPOSITION 3.3**

We prove the existence of a minimal configuration for $I_{p,\alpha}$, that is, $\{(x, d)\} = \{(x_1, \ldots, x_N), (d_1, \ldots, d_N)\}$. Let $\{(x_n, d_n)\}_n$ be a minimizing sequence of configuration of $I_{p,\alpha}$, that is,
\[
\inf_{w \in H^1(\Omega^0_{\rho}, S^1) \backslash \{0\}} \int_{\Omega^0_{\rho}} \frac{1}{2} \alpha |\nabla w|^2 - I_{p,\alpha} = \inf_{w \in B(x_1, \rho)} \alpha |\nabla w|^2 - I_{p,\alpha}.
\]

Here, $\Omega^0_{\rho} = \Omega' \setminus \bigcup B(x_i, \rho)$. Up to a subsequence, we have $N_n = N = \text{Cst}$, $d_n = d = \text{Cst}$, and $x_n \to x$ for all $i$ such that $\min_{i \neq j} |x_i - x_j| \geq 8\rho$.

Consider $w_n \in H^1(\Omega^0_{\rho}, S^1)$ a minimal map (Proposition 3.2). Since $w_n$ is bounded independently of $n$ in $H^1(\Omega^0_{\rho})$, up to a subsequence, we have $w_n \to w_0$ in $H^1(\Omega^0_{\rho})$, $\Omega^0_{\rho} = \Omega' \setminus \bigcup B(x_i, \rho)$. Clearly, the following properties hold:

- $w_0 \in H^1_{\text{loc}}(\Omega^0_{\rho}, S^1)$ and $w_0 = g$ in $\Omega^0_{\rho} \setminus \Omega$.
- For all compact $K \subset \Omega^0_{\rho}$, we have
  \[
  \frac{1}{2} \int_K \alpha |\nabla w_0|^2 \leq \liminf \frac{1}{2} \int_K \alpha |\nabla w_n|^2 \leq I_{p,\alpha}.
  \]

Thus $w_0 \in H^1_{\text{loc}}(\Omega^0_{\rho}, S^1)$ and $\int_{\Omega^0_{\rho}} \alpha |\nabla w_0|^2 \leq I_{p,\alpha}$.

Now, it suffices to check that $\deg_{\partial B(x_i, \rho)}(w_0) \in \mathbb{N}^*$ for all $i$. Since $w_0$ is $S^1$-valued, this fact is equivalent to $\deg_{\partial B(x_i, \rho')}(w_0) \in \mathbb{N}^*$ for all $i$ and for all $\rho' \in (\rho, 2\rho)$.

In view of the facts, we have the following:

- For $\rho' \in (\rho, 2\rho)$,
  \[
  w'_n = w_n|_{\Omega \setminus \bigcup B(x_i, \rho')} - w'_0 = w_0|_{\Omega \setminus \bigcup B(x_i, \rho')}.
  \]

The set
\[
T' := \{w' \in H^1(\Omega' \setminus \bigcup B(x_i, \rho'), S^1) \mid \deg_{\partial B(x, \rho')}(w') = d_i \forall i \in \{1, \ldots, N\}\}
\]

is closed under the $H^1$-weak convergence (see Appendix A or [Las96]).
Given these, and since $w_n' \in I'$, we obtain that $w_0' \in I'$. Therefore
\[
\{x, d\} = \{(x_1, \ldots, x_N), (d_1, \ldots, d_N)\}
\]
is a minimal configuration for $I_{\rho, \alpha}$.

We now prove the existence of a minimal configuration for $J_{\rho, \alpha}$. Let $(x_n)_n$ be a minimizing sequence of configuration for $I_{\rho, \alpha}$, that is, $\tilde{J}_{\rho, \alpha}(x_n, 1) \to J_{\rho, \alpha}$. Up to a subsequence, one may assume that there is $x = (x_1, \ldots, x_d) \in \Omega^d$ such that $x_i^n - x_i$, $|x_i - x_j| \geq 8\rho$ and $\text{dist}(x_i, \partial \Omega) \geq 8\rho$.

Let $\eta_n = 8 \max |x_i^n - x_i|$. There is a smooth diffeomorphism $\phi_n : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying
\[
\begin{cases}
\phi_n \equiv \text{Id}_{\mathbb{R}^2} & \text{in } \mathbb{R}^2 \setminus \bigcup B(x_i^n, \rho + \eta_n^{1/2}), \\
\phi_n[x_i + (1 + \eta_n)x] = x_i^n + x & \text{for } x \in B(0, \rho), \\
\|\phi_n - \text{Id}_{\mathbb{R}^2}\|_{C^1(\mathbb{R}^2)} = o_n(1).
\end{cases}
\]

For example, we can consider $\phi_n = \text{Id}_{\mathbb{R}^2} + H_n$ with
\[
\begin{cases}
H_n = 0 & \text{in } \mathbb{R}^2 \setminus \bigcup B(x_i^n, \rho + \eta_n^{1/2}) \\
H_n[x_i + (1 + \eta_n)x] = [1 - \psi_n(|x|)](x_i^n - x_i - \eta_n x) & \text{for } x \in B \left(0, \frac{\rho + \eta_n^{1/2}}{1 + \eta_n} \right).
\end{cases}
\]
Here, $\psi_n : \mathbb{R}^+ \to [0, 1]$ is a smooth function satisfying
\[
\psi_n(r) = \begin{cases}
0 & \text{if } r \leq \rho, \\
1 & \text{if } r \geq \rho + \eta_n^{1/2}/2,
\end{cases}
\quad |\psi_n'| = O(\eta_n^{1/2}).
\]

For $w_n \in J_{\rho}(x_n, 1)$ a minimal map, we consider
\[
\tilde{w}_n : \Omega \setminus \bigcup_i B(x_i, (1 + \eta_n)\rho) \to \mathbb{S}^1, \quad x \mapsto w_n[\phi_n(x)].
\]
Clearly, $\tilde{w}_n$ is well defined, and we have
\[
\int_{\Omega \setminus \bigcup_i B(x_i, (1 + \eta_n)\rho)} \alpha |\nabla \tilde{w}_n|^2 = \int_{\Omega \setminus \bigcup_i B(x_i^n, \rho)} \alpha |\nabla w_n|^2 + o_n(1),
\]
\[
\tilde{w}_n[x_i + (1 + \eta_n)\rho e^{i\theta}] = w_n[\phi(x_i + (1 + \eta_n)\rho e^{i\theta})] = w_n[x_i^n + \rho e^{i\theta}] = e^{i(\theta + \theta_i)}.
\]
We can extend \( \tilde{w}_n \) in \( \bigcup_j B(x_i, (1 + \eta_n)\rho) \setminus \overline{B(x_i, \rho)} \) by \( \tilde{w}_n(x_i + re^{i\theta}) = e^{i(\theta + \theta_i)}, \rho < r < (1 + \eta_n)\rho \). Clearly, we have \( \tilde{w}_n \in J_{\rho, \alpha}(x, 1) \) and

\[
\frac{1}{2} \int_{\Omega \setminus \bigcup_j B(x_i, \rho)} \alpha |\nabla \tilde{w}_n|^2 = J_{\rho, \alpha} + o_n(1).
\]

Thus considering \( w \in J_{\rho, \alpha}(x, 1) \) a minimizer of \( \frac{1}{2} \int_{\Omega \setminus \bigcup_j B(x_i, \rho)} \alpha |\nabla \cdot w|^2 \), we obtain

\[
\frac{1}{2} \int_{\Omega \setminus \bigcup_j B(x_i, \rho)} \alpha |\nabla w|^2 \leq \frac{1}{2} \int_{\Omega \setminus \bigcup_j B(x_i, \rho)} \alpha |\nabla \tilde{w}_n|^2 = J_{\rho, \alpha} + o_n(1).
\]

Letting \( n \to \infty \), we deduce that the configuration \( x = (x_1, \ldots, x_d) \) is minimal.

**APPENDIX C. PROOF OF PROPOSITION 3.4**

As explained in Section 3.2, Proposition 3.4 is easily established when either \( N = 1 \) or the points are well separated. It remains to consider the case where \( N \geq 2 \) and there are \( i \neq j \) such that \( |x_i - x_j| \leq 4\eta_{\text{stop}} \).

### 3.1. The separation process.

We assume that \( N \geq 2 \) and that the points are not well separated. Our purpose is to compare the energy of \( \tilde{w}_n \) with the energy of \( \tilde{w}_n \). To this purpose, we decompose \( \Omega \) into several regions and compare energies in each region. These regions are constructed recursively using the following version of Theorem IV.1 in [BBH94].

**Lemma C.1.** Let \( N \geq 2 \), \( x_1, \ldots, x_N \in \mathbb{R}^2 \), and \( \eta > 0 \). There are \( \kappa \in \{g^0, \ldots, g^{N-1}\} \) and \( \{y_1, \ldots, y_{\kappa N}\} \subset \{x_1, \ldots, x_N\} \) such that

\[
\bigcup_{i=1}^{\kappa N} B(x_i, \eta) \subset \bigcup_{i=1}^{N} B(y_i, \kappa \eta), \quad \text{and} \quad |y_i - y_j| \geq 8\kappa \eta \quad \text{for } i \neq j.
\]

We let \( x^{k_i}_1, \ldots, x^{k_i}_N \) be the initial points \( x_1, \ldots, x_N \), and \( N_0 = N \) be the initial number of points. For \( k \geq 1 \) (here, \( k \) is an iteration in the construction of the regions), we let \( N_k \) be the number of selected points at Step \( k \), and we denote by \( x^k_1, \ldots, x^{k}_{N_k} \) these selected points.

The recursive construction is made in such a way that \( N_k > N_{k+1} \) and \( N_k \geq 1 \) for all \( k \geq 1 \). The process will stop at the end of Step \( k \) if and only if one of the following conditions obtains:

**Rule 1:** There is a unique point in the selection (i.e., \( N_k = 1 \));

**Rule 2:** We have \( \min_{i \neq j} |x^k_i - x^k_j| > 4\eta_{\text{stop}} \).

**Step \( k, k \geq 1 \):** Let \( \eta'_k = \frac{1}{4} \min_{i \neq j} |x^{k-1}_i - x^{k-1}_j| \). Using Lemma C.1, we have \( \kappa_k \in \{g^1, \ldots, g^{N_{k-1}}\} \) and \( \{x^k_1, \ldots, x^k_{N_k}\} \subset \{x^{k-1}_1, \ldots, x^{k-1}_{N_{k-1}}\} \), such that

\[
\bigcup_{i=1}^{N_{k-1}} B(x^{k-1}_i, \eta'_k) \subset \bigcup_{i=1}^{N_k} B(x^k_i, \kappa_k \eta'_k) \quad \text{and} \quad |x^k_i - x^k_j| \geq 8\kappa_k \eta'_k \quad \text{for } i \neq j.
\]
We denote \( \eta_k = 2\kappa_k\eta_k' \). We stop the construction if \( N_k = 1 \) (Rule 1) or if \( \frac{1}{4} \min |x_i^k - x_j^k| > \eta_{\text{stop}} \) (Rule 2).

In Figures C.1 and C.2, both stop-conditions are presented.

**Remark C.2.**

(i) From the definitions of \( \eta_k' \) and \( \eta_k \), we have \( N_k < N_{k-1} \) and \( \eta_{k-1} \leq \eta_k' < \eta_k \).

(ii) The balls \( B(x_j^k, 2\eta_k) \) are disjoint.

(iii) Denoting \( \Lambda_j^k \subseteq \{1, \ldots, N_{k-1}\} \) the set of indices \( i \) such that \( x_i^{k-1} \in B(x_j^k, \kappa_k\eta_k') \), then for \( i \in \Lambda_j^k \) we have \( B(x_i^{k-1}, \eta_k') \subseteq B(x_j^k, \kappa_k\eta_k') \). Furthermore, by construction, \( |x_i^{k-1} - x_j^k| \geq 4\eta_k' \).

### 3.2. The separation process gives a natural partition of \( \Omega \)

Let \( \Omega, g, x_1, \ldots, x_N, d \) and \( \rho \), and \( \eta_{\text{stop}} \) be as in Section 3.2 with \( N \geq 2 \), and such that the points are not well separated.

We apply the separation process. The process stops after \( K \) steps, \( 1 \leq K \leq N - 1 \).

We denote by \( \{y_1, \ldots, y_{N'}\} \subseteq \{x_1, \ldots, x_N\} \) the selection that we obtain with the separation process, that is, \( y_j = x_j^k \) and \( N' = N_k \). We define

\[
\eta = \begin{cases} 
9^N \cdot \eta_{\text{stop}} & \text{if } N' = 1, \\
\min \left\{ 9^N \cdot \eta_{\text{stop}}, \frac{1}{4} \min |y_i - y_j| \right\} & \text{if } N' > 1,
\end{cases}
\]

so that \( \eta \geq \max(\eta_k', \eta_{\text{stop}}) \).

We denote

\[
D_{j,k} = B(x_j^k, \eta_k) \setminus \bigcup_{x_i^{k-1} \in B(x_j^k, \eta_k), \ k \in \{1, \ldots, K\}, \ j \in \{1, \ldots, N_k\}} B(x_i^{k-1}, \eta_k'),
\]

\[
R_{j,k} = B(x_j^k, \eta_k') \setminus B(y_j, \eta_k), \quad k \in \{0, \ldots, K-1\}, \ j \in \{1, \ldots, N_k\},
\]

\[
R_j = B(y_j, \eta) \setminus \overline{B(y_j, \eta_k)}, \quad j \in \{1, \ldots, N'\},
\]

and

\[
D = \Omega \setminus \bigcup_{j \in \{1, \ldots, N'\}} \overline{B(y_j, \eta)}.
\]

Note that by construction of \( \eta_k', \eta_k \), and \( x_i^k \), the following properties are satisfied:

(C.5) The balls \( B(x_i^{k-1}, 2\eta_k') \) are disjoint;

(C.6) \( 2 \cdot 9^\eta_k' \leq \eta_k \leq 9^N \eta_k' \).

Therefore,

\[
\Omega_\rho = D \cup \bigcup_{j,k} \overline{D_{j,k}} \cup \bigcup_{j,k} R_{j,k} \cup \bigcup_{j} \overline{R_j} \quad \text{with disjoint unions.}
\]
\(4\eta_{\text{stop}} \geq 4\eta_1\)  

(a) The initial balls  

(b) The first step: a selection of two centers  

(c) The process stops at the end of the first step since there are two well-separated balls.

**Figure C.1.** The process stops when we obtain well-separated balls.

### 3.3. Construction of test functions.

**Construction of test functions in \(D\) and \(D_{j,k}\).**

**Lemma C.3.**

(1) Let \(\eta > 0\). There exists \(C_1(\eta) > 0\) (depending on \(\Omega, g, \) and \(\eta\)) such that if \(x_1, \ldots, x_N \in \Omega\) satisfy \(\min_{i \neq j} |x_i - x_j|\), if \(\min_{i} \text{dist}(x_i, \partial \Omega) > 4\eta\), and if \(d_1, \ldots, d_N \in \mathbb{N}^*\) are such that \(\sum d_i = d\), then there exists \(w \in \)
Figure C.2. The process stops when we obtain a unique ball.

\[ H^1_\partial (\Omega \setminus \bigcup_{i} \overline{B(x_i, \eta)}, S^1) \] such that \( w(x) = (x - x_i)^{d_i} / \eta^{d_i} \) on \( \partial B(x_i, \eta) \) and
\[
\int_{\Omega \setminus \bigcup_{i} \overline{B(x_i, \eta)}} |\nabla w|^2 \leq C_1(\eta).
\]
Moreover, \( C_1 \) can be considered decreasing with respect to \( \eta \).

(2) Let \( \eta > 0, \kappa \geq 8, d_0, d_1, \ldots, d_N \in \mathbb{N}^* \) be such that \( \sum_{i=1}^{N} d_i = d_0 \).
Then, there is \( C_2(\kappa, d_0) \) such that for \( x_1, \ldots, x_N \in B(0, \kappa \eta) \) satisfying...
(a) The macroscopic perforated domain and the first mesoscopic rings

(b) A mesoscopic ring and a mesoscopic perforated domain
min_{i \neq j} |x_i - x_j| \geq 4\eta, we can associate a map

\[ w \in H^1 \left( B(0, 2\kappa \eta) \setminus \bigcup B(x_i, \eta), \mathbb{S}^1 \right) \]

such that

\[ w(x) = \begin{cases} 
  \frac{x^{d_0}}{(2\kappa \eta)^d_0} & \text{on } \partial B(0, 2\kappa \eta), \\
  \frac{(x - x_i)^{d_i}}{\eta^{d_i}} & \text{on } \partial B(x_i, \eta),
\end{cases} \]

and

\[ \int_{B(0, 2\kappa \eta) \setminus \bigcup B(x_i, \eta)} |\nabla w|^2 \leq C_2(\kappa, d_0). \]

Moreover, \( C_2 \) can be considered increasing with respect to \( \kappa, d_0 \).

**Proof.** In order to prove (1), we consider a function such as the test function defined in \( \Omega_\eta := \Omega \setminus \bigcup B(x_i, \eta) \) by

\[ w = e^{iH} \prod_i \frac{(x - x_i)^{d_i}}{|x - x_i|^{d_i}}, \text{ with } H \text{ such that} \]

\[
\begin{align*}
H : \Omega_\eta &\to \mathbb{R}, \\
H &\equiv 0 \text{ in } \{ \text{dist}[x, \partial \Omega_\eta] \geq \eta \}, \\
-\Delta H &\equiv 0 \text{ in } \{ \text{dist}[x, \partial \Omega_\eta] < \eta \}, \\
\hat{w} &\in H^1_0(\Omega_\eta, \mathbb{S}^1), \\
w(x) &\equiv \frac{(x - x_i)^{d_i}}{\eta^{d_i}} \text{ on } \partial B(x_i, \eta).
\end{align*}
\]

Assertion (2) was essentially established in [HS95, Section 3], whose argument we here adapt. By conformal invariance, we may assume that \( \eta = 1 \). We let

\[ w(x) = \prod_i \frac{[x + 2x_i(|x|/\kappa - 2)]^{d_i}}{|x + x_i(|x|/\kappa - 2)|^{d_i}} \text{ in } B(0, 2\kappa) \setminus \overline{B(0, 3\kappa/2)}, \]

\[ \prod_i \frac{(x - x_i)^{d_i}}{|x - x_i|^{d_i}} \text{ in } B(0, 3\kappa/2) \setminus \bigcup B(x_i, 3/2), \]

\[ \prod_i \frac{(x - x_i)^{d_i}}{|x - x_i|^{d_i}} e^{i(2|x - x_i| - 2)\varphi_i} \text{ in } B(x_i, 3/2) \setminus \bigcup B(x_i, 1); \]

here, \( \varphi_i \in C^\infty(B(x_i, 3/2), \mathbb{R}) \) is defined by

\[ e^{i\varphi_i} = \prod_{j \neq i} \frac{(x - x_j)^{d_j}}{|x - x_j|^{d_j}} \text{ and } \varphi_i(x_i) \in [0, 2\pi). \]
Clearly, \(\|\varphi_i\|_{H^1(B(x_i, 3/2), B(x_i, 1))}\) is bounded by a constant which depends only on \(d_0\).

By (C.1) and Lemma C.3 (1), one may find a map \(w_0 \in H^1(D, S^1)\) such that

\[
\begin{cases}
\vartheta & \text{on } \partial \Omega, \\
w_0(x) = \frac{(x - y_j)^{d_j}}{\eta^{d_j}} & \text{on } \partial B(y_j, \eta),
\end{cases}
\]

where \(d_j = \sum_{x_i \in B(y_j, \eta)} d_i\), satisfying also

\[
(C.8) \quad \int_D |\nabla w_0|^2 \leq C_1(\eta) \leq C_1(\eta_{\text{stop}}).
\]

For each \(D_{j,k}\), combining (C.2), (C.5), and (C.6), and using Lemma C.3 (2), there exists a map \(w_{j,k} \in H^1(D_{j,k}, S^1)\) such that

\[
w_{j,k}(x) = \begin{cases} 
\frac{(x - x^j)^{d_{j,k}}}{\eta_k} & \text{for } x \in \partial B(x^j_k, \eta_k), \\
\frac{(x - x^{k-1})^{d_{j,k-1}}}{\eta_k^{d_{j,k-1}}} & \text{for } x \in \partial B(x^{k-1}_j, \eta^*_k).
\end{cases}
\]

Here, \(d_{j,k} = \sum_{x_i \in B(x^j_k, \eta_k)} d_i\), and

\[
(C.9) \quad \int_{D_{j,k}} |\nabla w_{j,k}|^2 \leq C_2(2\kappa_k, d_{j,k}) \leq C_2(2 \cdot 9^{d-1}, d).
\]

**Construction of the test functions in the rings \(R_j\) and \(R_{j,k}\).** For \(R > r > 0\) and \(x_0 \in \mathbb{R}^2\), we denote \(\mathcal{R}(x_0, R, r) := B(x_0, R) \setminus B(x_0, r)\). For \(\alpha \in L^\infty(\mathbb{R}^2, [b^2, 1])\), we define

\[
(C.10) \quad \mu_\alpha(\mathcal{R}(x_0, R, r), \tilde{d}) = \inf_{w \in H^1(\mathcal{R}(x_0, R, r), S^1)} \left\{ \frac{1}{2} \int_{\mathcal{R}(x_0, R, r)} \alpha |\nabla w|^2 \right\}
\]

and

\[
(C.11) \quad \mu_\alpha^{\text{Dir}}(\mathcal{R}(x_0, R, r), \tilde{d}) = \inf_{w \in H^1(\mathcal{R}(x_0, R, r), S^1), w\neq\text{const}} \left\{ \frac{1}{2} \int_{\mathcal{R}(x_0, R, r)} \alpha |\nabla w|^2 \right\}.
\]

In the special case \(\alpha = U^2_\varepsilon\), we denote

\[
\mu_\varepsilon(\mathcal{R}(x_0, R, r), \tilde{d}) = \mu_{U^2_\varepsilon}(\mathcal{R}(x_0, R, r), \tilde{d})
\]

and

\[
\mu_\varepsilon^{\text{Dir}}(\mathcal{R}(x_0, R, r), \tilde{d}) = \mu_{U^2_\varepsilon}^{\text{Dir}}(\mathcal{R}(x_0, R, r), \tilde{d}).
\]
Note that the minimization problems (C.10) and (C.11) admit solutions; this is obtained by adapting the proof of Proposition 3.2.

We present now an adaptation of a result of Sauvageot, Theorem 2 in [Sau09].

**Proposition C.4.** There is \( C_b > 0 \) depending only on \( b \in (0, 1) \) such that for \( R > r > 0 \) and \( \alpha \in L^\infty(\mathbb{R}^2, \mathbb{R}) \) satisfying \( 1 \geq \alpha \geq b^2 \), we have

\[
\mu_{\alpha}^\text{Dir}(\mathcal{R}(x_0, R, r), \tilde{d}) \leq \mu_\alpha(\mathcal{R}(x_0, R, r), \tilde{d}) + \tilde{d}^2 C_b.
\]

**Proof.** This result was obtained by Sauvageot with \( \alpha \in W^{1,\infty}(\mathbb{R}^2, [b^2, 1]) \). We may extend this estimate to \( \alpha \in L^\infty(\mathbb{R}^2, [b^2, 1]) \).

Indeed, let \((\rho_t)_{t \geq 0} \) be a classical mollifier, namely \( \rho_t(x) = t^{-2} \rho(x/t) \) with \( \rho \in C^\infty(\mathbb{R}^2, [0, 1]) \), \( \text{Supp} \rho \subset B(0, 1) \), and \( \int_{\mathbb{R}^2} \rho = 1 \).

Set \( \alpha_t = \alpha \ast \rho_t \in W^{1,\infty}(B(x_0, R), [b^2, 1]) \). We have

\[
\lim_{t \to 0} \mu_{\alpha_t}(\mathcal{R}(x_0, R, r), \tilde{d}) = \mu_\alpha(\mathcal{R}(x_0, R, r), \tilde{d})
\]

and

\[
\lim_{t \to 0} \mu_{\alpha_t}^\text{Dir}(\mathcal{R}(x_0, R, r), \tilde{d}) = \mu_\alpha^\text{Dir}(\mathcal{R}(x_0, R, r), \tilde{d}).
\]

We prove (C.12), and equality (C.13) follows with the same lines.

Let \( w \) be a minimizer of \( \mu_\alpha(\mathcal{R}(x_0, R, r), \tilde{d}) \). By using the Dominated Convergence Theorem, since \( \alpha_t \to \alpha \) in \( L^1(B(x_0, R)) \), we obtain that \( \alpha_t |\nabla w|^2 \to \alpha |\nabla w|^2 \) in \( L^1(\mathcal{R}(x_0, R, r)) \) as \( t \to 0 \). Consequently,

\[
\limsup_{t \to 0} \mu_{\alpha_t}(\mathcal{R}(x_0, R, r), \tilde{d}) \leq \mu_\alpha(\mathcal{R}(x_0, R, r), \tilde{d}).
\]

On the other hand, let \( w_t \) be a minimizer of \( \mu_{\alpha_t}(\mathcal{R}(x_0, R, r), \tilde{d}) \), and let \( t_n \to 0 \). Up to passing to a subsequence, \( w_{t_n} \to w_0 \) in \( H^1(\mathcal{R}(x_0, R, r)) \) as \( n \to \infty \), and \( \sqrt{\alpha_{t_n}} \nabla w_{t_n} \to \sqrt{\alpha} \nabla w_0 \) in \( L^2(\mathcal{R}(x_0, R, r)) \).

Since the class \( \mathcal{I} := \{ w \in H^1(\mathcal{R}(x_0, R, r), \mathbb{R}) \mid \text{deg}_{\partial B(x_0, R)}(w) = \tilde{d} \} \) is closed under the \( H^1 \)-weak convergence (see Appendix A or [Las96]), we obtain that \( w_0 \in \mathcal{I} \). Consequently, we have

\[
\liminf_{t \to 0} \mu_{\alpha_t}(\mathcal{R}(x_0, R, r), \tilde{d}) \geq \mu_\alpha(\mathcal{R}(x_0, R, r), \tilde{d}).
\]

Thus the proof of (C.12) is complete.

Therefore, without loss of generality, we may assume that \( \alpha \) is Lipschitz.

One may easily prove that if \( R \leq 4r \), then \( \mu_\alpha^\text{Dir}(\mathcal{R}(x_0, R, r), \tilde{d}) \leq 2\tilde{d}^2\pi \ln 4 \).

Thus we assume that \( R > 4r \). Clearly, it suffices to obtain the result for \( \tilde{d} = 1 \) and \( x_0 = 0 \).
Let \( w \) be a global minimizer of \( \mu_\alpha(\mathbb{R}(x_0, R/2, 2r), 1) \). As explained in Appendix A, denoting \( x/|x| = e^{i\theta} \), one may write
\[
w = e^{i(\theta + \phi)} \quad \text{for some} \quad \phi \in H^2(\mathbb{R}(x_0, R/2, 2r), \mathbb{R}).
\]

Now, we switch to polar coordinates.

Consider
\[
I = \left\{ \rho \in [2r, R/2] \mid \int_0^{2\pi} \alpha|\nabla (\theta + \phi)|^2(\rho, \theta) \, d\theta \leq \frac{1}{\rho^2} \int_0^{2\pi} \alpha(\rho, \theta) \, d\theta \right\}.
\]

Then \( I \) is closed (since \( \phi \in H^2 \)). On the other hand, \( I \) is non-empty, by the mean value theorem.

Let \( r_1 = \min I \) and \( r_2 = \max I \). We may assume that \( \phi(r_2, 0) = 0 \) and \( \phi(r_1, 0) = \theta_0 \). We construct a test function:
\[
\phi'(\rho, \theta) =
\begin{cases}
0 & \text{if } 2r_2 \leq \rho \leq R, \\
\frac{2r_2 - \rho}{r_2} \phi(r_2, \theta) & \text{if } r_2 \leq \rho \leq 2r_2, \\
\phi(\rho, \theta) & \text{if } r_1 \leq \rho \leq r_2, \\
\frac{2\rho - r_1}{r_1} \phi(r_1, \theta) + 2\frac{r_1 - \rho}{r_1} \theta_0 & \text{if } r_1/2 \leq \rho \leq r_1, \\
\theta_0 & \text{if } \rho \leq r_1/2.
\end{cases}
\]

As explained in [Sau09], there is \( C \) depending only on \( b \) such that
\[
\frac{1}{2} \int_{\mathbb{R}(0, R/2, 2r)} \alpha(|\nabla (\theta + \phi')|^2 - |\nabla (\theta + \phi)|^2) \, d\rho \leq C.
\]

Thus the result follows. \( \square \)

As a direct consequence of Proposition C.4 (the first two assertions of the next proposition are direct), we have the following result.

**Proposition C.5.** Let \( \alpha \in L^\infty(\mathbb{R}^2, [b^2, 1]) \), \( R > r > r_1 > 0 \), \( \tilde{d} \in \mathbb{R} \) and \( x_0 \in \mathbb{R}^2 \); we have
\begin{enumerate}
\item \( \mu_\alpha(\mathbb{R}(x_0, R, r), \tilde{d}) = \tilde{d}^2 \mu_\alpha(\mathbb{R}(x_0, R, r), 1) \);
\item \( b^2 \pi \ln(R/r) \leq \mu_\alpha(\mathbb{R}(x_0, R, r), 1) \leq \pi \ln(R/r) \);
\item \( \mu_\alpha(\mathbb{R}(x_0, R, r), 1) \leq \mu_\alpha(\mathbb{R}(x_0, R, r_1), 1) + \mu_\alpha(\mathbb{R}(x_0, r_1, r), 1) + 2C_b \),
\end{enumerate}
where \( C_b \) is given by Proposition C.4 and depends only on \( b \).

We turn to the construction of test functions in \( R_j \) and \( R_{j,k} \).
Using Proposition C.4, there is $C_b$ depending only on $b \in (0,1)$ such that for $\alpha \in L^\infty(\Omega, [b^2, 1])$ and for all $k \in \{1, \ldots, K - 1\}$, $j \in \{1, \ldots, N_k\}$, there is $w_{\alpha,j,k} \in H^1(R_{j,k}, S^1)$ such that

$$w_{\alpha,j,k}(x) = \begin{cases} (x - x^j_k)d_{j,k} \eta_{d_{j,k}+1} & \text{for } x \in \partial B(x^j_k, \eta_{d_{j,k}+1}), \\ \gamma_{\alpha,j,k} \frac{(x - y^j_k)d_{j,k}}{\eta_{d_{j,k}}} & \text{for } x \in \partial B(x^j_k, \eta_k) \text{ where } \gamma_{\alpha,j,k} \in S^1, \end{cases}$$

and such that for all $w \in H^1(R_{j,k}, S^1)$ satisfying $\deg_{\partial B(x^j_k, \eta_k)}(w) = \tilde{d}_{j,k}$, one has

$$\int_{R_{j,k}} |\nabla w_{\alpha,j,k}|^2 \leq \int_{R_{j,k}} |\nabla w|^2 + 2C_b \tilde{d}_{j,k}^2 \leq \int_{R_{j,k}} |\nabla w|^2 + 2d^2C_b.$$  (C.14)

We now consider the rings $R_j$. For $j \in \{1, \ldots, N'\}$, denote $\tilde{d}_j = \sum_{x_i \in B(y_j, \eta)} d_i$. Using Proposition C.4, for $j \in \{1, \ldots, N'\}$, we obtain the existence of $w_{\alpha,j} \in H^1(R_j, S^1)$ such that

$$w_{\alpha,j}(x) = \begin{cases} (x - y^j)d \eta^d & \text{for } x \in \partial B(y_j, \eta), \\ \gamma_{\alpha,j} \frac{(x - y^j)d}{\eta_k^d} & \text{for } x \in \partial B(y_j, \eta_k) \text{ where } \gamma_{\alpha,j} \in S^1, \end{cases}$$

and such that for all $w \in H^1(R_j, S^1)$ satisfying $\deg_{\partial B(\eta, \eta_j)}(w) = \tilde{d}_j$, one has

$$\int_{R_j} |\nabla w_{\alpha,j}|^2 \leq \int_{R_j} |\nabla w|^2 + 2d^2C_b.$$  (C.15)

### 3.4. Proof of Proposition 3.4
Note that there are at most $d^2$ regions $D_{j,k}$, at most $d^2$ rings $R_{j,k}$, and at most $d$ rings $R$. Consequently, denoting

$$C_4(\eta_{\text{stop}}) = C_1(\eta_{\text{stop}}) + d^2C_2(2 \cdot 9^{d-1}, d) + 4d^4C_b,$$

and using (C.7), (C.8), (C.9), (C.14), and (C.15), one may construct a test function $w_{\alpha} \in I_\rho$ (up to multiplying by some $S^1$-constants each function previously constructed) such that for all $w \in I_\rho$, one has

$$\int_{\Omega_\rho} |\nabla w_{\alpha}|^2 \leq \int_{\Omega_\rho} |\nabla w|^2 + C_4.$$  (C.16)

Clearly, (C.16) allows us to prove Proposition 3.4 with $C_0 = C_4/2$.  

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APPENDIX D. PROOF OF PROPOSITION 3.6

4.1. Description of the special solution $U_\varepsilon$. From Proposition 1.4, we know that, far away, $\partial \omega_\varepsilon$, $U_\varepsilon$ is uniformly close to $a_\varepsilon$. Here we prove that, in a neighborhood of $\partial \omega_\varepsilon$, $U_\varepsilon$ is very close to a cell regularization of $a_\varepsilon$.

Let

$$a^\lambda : Y = \left( -\frac{1}{2}, \frac{1}{2} \right) \times \left( -\frac{1}{2}, \frac{1}{2} \right) \rightarrow \{ b, 1 \}$$

with $x \rightarrow \begin{cases} b & \text{if } x \in \omega^\lambda = \lambda \cdot \omega, \\ 1 & \text{otherwise}. \end{cases}$

Consider $V_\varepsilon$ the unique minimizer of

$$(D.1) \quad E^\lambda_\varepsilon(V, Y) = \frac{1}{2} \int_Y |\nabla V|^2 + \frac{1}{2\varepsilon^2} (a^\lambda - V^2)^2, \quad V \in H_1^1(Y, \mathbb{R}).$$

Lemma D.1. We have the existence of $C, \gamma > 0$ such that for $\varepsilon > 0$ and $x \in Y$,

$$|U_\varepsilon[y_{i,j}^\varepsilon + \delta^j x] - V_{\varepsilon/\delta^j}(x)| \leq C e^{-\gamma \delta^j/\varepsilon}.$$

Thus in the periodic case, we have $U_\varepsilon$ which is almost a $\delta \cdot (\mathbb{Z} \times \mathbb{Z})$-periodic function in $\omega^{\varepsilon}_{\text{incl}}$, in the sense that

$$|U_\varepsilon(x) - U_\varepsilon[x + (\delta k, \delta \ell)]| \leq C e^{-\gamma \delta^j/\varepsilon} \quad \text{if} \quad x, x + (\delta k, \delta \ell) \in \omega^{\varepsilon}_{\text{incl}}$$

Proof.

Step 1. We first prove that, for all $s > 0$ and for sufficiently small $\varepsilon$, we have $U_\varepsilon^2 \geq (b^2 + 1)/2 - s$ in $\Omega \setminus \omega_\varepsilon$. The same argument leads to $U_\varepsilon^2 \leq (b^2 + 1)/2 + s$ in $\omega_\varepsilon$ and for sufficiently small $\varepsilon$: thus, $V_\varepsilon^2 \geq (b^2 + 1)/2 - s$ in $Y \setminus \omega^\lambda$ and $V_\varepsilon^2 \leq (b^2 + 1)/2 + s$ in $\omega^\lambda$.

From Proposition 1.4, it suffices to prove that for

$$R = -\alpha^{-1} \ln \frac{C}{1 - \sqrt{(1 + b^2)/2}},$$

we have $U_\varepsilon^2 \geq (b^2 + 1)/2 - s$ in $\{ x \in \Omega \setminus \omega_\varepsilon \mid \text{dist}(x, \partial \omega_\varepsilon) < R \varepsilon \}$ (for sufficiently small $\varepsilon$). Here $C > 1$, $\alpha > 0$ are given by (1.5).

We fix $0 < s < 1$ and we let $z_\varepsilon = y_{i,j}^\varepsilon + \lambda \delta^j z_0^\varepsilon \in \partial \omega_\varepsilon$, $z_0^\varepsilon \in \partial \omega$. For $x \in B(z_\varepsilon, \lambda \delta^{p+1})$, we write $x = z_\varepsilon + \varepsilon \tilde{x}$ with $\tilde{x} \in B(0, \lambda \delta^{p+1}/\varepsilon)$. Here, $P = j = 1$ and $y_{i,j}^\varepsilon \in \delta \mathbb{Z} \times \delta \mathbb{Z}$ if we are in the periodic situation.

We define

$$\tilde{U}_\varepsilon(\tilde{x}) : B \left( 0, \frac{\lambda \delta^{p+1}}{\varepsilon} \right) \rightarrow [b, 1], \quad \tilde{x} \rightarrow U_\varepsilon(z_\varepsilon + \varepsilon \tilde{x}).$$
It is easy to check that

\[
\begin{align*}
-\Delta \tilde{U}_\varepsilon &= \tilde{U}_\varepsilon (\tilde{a}_\varepsilon^2 - \tilde{U}_\varepsilon^2) \quad \text{in } B \left( 0, \frac{\lambda \delta^{p+1}_\varepsilon}{\varepsilon} \right), \\
\tilde{U}_\varepsilon &\in H^1 \cap L^\infty \left( B \left( 0, \frac{\lambda \delta^{p+1}_\varepsilon}{\varepsilon} \right), [b, 1] \right),
\end{align*}
\]

where

\[
\tilde{a}_\varepsilon = \begin{cases} 
\dfrac{\omega_\varepsilon - z_\varepsilon}{\varepsilon} & \text{in } B \left( 0, \frac{\lambda \delta^{p+1}_\varepsilon}{\varepsilon} \right), \\
1 & \text{in } (\mathbb{R}^2 \setminus \omega_\varepsilon) - \dfrac{z_\varepsilon}{\varepsilon} \cap B \left( 0, \frac{\lambda \delta^{p+1}_\varepsilon}{\varepsilon} \right).
\end{cases}
\]

Clearly,

\[
\dfrac{\omega_\varepsilon - z_\varepsilon}{\varepsilon} \cap B \left( 0, \frac{\lambda \delta^{p+1}_\varepsilon}{\varepsilon} \right) = \left[ \dfrac{\lambda \delta^j}{\varepsilon} \cdot (\omega - z_\varepsilon^0) \right] \cap B \left( 0, \frac{\lambda \delta^{p+1}_\varepsilon}{\varepsilon} \right) = \dfrac{\lambda \delta^j}{\varepsilon} \cdot \left[ (\omega - z_\varepsilon^0) \cap B(0, \delta^{p+1-j}) \right],
\]

and thus

\[
\dfrac{(\mathbb{R}^2 \setminus \omega_\varepsilon) - z_\varepsilon}{\varepsilon} \cap B \left( 0, \frac{\lambda \delta^{p+1}_\varepsilon}{\varepsilon} \right) = \dfrac{\lambda \delta^j}{\varepsilon} \cdot \left[ (\mathbb{R}^2 \setminus \omega) - z_\varepsilon^0 \right] \cap B(0, \delta^{p+1-j}).
\]

Note that \(\lambda \delta^{p+1}_\varepsilon / \varepsilon \to \infty\) and \(\delta^{p+1-j} \to 0\); thus by smoothness of \(\omega\), up to a subsequence, we have

\[
\dfrac{\lambda \delta^j}{\varepsilon} \cdot \left[ (\mathbb{R}^2 \setminus \omega) - z_\varepsilon^0 \right] \cap B(0, \delta^{p+1-j}) \to R_{\theta_0}(\mathbb{R} \times \mathbb{R}^+).
\]

Here, \(R_{\theta_0}\) is the vectorial rotation of angular \(\theta_0 \in [0, 2\pi)\).

For the sake of simplicity, we assume that \(\theta_0 = 0\). From (D.2) and standard elliptic estimates, we obtain that \(\tilde{U}_\varepsilon\) is bounded in \(W^{2,p}(B(0,R))\) for \(p \geq 2\), \(R > 0\). Thus up to considering a subsequence, we obtain that \(\tilde{U}_\varepsilon \to \tilde{U}_b\) in \(C^1_{\text{loc}}(\mathbb{R}^2)\) (\(\varepsilon \to 0\)), where \(\tilde{U}_b \in C^1(\mathbb{R}^2, [b, 1])\) is a solution of

\[
\begin{align*}
-\Delta \tilde{U}_b &= \tilde{U}_b (1 - \tilde{U}_b^2) \quad \text{in } \mathbb{R} \times \mathbb{R}^+, \\
-\Delta \tilde{U}_b &= \tilde{U}_b (b^2 - \tilde{U}_b^2) \quad \text{in } \mathbb{R} \times \mathbb{R}^-, \\
\tilde{U}_b &\in C^1(\mathbb{R}^2) \cap H^2_{\text{loc}}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2).
\end{align*}
\]
It is proved in [Kac09, Theorem 2.2] that (D.3) admits a unique positive solution. Moreover, \( \hat{U}_b(x, y) = U_b(y) \) (\( \hat{U}_b \) is independent of its first variable), and \( U_b \) is the unique solution of
\[
\begin{align*}
- U_b'' &= U_b(1 - U_b^2) \quad \text{in } \mathbb{R}^+, \\
- U_b'' &= U_b(b^2 - U_b^2) \quad \text{in } \mathbb{R}^-, \\
U_b &\in C^1(\mathbb{R}, \mathbb{R}), \quad U_b(0) > 0, \\
\lim_{\pm \infty} U_b &= 1, \quad \lim_{- \infty} U_b = b.
\end{align*}
\]

Note that since the limit is unique, the convergence is valid for the entire sequence.

This solution \( U_b \) may be explicitly obtained by looking for \( U_b \) under the form
\[
U_b(y) = \begin{cases}
\frac{A e^\sqrt{2} y - 1}{A e^\sqrt{2} y + 1} & \text{if } y \geq 0, \\
\frac{b B e^{b \sqrt{2} y} - 1}{b B e^{b \sqrt{2} y} + 1} & \text{if } y < 0.
\end{cases}
\]

We get
\[
B = \frac{-3b^2 + 1 + 2b \sqrt{2(b^2 + 1)}}{1 - b^2},
\]
\[
A = \frac{B(1 + b) + 1 - b}{B(1 - b) + 1 + b}
\]
and
\[
U_b(0) = \frac{b B - 1}{B + 1} = \frac{1 + b^2 + b \sqrt{2(b^2 + 1)}}{2b + \sqrt{2(b^2 + 1)}}.
\]

Since \( U_b(0)^2 = (b^2 + 1)/2 \) and \( U_b \) is an increasing function, for \( y \geq 0, \) \( U_b(y)^2 \geq (b^2 + 1)/2. \) From the convergence \( \hat{U}_\epsilon \rightarrow \hat{U}_b \) in \( L^\infty(B(0, R)) \), we obtain that, for \( \epsilon \) sufficiently small, \( \hat{U}_\epsilon^2 \geq (b^2 + 1)/2 - s \) in \( B(0, R) \cap \{\lambda \delta^j/\epsilon \cdot [((\mathbb{R}^2 \setminus \omega) - \epsilon^2)]\}. \)

**Step 2** Fix \( j \in \{1, \ldots, P\} \) such that \( \mathcal{M}^j_\delta \neq \emptyset \), and fix \( y^\epsilon_{i,j} \in \hat{\mathcal{M}}^j_\delta \). Note that if we are in the periodic case, then \( j = 1 \), and we fix \( y_{k,\ell} = (\delta k, \delta \ell) \in \delta \mathbb{Z} \times \delta \mathbb{Z} \) such that \( y_{k,\ell} + \delta \cdot Y \subset \Omega. \)

We denote \( \xi := \delta^j/\epsilon. \) For \( x \in Y \), consider \( W(x) = V_\xi(x) - U_\epsilon(y^\epsilon_{i,j} + \delta^j x) \), which satisfies (using \( (1.5) \))
\[
\begin{align*}
- \xi^2 \Delta W(x) &= W(x) \left\{ \alpha \lambda(x)^2 - [V_\xi(x)^2 \\
&\quad + U_\epsilon(y^\epsilon_{i,j} + \delta^j x)V_\xi(x) + U_\epsilon(y^\epsilon_{i,j} + \delta^j x)^2] \right\} \quad \text{in } Y, \\
0 \leq W \leq Ce^{-\gamma/\xi} \quad \text{on } \partial Y.
\end{align*}
\]
Here, $\gamma = \alpha \cdot \text{dist}(\partial Y, \omega)$, $C$ and $\alpha$ are given by \eqref{eq:1.5}.

By \textit{Step 1}, taking $s = b^2$, for sufficiently small $\varepsilon$, we have for $x \in Y \setminus \omega$

\[
U_\varepsilon^2(y_{i,j}^\varepsilon + \delta^i x), V_\varepsilon^2(x) \geq \max \left( b^2, \frac{1 - b^2}{2} \right) \geq \frac{1}{3}.
\]

Thus, using the weak maximum principle, we find that $W \geq 0$ in $Y$. Consequently, since $W$ is subharmonic, we deduce that $W \leq C e^{-\gamma/\xi}$. □

4.2. Behavior of almost minimizers of $I_{\rho,\varepsilon}$. We recall that, for $x_0 \in \mathbb{R}^2$ and $R > r > 0$, we denoted $\mathcal{R}(x_0, R, r) := B(x_0, R) \setminus B(x_0, r)$.

4.2.1. Useful results for the periodic situation. We establish three preliminary results for the periodic situation represented in Figure 1.1. Thus, in this subsection we assume that $U_\varepsilon$ is the unique global minimizer of $E_\varepsilon$ in $H^1$ with the periodic pinning term $a_\varepsilon$ represented in Figure 1.1.

Energetic estimates in rings and global energetic upper bounds. From Lemma D.1 ($U_\varepsilon$ is close to a periodic function), we obtain the following result.

**Lemma D.2.** For all $1 \geq R > r \geq \varepsilon$, $x, x_0 \in \mathbb{R}^2$ such that $B(x_0, R) \subset \Omega^{\text{incl}}$ and $x - x_0 \in \delta \cdot \mathbb{Z}^2$, we have

\[
\mu_\varepsilon(\mathcal{R}(x, R, r), 1) \geq \mu_\varepsilon(\mathcal{R}(x_0, R, r), 1) - o_\varepsilon(1).
\]

Adding the condition that $B(x, R) \subset \Omega^{\text{incl}}$, we have

\[
|\mu_\varepsilon(\mathcal{R}(x, R, r), 1) - \mu_\varepsilon(\mathcal{R}(x_0, R, r), 1)| \leq o_\varepsilon(1).
\]

Moreover, the $o_\varepsilon(1)$ may be considered independent of $x, x_0, R, r$.

**Lemma D.2** implies easily the following estimate.

**Proposition D.3.** Let $\eta > 0$ and $\eta > \rho \geq \varepsilon$. Then there exists $C > 0$, depending on $\Omega, \Omega', g, b$, and $\eta$, such that for $x_0 \in \mathbb{R}^2$ we have

\[
I_{\rho,\varepsilon} \leq d\mu_\varepsilon(\mathcal{R}(x_0, \eta, \rho), 1) + C(\eta),
\]

where $C(\eta)$ is independent of $x_0$ and $\rho$.

From Lemma D.1, we get the almost periodicity of $\mu_\varepsilon(\mathcal{R}(\cdot, R, r), 1)$ with respect to a $\delta \times \delta$-grid (expressed in Lemma D.1). Therefore, the “best points” to minimize $\mu_\varepsilon(\mathcal{R}(\cdot, R, r), 1)$ should be an almost periodic set.

Another important result is the next proposition. It expresses that the center of an inclusion is not too far from a good point to minimize $\mu_\varepsilon(\mathcal{R}(\cdot, R, r), 1)$. This proposition may be seen as a first step in the proof of the pinning effect by $\omega_\varepsilon$. 

\[
\text{where } C(\eta) \text{ is independent of } x_0 \text{ and } \rho.
\]
Proposition D.4. There is $C_*$ which depends only on $\omega$, $b$, and $\Omega$ such that for sufficiently small $\varepsilon$, for $x \in \Omega$ and $x_{\text{per}} \in B(x, \delta \sqrt{\pi}/2) \cap (\delta \mathbb{Z} \times \delta \mathbb{Z}) \cap \omega$, we have, for $1 > R > r > \varepsilon$,

\[
\mu_\varepsilon(\mathcal{R}(x_{\text{per}}, R, r), 1) \leq \mu_\varepsilon(\mathcal{R}(x, R, r), 1) + C_*.
\]

(D.4)

Proof. If $R \leq 10^2r$, then the result is obvious with $C_* = 2\pi \ln 10$. Thus we assume that $R > 10^2r$.

We split the proof into three cases:

Case 1: $r \geq \delta$,

Case 2: $\delta \geq R > r \geq \lambda \delta$,

Case 3: $R \leq \lambda \delta$.

Assume for the moment that

\[
(D.5) \quad \exists \tilde{C}_* > 0 \text{ such that } (D.4) \text{ holds in the three previous cases with } C_* = \tilde{C}_*.
\]

For the general case, we divide $\mathcal{R}(x, R, r)$ into $R_1(x) \cup R_2(x) \cup R_3(x)$ with

\[
R_1(x) = \mathcal{R}(x, R, \min\{\max(\delta, r), R\}),
\]

\[
R_2(x) = \mathcal{R}(x, \min\{\max(\delta, r), R\}, \min\{R, \max(\lambda \delta, r)\}),
\]

\[
R_3(x) = \mathcal{R}(x, \min\{R, \max(\lambda \delta, r)\}, r).
\]

Remark D.5.

(1) For $k \in \{1, 2, 3\}$, we have $\emptyset \subseteq R_k \subseteq \mathcal{R}(x, R, r)$.

(2) It is easy to check that

- $[R_1 = \mathcal{R}(x, R, r) \iff r \geq \delta \text{ (Case 1)}]$ and $[R_1 = \emptyset \iff R \leq \delta]$,

- $[R_2 = \mathcal{R}(x,\min\{\max(\delta, r), R\}, \min\{R, \max(\lambda \delta, r)\})$ and $[R_2 = \emptyset \iff \{\lambda = 1 \text{ or } r \geq \delta \text{ or } R \leq \lambda \delta\}]$,]

- $[R_3 = \mathcal{R}(x, R, r) \iff R \leq \lambda \delta \text{ (Case 3)}]$ and $[R_3 = \emptyset \iff r \geq \lambda \delta]$.

(3) If $\lambda \equiv 1$, then Case 2 never occurs, and $R_2 = \emptyset$.

Therefore we have (using Propositions C.4, C.5 (3), and (D.5)) the following:

\[
\mu_\varepsilon(\mathcal{R}(x, R, r), 1) \geq \mu_\varepsilon(R_1(x), 1) + \mu_\varepsilon(R_2(x), 1) + \mu_\varepsilon(R_3(x), 1)
\]

\[
\geq \mu_\varepsilon(R_1(x_{\text{per}}), 1) + \mu_\varepsilon(R_2(x_{\text{per}}), 1) + \mu_\varepsilon(R_3(x_{\text{per}}), 1) - 3\tilde{C}_* \quad \text{by (D.5)}
\]

\[
\geq \mu_\varepsilon^{\text{Dir}}(R_1(x_{\text{per}}), 1) + \mu_\varepsilon^{\text{Dir}}(R_2(x_{\text{per}}), 1) + \mu_\varepsilon^{\text{Dir}}(R_3(x_{\text{per}}), 1)
\]

\[
- 3(\tilde{C}_* + C_b) \quad \text{by Proposition C.4}
\]

\[
\geq \mu_\varepsilon(\mathcal{R}(x_{\text{per}}, R, r), 1) - 3(\tilde{C}_* + C_b).
\]

The last line is obtained by constructing a test function. Therefore, it suffices to take $C_* := 3(\tilde{C}_* + C_b)$. 
We now turn to the proof of (D.5) in Case 1, Case 2, and Case 3. Recall that we assumed that \( R > 10^{-2}r \).

We first treat Case 1 \( (R > r \geq \delta) \):

\[
\mu_\varepsilon(\mathcal{R}(x, R, r), 1) \geq \mu_\varepsilon^{\text{Dir}}(\mathcal{R}(x, R, r), 1) - C_b
\]

by Proposition C.4

\[
\geq \mu_\varepsilon^{\text{Dir}}(\mathcal{R}(x, 10R, 10^{-1}r), 1) - 2\pi \ln 10 - C_b
\]

because \( \mathcal{R}(x, 10R, 10^{-1}r) \subset \mathcal{R}(x, 10R, 10^{-1}r) \)

\[
\geq \mu_\varepsilon(\mathcal{R}(x_{\text{per}}, R, 10r), 1) - 2\pi \ln 10 - C_b
\]

by Proposition C.4.

Thus we may take \( \tilde{C}_\ast = 3\pi \ln 10 + 2C_b \).

For Case 2, we note that from Remark D.5 (3), we may assume that \( \lambda \to 0 \).

On the one hand, it is clear that

\[
\mu_\varepsilon(\mathcal{R}(x_{\text{per}}, R, r), 1) \leq \pi \ln \frac{R}{r}.
\]

On the other hand, if we let

\[
\alpha_\varepsilon : \mathbb{R}^2 \to \{b^2, 1\}
\]

\[
x \mapsto \begin{cases} 
  b^2 & \text{if } x \in \bigcup_{M \in \mathbb{Z}^2} B(\delta M, \lambda \delta), \\
  1 & \text{otherwise},
\end{cases}
\]

we have from Proposition 1.4 that \( \alpha_\varepsilon \leq U_\varepsilon^2 + V_\varepsilon \) with \( \|V_\varepsilon\|_\infty = o(\varepsilon^2) \).

If \( \mathcal{R}(x, R, r) \cap \{\alpha_\varepsilon = b^2\} = \emptyset \), then \( \mu_\varepsilon(\mathcal{R}(x, R, r), 1) \geq \pi \ln(R/r) + o(\varepsilon^2 \ln \lambda) \). Thus the result holds with \( \tilde{C}_\ast = 1 \) (for sufficiently small \( \varepsilon \)).

Otherwise, we have \( \mathcal{R}(x, R, r) \cap \{\alpha_\varepsilon = b^2\} \neq \emptyset \). In this situation, because \( R \leq \delta \), we get that \( \mathcal{R}(x, R, r) \cap \{\alpha_\varepsilon = b^2\} \) is a union of at most four connected components. Therefore, \( S = \{\rho \in (r, R) | \partial B(x, \rho) \cap \{\alpha_\varepsilon = b^2\}\} \neq \emptyset \) is a union of at most four segments whose length is lower than \( 8\lambda \delta \). Consequently, denoting \( \hat{S} = \bigcup_{i=1}^{k} [s_i, t_i] \) (with \( s_i < s_{i+1} \)), for \( w_\ast \in H^1(\mathcal{R}(x, R, r), S^1) \) which minimizes \( \mu_\varepsilon(\mathcal{R}(x, R, r), 1) \), we have

\[
\frac{1}{2} \int_{\mathcal{R}(x, R, r)} U_\varepsilon^2 |\nabla w_\ast|^2 + o_\varepsilon(1) \geq \frac{1}{2} \int_{\mathcal{R}(x, R, r)} \alpha_\varepsilon |\nabla w_\ast|^2
\]

\[
\geq \sum_{i=0}^{k} \frac{1}{2} \int_{t_i}^{s_{i+1}} \frac{d\rho}{\rho} \int_{s_i}^{s_{i+1}} |\partial_\theta w_\ast|^2 \quad (t_0 = r \text{ and } s_{k+1} = R)
\]

\[
\geq \pi \sum_{i=0}^{k} \ln \frac{s_{i+1}}{t_i} = \pi \ln \frac{R}{r} - \pi \sum_{i=1}^{k} \ln \frac{t_i}{s_i}.
\]
Since $\lambda \delta \leq s_i \leq t_i \leq s_i + 8\lambda \delta$, we have $1 \leq t_i / s_i \leq 1 + 8\lambda \delta / s_i \leq 9$. Therefore, we may take $\tilde{C}_* = 4\pi \ln 9 + 1$.

We now treat the last case. Since $R \leq \lambda \delta$ and $x_{\text{per}} \in (\delta Z \times \delta Z) \cap \omega$, there is $\tilde{C}_*$ such that we have (for sufficiently small $\epsilon$) $\mu_\epsilon(\mathcal{R}(x_{\text{per}}, R, r), 1) \leq \pi b^2 \ln(R/r) + \tilde{C}_*$. On the other hand, we have $\mu_\epsilon(\mathcal{R}(x, R, r), 1) \geq \pi b^2 \ln(R/r)$, due to Proposition C.5 (2). Therefore, the estimate in the third case is proved. \qed

**Estimates for almost minimizers.** In this subsection, we establish a fundamental result. We fix an almost minimal configuration $\{x, 1\}$ for $I_{\rho, \epsilon}$ (the existence of such a configuration is proved in Section D.3), and a map which almost minimizes $\frac{1}{2} \int_{\mathcal{R}(x, R, r)} U_\epsilon^2 |\nabla \cdot |^2$. Then the map almost minimizes the weighted Dirichlet functional $\frac{1}{2} \int_{\mathcal{R}(x, R, r)} U_\epsilon^2 |\nabla \cdot |^2$, $10^{-2} \min_{i \neq j} |x_i - x_j| > \rho > \rho$.

**Lemma D.6.**

(1) Let $x \in \mathbb{R}^2$, $0 < r < R$, $\alpha \in L^\infty(\mathbb{R}^2, [b^2, 1])$, $C_0 > 0$, and a map $w \in H^1(\mathcal{R}(x, R, r), S^1)$ be such that $\deg_{\partial \mathcal{B}(x, R)}(w) = 1$ and

$$\frac{1}{2} \int_{\mathcal{R}(x, R, r)} \alpha |\nabla w|^2 - \mu_\alpha(\mathcal{R}(x, R, r), 1) \leq C_0.$$ 

Then for all $r', R'$ such that $r < r' < R < R'$, one has

$$\frac{1}{2} \int_{\mathcal{R}(x, R', r')} \alpha |\nabla w|^2 - \mu_\alpha(\mathcal{R}(x, R', r'), 1) \leq 4C_0 + C_0,$$

where $C_0$ depends only on $b$ and is given by Proposition C.4.

(2) Let $x_1, \ldots, x_d \in \Omega (x_i \neq x_j$ for $i \neq j)$, $d_i = 1$, $\epsilon < \rho < 10^{-2} \eta$, $\eta := 10^{-2} \cdot \min\{ |x_i - x_j|, \text{dist}(x_i, \partial \Omega) \}$, $C_0 > 0$, and $w \in H^1(\Omega', S^1)$ be such that

$$\frac{1}{2} \int_{\Omega'} U_\epsilon^2 |\nabla w|^2 \leq I_{\rho, \epsilon} + C_0.$$ 

Then for $\rho \leq r < R < \eta$, one has for all $i$ that

$$\frac{1}{2} \int_{\mathcal{R}(x_i, R, r)} U_\epsilon^2 |\nabla w|^2 - \mu_\epsilon(\mathcal{R}(x_i, R, r), 1) \leq C_0 + C(\eta);$$

here, $C(\eta)$ depends only on $b, g, \Omega, \Omega'$, and $\eta$.

(3) Under the hypotheses of (2), for $\eta > \rho_0 > \rho$, we also have

$$\frac{1}{2} \int_{\Omega_{\rho_0}} U_\epsilon^2 |\nabla w|^2 \leq C(\rho_0, C_0);$$

here, $C(\rho_0, C_0)$ depends only on $b, g, \Omega, \Omega'$, $C_0$, $\rho_0$, and $\eta$. 
Proof. Using the third part of Proposition C.5, we have

\[
\frac{1}{2} \int_{\mathcal{R}(x,R,r)} \alpha |\nabla w|^2 \leq \mu_\alpha(\mathcal{R}(x,R,R'), 1) + \mu_\alpha(\mathcal{R}(x,R', r'), 1) + \mu_\alpha(\mathcal{R}(x,r', r), 1) + 4C_b + C_0.
\]

We easily obtain

\[
\frac{1}{2} \int_{\mathcal{R}(x,R,r)} \alpha |\nabla w|^2 \geq \mu_\alpha(\mathcal{R}(x,R,R'), 1) + \frac{1}{2} \int_{\mathcal{R}(x,R', r')} \alpha |\nabla w|^2 + \mu_\alpha(\mathcal{R}(x,r', r), 1),
\]

which proves the first assertion.

The second assertion is obtained by using the same argument combined with Proposition D.3; the last assertion is a straightforward consequence of Proposition D.3 and both previous assertions. \(\square\)

4.2.2. Lower bound on circles. In this subsection, we prove an estimate for the minimization of weighted one-dimensional Dirichlet functionals. In the following, this estimate will be used to get lower bounds in rings.

Lemma D.7. Let \(\theta_0 \in (0, 2\pi)\), and let \(\alpha \in L^\infty([0,2\pi], \{b^2, 1\})\) be such that \(\mathcal{H}^1(\{\alpha = b^2\}) = \theta_0\).

Let \(\varphi \in H^1([0,2\pi], \mathbb{R})\) such that \(\varphi(2\pi) - \varphi(0) = 2\pi\). The following lower bound holds:

\[
\frac{1}{2} \int_0^{2\pi} \alpha(\theta) |\partial_\theta \varphi(\theta)|^2 d\theta \geq \frac{2\pi^2}{\int_0^{2\pi} \frac{1}{\alpha}} - \frac{2\pi^2}{2\pi + \theta_0(b^{-2} - 1)}.
\]

Here, \(\mathcal{H}^1\) is the one-dimensional Hausdorff measure.

Proof. The proof of this lower bound is based on the computation of the minimal energy.

It is quite easy to check that a minimal function \(\varphi_{\text{min}} \in H^1([0,2\pi], \mathbb{R})\) for \(\frac{1}{2} \int_0^{2\pi} \alpha(\theta) |\partial_\theta \varphi(\theta)|^2 d\theta\) under the constraint \(\varphi(2\pi) - \varphi(0) = 2\pi\) exists and satisfies \(\partial_\theta(\alpha \partial_\theta \varphi_{\text{min}}) = 0\). Thus, \(\partial_\theta \varphi_{\text{min}} = \text{Cst}/\alpha\) with \(\text{Cst} = 2\pi/\int_0^{2\pi} \alpha^{-1}\), and therefore,

\[
\frac{1}{2} \int_0^{2\pi} \alpha(\theta) |\partial_\theta \varphi(\theta)|^2 d\theta \geq \frac{1}{2} \int_0^{2\pi} \alpha(\theta) |\partial_\theta \varphi_{\text{min}}(\theta)|^2 d\theta = \frac{2\pi^2}{\int_0^{2\pi} \frac{1}{\alpha}} - \frac{2\pi^2}{2\pi + \theta_0(b^{-2} - 1)}, \quad \square
\]
4.3. Proof of the first part of Proposition 3.6. Let $x_i^n, \ldots, x_N^n \in \Omega$ such that $|x_i^n - x_j^n| \geq 8\rho$ and $d_1, \ldots, d_N > 0, \sum d_i = d$ (up to a subsequence, the degrees may be considered independent of $n$).

Assume that

\[(D.6)\quad \text{There exists } i_0 \in \{1, \ldots, N\} \text{ such that } d_{i_0} \neq 1,\]

or there exist $i \neq j$ such that $|x_i^n - x_j^n| \rightarrow 0$.

Up to passing to a subsequence, there are $a_1, \ldots, a_M \in \hat{\Omega}$ and $\{\Lambda_1, \Lambda_M\}$ a partition of $\{1, \ldots, N\}$ such that $i \in \Lambda_\ell \iff x_i^n \rightarrow a_\ell$. For the sake of simplicity, we drop the superscript $n$ for the points, that is, we write $x_i$ instead of $x_i^n$.

We let

\[\rho_0 := 10^{-2} \min_{k \neq \ell} \min |a_k - a_\ell|, \text{dist}(\partial \Omega, \partial \Omega')\],

\[\min_{k \neq \ell} |a_k - a_\ell| = +\infty \text{ if } M = 1.\]

Note that since $d_i > 0$, $(D.6)$ is equivalent to

\[(D.7)\quad \text{There exists } \ell_0 \in \{1, \ldots, M\} \text{ such that } d_{\ell_0} = \sum_{i \in \Lambda_{\ell_0}} d_i > 1.\]

We intend to prove that $(D.7)$ is not possible for almost minimal configurations. In order to do this, for $\ell \in \{1, \ldots, M\}$, we obtain a lower bound for the weighted Dirichlet functional defined around $a_\ell$. Then, using Proposition D.3, we will conclude.

For $\ell \in \{1, \ldots, M\}$, there are two cases: $\text{Card}(\Lambda_\ell) > 1$, and $\text{Card}(\Lambda_\ell) = 1$. In the first case, we apply the separation process (defined in Section C.1) in $\Omega_\ell^0 = B(a_\ell, 2\rho_0) \setminus \bigcup_{i \in \Lambda_\ell} B(x_i, \rho)$ with $\eta_{\text{stop}} = 10^{-2}\rho_0$. By construction, the process stops after $K$ steps. For $k \in \{1, \ldots, K\}$, we use the following denotations:

- $\{x_1^k, \ldots, x_N^k\}$ $(x_i^k = x_i, i \in \Lambda_\ell)$, for the selection of points made in Step $k$
- $\eta_k^i (\eta_k^i = \frac{1}{4} \min_{i \neq j} |x_i^{k-1} - x_j^{k-1}|)$, for the radius of the intermediate balls in Step $k$
- $\eta_k (\eta_k = 2\kappa_\ell \eta_k^\ell, \kappa_\ell \in \{9^0, \ldots, 9^d\} \text{ and } \eta_0 = \rho)$, for the radius of the final balls in Step $k$

Since for $i, j \in \Lambda_\ell$ we have $|x_i - x_j| \rightarrow 0$, then, in the end of the process (after $K$ steps), we obtain a unique $x_\ell^k = y_\ell \in \{x_i \mid i \in \Lambda_\ell\}$ in the final selection of points and $\eta_k \rightarrow 0$.

From (C.3) and (C.4), for $k \in \{0, \ldots, K - 1\}, j \in \{1, \ldots, N_k\}$, the following rings are mutually disjoint (denoting $\eta_0 = \rho$):

\[R_{\ell}^k = \mathcal{R}(y_\ell, \rho_0, \eta_k) \text{ and } R_{j, k} = \mathcal{R}(x_j^k, \eta_k^j, \eta_k)\].

We make the following assignments:

- For $k \in \{0, \ldots, K - 1\} \text{ and } j \in \{1, \ldots, N_k\}$, let $\hat{d}_{j, k} := \sum_{x_i \in B(x_j^k, \eta_k^j)} d_i$;
- For $n \geq 1$, let $x_0 = x_0(n) \in (\delta \mathbb{Z} \times \delta \mathbb{Z}) \cap \omega_{\varepsilon_n}$ be such that $B(x_0, 2\rho_0) \subset$$\Omega$. 

By combining Lemma D.2 with Proposition D.4, we get that, for sufficiently large \( n \) and for \( \rho_0 \geq R > r \geq \rho \),

\[
\mu_{\tilde{e}_n}(\mathcal{R}(x_0, R, r), 1) \leq \inf_{x \in \Omega} \mu_{\tilde{e}_n}(\mathcal{R}(x, R, r), 1) + C_\nu + 1.
\]

For \( w \in H_0^1(\Omega', S^1) \), we have

\[
(\text{D.9}) \quad \frac{1}{2} \int_{\Omega''_n} U_{\tilde{e}_n}^2 |\nabla w|^2 \geq \frac{1}{2} \int_{\Omega''_n} U_{\tilde{e}_n}^2 |\nabla w|^2 + \sum_{k=0}^{K-1} \sum_{j=1}^{N_k} \frac{1}{2} \int_{R_{j,k}} U_{\tilde{e}_n}^2 |\nabla w|^2
\]

\[
\geq \frac{1}{2} \int_{\mathcal{R}(x_0, \rho_0, \eta_K)} U_{\tilde{e}_n}^2 |\nabla w|^2 + \sum_{k=0}^{K-1} \sum_{j=1}^{N_k} \mu_{\tilde{e}_n}(\mathcal{R}(x_0, \eta_{k+1}, \eta_k), \tilde{d}_{j,k})
\]

\[
\geq \frac{1}{2} \int_{\mathcal{R}(x_0, \rho_0, \eta_K)} U_{\tilde{e}_n}^2 |\nabla w|^2 + \sum_{k=0}^{K-1} \sum_{j=1}^{N_k} \mu_{\tilde{e}_n}(\mathcal{R}(x_0, \eta_{k+1}, \eta_k), \tilde{d}_{j,k}) - o(1) \quad \text{by (D.8)}
\]

\[
(\text{D.10}) \quad \frac{1}{2} \int_{\mathcal{R}(x_0, \rho_0, \rho)} U_{\tilde{e}_n}^2 |\nabla w|^2 \geq \frac{1}{2} \int_{\mathcal{R}(x_0, \rho_0, \rho)} U_{\tilde{e}_n}^2 |\nabla w|^2
\]

\[
\geq (\tilde{d}_K - \tilde{d}_\ell) \pi b^2 |\ln \eta_K| - o(1) \quad \text{by Proposition C.5 (3)}.
\]

In the second case (\( \text{Card}(A_\ell) = 1 \)), the computations are direct:

\[
(\text{D.9}) \quad \frac{1}{2} \int_{\Omega''_n} U_{\tilde{e}_n}^2 |\nabla w|^2 \geq \frac{1}{2} \int_{\mathcal{R}(x_0, \rho_0, \rho)} U_{\tilde{e}_n}^2 |\nabla w|^2
\]

\[
\geq (\tilde{d}_K - \tilde{d}_\ell) \pi b^2 |\ln \eta_K| - o(1).
\]

Summing the lower bounds (D.9) and (D.10) over \( \ell \), and applying Proposition D.3, we obtain that if (D.7) occurs, then the configuration \( \{x, \mathbf{d}\} \) cannot be almost minimal because \( \eta_K, \rho \to 0 \) and \( \tilde{d}_\ell > 1 \). Therefore (D.7) cannot occur for almost minimal configurations.

### 4.4. Proof of the second part of Proposition 3.6.

We now prove the second part of Proposition 3.6: we establish the repelling effect of \( \partial \Omega \) on the centers \( x_i \).

Let \( x^n_1, \ldots, x^n_d \in \Omega \) and \( \rho = \rho(\tilde{e}_n) \to 0 \) be such that \( |x^n_i - x^n_j| \geq 8\rho \) \( (i \neq j) \) and \( \text{dist}(x^n_i, \partial \Omega) \to 0 \). From the previous subsection, we may assume that there is \( \eta_0 > 0 \) (independent of \( n \)) such that

\[
\min \{ \min_{i \neq j} |x^n_i - x^n_j|, \text{dist}(\Omega, \partial \Omega') \} \geq 10^2 \eta_0.
\]

Up to passing to a subsequence, we may assume that \( x^n_i \to a_i \in \Omega \) with \( a_i \neq a_j \) for \( i \neq j \), and that \( \eta = \max\{\sqrt{\text{dist}(x^n_i, \partial \Omega)}, \rho\} \to 0 \).
For the sake of simplicity, we assume that for \( i = 2, \ldots, d \) we have \( a_i \in \Omega \). If this condition is not satisfied, then a direct adaptation of the following argument may be done. We assume that \( \eta_0 \) is such that for \( i = 2, \ldots, d \), we have \( \text{dist}(x^n_0, \partial \Omega) \geq 10^2 \eta_0 \).

We fix \( x_0 = x_0(\varepsilon_\nu) \in \Omega \) such that \( x_0 - x^n_0 \in \delta \mathbb{Z} \times \delta \mathbb{Z} \), \( \text{dist}(x^n_0, \partial \Omega) \geq 10^2 \eta_0 \), and \( \min_{i=1,\ldots,d} |x_0 - x^n_i| \geq 10^2 \eta_0 \).

We intend to prove that, for \( w \in H^1_\partial (\Omega) \backslash \bigcup_i B(x^n_i, \rho), S^1 \), we have

\[
\frac{1}{2} \int_{\mathbb{R}(x^n_0, \rho)} U_{\varepsilon_n}^2 |\nabla w|^2 - \mu_{\varepsilon_n}(\mathbb{R}(x_0, \sqrt{\eta}), 1) - \infty.
\]

**Remark D.8.** The estimate (D.11) implies that \( \{x^n_1, \ldots, x^n_d\} \) cannot be an almost minimal configuration of points.

Indeed, we may construct a suitable test function \( \tilde{w} \) as follows:

**Construction D.9 (The test function \( \tilde{w} \)).**

\[
\tilde{w} \in H^1_\partial \left( \Omega \backslash \left( \bigcup_{i=2}^d B(x^n_i, \rho) \right), S^1 \right).
\]

- For \( i = 2, \ldots, d \), we define \( \tilde{w}|_{\mathbb{R}(x^n_0, \rho)} \) by taking a minimal map for \( \frac{1}{2} \int_{\mathbb{R}(x^n_0, \rho)} U_{\varepsilon_n}^2 |\nabla \cdot |^2 \) in \( H^1(\mathbb{R}(x^n_0, \rho), S^1) \) with the boundary conditions \( \tilde{w}(x^n_0 + \eta_0 e^{i\theta}) = e^{i\theta} \) and \( \tilde{w}(x^n_0 + \rho e^{i\theta}) = C_{\text{st}} e^{i\theta}, C_{\text{st}} \in S^1 \). From Proposition C.4, we have

\[
\frac{1}{2} \int_{\mathbb{R}(x^n_0, \rho)} U_{\varepsilon_n}^2 |\nabla \tilde{w}|^2 \leq \mu_{\varepsilon_n}(\mathbb{R}(x^n_0, \rho), 1) + C_{\text{b}}.
\]

- We divide \( \mathbb{R}(x_0, \eta_0, \rho) \) into \( \mathbb{R}(x_0, \eta_0, \sqrt{\eta}), \mathbb{R}(x_0, \sqrt{\eta}, \eta), \) and \( \mathbb{R}(x_0, \eta, \rho) \).

In each of these rings, we consider the minimal maps for \( \frac{1}{2} \int_{\text{ring}} U_{\varepsilon_n}^2 |\nabla \cdot |^2 \) with the boundary conditions \( \tilde{w}(x_0 + Re^{i\theta}) = e^{i\theta} \) and \( \tilde{w}(x_0 + re^{i\theta}) = C_{\text{st}} e^{i\theta}, C_{\text{st}} \in S^1 \), where \( \text{ring} \in \{\mathbb{R}(x_0, \eta_0, \sqrt{\eta}), \mathbb{R}(x_0, \sqrt{\eta}, \eta), \mathbb{R}(x_0, \eta, \rho)\} \), \( r < R \) and \( \text{ring} = \mathbb{R}(x_0, R, r) \).

Up to considering suitable rotations, we glue these functions to get an \( S^1 \)-valued map \( \tilde{w}|_{\mathbb{R}(x_0, \eta_0, \rho)} \in H^1(\mathbb{R}(x_0, \eta_0, \rho), S^1) \) which is such that \( \tilde{w}(x_0 + \eta_0 e^{i\theta}) = e^{i\theta} \) and (from Proposition C.4)

\[
\frac{1}{2} \int_{\text{ring}} U_{\varepsilon_n}^2 |\nabla \tilde{w}|^2 \leq \mu_{\varepsilon_n}(\text{ring}, 1) + C_{\text{b}},
\]

with \( \text{ring} \in \{\mathbb{R}(x_0, \eta_0, \sqrt{\eta}), \mathbb{R}(x_0, \sqrt{\eta}, \eta), \mathbb{R}(x_0, \eta, \rho)\} \).
We extend \( \tilde{w} \) in \( \Omega \setminus \left( B(x_0, \eta_0) \cup \bigcup_{i=2}^{d} B(x^n_i, \eta_0) \right) \) using Lemma C.3 (1). Then, we finally obtain \( \tilde{w} \in H^1_{\rho} (\Omega' \setminus \left( B(x_0, \rho) \cup \bigcup_{i=2}^{d} B(x^n_i, \rho) \right), S^1) \).

For \( w_n \in H^1_{\rho} (\Omega' \setminus \bigcup_i B(x^n_i, \rho), S^1) \), from Lemma D.2, (D.11), and by construction of \( \tilde{w} \), we have easily that
\[
\int_{\Omega' \setminus \bigcup_i B(x^n_i, \rho)} U_{\tilde{\epsilon} n}^2 |\nabla w_n|^2 - \int_{\Omega' \setminus (B(x_0, \rho) \cup \bigcup_{i=2}^{d} B(x^n_i, \rho))} U_{\tilde{\epsilon} n}^2 |\nabla \tilde{w}|^2 \to +\infty,
\]
which implies that \( \{x^n_1, \ldots, x^n_d\} \) cannot be an almost minimal configuration of points.

We now turn to the proof of (D.11). We argue by contradiction and assume that there is \( w_* = w_n^* \in H^1_{\rho} (\Omega' \setminus \bigcup_i B(x^n_i, \rho), S^1) \) such that
\[
(\text{D.12}) \quad \frac{1}{2} \int_{\mathcal{R}(x^n_1, \sqrt{\eta}, \eta)} U_{\tilde{\epsilon} n}^2 |\nabla w_*|^2 \leq \mu_{\tilde{\epsilon} n} (\mathcal{R}(x, \sqrt{\eta}, \eta), 1) + O(1) \tag{D.12}
\]
In particular (using Lemma D.2), we have
\[
\frac{1}{2} \int_{\mathcal{R}(x^n_1, \sqrt{\eta}, \eta)} U_{\tilde{\epsilon} n}^2 |\nabla w_*|^2 = \mu_{\tilde{\epsilon} n} (\mathcal{R}(x^n_1, \sqrt{\eta}, \eta), 1) + O(1). \tag{D.13}
\]
The key ingredient to get a contradiction is the fact that the map \( w_* \) is almost constant in the “half” ring \( \mathcal{R}(x^n_1, \sqrt{\eta}, \eta) \setminus \Omega \).

By smoothness of \( \Omega \), we may assume that the cone
\[
K_{x^n_1, \sqrt{\eta}} := \{ x = x^n_1 + \rho e^{i\theta} | \theta \in [0, \pi/2], \eta \leq \rho \leq \sqrt{\eta} \}
\]
does not intersect \( \Omega \): \( K_{x^n_1, \sqrt{\eta}} \cap \Omega = \emptyset \).

We consider the map
\[
w_0(x^n_1 + \rho e^{i\theta}) = \begin{cases}
  e^{i\theta} & \text{if } \theta \in [0, \pi/2], \\
  1 & \text{otherwise}
\end{cases}
\]
with \( r > 0 \), such that \( w_0 \in H^1 (\mathcal{R}(x^n_1, \sqrt{\eta}, \eta), S^1) \) and \( \text{deg}_{\partial B(x^n_1, \sqrt{\eta})} (w_0) = 1 \).

We define the map \( w_{\text{test}} = w_n^d w_0 \in H^1 (\mathcal{R}(x^n_1, \sqrt{\eta}, \eta), S^1) \) for \( d \in \mathbb{N}^* \) (to be fixed later) and \( \text{deg}_{\partial B(x^n_1, \sqrt{\eta})} (w_{\text{test}}) = \tilde{d} + 1 \).

Thus, we have
\[
\frac{1}{2} \int_{\mathcal{R}(x^n_1, \sqrt{\eta}, \eta)} U_{\tilde{\epsilon} n}^2 |\nabla w_{\text{test}}|^2 \geq \mu_{\tilde{\epsilon} n} (\mathcal{R}(x^n_1, \sqrt{\eta}, \eta), \tilde{d} + 1) = (\tilde{d} + 1)^2 \mu_{\tilde{\epsilon} n} (\mathcal{R}(x^n_1, \sqrt{\eta}, \eta), 1).
\]
On the other hand, letting \( \varphi_\ast, \varphi_0 : \mathcal{R}(x^n_1, \sqrt{\eta}, \eta) \to \mathbb{R} \) be such that \( w_\ast = e^{t\varphi_\ast} \) and \( w_0 = e^{t\varphi_0} \) (note that \( \varphi_\ast, \varphi_0 \) are locally defined, and with gradients that are globally defined and in \( L^2(\mathcal{R}(x^n_1, \sqrt{\eta}, \eta, \mathbb{R})) \)), using (D.12) gives us

\[
\frac{1}{2} \int_{\mathcal{R}(x^n_1, \sqrt{\eta}, \eta)} U^n_{\varphi_\ast} | \nabla w_{\text{test}} |^2 = \frac{1}{2} \int_{\mathcal{R}(x^n_1, \sqrt{\eta}, \eta)} U^n_{\varphi_\ast} | \nabla \varphi_\ast + \nabla \varphi_0 |^2 \\
= \frac{1}{2} \int_{\mathcal{R}(x^n_1, \sqrt{\eta}, \eta)} U^n_{\varphi_\ast} | \nabla \varphi_\ast |^2 + \frac{1}{2} \int_{\mathcal{R}(x^n_1, \sqrt{\eta}, \eta)} U^n_{\varphi_\ast} | \nabla \varphi_0 |^2 \\
+ \int_{\mathcal{R}(x^n_1, \sqrt{\eta}, \eta)} U^n_{\varphi_\ast} \nabla \varphi_\ast \cdot \nabla \varphi_0 \\
\leq \tilde{d}^2 \mu_{\varphi_\ast}(\mathcal{R}(x^n_1, \sqrt{\eta}, \eta), 1) + 2\pi | \ln \eta | + \tilde{d} \int_{\mathcal{R}(x^n_1, \sqrt{\eta}, \eta)} U^n_{\varphi_\ast} \nabla \varphi_\ast \cdot \nabla \varphi_0 + O(1).
\]

Since \( w_\ast = g \) in \( \mathcal{R}(x^n_1, \sqrt{\eta}, \eta) \setminus \tilde{\Omega} \) and \( \| \nabla \varphi_0 \|_{L^2(\mathcal{R}(x^n_1, \sqrt{\eta}, \eta) \cap \tilde{\Omega})} = 0 \), we have (using the Cauchy-Schwarz inequality) that

\[
\int_{\mathcal{R}(x^n_1, \sqrt{\eta}, \eta)} U^n_{\varphi_\ast} | \nabla \varphi_\ast | | \nabla \varphi_0 | = \int_{\mathcal{R}(x^n_1, \sqrt{\eta}, \eta) \setminus \tilde{\Omega}} U^n_{\varphi_\ast} | \nabla \varphi_\ast | | \nabla \varphi_0 | = O(\sqrt{| \ln \eta |}).
\]

Therefore, we obtain

\[
\tilde{d}^2 \mu_{\varphi_\ast}(\mathcal{R}(x^n_1, \sqrt{\eta}, \eta), 1) + 2\pi | \ln \eta | + O(\sqrt{| \ln \eta |}) \geq (\tilde{d} + 1)^2 \mu_{\varphi_\ast}(\mathcal{R}(x^n_1, \sqrt{\eta}, \eta), 1),
\]

which implies that

\[
2\pi | \ln \eta | + O(\sqrt{| \ln \eta |}) \geq (2\tilde{d} + 1) \mu_{\varphi_\ast}(\mathcal{R}(x^n_1, \sqrt{\eta}, \eta), 1) \geq (2\tilde{d} + 1)b^2 \pi | \ln \eta |.
\]

Clearly, we obtain a contradiction by taking \( \tilde{d} > (2 - b^2)/(2b^2) \). Thus, by using Remark D.8, the second part of Proposition 3.6 is proved.

4.5. Proof of the third part of Proposition 3.6. In this subsection, we prove the third part of Proposition 3.6: the attractive effect of the inclusions.

Assume that there exist \( C_0 > 0 \), sequences \( \varepsilon_n, \rho \uparrow 0 \), \( \rho = \rho(\varepsilon_n) \geq \varepsilon_n \) such that \( \rho / (\lambda \delta) \to 0 \), and distinct points \( x^n_1, \ldots, x^n_n \), satisfying

\[
(D.13) \quad \inf_{w \in H^2_0(\Omega^n_\rho, S_1)} \frac{1}{2} \int_{\Omega^n_\rho} U^n_{\varphi_\ast} | \nabla w |^2 - I_{\rho, \varepsilon_n} \leq C_0.
\]

We denote \( x_n = (x^n_1, \ldots, x^n_n) \). From the first and second assertion, there exists \( \eta_0 > 0 \) (independent of \( n \)) such that

\[
\min \{ \min_{i \neq j} | x^n_i - x^n_j |, \min_{i} \text{dist}(x^n_i, \partial \Omega) \} \geq 10^2 \cdot \eta_0 > 0.
\]
We want to prove that there is some \( c > 0 \) such that for \( i = 1, \ldots, d \) we have (for large \( n \)) that \( B(x^n_i, c\lambda\delta) \subset \omega_{\varepsilon_n} \).

To this end, we argue by contradiction, and assume that either \( x^n_1 \notin \omega_{\varepsilon_n} \), or \( x^n_1 \in \omega_{\varepsilon_n} \) and \( \text{dist}(x^n_1, \partial \omega_{\varepsilon_n})/(\lambda\delta) \to 0 \).

We intend to prove that, letting \( y_n \in \delta \cdot (\mathbb{Z} \times \mathbb{Z}) \) be such that \( x^n_1, y_n \in Y^{\delta}_{k, \ell} \), we then have that

\[
(D.14) \quad \hat{\mu}_{\rho, \varepsilon_n}(x_n, 1) - \hat{\mu}_{\rho, \varepsilon_n}((y_n, x^n_2, \ldots, x_d^n), 1) \to \infty.
\]

Up to a subsequence, we may assume that \( \lim_n (\text{dist}(x^n_1, \omega_{\varepsilon_n})/(\lambda\delta)) \) exists. We divide the proof into two steps:

**Step 1** \( x^n_1 \notin \omega_{\varepsilon_n} \) and \( \text{dist}(x^n_1, \omega_{\varepsilon_n})/(\lambda\delta) \to c \in (0, \infty] \), then (D.14) holds;

**Step 2** \( \text{dist}(x^n_1, \omega_{\varepsilon_n})/(\lambda\delta) \to 0 \), then (D.14) holds.

To prove **Step 1**, assume that \( x^n_1 \notin \omega_{\varepsilon_n} \), \( \text{dist}(x^n_1, \omega_{\varepsilon_n})/(\lambda\delta) \to c \in (0, \infty] \) and

\[
\inf_{\sigma \in H^1_0(\Omega; \mathbb{S}^1)} \frac{1}{2} \int_{\Omega} \frac{U_{\varepsilon_n}^2}{|\nabla \sigma|^2} - I_{\rho, \varepsilon_n} \leq C_0.
\]

Denote by \( w_n \) a minimizer for \( \hat{\mu}_{\rho, \varepsilon_n}(x_n, 1) \) (see Proposition 3.2). Using Lemma D.6 (2), for \( \rho \leq r < R < \eta_0 \), we have

\[
\frac{1}{2} \int_{B(x^n_1, R)} \frac{U_{\varepsilon_n}^2}{|\nabla w_n|^2} - \mu_{\varepsilon_n}(B(x^n_1, R) \setminus B(x^n_1, r), 1) \leq C_0 + C(\eta_0).
\]

Let \( \kappa \in (0, 10^{-2} \cdot c) \) be such that \( B(0, 10^{3}\kappa) \subset \omega \subset Y \) and \( \text{dist}(\omega, \partial Y) \geq 10^{2}\kappa \).

From Lemma D.6 (Assertions (2) and (3)), we have

\[
\hat{\mu}_{\rho, \varepsilon_n}(x_n, 1) = \sum_{i=1}^{d} \mu_{\varepsilon_n}(\mathcal{R}(x^n_i, \eta_0, \rho), 1) + \mathcal{O}(1)
\]

and

\[
\hat{\mu}_{\rho, \varepsilon_n}((y_n, x^n_2, \ldots, x_d^n), 1) = \mu_{\varepsilon_n}(\mathcal{R}(y_n, \eta_0, \rho), 1) + \sum_{i=2}^{d} \mu_{\varepsilon_n}(\mathcal{R}(x^n_i, \eta_0, \rho), 1) + \mathcal{O}(1).
\]

Recall that \( y_n \in \delta \cdot 2^2 \) is such that \( x^n_1, y_n \in Y^{\delta}_{k, \ell} \). Since \( |x^n_1 - y_n| \leq \delta \), using Lemma C.3 (2) and Propositions C.4, C.5 (3), we have

\[
\mu_{\varepsilon_n}(\mathcal{R}(y_n, \eta_0, \rho), 1) = \mu_{\varepsilon_n}(\mathcal{R}(x^n_1, \eta_0, \delta), 1) + \mu_{\varepsilon_n}(\mathcal{R}(y_n, \kappa\delta, \rho), 1) + \mathcal{O}(1).
\]
Therefore,
\[\hat{J}_{\rho,\varepsilon}(x, 1) - \hat{J}_{\rho,\varepsilon}(y, 1) = \mu_{\varepsilon}(\mathcal{R}(x, \kappa\delta, \rho), 1) - \mu_{\varepsilon}(\mathcal{R}(y, \kappa\delta, \rho), 1) + O(1).\]
Thus it suffices to estimate the energies in the rings with radii \(\kappa\delta\) and \(\rho\). Using (1.5), we have
\[\mu_{\varepsilon}(\mathcal{R}(y, \kappa\delta, \rho), 1) = \pi \ln \frac{\lambda \delta}{\rho} + O(1).\]
In order to estimate \(\mu_{\varepsilon}(\mathcal{R}(x, \kappa\delta, \rho), 1)\), we divide the argument according to the asymptotic of \(\lambda\). If \(\lambda \equiv 1\), then \(c \in (0, \infty)\) and thus \(\text{dist}(B(x_i, c\delta/3), \omega_{\varepsilon}) \geq c\delta/3\). Consequently, from Proposition 1.4, we have
\[\mu_{\varepsilon}(\mathcal{R}(x, \kappa\delta, \rho), 1) = \pi \ln \frac{\delta}{\rho} + O(1).\]
Therefore (D.14) holds.
We now turn to Step 2. Arguing as in Step 1, it suffices to prove that
\[\mu_{\varepsilon}(\mathcal{R}(x, \kappa\delta, \rho), 1) - \mu_{\varepsilon}(\mathcal{R}(y, \kappa\delta, \rho), 1) \to \infty\]
for some fixed \(\kappa\)
and \(y \in \delta \cdot \mathbb{Z}^2\) such that \(x = Y_{k,\ell}^\delta\).
We let \(\kappa > 0\) (depending only on \(\omega\)) be such that
\[\kappa < 10^{-2} \cdot \text{dist}(\omega, \partial Y) \text{ and } B(0, 10^2 \cdot \kappa) \subset \omega.\]
In order to prove (D.18), we divide the annular $\mathcal{R}(x_1^n, \kappa \delta, \rho)$ into three regions:

$$\mathcal{R}(y_n, \kappa \delta, \rho) = \mathcal{R}(x_1^n, \kappa \delta, \kappa \lambda \delta) \bigcup \mathcal{R}(x_1^n, \kappa \lambda \delta, r_n) \bigcup \mathcal{R}(x_1^n, r_n, \rho)$$

with

$$r_n = \max \{ \varepsilon_n^{1/4}, \rho, \sqrt{\lambda \delta \cdot \text{dist}(x_n, \partial \omega_n)} \} + \sqrt{\varepsilon_n}.$$

We prove that $\mu_{\varepsilon_n}(\mathcal{R}(x_1^n, \kappa \lambda \delta, r_n), 1)$ is too large.

To do so, we consider $K_n$ the cone:

- of vertex $x_1^n$ and aperture $\pi/2$,
- which admits the line $(x_1^n, \Pi_{\partial \omega_n}(x_1^n))$ for symmetry axis,
- such that $K_n \cap \omega_{x_n} \cap \mathcal{R}(x_1^n, \kappa \lambda \delta, r_n) = \emptyset$.

Here, $\Pi_{\partial \omega_n}(x_1^n)$ is the orthogonal projection of $x_1^n$ on $\partial \omega_n$.

Note that since $\text{dist}(x_1^n, \omega_{x_n})/(\lambda \delta) \to 0$, for large $n$ and small $\kappa$ (independently of $n$), by smoothness of $\omega$, $K_n$ is well defined (see Figure D.1).

![Figure D.1](image)

**Figure D.1.** The domain $\mathcal{R}(x_1^n, \kappa \lambda \delta, r_n) \cap K_n$

We have $U_{\varepsilon_n} = 1 + V_n$ in $\mathcal{R}(x_1^n, \kappa \lambda \delta, r_n) \cap K_n$, where $\|V_n\|_{L^\infty} = o(\varepsilon_n^2)$. Thus, if we define

$$\alpha_n = \begin{cases} 1 & \text{in } K_n, \\ b^2 & \text{otherwise,} \end{cases}$$

then, from Lemma D.7 with $\theta_0 = 3\pi/2$, for $w \in H^1(\mathcal{R}(x_1^n, \kappa \lambda \delta, r_n), \mathbb{S}^1)$ such that $\text{deg}_{\partial \mathcal{B}(x_1^n, r_n)}(w) = 1$, we have

$$\frac{1}{2} \int_{\mathcal{R}(x_1^n, \kappa \lambda \delta, r_n)} \alpha_n |\nabla w|^2 \geq b^2 \frac{4\pi}{b^2 + 3} \ln \frac{\kappa \lambda \delta}{r_n}.$$
Clearly, from construction, $U_{\varepsilon n}^2 \geq \alpha_n + o(\varepsilon_n^2)$ in $\mathcal{R}(x_1^n, \kappa \lambda \delta, r_n)$; thus if $w_n$ is a minimal map for $\mu_{\varepsilon n}(\mathcal{R}(x_1^n, \kappa \lambda \delta, r_n), 1)$, we then have

$$\frac{1}{2} \int_{\mathcal{R}(x_1^n, \kappa \lambda \delta, r_n)} U_{\varepsilon n}^2 |\nabla w_n|^2 \geq b^2 \frac{4\pi}{b^2 + 3} \ln \frac{\kappa \lambda \delta}{r_n} + o_n(1).$$

The computations now become direct:

(D.19) \quad \mu_{\varepsilon n}(\mathcal{R}(x_1^n, \kappa \delta, \rho), 1)

$$= \mu_{\varepsilon n}(\mathcal{R}(x_1^n, \kappa \delta, 3\lambda \delta), 1) + \mu_{\varepsilon n}(\mathcal{R}(x_1^n, \kappa \lambda \delta, r_n), 1)
\quad + \mu_{\varepsilon n}(\mathcal{R}(x_1^n, r_n, \rho), 1) + O(1)
\geq \pi |\ln \lambda| + b^2 \frac{4\pi}{b^2 + 3} \ln \frac{\lambda \delta}{r_n} + b^2 \pi \ln \frac{r_n}{\rho} + O(1).$$

Therefore, (D.18) is a direct consequence of (D.16) and (D.19) since $\lambda \delta/r_n \to +\infty$.

**APPENDIX E. PROOF OF PROPOSITIONS 3.10 AND 3.12**

This appendix is dedicated to results specific to the diluted pinning term ($\lambda \to 0$). The first result (Lemma E.1) is the key ingredient in this appendix.

**5.1. An important effect of the dilution of inclusions.** We first state a result which establishes that a "sufficiently large" circle has a small intersection with $\omega_{\varepsilon}$ if $\lambda \to 0$.

**Lemma E.1.** We denote by $C_{\rho}$ a circle with radius $\rho$.

1. Assume that the pinning term is periodic. Once $\lambda \leq 1/(8\pi)$, for $\rho \geq \delta/3$ we have $\mathcal{H}^1(C_{\rho} \cap \omega_{\varepsilon}) \leq 16\pi^2 \delta \rho$.

2. Assume that the pinning term is not periodic, and recall that the inclusions with size $\lambda \delta^j$ make $\omega_{\varepsilon}^j = \bigcup_{i \in M_{\varepsilon}^j} \{ y_{i,j}^\varepsilon + \lambda \delta^j \cdot \omega \}, j \in \{1, \ldots, P\}$.

Once $\lambda \leq 1/(8\pi)$, for $\rho \geq \delta^j/3$ we have $\mathcal{H}^1(C_{\rho} \cap \omega_{\varepsilon}^j) \leq 16\pi^2 \lambda \rho$.

Here $\mathcal{H}^1(\cdot)$ is the one-dimensional Hausdorff measure.

**Proof.** In order to unify the notation, we fix $j = 1$ and denote $\omega_{\varepsilon} = \omega_{\varepsilon}^1$ if we are in the periodic case (and $j \in \{1, \ldots, P\}$ if we are in the non-periodic case).

Assume that $C_{\rho} \cap \omega_{\varepsilon} \neq \emptyset$ and let

$$S_j := \begin{cases} 
\{ \tilde{Y}_\varepsilon = (\delta k, \delta \ell) + \delta \cdot Y \mid (k, \ell) \in \mathbb{Z}^2, \tilde{Y}_\varepsilon \in \Omega \text{ and } \tilde{Y}_\varepsilon \cap C_{\rho} \neq \emptyset \} 
\quad \text{in the periodic case}, \\
\{ \tilde{Y}_\varepsilon = B(y_{i,j}^\varepsilon, \delta^j) \mid y_{i,j}^\varepsilon \in \hat{M}_{\varepsilon}^j \text{ and } \tilde{Y}_\varepsilon \cap C_{\rho} \neq \emptyset \} 
\quad \text{in the non-periodic case}. 
\end{cases}$$
For $\hat{Y}_\varepsilon \in S_j$, we denote by $\hat{\omega}_\varepsilon$ the connected component of $\omega_\varepsilon^j$ which is included in $\hat{Y}_\varepsilon$, and by $z$ the center of $\hat{Y}_\varepsilon$, where

$$\hat{Y}_\varepsilon = \begin{cases} \varepsilon + \delta \cdot Y & \varepsilon \in \delta \cdot \mathbb{Z}^2 \text{ in the periodic case,} \\ B(z, \delta^j) & \varepsilon \in \hat{M}^j \text{ in the non-periodic case.} \end{cases}$$

We first treat the case where $C_\rho \subset \hat{Y}_\varepsilon \subset S_j$: since $\rho \geq \delta^j/3$ and $\hat{\omega}_\varepsilon \subset B(z, \lambda \delta^j)$ (because $\omega \subset B(0, 1)$), we have

$$\mathcal{H}^1(C_\rho \cap \hat{\omega}_\varepsilon) = \mathcal{H}^1(C_\rho \cap \hat{\omega}_\varepsilon^j) \leq \mathcal{H}^1(\delta B(z_j, \lambda \delta^j)) = 2\pi \lambda \delta^j \leq 6\pi \lambda \rho.$$ 

Otherwise, for $\hat{Y}_\varepsilon \in S_j$, we have $C_\rho \not\subset \hat{Y}_\varepsilon$, and thus

$$\mathcal{H}^1(C_\rho \cap \hat{\omega}_\varepsilon) \leq \mathcal{H}^1(C_\rho \cap B(z_j, \lambda \delta^j)) \leq 2\pi \lambda \delta^j$$

and

$$\mathcal{H}^1(C_\rho \cap \hat{Y}_\varepsilon \setminus \hat{\omega}_\varepsilon) \geq \delta^j \cdot \left( \frac{1}{2} - 2\pi \lambda \right).$$

The last estimate comes from the fact that $C_\rho \not\subset \hat{Y}_\varepsilon$. Thus $\mathcal{H}^1(C_\rho \cap \hat{Y}_\varepsilon \setminus \hat{\omega}_\varepsilon)$ is at least a radius of $\hat{Y}_\varepsilon$ (or half of side length of $\hat{Y}_\varepsilon$ if we are in the periodic case) minus the previous upper bound. Thus we obtain (for $\lambda \leq 1/8\pi$) that

$$\mathcal{H}^1(C_\rho \cap \hat{\omega}_\varepsilon) \leq 2\pi \lambda \frac{\mathcal{H}^1(C_\rho \cap \hat{Y}_\varepsilon \setminus \hat{\omega}_\varepsilon)}{1/2 - 2\pi \lambda} \leq 8\pi \lambda \mathcal{H}^1(C_\rho \cap \hat{Y}_\varepsilon \setminus \hat{\omega}_\varepsilon).$$

Consequently,

$$\mathcal{H}^1(C_\rho \cap \hat{\omega}_\varepsilon) = \sum_{\hat{Y}_\varepsilon \in S_j} \mathcal{H}^1(C_\rho \cap \hat{\omega}_\varepsilon) \leq 8\pi \lambda \sum_{\hat{Y}_\varepsilon \in S_j} \mathcal{H}^1(C_\rho \cap \hat{Y}_\varepsilon \setminus \hat{\omega}_\varepsilon)$$

$$\leq 8\pi \lambda \mathcal{H}^1(C_\rho) = 16\pi^2 \lambda \rho.$$ 

\[\square\]

### 5.2. Proof of Proposition 3.10

We are now in a position to prove Proposition 3.10. The proof is done in three steps.

Let $\varepsilon_0 > 0$, $\rho = \rho(\varepsilon_0) > 0$, and $\rho \geq \varepsilon_0$, and let $x_n$ be a quasi-minimizer for $J_{\rho, \varepsilon_0}$ (defined in Notation 3.9). From Corollaries 3.7 and 3.8, up to passing to a subsequence, there are $\eta_0 > 0$ and $a = (a_1, \ldots, a_d) \in \Omega^d$ such that $x_n^\varepsilon - a_i$ and $\min\{|a_i - a_j|, \text{dist}(a_i, \partial \Omega)\} > 10^2 \eta_0$.

We prove that $W_\vartheta(a_1, \ldots, a_d) = \min_{b_1, \ldots, b_d \in \Omega} W_\vartheta(b_1, \ldots, b_d)$. We argue by contradiction and assume that, up to considering a smaller value for $\eta_0$ if necessary, we have the existence of $b = (b_1, \ldots, b_d) \in \Omega^d$ such that

$$\min\{|b_i - b_j|, \text{dist}(b_i, \partial \Omega)\} > 10^2 \eta_0 \quad \text{and} \quad W_\vartheta(b) < W_\vartheta(a) - 10^2 \eta_0.$$
**Step 1.** We estimate the energies in perforated domains with a fixed perforation size. The goal of this step is to prove the existence of small $\rho_0$ (independent of $n$) such that we have, for $c \in \{a, b\}$ and $x \in \Omega^d$ satisfying $\max_i |x_i - c_i| \leq \rho_0$,

$$0 \leq \hat{J}_{\rho_0, 1}(x) - \hat{J}_{\rho_0, n}(x) \leq 2\eta_0. \tag{E.1}$$

From [CM96, equation (15) and Lemma 2], we may fix $\rho_0 > 0$ independent of $n$ such that $\rho_0 < \eta_0$; and for $c \in \{a, b\}$ we have, for all $x \in \Omega^d$ such that $\max_i |x_i - c_i| \leq \rho_0$,

$$\hat{J}_{\rho_0, 1}(x) - \hat{J}_{\rho_0, n}(x) \leq \eta_0,$$

where

$$|\hat{J}_{\rho_0, 1}(x) - \pi d| |\ln \rho_0| - W_{\rho}(x) | \leq \eta_0,$$

and

$$|W_{\rho}(c) - W_{\rho}(x)| \leq \eta_0.$$

For $c \in \{a, b\}$ and $x \in \Omega^d$ such that $\max_i |x_i - c_i| \leq \rho_0$:

- We let $\theta_x = \sum_{i=1}^d \theta_{x_i}$, where $\theta_{x_i} \in (-\pi, \pi]$, $(x - x_i)/|x - x_i| = e^{i\theta_{x_i}}$ for $x \neq x_i$ ($\theta_{x_i}$ is the main determination of the argument of $x - x_i$).
- We fix $\phi_0^x \in C^\infty(\partial \Omega, \mathbb{R})$ such that $e^{i\theta_{x_i}} = ge^{-i\theta_{x_i}}$. Since $\deg_{\partial \Omega}(ge^{-i\theta_{x_i}}) = 0$, and since $ge^{-i\theta_{x_i}} \in C^\infty(\partial \Omega, \mathbb{R})$, clearly we have that $\phi_0^x \in C^\infty(\partial \Omega, \mathbb{R})$ is well defined [BBH93].
- We let $\phi_* = \phi_0^x$, $\phi = \phi_0 \in H^1$ be the solutions of

$$\begin{cases}
-\Delta \phi_* = 0 & \text{in } \Omega \setminus \bigcup \overline{B(x_i, \rho_0)}, \\
\phi_* = \phi_0 & \text{on } \partial \Omega, \\
\partial_v \phi_* = - \sum_{j \neq i} \partial_v \theta_{x_j} & \text{on } \partial B(x_i, \rho_0), \ i = 1, \ldots, d,
\end{cases} \tag{E.2}$$

and

$$\begin{cases}
-\div(U_2^2 \nabla \phi) = \div(U_1^2 \nabla \theta_x) & \text{in } \Omega \setminus \bigcup \overline{B(x_i, \rho_0)}, \\
\phi = \phi_0 & \text{on } \partial \Omega, \\
\partial_v \phi = - \sum_{j \neq i} \partial_v \theta_{x_j} & \text{on } \partial B(x_i, \rho_0), \ i = 1, \ldots, d.
\end{cases}$$

- We let $\psi = \phi - \phi_*$ be the solution of

$$\begin{cases}
-\div(U_2^2 \nabla \psi) = \div[(U_2^2 - 1)(\nabla \theta_x + \nabla \phi_*))] & \text{in } \Omega \setminus \bigcup \overline{B(x_i, \rho_0)}, \\
\psi = 0 & \text{on } \partial \Omega, \\
\partial_v \psi = 0 & \text{on } \partial B(x_i, \rho_0), \ i = 1, \ldots, d. \tag{E.3}
\end{cases}$$
Remark E.2.

(1) From Proposition 3.2, the functions $\phi_\ast$, $\phi$ are such that $w_\ast \equiv e^{i(\theta_x + \phi_\ast)}$, $w = e^{i(\theta_x + \phi)} \in I_{\rho_0}(x)$, and they satisfy

$$
\hat{J}_{\rho_0,1}(x) = \frac{1}{2} \int_{\Omega(x) \cup B(x_i, \rho_0)} |\nabla w_\ast|^2 = \frac{1}{2} \int_{\Omega(x, \rho_0)} |\nabla (\theta_x + \phi_\ast)|^2
$$

and

$$
\hat{J}_{\rho_0,2}(x) = \frac{1}{2} \int_{\Omega(x) \cup B(x_i, \rho_0)} U^2 \nabla w = \frac{1}{2} \int_{\Omega(x, \rho_0)} U^2 |\nabla (\theta_x + \phi)|^2.
$$

(2) $\nabla \theta_x$, $\nabla \phi$, $\nabla \phi_\ast$ are bounded independently of $x$, and we have $\varepsilon_n$ in $L^2(\Omega \cup B(x_i, \rho_0))$.

(3) We have $|\nabla \phi_\ast|$ which is bounded in $L^\infty(\Omega \cup B(x_i, \rho_0))$; thus, $|\nabla \phi_\ast| \leq C_0$, with $C_0$ independent of $x$. Indeed, with standard result of elliptic interior regularity, we have

$$
\|\phi_\ast\|_{C^2(\partial B(x_i, \rho_0))}, \|\phi_\ast\|_{C^2(\partial B(x_i, \rho_0))} \leq C_0',
$$

and thus, from global elliptic regularity, we have

$$
\|\nabla \phi_\ast\|_{L^\infty(\Omega \cup B(x_i, \rho_0))}, \|\nabla \phi_\ast\|_{L^\infty(\partial B(x_i, \rho_0))} \leq C_0'.
$$

We let $\Omega_{\rho_0} = \Omega_{\rho_0}(x) := \Omega \cup B(x_i, \rho_0)$. We are now in a position to prove that $\int_{\Omega_{\rho_0}} |\nabla \psi|^2 \to 0$ when $n \to \infty$ uniformly on $x$. This estimate will easily imply (E.1). Indeed, from Remark E.2, we have

$$
0 \leq \hat{J}_{\rho_0,1}(x) - \hat{J}_{\rho_0,2}(x)
$$

$$
= \frac{1}{2} \int_{\Omega_{\rho_0}} U^2 \left[ |\nabla (\theta_x + \phi_\ast)|^2 - |\nabla (\theta_x + \phi)|^2 \right] + \frac{1}{2} \int_{\Omega_{\rho_0}} (1 - U^2) |\nabla (\theta_x + \phi_\ast)|^2
$$

$$
\leq \tilde{C}_0 \left( \|\nabla \psi\|_{L^2(\Omega_{\rho_0})} + \|1 - U^2\|_{L^2(\Omega_{\rho_0})} \right) \quad \text{by Cauchy-Schwarz inequality}
$$

$$
\to 0.
$$

Consequently, we obtain

$$
0 \leq \hat{J}_{\rho_0,1}(x) - \hat{J}_{\rho_0,2}(x) \leq \hat{J}_{\rho_0,1}(x) - \hat{J}_{\rho_0,2}(x) \leq \eta_0 \leq \eta_0 + o_n(1) \leq 2\eta_0,
$$

which is exactly (E.1).

Thus it remains to establish that $\int_{\Omega_{\rho_0}} |\nabla \psi|^2 \to 0$ when $n \to \infty$ uniformly on $x$. From (E.2), (E.3), and by using integrations by parts, we have

$$
\int_{\Omega_{\rho_0}} U^2 |\nabla \psi|^2 = \int_{\Omega_{\rho_0}} \text{div}[ (U^2 - 1)(\nabla \theta_x + \nabla \phi_\ast) \psi]
$$

$$
= \int_{\Omega_{\rho_0}} (1 - U^2)(\nabla \theta_x + \nabla \phi_\ast) \cdot \nabla \psi.
$$
From the $L^2$-bound on $\nabla \psi$ and the $L^\infty$-bounds on $\nabla \phi_\epsilon$, $\nabla \theta_\epsilon$, we have (with $C_0$ independent of $x$) that

\[
\int_{\Omega_{\rho_0}} U_{\epsilon_n}^2 |\nabla \psi|^2 \leq \left( \int_{\Omega_{\rho_0}} |1 - U_{\epsilon_n}^2| |\nabla \theta_\epsilon + \nabla \phi_\epsilon|^2 \right)^{1/2} \left( \int_{\Omega_{\rho_0}} |\nabla \psi|^2 \right)^{1/2} \leq C_0 \|1 - U_{\epsilon_n}^2\|_{L^2(\Omega_{\rho_0})} \quad \text{by Remark E.2 (2) and (3)}.
\]

From Proposition 1.4, it is clear that $\|1 - U_{\epsilon_n}^2\|_{L^2(\Omega_{\rho_0})} \lesssim \|1 - U_{\epsilon_n}^2\|_{L^2(\Omega)} = O(\lambda)$.

Therefore $\int_{\Omega \cup \overline{B(x_i, \rho)}} |\nabla \psi|^2 \to 0$ when $n \to \infty$ uniformly on $x_i$, and so (E.1) holds.

**Step 2.** Here, we introduce the key ingredient to study the energies in $\mathcal{R}(x_i, \rho_0, \max(\delta, \lambda^2))$.

Let $\kappa = \max(\sqrt{\delta}, \lambda)$, let $x_n$ be a quasi minimizer for $J_{\rho, \epsilon}$, and let $w_n = e^{i \varphi_n}$ be a minimizer of $\overline{J}_{\rho, \epsilon_n}(x_n)$ (where $\varphi_n$ is locally defined and its gradient is globally defined in $\Omega \setminus \bigcup B(x_i, \rho)$).

We prove that there is $r \in (\kappa^2, \kappa)$ such that

\[
\frac{1}{2} \int_0^{2\pi} U_{\epsilon_n}^2 \left| \partial_\theta \varphi_n (x_i^n + r e^{i \theta}) \right|^2 d\theta \leq \pi + \frac{1}{\sqrt{| \ln r |}} \quad \text{for } i = 1, \ldots, d.
\]

This estimate is obtained via a mean value argument. We first prove that

\[
\mu_{\epsilon_n}(\mathcal{R}(x_i^n, \kappa, \kappa^2), 1) = \mu_1(\mathcal{R}(x_i^n, \kappa, \kappa^2), 1) + o_n(1), \quad i = 1, \ldots, d.
\]

Indeed, we let $\omega'$ be a smooth open set such that $\bar{\omega} \subset \omega'$ and $\overline{\omega'} \subset B(0, 1)$. We define

\[
\alpha'_\epsilon = \begin{cases} b^2 & \text{in } \delta \mathbb{Z} \times \delta \mathbb{Z} + \lambda \delta \cdot \omega', \\ 1 & \text{otherwise}. \end{cases}
\]

From Proposition 1.4, we have $\alpha'_\epsilon \leq U_{\epsilon_n}^2 + V_{\epsilon_n}$ in $\mathbb{R}^2$ with $\|V_{\epsilon_n}\|_{L^\infty} = O(\epsilon^2)$.

For $\rho \geq \delta$ and $x \in \mathbb{R}^2$, from Lemma E.1, we have $\mathcal{H}^1([\{ \alpha'_\epsilon = b^2 \} \cap \partial B(x, \rho)]) \leq 16 \pi^2 | \ln \lambda |$. Therefore, using Lemma D.7, we obtain

\[
\mu_1(\mathcal{R}(x_i^n, \kappa, \kappa^2), 1) = \mu_1(\mathcal{R}(x_i^n, \kappa, \kappa^2), 1) + o_n(1) \quad \text{(by Lemma D.7)}
\]

\[
\leq \mu_{\epsilon_n}(\mathcal{R}(x_i^n, \kappa, \kappa^2), 1) + o_n(1) \quad (\alpha'_\epsilon \leq U_{\epsilon_n}^2 + V_{\epsilon_n})
\]

\[
\leq \mu_1(\mathcal{R}(x_i^n, \kappa, \kappa^2), 1) + o_n(1) \quad (U_{\epsilon_n}^2 \leq 1).
\]

Because $1 \geq \kappa \geq \lambda$, we obtain

\[
\mu_{\epsilon_n}(\mathcal{R}(x_i^n, \kappa, \kappa^2), 1) = \mu_1(\mathcal{R}(x_i^n, \kappa, \kappa^2), 1) + o_n(1) = \pi | \ln \kappa | + o_n(1).
\]
Therefore, from Corollary 3.8 and Lemma D.6 (2), we obtain
\[ \frac{1}{2} \int_{\mathcal{R}(x^n, \kappa, k^2)} U_{\xi_n}^2 |\nabla w_n|^2 = \pi |\ln \kappa| + \mathcal{O}(1), \quad i = 1, \ldots, d. \]
We (easily) deduce that
\[ (E.5) \quad \frac{1}{2} \int_{\mathcal{R}(x^n, \kappa, k^2)} U_{\xi_n}^2 |\nabla w_n|^2 \leq \pi d |\ln \kappa| + \mathcal{O}(1). \]
On the other hand, combining again (for \( s \in (\kappa^2, \kappa) \)) the estimates
\[ \mathcal{H}^1(\{\alpha_i' = b^2\} \cap \partial B(x, s)) \leq 16\pi^2 \lambda s \quad \text{and} \quad \alpha_i' \leq U_\xi + \mathcal{O}(\varepsilon^2) \]
with Lemma D.7, we have
\[ \frac{1}{2} \int_0^{2\pi} U_{\xi_n}^2 |\partial_\theta \varphi_n(x^n_i + se^{i\theta})|^2 d\theta \geq \pi d + \frac{1}{\sqrt{|\ln \kappa|}} - \mathcal{O}(\lambda), \quad \forall s \in (\kappa^2, \kappa), \]
where \( \mathcal{O}(\lambda) \) is independent of \( s \).
Assume that for all \( r \in (\kappa^2, \kappa), (E.4) \) is not satisfied. Then we obtain that, for \( s \in (\kappa^2, \kappa), \)
\[ \sum_{i=1}^{d} \frac{1}{2} \int_0^{2\pi} U_{\xi_n}^2 |\partial_\theta \varphi_n(x^n_i + se^{i\theta})|^2 d\theta > \pi d + \frac{1}{\sqrt{|\ln \kappa|}} - \mathcal{O}(\lambda), \]
and consequently
\[ \frac{1}{2} \int_{\mathcal{R}(x^n, \kappa, k^2)} U_{\xi_n}^2 |\nabla w_n|^2 \geq |\ln \kappa| \left( \pi d + \frac{1}{\sqrt{|\ln \kappa|}} - \mathcal{O}(\lambda) \right) \]
\[ = \pi d |\ln \kappa| + \frac{1}{\sqrt{|\ln \kappa|}} + o_\varepsilon(1). \]
Clearly, this lower bound contradicts (E.5).
We are now in a position to estimate the energy in \( \mathcal{R}(x^n, \rho_0, r) \). In order to use Proposition 13 in [DM11], we let
\[ \bullet \ h_i^n : S^1 \to S^1, \ h_i^n(e^{i\theta}) = w_n(x^n_i + re^{i\theta}), \]
\[ \bullet \ \bar{U}_{n,i} = \bar{U}_i : S^1 - [b, 1], \ \bar{U}_i(e^{i\theta}) = U_{\xi_n}(x^n_i + re^{i\theta}). \]
We denote by \( \partial_\tau \) the tangential derivative on \( S^1 \). We have \( h_i^n \wedge \partial_\tau h_i^n(\cdot) = \partial_\tau \left[ \varphi_n(x^n_i + r \cdot) \right] \).
Thus, from (E.4) we have \( \| \bar{U}_i h_i^n \wedge \partial_\tau h_i^n \|_{L^2(S^1)} \leq 2\pi + 2/\sqrt{|\ln \kappa|} \). Consequently,
\[ \int_{S^1} \bar{U}_i^2 |h_i^n \wedge \partial_\tau h_i^n - 1|^2 = \int_{S^1} \bar{U}_i^2 [ |h_i^n \wedge \partial_\tau h_i^n|^2 + 1 - 2h_i^n \wedge \partial_\tau h_i^n] \]
\[ \leq 2\pi + \frac{2}{\sqrt{|\ln \kappa|}} + \int_{S^1} \bar{U}_i^2 - 2 \int_{S^1} \bar{U}_i^2 h_i^n \wedge \partial_\tau h_i^n. \]
It is easy to prove that $\tilde{U}^2_i \rightarrow 1$ in $L^2(\mathbb{S}^1)$ and that
\[
\left| 2\pi - \sum_{i} \tilde{U}^2_i h^m_i \wedge \partial_T h^m_i \right| = \left| \sum_{i} (\tilde{U}^2_i - 1) h^m_i \wedge \partial_T h^m_i \right| \\
\leq \|\tilde{U}^2_i - 1\|_{L^2(\mathbb{S}^1)} \|h^m_i \wedge \partial_T h^m_i\|_{L^2(\mathbb{S}^1)} \rightarrow 0.
\]
Thus $\sum_{i} \tilde{U}^2_i h^m_i \wedge \partial_T h^m_i - 1 \leq 2\pi + 2/\sqrt{|\ln \kappa|} + 2\pi - 4\pi + o_n(1) = o_n(1)$, and therefore $h^m_i \wedge \partial_T h^m_i - 1$ in $L^2(\mathbb{S}^1)$. Consequently, up to passing to a subsequence, we have the existence of $\alpha_i \in \mathbb{S}^1$ such that $\alpha_i^{-1} h^m_i e^{it\theta} \rightarrow 1$ in $H^1(\mathbb{S}^1)$.

Arguing as in Proposition 13 in [DM11], we have
\[
\inf_{\substack{w \in H^1(\mathbb{R}(x^m_i, \rho_0, r), \mathbb{S}^1) \\wedge \partial_T h^m_i, \w(x^m_i + r \rho_0 e^{it\theta}) = \alpha_i e^{it\theta} \\w(x^m_i + r e^{it\theta}) = \alpha_i e^{it\theta}}} \frac{1}{2} \iint_{\mathbb{R}(x^m_i, \rho_0, r)} |
\nabla w|^2 \, dx \, dy = \inf_{\substack{w \in H^1(\mathbb{R}(x^m_i, \rho_0, r), \mathbb{S}^1) \\w(x^m_i + \rho_0 e^{it\theta}) = \alpha_i e^{it\theta} \\w(x^m_i + r e^{it\theta}) = \alpha_i e^{it\theta}}} \frac{1}{2} \iint_{\mathbb{R}(x^m_i, \rho_0, r)} |
\nabla w|^2 \, dx \, dy + o_n(1) = \pi \ln \frac{\rho_0}{r} + o_n(1).
\]

**Step 3.** We conclude by constructing a map $\tilde{w}_n \in F_\rho(y_n)$, with $y_n$ such that $\max |y^n_i - b_i| \leq \delta$ satisfies
\[
(E.6) \quad \frac{1}{2} \iint_{\Omega \setminus B(y^n_i, \rho)} U_{\tilde{h}_n}^2 \|\nabla \tilde{w}_n\|^2 \, dx \, dy + \eta_0 \leq \hat{f}_{\rho, \varepsilon_n}(x_n).
\]
Clearly, (E.6) is in contradiction with the assumption $F_\rho, \varepsilon_n \rightarrow \hat{f}_{\rho, \varepsilon_n}(x_n) \rightarrow 0$. Then, this contradiction will imply that $a = \lim x_n$ minimizes $W_\rho$.

We let $y_n$ be such that $\max |y^n_i - b_i| \leq \delta$ and $x^n_i - y^n_i \in \delta \bar{Z} \times \delta \bar{Z}$. We define
\[
\tilde{w}_n(x) = \begin{cases} 
\tilde{w}^n_p(x) & \text{if } x \in \Omega \setminus \bigcup B(y^n_i, \rho_0), \\
C_{\tilde{t}, n} \tilde{w}^i(x - y^n_i + x^n_i) & \text{if } x \in \mathbb{R}(y^n_i, \rho_0, r), \\
C_{\tilde{t}, n} \tilde{w}_n(x - y^n_i + x^n_i) & \text{if } x \in \mathbb{R}(y^n_i, r, \rho).
\end{cases}
\]

Here, we note the following:
- $\tilde{w}^n_p$ is a minimizer of $\hat{f}_{\rho_0, 1}(y_n)$;
- $\tilde{w}^i$ is a minimizer of
\[
\inf_{\substack{w \in H^1(\mathbb{R}(x^m_i, \rho_0, r), \mathbb{S}^1) \\w(x^m_i + \rho_0 e^{it\theta}) = \alpha_i e^{it\theta} \\w(x^m_i + r e^{it\theta}) = \alpha_i e^{it\theta}}} \frac{1}{2} \iint_{\mathbb{R}(x^m_i, \rho_0, r)} |
\nabla w|^2 \, dx \, dy,
\]
with $\alpha_i, h^m_i$ as defined in Step 2 (recall that $h^m_i(e^{it\theta}) \rightarrow \alpha_i e^{it\theta}$ in $H^1(\mathbb{S}^1)$ as $n \rightarrow \infty$).
\[ w_n \text{ is the minimizer of } \tilde{J}_{\rho,\epsilon}(x_n) \text{ used in Step 2;} \]
\[ \text{Cst}_{i,n} \in S^1 \text{ is a constant such that } \tilde{w}_n \in H^1(\Omega \setminus \bigcup B(y^n_i, \rho), S^1). \]

We now compare the energies of \( \tilde{w}_n \) and \( w_n \):

\[
\int_{\Omega \setminus \bigcup B(y^n_i, \rho)} U^2_{\epsilon_n} |\nabla \tilde{w}_n|^2 = \int_{\Omega \setminus \bigcup B(y^n_i, \rho_0)} U^2_{\epsilon_n} |\nabla \tilde{w}_n|^2 + \int_{\cup_{i} \mathcal{X}(x^n_i, \rho_0, r)} U^2_{\epsilon_n} |\nabla \tilde{w}_n|^2
\]

From Step 1 (the definition of \( \rho_0 \) and estimate (E.1)), we have

\[
\frac{1}{2} \int_{\Omega \setminus \bigcup B(y^n_i, \rho_0)} U^2_{\epsilon_n} |\nabla \tilde{w}_n|^2 \leq \pi d \ln \rho_0 + W_\delta(y_n) + \eta_0 + o_n(1)
\]

\[
\leq \pi d \ln \rho_0 + W_\delta(x_n) - 10 \eta_0
\]

\[
\leq \frac{1}{2} \int_{\Omega \setminus \bigcup B(x^n_i, \rho_0)} U^2_{\epsilon_n} |\nabla w_n|^2 - 2 \eta_0.
\]

From Step 2, we let

\[
\alpha'_\epsilon = \begin{cases} 
  b^2 & \text{in } \{ \delta \mathbb{Z} \times \delta \mathbb{Z} + B(0, \lambda \delta) \}, \\
  1 & \text{otherwise},
\end{cases}
\]

and so we have (because \( r \geq \lambda^2 \)) that

\[
\frac{1}{2} \int_{\cup_{i} \mathcal{X}(x^n_i, \rho_0, r)} U^2_{\epsilon_n} |\nabla \tilde{w}_n|^2 = \pi d \ln \frac{\rho_0}{r} + o_n(1) \quad \text{(by Step 2)}
\]

\[
\leq \frac{1}{2} \int_{\cup_{i} \mathcal{X}(x^n_i, \rho_0, r)} \alpha'_\epsilon |\nabla w_n|^2 + o_n(1) \quad \text{(by Lemma D.7)}
\]

\[
\leq \frac{1}{2} \int_{\cup_{i} \mathcal{X}(x^n_i, \rho_0, r)} U^2_{\epsilon_n} |\nabla w_n|^2 + o_n(1) \quad \text{(} \alpha'_\epsilon \leq U^2_{\epsilon} + V_\delta).\]

From Lemma D.1, we have

\[
\frac{1}{2} \int_{\cup_{i} \mathcal{X}(x^n_i, r, \rho)} U^2_{\epsilon_n} |\nabla \tilde{w}_n|^2 = \frac{1}{2} \int_{\cup_{i} \mathcal{X}(x^n_i, r, \rho)} U^2_{\epsilon_n} |\nabla w_n|^2 + o_n(1).
\]

Therefore, we obtain (E.6), and consequently Proposition 3.10 holds.

**5.3. Proof of Proposition 3.12.** The strategy to prove Proposition 3.12 is as follows:

Step 1: We let \( \kappa = \max(\lambda, \delta) \). We first characterize almost minimal configurations for \( I_{\kappa,\epsilon} \) (i.e., the domain \( \Omega \) is perforated by discs with radius \( \kappa \)).
Step 2: For \( \mu_\varepsilon(\mathcal{R}(\cdot, \kappa, \lambda \delta^{3/2}), 1) \), we make the description of almost minimal points \((x_\varepsilon)\) below.

Step 3: We estimate \( \inf_{x_0 \in \mathbb{R}^d} \mu_\varepsilon(\mathcal{R}(x_0, \lambda \delta^{3/2}, \rho), 1) \), and we conclude.

**Step 1.** We study almost minimal configurations for \( I_{K, \varepsilon} \), \( \kappa = \max(\lambda, \delta) \).

We prove that \( \{x, d\} = \{(x_1^\varepsilon, d_1), \ldots, (x_N^\varepsilon, d_N)\} \) is an almost minimal configuration for \( I_{K, \varepsilon} \) if and only if we have \( N = d \), \( d_i = 1 \) and there is \( \eta_0 \) such that dist\((x_i^\varepsilon, \partial \Omega), |x_i^\varepsilon - x_j^\varepsilon| \geq \eta_0 \).

First note that, for \( \eta_0 > 0 \) and \( x = (x_1^\varepsilon, \ldots, x_d^\varepsilon) \in \Omega^d \) such that

\[
\min \{ \text{dist}(x_i^\varepsilon, \partial \Omega), |x_i^\varepsilon - x_j^\varepsilon| \} \geq \eta_0,
\]

we have easily

\[
I_{K, \varepsilon} \leq \hat{I}_{K, \varepsilon}(x, 1) \leq \pi d |\ln \kappa| + C(\eta_0),
\]

with \( C(\eta_0) \) which is independent of \( \varepsilon \).

We consider \( \{x, d\} \) which is almost minimal for \( I_{K, \varepsilon} \). We argue as in the proof of Proposition 3.6 (Assertions (1) and (2); see Subsections D.3 and D.4).

We use the separation process defined in Subsection C.1 and the associated natural partition of \( \Omega \) := \( \Omega \setminus \bigcup B(x_i^\varepsilon, \kappa) \) (see Subsection C.2).

Here, the key ingredients are Lemmas D.7 and E.1 (which replace the periodic structure of the pinning term). Combining both lemmas, we get that if \( R > r \geq \kappa \), then

\[
\mu_\varepsilon(\mathcal{R}(x_0, R, r), 1) = \pi \ln \frac{R}{r} + O \left( \lambda \ln \frac{R}{r} \right).
\]

The rings \( \mathcal{R}(x_0, R, r) \) which occur in the partition of \( \Omega \) are such that \( C(\Omega) \geq R > r \geq \kappa \), and thus \( R/r = O(\kappa^{-1}) \). These bounds imply that \( O(\lambda \ln(R/r)) = o_\varepsilon(1) \), and consequently \( \mu_\varepsilon(\mathcal{R}(x_0, R, r), 1) = \pi \ln(R/r) + o_\varepsilon(1) \).

Therefore, if \( \{x, d\} \) is an almost minimal configuration for \( I_{K, \varepsilon} \), then \( N = d \), \( d_i = 1 \), and there is \( \eta_0 > 0 \) such that \( \min \{ \text{dist}(x_i^\varepsilon, \partial \Omega), |x_i^\varepsilon - x_j^\varepsilon| \} \geq \eta_0 \). These facts are proved by arguing by contradiction exactly as in Subsections D.3 and D.4, and by using (E.7).

In addition, if \( \{x, 1\} \) is an almost minimal configuration for \( I_{K, \varepsilon} \), then the arguments of Subsections D.3 and D.4 in conjunction with (E.7) yield

\[
|\hat{I}_{K, \varepsilon}(x, 1) - \pi d |\ln \kappa| | \leq C(\eta_0).
\]

Here, \( \eta_0 \) is obtained in the previous paragraph, and so we get \( I_{K, \varepsilon} = \pi d |\ln \kappa| + O(1) \).

Conversely, from (E.7), for \( \eta_0 > 0 \) and \( x_1^\varepsilon, \ldots, x_d^\varepsilon \in \Omega \) such that dist\((x_i^\varepsilon, \partial \Omega), |x_i^\varepsilon - x_j^\varepsilon| \geq \eta_0 \), we have that \( \{x_1^\varepsilon, \ldots, x_d^\varepsilon\} \) is almost minimal for \( I_{K, \varepsilon} \).
Step 2. We study almost minimal configurations for \( \mu_\epsilon(\mathcal{R}(\cdot, \kappa, \lambda \delta^{3/2}), 1) \).

For \( j \in \{1, \ldots, P\} \), we set \( \omega^j_\epsilon := \bigcup_{i \in \mathcal{M}^j_\epsilon} \{y^i_{\epsilon,j} + \lambda \delta^j \cdot \omega\} \). We recall that the set of centers of connected components of \( \omega^j_\epsilon \) is \( \tilde{\mathcal{M}}^j_\epsilon := \{y^i_{\epsilon,j} \mid i \in \mathcal{M}^j_\epsilon\} \).

Let \( x^0_\epsilon \in \omega_\epsilon \) and \( c > 0 \) (independent of \( \epsilon \)) be such that \( B(x^0_\epsilon, c \lambda \delta) \subset \omega^1_\epsilon \).

On the one hand, we may easily prove that

\[
\mu_\epsilon(\mathcal{R}(\omega^0_\epsilon, \delta, \lambda \delta^{3/2}), 1) = \pi b^2 |\ln \delta^{1/2}| + \pi |\ln \lambda| + O(1).
\]

On the other hand, applying Lemmas D.7 and E.1, we have

\[
\mu_\epsilon(\mathcal{R}(x^0_\epsilon, \kappa, \delta), 1) = \pi[1 + O(\lambda)]\ln \frac{\kappa}{\delta}.
\]

Therefore, from (E.8) and (E.9), we get

\[
\mu_\epsilon(\mathcal{R}(x^0_\epsilon, \kappa, \lambda \delta^{3/2}), 1) = \pi \left[ \frac{b^2}{2} + 1 + O(\lambda) \right] |\ln \delta| + \pi \ln \frac{\kappa}{\lambda} + O(1).
\]

We intend to prove that this situation \( (B(x^0_\epsilon, c \lambda \delta) \subset \omega^1_\epsilon) \) is the only way to get an almost minimal energy. More precisely, we prove that for a fixed constant \( C_0 > 0 \), if we have \((x_\epsilon) \subset \Omega \) which is such that

\[
\mu_\epsilon(\mathcal{R}(x_\epsilon, \kappa, \lambda \delta^{3/2}), 1) \leq \inf_{x_0 \in \Omega} \mu_\epsilon(\mathcal{R}(x_0, \kappa, \lambda \delta^{3/2}), 1) + C_0,
\]

then there is \( c > 0 \) independent of \( \epsilon \) such that, for sufficiently small \( \epsilon \), we have \( B(x_\epsilon, c \lambda \delta) \subset \omega_\epsilon \), that is, \( B(x_\epsilon, c \lambda \delta) \subset y^i_{\epsilon,j} + \lambda \delta \cdot \omega \) with \( y^i_{\epsilon,j} \in \tilde{\mathcal{M}}^j_\epsilon \).

We let \( C_0 > 0 \) and \((x_\epsilon) \subset \Omega \) be such that (E.11) holds.

Up to passing to a sequence \( \epsilon_n \to 0 \), dropping the subscript \( \epsilon \) (we write \( \epsilon \) instead of \( \epsilon_n \)), we may assume that one of the following cases occurs:

Case 0: \( \exists c > 0 \) such that \( B(x_\epsilon, c \lambda \delta) \subset \omega_\epsilon \);

Case 1: \( x_\epsilon \notin \bigcup_{j=1}^{P} \bigcup_{i \in \mathcal{M}} B(y^i_{\epsilon,j}, \delta^j) \);

Case 2: \( x_\epsilon \in \bigcup_{j=2}^{P} \bigcup_{i \in \mathcal{M}} B(y^i_{\epsilon,j}, \delta^j) \);

Case 3: \( \{x_\epsilon \in \bigcup_{i \in \mathcal{M}} B(y^i_{\epsilon,1}, \delta) \setminus \omega^1_\epsilon \} \) or \( \{x_\epsilon \in \omega^1_\epsilon \) and \( \text{dist}(x_\epsilon, \hat{\omega}^1_\epsilon)/\lambda \delta \to 0 \). We want to prove that only Case 0 occurs if (E.11) holds.

Case 1: Since for \( j = 1, \ldots, P \) we have \( \text{dist}(x_\epsilon, \omega^j_\epsilon) \geq \delta^j/2 \), from Lemmas D.7 and E.1, it is direct to prove that

\[
\mu_\epsilon(\mathcal{R}(x_\epsilon, \kappa, \lambda \delta^{3/2}), 1) \geq \pi [1 + O(\lambda)]\ln \frac{\kappa}{\lambda \delta^{3/2}} = \pi \left[ \frac{3}{2} + O(\lambda) \right] |\ln \delta| + \pi \ln \frac{\kappa}{\lambda}.
\]

Using (E.10), we get

\[
\mu_\epsilon(\mathcal{R}(x_\epsilon, \kappa, \lambda \delta^{3/2}), 1) - \inf_{x_0 \in \Omega} \mu_\epsilon(\mathcal{R}(x_0, \kappa, \lambda \delta^{3/2}), 1) \to +\infty.
\]
Therefore, if \((x_\varepsilon)_\varepsilon\) satisfies (E.11), then Case 1 does not occur.

Case 2: Up to passing to a subsequence, we may assume there is \(j_0 \in \{2, \ldots, P\}\) such that \(x_\varepsilon \in \bigcup_{j \neq j_0} B(y_0^j, \lambda \delta^{3/2})\) (\(j_0\) is independent of \(\varepsilon\)).

We define \(\kappa' := \max\{\delta^h, \lambda \delta^{3/2}\}\), and we let \(y_0 = y_{i,j_0}^\varepsilon \in \mathcal{M}_{j_0}^\varepsilon\) be such that \(x_\varepsilon \in B(y_0, \delta^h)\) (\(y_0\) is dependent on \(\varepsilon\)).

We first assume that \(x_\varepsilon \notin \mathcal{W}_\varepsilon\), and we let

\[
\kappa = \max\{\lambda \delta^{3/2}, \text{dist}(x_\varepsilon, \partial \omega_j^h) - \lambda \delta^h\}
\]

(note that the remainder situation—\(x_\varepsilon \in \mathcal{W}_\varepsilon\)—is easier). Here, we have \(\kappa' \geq \kappa\).

In order to estimate \(\mu_\varepsilon(\mathcal{R}(x_\varepsilon, \kappa, \lambda \delta^{3/2}), 1)\), we divide \(\mathcal{R}(x_\varepsilon, \kappa, \lambda \delta^{3/2})\) into

\[
\mathcal{R}(x_\varepsilon, \kappa, \kappa' + 2\lambda \delta^h) \cup \mathcal{R}(x_\varepsilon, \kappa' + 2\lambda \delta^h, \kappa) \cup \mathcal{R}(x_\varepsilon, \kappa, \lambda \delta^{3/2})
\]

The following situations obtain:

- When \(\kappa' = \kappa\), then the ring \(\mathcal{R}(x_\varepsilon, \kappa', 2\lambda \delta^h, \kappa + 2\lambda \delta^h)\) is omitted.
- When \(\kappa = \lambda \delta^{3/2}\), then the ring \(\mathcal{R}(x_\varepsilon, \kappa, \lambda \delta^{3/2})\) is omitted.

The key tool is the dilution of the inclusions that may be expressed as follows: because \(x_\varepsilon \in B(y_0, \delta^h)\), if \(\partial B(x_\varepsilon, s) \cap \omega_j^h \neq \emptyset\) (with \(s \geq \kappa\)), then either \(s \geq \delta^h + \delta^h/2\) or \(j = j_0\), \(\partial B(x_\varepsilon, s) \cap \omega_j^h = \partial B(x_\varepsilon, s) \cap \{y_0 + \lambda \delta^h \cdot \omega\}\) and \(s \leq \delta^h + \lambda \delta^h\).

From the dilution of the inclusions and Lemmas D.7 and E.1, we have (because \(\kappa > \kappa' + 2\lambda \delta^h > \delta^h + \lambda \delta^h\)) that

\[
\mu_\varepsilon(\mathcal{R}(x_\varepsilon, \kappa, \kappa' + 2\lambda \delta^h), 1) \geq \pi [1 + O(\lambda)] \ln \frac{\kappa}{\kappa' + 2\lambda \delta^h}.
\]

We now consider the second ring, and thus assume \(\kappa > \kappa\). It is clear we have \(\text{dist}[\mathcal{R}(x_\varepsilon, \kappa', 2\lambda \delta^h, \kappa + 2\lambda \delta^h), B(y_0, \lambda \delta^h)] > 0\). Moreover, if for some \(j\) we have \(\mathcal{R}(x_\varepsilon, \kappa', 2\lambda \delta^h, \kappa + 2\lambda \delta^h) \cap \omega_j^h \neq \emptyset\), then \(\text{dist}(x_\varepsilon, \omega_j^h) \geq \delta^h/2\) (because \(x_\varepsilon \in B(y_0, \delta^h)\)). Therefore, using Proposition 1.4 and Lemmas D.7 and E.1, we get

\[
\mu_\varepsilon(\mathcal{R}(x_\varepsilon, \kappa', 2\lambda \delta^h, \kappa + 2\lambda \delta^h), 1) \geq \pi [1 + O(\lambda)] \ln \frac{\kappa' + 2\lambda \delta^h}{\kappa + 2\lambda \delta^h}.
\]

It is obvious that

\[
\mu_\varepsilon(\mathcal{R}(x_\varepsilon, \kappa, \lambda \delta^{3/2}), 1) \leq \pi \ln \frac{\kappa + 2\lambda \delta^h}{\kappa} \leq \pi \ln (1 + 2\delta^h - 3/2) = o_\varepsilon(1)
\]

We now consider the last ring, and we assume that \(\kappa > \lambda \delta^{3/2}\). By the definition of \(\kappa\) and from Proposition 1.4, we have

\[
\mu_\varepsilon(\mathcal{R}(x_\varepsilon, \kappa, \lambda \delta^{3/2}), 1) \geq \pi \ln \frac{\kappa}{\lambda \delta^{3/2}} - o_\varepsilon(1).
\]
Summing these lower bounds, we have

\[
\mu_\epsilon(\mathcal{R}(x_\epsilon, \kappa, \lambda \delta^{3/2}), 1) \geq \pi \left[ 1 + O(\lambda) \right] \left[ \ln \frac{\kappa}{\kappa' + 2\lambda \delta \lambda_0} + \ln \frac{\kappa'}{\kappa' + 2\lambda \delta \lambda_0} + \ln \frac{\kappa}{\lambda \delta^{3/2}} + o_\epsilon(1) \right]
\]

\[
= \pi \left[ \frac{3}{2} + O(\lambda) \right] \left| \ln \delta \right| + \pi \ln \frac{K}{\lambda} + o_\epsilon(1),
\]

and therefore \( \mu_\epsilon(\mathcal{R}(x_\epsilon, \kappa, \lambda \delta^{3/2}), 1) - \inf_{x_0 \in \Omega} \mu_\epsilon(\mathcal{R}(x_0, \kappa, \lambda \delta^{3/2}), 1) \to +\infty \) (from (E.10)).

We now assume that \( x_\epsilon \in \Omega_\epsilon \). Because \( j_0 \geq 2 \) and \( x_\epsilon \in B(y_0, \lambda \delta \lambda_0) \), we have \( B(y_0, 2\lambda \delta \lambda_0) \cap \mathcal{R}(x_\epsilon, \kappa, \lambda \delta^{3/2}) \neq \emptyset \). Therefore, from the dilution of the inclusion, if there is \( \omega_\epsilon \), a connected component of \( \omega_\epsilon^j \), such that \( \mathcal{R}(x_\epsilon, \kappa, \lambda \delta^{3/2}) \cap \omega_\epsilon \), then \( \text{dist}(x_\epsilon, \omega_\epsilon) \geq \delta / 2 \). Consequently, from Proposition 1.4 and Lemmas D.7 and E.1, we have

\[
\mu_\epsilon(\mathcal{R}(x_\epsilon, \kappa, \lambda \delta^{3/2}), 1) \geq \pi \left[ 1 + O(\lambda) \right] \ln \frac{K}{\lambda \delta^{3/2}}
\]

\[
= \pi \left[ \frac{3}{2} + O(\lambda) \right] \left| \ln \delta \right| + \pi \ln \frac{K}{\lambda} + o_\epsilon(1).
\]

From (E.10), we obtain that

\[
\mu_\epsilon(\mathcal{R}(x_\epsilon, \kappa, \lambda \delta^{3/2}), 1) - \inf_{x_0 \in \Omega} \mu_\epsilon(\mathcal{R}(x_0, \kappa, \lambda \delta^{3/2}), 1) \to +\infty.
\]

We deduce that if \( (x_\epsilon) \) satisfies (E.11), then Case 2 does not occur.

Case 3: We let \( y_0 := y_\epsilon^{(1)} \in \widehat{M}_\epsilon \) be such that \( x_\epsilon \in B(y_0, \delta) \).

We split this case in two parts: \( \text{dist}(x_\epsilon, \omega_\epsilon^{(1)}) \geq \delta / 3 \) and \( \text{dist}(x_\epsilon, \omega_\epsilon^{(1)}) \leq \delta / 3 \).

If \( \text{dist}(x_\epsilon, \omega_\epsilon^{(1)}) \geq \delta / 3 \), then we have (using Proposition 1.4 and Lemmas D.7 and E.1) that

\[
\mu_\epsilon(\mathcal{R}(x_\epsilon, \kappa, \lambda \delta^{3/2}), 1) \geq \mu_\epsilon(\mathcal{R}(x_\epsilon, \kappa, 2\delta / 3), 1) + \mu_\epsilon(\mathcal{R}(x_\epsilon, 10^{-1} \delta, \lambda \delta^{3/2}), 1)
\]

\[
\geq \pi (1 + O(\lambda)) \ln \frac{K}{\delta} + \pi |\ln(\lambda \delta^{1/2})| - O(1)
\]

\[
= \pi \left[ \frac{3}{2} + O(\lambda) \right] \left| \ln \delta \right| + \pi \ln \frac{K}{\lambda} - O(1).
\]

Therefore, if \( (x_\epsilon) \) satisfies (E.11), then \( \{ \text{dist}(x_\epsilon, \omega_\epsilon^{(1)}) \geq \delta / 3 \} \), and Case 3 does not occur.
We now assume that \( \text{dist}(x, \omega^1) \leq \delta/3 \). We have

\[
\mathcal{R}(y_0, 10\kappa, 10\delta) \subset \mathcal{R}(x, 10^2\kappa, 10^{-1}\delta),
\]

and thus (using Proposition C.4) we get

\[
\mu_\varepsilon(\mathcal{R}(x, \delta/2, \lambda\delta^{3/2}), 1) - \mu_\varepsilon(\mathcal{R}(y_0, \delta/2, \lambda\delta^{3/2}), 1) \rightarrow +\infty.
\]

We therefore have the existence of \( H_\varepsilon \rightarrow +\infty \) as \( \varepsilon \rightarrow 0 \), such that

\[
\mu_\varepsilon(\mathcal{R}(x, \delta/2, \lambda\delta^{3/2}), 1) \geq \mu_\varepsilon(\mathcal{R}(y_0, \delta/2, \lambda\delta^{3/2}), 1) + H_\varepsilon
\]

where the last inequality is due to Proposition C.5 (3). Consequently,

\[
\mu_\varepsilon(\mathcal{R}(x, \delta/2, \lambda\delta^{3/2}), 1) - \inf_{x_0 \in \Omega} \mu_\varepsilon(\mathcal{R}(x_0, \lambda\delta^{3/2}), 1) \rightarrow +\infty,
\]

and since \( (x_\varepsilon)_\varepsilon \) satisfies (E.11), Case 3 does not occur. Therefore, only Case 0 occurs if \( (x_\varepsilon)_\varepsilon \) satisfies (E.11).

**Step 3.** We study \( \inf_{x_0 \in \mathbb{R}^2} \mu_\varepsilon(\mathcal{R}(x_0, \lambda\delta^{3/2}, \rho), 1) \), and thereby conclude. It is obvious that

\[
\inf_{x_0 \in \mathbb{R}^2} \mu_\varepsilon(\mathcal{R}(x_0, \lambda\delta^{3/2}, \rho), 1) = \pi b^2 \ln(\lambda\delta^{3/2}/\rho) + o_\varepsilon(1).
\]

On the one hand, from the previous steps, for \( \eta_0, c > 0 \) and a configuration of points/degrees \( \{x_\varepsilon, 1\} = \{(x^\varepsilon_1, 1), \ldots, (x^\varepsilon_d, 1)\} \) such that \( |x^\varepsilon_i - x^\varepsilon_j|, \text{dist}(x^\varepsilon_i, \partial\Omega) \geq \eta_0 \) and \( B(x^\varepsilon_i, c\lambda\delta) \subset \omega^1 \) for all \( i \neq j \), \( i, j \in \{1, \ldots, d\} \), we have \( I_{\rho, \varepsilon}(x_\varepsilon) = I_{\rho, \varepsilon} + \theta(1) \). On the other hand, for \( \{x_\varepsilon, d\} = \{(x^\varepsilon_1, d_1), \ldots, (x^\varepsilon_d, d_N)\} \) a configuration of points/degrees, we consider two possibilities:

- Either there is \( i \in \{1, \ldots, N\} \) such that \( d_i > 1 \);
- Or for \( \varepsilon > 0 \) we have \( \text{dist}(x^{\varepsilon_n}_i, \partial\Omega) \rightarrow 0 \) or \( \min_{\varepsilon \in [0, \varepsilon]} |x^{\varepsilon_n}_i - x^{\varepsilon_n}_j| \rightarrow 0 \).
In either case, the configuration of points/degrees cannot be almost minimal for $I_{\kappa,\varepsilon_n}$ and thus it cannot be almost minimal for $I_{\rho,\varepsilon_n}$. Moreover, if there exist $i$ and $\varepsilon_n \downarrow 0$ such that either $x_{i\varepsilon_n} \notin \omega_1 \varepsilon_n$ or $\text{dist}(x_{i\varepsilon_n}, \partial \omega_1 \varepsilon_n) \sim 0$, then $(x_{i\varepsilon_n})_n$ cannot be an almost minimal configuration for $\mu_{\varepsilon_n}(\mathbb{R}, \varepsilon_n, \lambda \delta^{3/2}, 1)$. Thus $\{x_{\varepsilon_n}, d\}$ cannot be an almost minimal configuration for $I_{\rho,\varepsilon_n}$.

Therefore, Assertions (1) and (2) of Proposition 3.12 hold. The rest of the proposition is obtained exactly as Corollary 3.8.

**APPENDIX F. PROOF OF PROPOSITION 4.15**

We use the unfolding operator (see [CDG08, Definition 2.1]), and define, for $\Omega_0 \subset \mathbb{R}^2$ a bounded open set, $p \in (1, \infty)$ and $\delta > 0$, the following:

$$T_\delta : L^p(\Omega_0) \to L^p(\Omega_0 \times \tilde{Y})$$

$$\phi \to T_\delta(\phi)(x, y) = \begin{cases} \phi \left( \delta \left( \frac{x}{\delta} \right) + \delta y \right) & \text{for } (x, y) \in \tilde{\Omega}^{\text{incl}}_\delta \times \tilde{Y}, \\ 0 & \text{for } (x, y) \in \Lambda_\delta \times \tilde{Y}, \end{cases}$$

and

$$\tilde{Y} = (0, 1) \times (0, 1), \quad \tilde{\Omega}^{\text{incl}}_\delta := \bigcup_{\tilde{Y}_\delta \in \tilde{\Omega}_\delta} \tilde{Y}_\delta^k,$$

$$\Lambda_\delta := \Omega_0 \setminus \tilde{\Omega}^{\text{incl}}_\delta, \quad \left[ \frac{x}{\delta} \right] := \left( \left[ \frac{x_1}{\delta} \right], \left[ \frac{x_2}{\delta} \right] \right) \in \mathbb{Z}^2.$$

Here, for $s \in \mathbb{R}$, $[s]$ is the integer part of $s$.

We use the following results:

(F.1) $T_\delta$ is linear and continuous, of norm at most 1 ([CDG08, Proposition 2.5]);

(F.2) $T_{\delta}(\phi \psi) = T_{\delta}(\phi) T_{\delta}(\psi)$ ([CDG08, equation (2.2)]);

(F.3) $\delta T_{\delta}(\nabla \phi)(x, y) = \nabla_y T_{\delta}(\phi)(x, y)$ for $\phi \in H^1(\Omega_0)$, ([CDG08, equation (3.1)]);

(F.4) $\int_{\tilde{\Omega}^{\text{incl}}_\delta} \phi = \int_{\Omega_0 \times \tilde{Y}} T_{\delta}(\phi)$ ([CDG08, Proposition 2.5 (i)]).

If $\phi_\delta \in H^1(\Omega_0)$ is such that $\phi_\delta \rightharpoonup \phi_0$ in $H^1$, then, up to a subsequence, there exists $\hat{\phi} \in L^2(\Omega_0, H^1_{\text{per}}(\tilde{Y}))$ such that

(F.5) $T_{\delta}(\phi_\delta) \rightharpoonup \phi_0$ and $T_{\delta}(\nabla \phi_\delta) \rightharpoonup \nabla \phi_0 + \nabla_y \hat{\phi}$ in $L^2(\Omega_0 \times \tilde{Y})$ ([CDG08, Theorem 3.5]).
Here, $H^1_{\text{per}}(\bar{Y})$ stands for the set of functions $\phi \in H^1(\bar{Y})$ such that the extending of $\phi$ by $\bar{Y}$-periodicity is in $H^1_{\text{loc}}(\mathbb{R}^2)$ (see [CD99, Section 3.4]).

In order to define properly the homogenized matrix $\mathcal{A}$, we recall a classical result (see Theorem 4.27 in [CD99]).

**Proposition F.1.** Let $H_0 \in L^\infty(\bar{Y}, [b^2, 1])$. For all $f \in (H^1_{\text{per}}(\bar{Y}))'$ such that $f$ annihilates the constants, there exists a unique solution $h \in H^1_{\text{per}}(\bar{Y})$ of

$$\text{div}(H_0 \nabla_y h) = f \quad \text{and} \quad \mathcal{M}_Y(h) = \int_{\bar{Y}} h = 0.$$ 

Using the previous theorem, for $j = 1, 2$, we denote by $\chi_j \in H^1_{\text{per}}(\bar{Y})$ the unique solution of

$$\text{div}(H_0 \nabla_y \chi_j) = \delta_y(H_0) \quad \text{and} \quad \mathcal{M}_Y(\chi_j) = 0.$$ 

With these auxiliary functions, we can give an explicit expression of $\mathcal{A}$ the homogenized matrix of $H_0(\bar{Y}) \text{Id}_{\mathbb{R}^2}$ (see Theorem 6.1 in [CD99]) as follows:

$$\mathcal{A} = \int_{\bar{Y}} H_0 \begin{pmatrix} 1 - \partial_{y_1}X_1 & -\partial_{y_1}X_2 \\ -\partial_{y_2}X_1 & 1 - \partial_{y_2}X_2 \end{pmatrix} = \int_{\bar{Y}} H_0(\text{Id}_{\mathbb{R}^2} - \nabla_y X), \quad X = (X_1, X_2).$$ 

For the convenience of the reader, we restate in larger detail Proposition 4.15.

**Proposition.** Let $\Omega_0 \subset \mathbb{R}^2$ be a smooth, bounded open set, and let $v_n \in H^2(\Omega_0, \mathbb{C})$ be such that

1. $|v_n| \leq 1$ and $\int_{\Omega_0} (1 - |v_n|^2)^2 \to 0$;
2. $\nabla v_n - v_\ast \in H^1(\Omega_0)$ and $v_\ast \in H^1(\Omega_0, \mathbb{S}^1)$;
3. there exist $H_n \in W^{1,\infty}(\Omega_0, [b^2, 1])$ and $\delta_n \downarrow 0$ such that $T_{\delta_n}(H_n) \to H_0$ in $L^2(\Omega_0 \times \bar{Y})$ with $H_0$ independent of $x \in \Omega_0$;
4. $-\text{div}(H_n \nabla v_n) = v_n f_n$, $f_n \in L^\infty(\Omega_0, \mathbb{R})$.

Then, $v_\ast$ is a solution of $-\text{div}(\mathcal{A} \nabla v_\ast) = (\mathcal{A} \nabla v_\ast \cdot \nabla v_\ast)v_\ast$. Here, $\mathcal{A}$ is the homogenized matrix of $H_0(\cdot/\delta) \text{Id}_{\mathbb{R}^2}$ given by

$$\mathcal{A} = \int_{\bar{Y}} H_0 \begin{pmatrix} 1 - \partial_{y_1}X_1 & -\partial_{y_1}X_2 \\ -\partial_{y_2}X_1 & 1 - \partial_{y_2}X_2 \end{pmatrix}.$$ 

**Proof.** In order to keep notation simple, we write, in what follows, $\delta$ rather than $\delta_n$.

Since $f_n$ is real valued, we have that $\text{div}(H_n \nabla v_n) \wedge v_n = 0$. From (F.1) and (F.2), we obtain

$$\text{div}_y [T_\delta(H_n)(x, y)T_\delta(\nabla v_n)(x, y)] \wedge T_\delta(v_n)(x, y) = 0 \quad \text{in } \Omega_0 \times \bar{Y}.$$
Note that, from the assumptions as well as (F.1) and (F.5), passing to a subsequence, there is \( \hat{\nu} \in L^2(\Omega_0, H^{-1}(\tilde{Y})) \) such that
\[
T_\delta(\nu_n)(x, y) \to v_*(x),
\]
\[
T_\delta(\nabla \nu_n)(x, y) - \nabla v_*(x) + \nabla_y \hat{\nu}(x, y),
\]
and
\[
T_\delta(H_n)(x, y) \to H_0(y)
\]
in \( L^2(\Omega_0 \times \tilde{Y}) \). Thus we obtain the convergence:
\[
\text{div}_y [T_\delta(H_n) \nabla v_n] \wedge T_\delta(v_n) \to \text{div}_y [H_0(\nabla v_* + \nabla_y \hat{\nu})] \wedge v_* \quad \text{in} \quad L^2(\Omega_0, H^{-1}(\tilde{Y})).
\]
Consequently, \( \text{div}_y [H_0(\nabla v_* + \nabla_y \hat{\nu})] \wedge v_* = 0 \). Since \( v_* \) is independent of \( y \in \tilde{Y} \), the previous assertion is equivalent to
\[
-\text{div}_y [H_0 \nabla_y (\hat{\nu} \wedge v_*)] = \sum_i \partial_y_i H_0(\hat{\nu} v_* \wedge v_*)
\]
which, in turn, is equivalent to (because \( H_0 \) is a real valued function) the following:
\[
-\text{div}_y [H_0 \nabla_y (\hat{\nu} \wedge v_*)] = \sum_i \partial_y_i H_0(\hat{\nu} v_* \wedge v_*)
\]
Hence, from Proposition F.1 and (F.6), we obtain
\[
(\text{F.8}) \quad \hat{\nu} \wedge v_* = -\sum_i \chi_i (\partial_i v_*, v_* \wedge v_) = -\chi \cdot (\nabla v_* \wedge v_), \quad \chi = (\chi_1, \chi_2).
\]
Let \( \psi \in \mathcal{D}(\Omega_0) \) and \( n \) be sufficiently large such that \( \text{Supp}(\psi) \subset \tilde{\Omega}_0^{\text{incl}} \). Since
\[
-\text{div}[H_n \nabla v_n \wedge v_n] = 0,
\]
we have
\[
\int_{\tilde{\Omega}_0^{\text{incl}}} H_n \nabla v_n \wedge v_n \cdot \nabla \psi = 0.
\]
This identity combined with (F.4) implies that
\[
\int_{\tilde{\Omega}_0^{\text{incl}}} \nabla v_n \wedge v_n \cdot \nabla \psi = 0.
\]
Therefore, using (F.3) and (F.5), we obtain
\[
0 = \int_{\Omega_0 \times \tilde{Y}} T_\delta[H_n(\nabla v_n \wedge v_n) \cdot \nabla \psi] = \int_{\Omega_0 \times \tilde{Y}} [T_\delta(H_n) \nabla v_n \wedge v_n] \cdot T_\delta(\nabla \psi)
\]
\[
- \int_{\Omega_0 \times \tilde{Y}} H_0[\nabla v_* \wedge v_* + \nabla_y (\hat{\nu} \wedge v_*)] \cdot \nabla \psi \quad \text{as} \quad n \to \infty.
\]
Here, \( \mathcal{A} = \int_{\tilde{Y}} H_0(\text{Id}_{\mathbb{R}^2} - \nabla_y \chi) \).
Thus \( -\text{div}(\mathcal{A}(\nabla v_* \wedge v_*)) = 0 \). Note that, since \( H_0 \) and \( \chi \) are independent of \( \chi \), we have that \( \mathcal{A} \) is a constant matrix. This fact, combined with the equation \( -\text{div}(\mathcal{A}(\nabla v_* \wedge v_*)) = 0 \) implies that \( v_* \) satisfies

(E.9) \[-\text{div}(\mathcal{A}\nabla v_*) = (\mathcal{A}\nabla v_* \cdot \nabla v_*)v_* .\]

Indeed, we can always consider \( \varphi_* \) that is locally defined in \( \Omega_0 \) and whose gradient is globally defined and in \( L^2(\Omega_0, \mathbb{R}^2) \) such that \( v_* = e^{i\varphi_*} \).

Since \( v_* \wedge \nabla v_* = \nabla \varphi_* \), we obtain that \( \text{div}(\mathcal{A}\nabla \varphi_*) = 0 \). Identity (E.9) follows from the equation of \( \varphi_* \) and the fact that \( \nabla \varphi_* = \nabla v_*/i v_* \).

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References


[Bre] H. Brezis, Equations de Ginzburg-Landau et singularités, Université Paris 6, Graduate Course.


G-L Energy with a Discontinuous and Rapidly Oscillating Pinning Term


L. LASSOUED, Sur quelques équations aux dérivées partielles non linéaires issues de la géométrie et de la physique, Ph.D. thesis (1996), Université de Paris 06.


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