Finite time Merton strategy under drawdown constraint: a viscosity approach

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Abstract

We consider the optimal consumption-investment problem under the drawdown constraint, i.e. the wealth process never falls below a fixed fraction of its running maximum. We assume that the risky asset is driven by the constant coefficients Black and Scholes model and we consider a general class of utility functions. On an infinite time horizon, Elie and Touzi [7] provided the value function as well as the optimal consumption and investment strategy in explicit form. In a more realistic setting, we consider here an agent optimizing its consumption-investment strategy on a finite time horizon. The value function interprets as the unique discontinuous viscosity solution of its corresponding Hamilton-Jacobi-Bellman equation. This leads to a numerical approximation of the value function and allows for a comparison with the explicit solution in infinite horizon.

Key words: consumption-investment strategy, drawdown constraint, viscosity solution, comparison principle.

MSC 1991 subject classifications: 91B28, 35K55, 65N12

1 Introduction

In this paper, we consider a fund manager who guaranties the investors that their wealth will stay above a fixed fraction of its current maximum. Under this so-called drawdown constraint, Grossman and Zhou [10] derived the explicit strategy of a manager maximizing the long term growth rate of the expected power utility of final wealth. This solution, deduced from the resolution of the corresponding Hamilton-Jacobi-Bellman equation, has been generalized to more general models by Cvitanic and Karatzas [6] through an elegant probabilistic argument. More recently, Elie and Touzi [7] considered a manager investing in a risky asset with Black Scholes dynamics, as well as distributing dividends to the investors, dividends which interpret economically as consumption. Working also in an infinite horizon framework, they derived explicitly the value function and the corresponding optimal strategy solving the classical Merton criterion maximizing its inter-temporal utility from consumption under a drawdown constraint. Their result is valid for a large class of utility functions and mainly relies on the use of a duality argument. They indeed determine the Fenchel transform of the value function by solving a linear ordinary differential equation with free boundaries.

We consider here the more realistic case where the fund manager handles the portfolio of investors over a fixed period T. We therefore study the problem of managing a portfolio subject to a drawdown constraint, with the purpose of maximizing the inter-temporal utility of consumption on a finite horizon T. We seek for a better comprehension of the influence of this fixed time horizon T on the behavior of the manager. In particular, we would like to understand after which sufficiently large horizon T, the manager can use the explicit optimal strategy derived in [7]. To our knowledge, this paper is the first contribution to the determination of an optimal investment strategy under drawdown constraint in a finite horizon framework.

In the absence of drawdown constraint, Merton [13, 14] derived explicit solutions to this problem for particular choices of utility functions, by solving the corresponding Hamilton-Jacobi-Bellman equations. By a duality argument, Cox and Huang [3] and Karatzas, Lehoczky and Shreeve [11] extended his results to a market with non Markovian price processes. Beyond the large number of articles considering the addition of imperfections to the market, we mention the work of El Karoui, Jeanblanc and Lacoste [8], who consider a related type of constraints on the strategy. They study the behavior of a manager maximizing its finite horizon utility of wealth under the constraint that the value of the portfolio stays above a fixed floor process. Allowing the fund manager to invest in American Puts, they derive an optimal strategy. We refer also to the work of El Karoui and Meziou [9] who consider a similar minimum floor constraint, but present a very different point of view. Instead of specifying the utility function of the manager, their optimisation relies on a stochastic dominance approach, for which they prove the existence of an optimal solution.

In contrast with the infinite horizon, no explicit form of the value function is available, since the additional dependence in time of the solution makes the computations in [7] untractable. The main difficulty relies in the obtention of the time dependent free boundary of the partial differential equation satisfied by the dual transform of the value function. The main result of this paper is to derive a PDE characterization of the value function associated to the finite time horizon maximization. The derivation of the associated PDE relies classically on the use of the dynamic programming principle. The boundary conditions of the PDE are given by a Dirichlet condition at maturity T and a Neumann condition when the process reaches its current maximum. Surprisingly, we do not require any Dirichlet condition on the semi real line where the drawdown constraint binds. Nevertheless, adding this Dirichlet condition allows to derive uniqueness of solution to the associated PDE in the viscosity sense under weaker assumptions. We first prove that the value function is a (discontinuous) viscosity solution of the corresponding Hamilton-Jacobi-Bellman equation. We then derive a comparison theorem for the associated PDE, which ensures the uniqueness of the solution within a particular class of functions. Since the consumption and investment controls are not bounded, the comparison result can not be obtained using classical penalization arguments. We overcame this difficulty by adapting the arguments of Zariphopoulou [15] where she studied a consumption-investment problem under general constraints. The comparison result then allows for a numerical approximation of the value function. We observe an exponential convergence of the value function in finite horizon to the explicit solution in infinite, result which is confirmed theoretically.

This paper is organized as follows. The problem is formulated in Section 2. The main results detailing properties of the value function and its characterization as the unique viscosity solution of the associated PDE are presented in Section 3. A corresponding approximation scheme and numerical results are provided in Section 4. The proofs of the viscosity property of the value function and the comparison result are respectively reported in Sections 5 and 6.

2 Problem formulation

We work in a similar framework as in [7] and we will try to follow their notations. We consider a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$ endowed with a Brownian motion $W = \{W_t,\, 0 \leq t \leq T\}$ with values in $\mathbb{R}$, and we denote by $\mathbb{F} := \{\mathcal{F}_t,\, 0 \leq t \leq T\}$. The financial market consists of a non-risky asset, with process normalized to unity, and a risky asset with price process defined by the Black and Scholes model

$$dS_t \;=\; \sigma S_t \left(dW_t + \lambda dt\right) ,$$

where $\sigma > 0$ is the volatility parameter, and $\lambda \in \mathbb{R}$ is a constant risk premium. For any continuous process $\{M_t, t \geq 0\}$, its current maximum is denoted M^*.

2.1 Consumption-investment strategies and the drawdown constraint

We next introduce the set of consumption-investment strategies whose induced wealth process X satisfies the drawdown constraint

$$X_t \;\geq\; \alpha X_t^* \quad \text{for every} \quad 0 \leq t \leq T\,, \;\; \text{a.s.}\,, \tag{2.1}$$

where α is some given parameter in the interval $[0, 1)$.
A consumption-investment strategy is an $\mathbb{F}-$adapted pair process $(C, \theta)_{0 \leq t \leq T}$ valued in $\mathbb{R}_+ \times \mathbb{R}$ satisfying the integrability condition

$$\int_0^T C_s\, ds + \int_0^T |\theta_s|^2\, ds \;<\; \infty \quad \text{a.s.}\;\; . \tag{2.2}$$

The wealth process induced by such a pair (C, θ) is therefore defined by

$$X_t^{x,C,\theta} \;=\; x - \int_0^t C_r dr + \int_0^t \sigma\theta_r \left(dW_r + \lambda dr\right)\,, \quad 0 \leq t \leq T\,, \tag{2.3}$$

where x is some given initial capital. We denote by $\mathcal{A}_\alpha(x)$ the collection of all such consumption-investment strategies whose corresponding wealth process satisfies the drawdown constraint (2.1).

Remark 2.1 As in the infinite horizon framework in [7], the manager can neither consume nor invest once the drawdown constraint binds. Thus, for a given initial wealth x and an admissible consumption-investment strategy $(C, \theta) \in \mathcal{A}_\alpha(x)$, we have

$$X_{.\vee\tau}^{x,C,\theta} \;=\; X_\tau^{x,C,\theta}\,, \quad \text{where } \tau \;:=\; \inf\left\{s \leq T \;:\; X_s^{x,C,\theta} = \alpha\{X^{x,C,\theta}\}_s^*\right\}\,. \tag{2.4}$$

The set of admissible consumption-investment strategies $\mathcal{A}_\alpha(x)$ contains in particular the strategies of the form

$$C_t \;=\; c_t\left[X_t - \alpha X_t^*\right] \quad \text{and} \quad \theta_t \;=\; \pi_t\left[X_t - \alpha X_t^*\right]\,, \tag{2.5}$$

where (c, π) is an $\mathbb{F}-$adapted pair process valued in $\mathbb{R}_+ \times \mathbb{R}$ satisfying the integrability condition

$$\int_0^T c_s ds + \int_0^T |\pi_s|^2 ds \;<\; \infty\,. \tag{2.6}$$

Indeed, as observed by Cvitanic and Karatzas [6], for any strategy of the form (2.5), the process $M := [X^{x,C,\theta} - \alpha(X^{x,C,\theta})^*][(X^{x,C,\theta})^*]^{\alpha/(1-\alpha)}$ has the following dynamics

$$dM_t = M_t[(\lambda\sigma\pi_t - c_t)dt + \sigma\pi_t dW_t]\,. \tag{2.7}$$

Noticing that $M^* = (1-\alpha)[(X^{x,C,\theta})^*]^{1/(1-\alpha)}$, the existence and uniqueness of $X^{x,C,\theta}$ satisfying (2.3) and (2.1) follows from the resolution of (2.7)

2.2 The finite horizon consumption-investment problem

Throughout this paper, we consider a utility function

$$U : \mathbb{R}_+ \to \mathbb{R} \qquad C^2, \text{ concave, satisfying } U(0) = 0\,,\ U'(0+) = \infty \text{ and } U'(\infty) = 0\,. \tag{2.8}$$

For a given initial capital $x > 0$ and an horizon T, the optimal consumption-investment Merton problem under drawdown constraint is defined by :

$$u_0 := \sup_{(C,\theta)\in\mathcal{A}_\alpha(x)} J_0(C,\theta) \quad \text{where} \quad J_0(C,\theta) := \mathbb{E}\left[\int_0^T e^{-\beta s} U(C_s)\, ds\right]. \tag{2.9}$$

In order to make use of the dynamic programming approach, we then need to introduce the dynamic version of this problem :

$$u(t,x,z) := \sup_{(C,\theta)\in\mathcal{A}_\alpha(t,x,z)} J(t,C,\theta) \text{ where } J(t,C,\theta) := \mathbb{E}\left[\int_t^T e^{-\beta s} U(C_s)\, ds\right], \tag{2.10}$$

where the pair (x,z) satisfying $x \le z$, stands for the initial condition of the state processes (X,Z) defined, for $s \ge t$, by

$$Z_s^{t,x,z,C,\theta} := z \vee \left\{X^{t,x,C,\theta}\right\}_s^* \text{ and } X_s^{t,x,C,\theta} = x - \int_t^s C_r dr + \int_t^s \sigma\theta_r\,(dW_r + \lambda dr)\,, \tag{2.11}$$

and $\mathcal{A}_\alpha(t,x,z)$ is the collection of all $\mathbb{F}-$adapted processes $(C_s,\theta_s)_{t\le s\le T}$ satisfying

$$\int_t^T C_s ds + \int_t^T |\theta_s|^2 ds < \infty \quad \text{a.s. }. \tag{2.12}$$

together with the drawdown constraint

$$X_s^{t,x,C,\theta} \ \ge\ \alpha Z_s^{t,x,z,C,\theta} \ \text{ a.s.}, \qquad t \le s \le T\,. \tag{2.13}$$

The value function u is thus defined for any triplets (t,x,z) in the closure $\overline{\mathcal{O}}_\alpha$ in $[0,T]\times\mathbb{R}_+\times\mathbb{R}_+$ of the domain

$$\mathcal{O}_\alpha := [0,T)\times\{(x,z)\ :\ 0<\alpha z<x<z\}\,.$$

Remark 2.2 For any $(t,x,z) \in \overline{\mathcal{O}}_\alpha$, $u(t,x,z)$ interprets as the value function of the consumption-investment problem given a wealth x, a running maximum z and an horizon $T-t$.

For any $y = (t,x,z) \in \overline{\mathcal{O}}_\alpha$ and $(C,\theta) \in \mathcal{A}_\alpha(y)$, we shall make use of the following notation

$$Y_s^{y,C,\theta} := (s, X_s^{t,x,C,\theta}, Z_s^{t,x,z,C,\theta})\,, \qquad \text{for any } s \ge t\,.$$

3 The main results

We denote by V the Fenchel-Lengendre transform of U given by

$$V(y) := \sup_{x\geq 0} \left(U(x) - xy\right), \tag{3.14}$$

and introduce the following notations

$$\gamma := \frac{2\beta}{\lambda^2}, \quad \text{and} \quad \delta := \frac{\gamma}{1-\alpha(1+\gamma)}.$$

We shall mainly work under the following Assumptions.

Assumption 3.1 $\quad \dfrac{\gamma}{1+\gamma} < 1-\alpha.$

Observe that this condition is in particular satisfied in the Merton case, $\alpha = 0$.

Assumption 3.2 $\quad p := \mathrm{AE}(U) = \lim_{x\to\infty} \frac{xU'(x)}{U(x)} < \dfrac{\gamma}{1+\gamma}.$

Since its introduction by Kramkov and Schachermayer in [12], p is called the asymptotic elasticity of the utility function U. According to them, a nice framework for utility maximization issues is obtained under the condition $p < 1$, which is a particular consequence of Assumption 3.2.

Assumption 3.3 $\quad \inf_{y>0} \left\{ \dfrac{1}{yV''(y)} \displaystyle\int_0^y \frac{-V'(s)}{s} \left(\frac{s}{y}\right)^{1+\delta} ds \right\} > 0\,.$

This last assumption has been introduced by Elie and Touzi [7] and appears to be necessary for the obtention of an explicit solution to the problem in infinite horizon. It is in particular automatically satisfied whenever the relative risk aversion of U is uniformly bounded from below.

These assumptions are almost similar to the one introduced in [7]. The only difference relies in the classical Merton Assumption 3.2 which is stronger than the one required in [7]. Therefore, they imply the existence of a solution u_∞ to the problem in infinite horizon. In particular, we deduce that u_∞ provides a natural upper-bound for the value function u :

$$u(t,x,z) \leq u_\infty(x,z)\,, \qquad (x,z) \in \overline{\mathcal{O}}_\alpha\,. \tag{3.15}$$

Remark 3.1 Since we aim at interpreting u as a (discontinuous) viscosity solution of a PDE, one may wonder the necessity of the regularity assumptions on the utility function U adopted in (2.8). The reason is simply the use of the explicit u_∞ as a regular upper-bound on u. As detailed in Lemma 3.3, $U(0) = 0$ allows the value function u to inherit continuity properties of u_∞ when the drawdown constraint nearly binds. These regularity properties are required for the proof of the general comparison result leading to Theorem 6.1. Nevertheless another version of the comparison result is obtained under weaker assumptions in Proposition 3.1 and discussed in Remark 6.1.

3.1 The PDE characterization

The dynamic programming equation is related to the second order operator defined for $\varphi \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ by

$$\mathcal{L}_T\,\varphi \;:=\; \sup_{C\geq 0,\theta\in\mathbb{R}} \mathcal{L}_T^{C,\theta}\varphi\,, \tag{3.16}$$

where, for any $C \geq 0$ and $\theta \in \mathbb{R}$, $\mathcal{L}_T^{C,\theta}\varphi$ is given by

$$\mathcal{L}_T^{C,\theta}\varphi \;:=\; -\beta\varphi + \varphi_t + U(C) + (\sigma\lambda\theta - C)\varphi_x + \frac{(\sigma\theta)^2}{2}\varphi_{xx}\,.$$

Observe that the above dynamic programming equation simplifies to

$$\mathcal{L}_T\,\varphi \;=\; -\beta\varphi + \varphi_t + V(\varphi_x) - \frac{\lambda^2}{2}\frac{\varphi_x^2}{\varphi_{xx}} \quad \text{whenever } \varphi \text{ is strictly concave in } x\,. \tag{3.17}$$

We next decompose the boundary of the domain of definition $\overline{\mathcal{O}}_\alpha$ of the value function u in the following four disjoint subsets

$$\begin{aligned}
\partial^0\mathcal{O}_\alpha &:= [0,T]\times\{(0,0)\}\,,\\
\partial^\alpha\mathcal{O}_\alpha &:= [0,T]\times\{(\alpha z,z)\;:z>0\}\,,\\
\partial^1\mathcal{O}_\alpha &:= [0,T)\times\{(z,z)\;:z>0\}\,,\\
\partial^T\mathcal{O}_\alpha &:= \{T\}\times\{(x,z)\;:0<\alpha z\leq x\leq z\}\,.
\end{aligned}$$

The purpose of this paper is to characterize u as the solution of the following dynamic programming equation

$$\left\{\begin{aligned}
-\mathcal{L}_T\,\varphi &= 0 \quad \text{on} \quad \mathcal{O}_\alpha\cup\partial^\alpha\mathcal{O}_\alpha\,,\\
-\varphi_z &= 0 \quad \text{on} \quad \partial^1\mathcal{O}_\alpha\,,\\
\varphi &= 0 \quad \text{on} \quad \partial^T\mathcal{O}_\alpha\cup\partial^0\mathcal{O}_\alpha\,.
\end{aligned}\right. \tag{3.18}$$

We now introduce the following classical notations. For any locally bounded function $v : \overline{\mathcal{O}}_\alpha \to \mathbb{R}$, we denote the corresponding lower and upper semi-continuous enveloppes of v by

$$v_*(y) \;:=\; \liminf_{\mathcal{O}_\alpha\ni y'\to y} v(y') \quad \text{and} \quad v^*(y) \;:=\; \limsup_{\mathcal{O}_\alpha\ni y'\to y} v(y')\,.$$

A viscosity solution of the PDE (3.18) is then defined in the following way.

Definition 3.1 *(i) A locally bounded function v is a (discontinuous) viscosity subsolution of* (3.18) *if $v^* \leq 0$ on $\partial^T\mathcal{O}_\alpha\cup\partial^0\mathcal{O}_\alpha$ and, for all $y_0 \in \overline{\mathcal{O}}_\alpha$ and $\varphi \in C^{1,2,1}(\overline{\mathcal{O}}_\alpha)$ such that $0 = (v^*-\varphi)(y_0) = \sup_{\overline{\mathcal{O}}_\alpha}(v^*-\varphi)$, we have*

$$-\mathcal{L}_T\varphi(y_0)\leq 0 \;\; \textit{if} \;\; y_0\in\mathcal{O}_\alpha\cup\partial^\alpha\mathcal{O}_\alpha \quad \textit{and} \quad \min\{-\mathcal{L}_T\varphi,-\varphi_z\}(y_0)\leq 0 \;\; \textit{if} \;\; y_0\in\partial^1\mathcal{O}_\alpha\,.$$

(ii) A locally bounded function v is a (discontinuous) viscosity supersolution of (3.18) *if $v_* \geq 0$ on $\partial^T \mathcal{O}_\alpha \cup \partial^0 \mathcal{O}_\alpha$ and, for all given $y_0 \in \overline{\mathcal{O}}_\alpha$ and $\varphi \in C^{1,2,1}(\overline{\mathcal{O}}_\alpha)$ such that $0 = (v_* - \varphi)(y_0) = \inf_{\overline{\mathcal{O}}_\alpha}(v_* - \varphi)$, we have*

$$-\mathcal{L}_T\varphi(y_0) \geq 0 \;\; \textit{if } y_0 \in \mathcal{O}_\alpha \quad \textit{and} \quad -\varphi_z(y_0) \geq 0 \;\; \textit{if } y_0 \in \partial^1 \mathcal{O}_\alpha \,.$$

(iii) A locally bounded function v is a (discontinuous) viscosity solution of (3.18) *if it is both a sub- and a supersolution.*

We now provide the main result of this paper.

Theorem 3.1 *The value function u is a viscosity solution of* (3.18)*. If furthermore, Assumptions 3.1, 3.2 and 3.3 hold, then u is the unique viscosity solution of* (3.18) *in the class of locally bounded functions v, right-continuous in the direction $\overrightarrow{e} := (0,1,1)$ on $\mathcal{O}_\alpha \cup \partial^1 \mathcal{O}_\alpha \cup \partial^\alpha \mathcal{O}_\alpha$, equal to 0 on $\partial^T \mathcal{O}_\alpha \cup \partial^0 \mathcal{O}_\alpha$ and satisfying the growth property*

$$v(t,x,z) \;\; \leq \;\; K(1+x^p)\,, \quad (t,x,z) \in \overline{\mathcal{O}}_\alpha\,, \quad \textit{for some } K > 0. \tag{3.19}$$

The proof of the first part of the theorem is reported in Section 5. We provide some properties of the value function u in section 3.2, including in particular the nullity of u on $\partial^T \mathcal{O}_\alpha \cup \partial^0 \mathcal{O}_\alpha$, the growth property (3.19), as well as the right-continuity of u in the direction $\overrightarrow{e}$ under Assumptions 3.1, 3.2 and 3.3. Finally a comparison result, ensuring uniqueness of viscosity solutions to the PDE (3.18) within the class of locally bounded functions, satisfying these particular growing and regularity properties, is presented in Section 6.

We conclude this section by stating another comparison result for the solution of the PDE (3.18) obtained under weaker assumptions on the utility function U. Indeed, as announced in Remark 3.1, the imposed regularity on U allows to use the explicit solution in infinite horizon derived in [7] as a regular upper bound to the value function u_∞, leading to the right-continuous in the direction $\overrightarrow{e}$ of u on the boundary $\partial^\alpha \mathcal{O}_\alpha$. Nevertheless, this regularity property is not needed for the obtention of a comparaison result as long as we consider a smaller class of functions forced to equal zero on the boundary $\partial^\alpha \mathcal{O}_\alpha$. The justification of this argument is provided in Remark 6.1 below.

Proposition 3.1 *Let U be a C^1, increasing, concave function satisfying $U(0) = 0$ as well as Assumption 3.2, and u be its associated value function. Then u is the unique viscosity solution of* (3.18) *in the class of locally bounded functions v, right-continuous in the direction $\overrightarrow{e} := (0,1,1)$ on $\mathcal{O}_\alpha \cup \partial^1 \mathcal{O}_\alpha$, equal to 0 on $\partial^T \mathcal{O}_\alpha \cup \partial^0 \mathcal{O}_\alpha \cup \partial^\alpha \mathcal{O}_\alpha$, and satisfying the growth property* (3.19)*.*

3.2 Properties of the value function

This section collects some properties of the value function u which, in addition to their self interest, will allow us to derive precise viscosity properties of u on the boundary of the domain $\overline{\mathcal{O}}_\alpha$ and to restrain the class of functions for which a comparison result is required.

Lemma 3.1 *The value function u satisfies*

$$u \geq 0 \text{ on } \overline{\mathcal{O}}_\alpha \quad \text{and} \quad u = 0 \text{ on } \partial^T \mathcal{O}_\alpha \cup \partial^0 \mathcal{O}_\alpha \cup \partial^\alpha \mathcal{O}_\alpha \,. \tag{3.20}$$

If Assumption 3.2 holds, then there exists $K > 0$ such that

$$u(y) \leq K\,(1 + x^p)\,, \quad y = (t, x, z) \in \overline{\mathcal{O}}_\alpha \,. \tag{3.21}$$

Proof. Observe first that u inherits the positivity of U. Recalling (2.4), we remark that there is no non-trivial admissible strategy on $\partial^T \mathcal{O}_\alpha \cup \partial^0 \mathcal{O}_\alpha \cup \partial^\alpha \mathcal{O}_\alpha$ and derive (3.20). Under Assumption 3.2, the asymptotic elasticity p of U is strictly smaller than one. We then deduce from Lemma 6.5 in [12] the existence of $K > 0$ such that

$$U(x) \quad \leq \quad K\left(1 + \frac{x^p}{p}\right), \quad x \geq 0\,. \tag{3.22}$$

But, in the absence of drawdown constraint, the value function u^* associated to the power utility function $x \mapsto x^p/p$ is well known to satisfy

$$u^*(t, x) \quad \leq \quad K' x^p, \quad t \geq 0\,, \;\; x \geq 0\,, \tag{3.23}$$

where K' is also a positive constant. Since the set of admissible strategies in the presence of drawdown constraint is smaller that the one of the classical Merton set-up, we deduce (3.21) from (3.22) and (3.23). □

Lemma 3.2 *The value function u is non-decreasing in its second variable x and non-increasing in its third variable z.*

Proof. Take (t, x, z, z') such that $(t, x, z) \in \overline{\mathcal{O}}_\alpha$, $(t, x, z') \in \overline{\mathcal{O}}_\alpha$ and $z' \leq z$. Since

$$X^{t,x,C,\theta} \;\geq\; \alpha Z^{t,x,z,C,\theta} \;\geq\; \alpha Z^{t,x,z',C,\theta}\,, \quad (C, \theta) \in \mathcal{A}_\alpha(t, x, z)\,,$$

we have $\mathcal{A}_\alpha(t, x, z) \subset \mathcal{A}_\alpha(t, x, z')$, which naturally leads to $u(t, x, z) \leq u(t, x, z')$. Similar arguments easily lead to the non-decreasing property of u in x. □

We now derive some regularity and concavity properties of the value function u in the direction $\overrightarrow{e} = (0, 1, 1)$.

Lemma 3.3 *The following holds.*

(i) *For any $y \in \overline{\mathcal{O}}_\alpha$, the function $h \mapsto u[y + h\overrightarrow{e}]$ is concave on $\mathbb{R}_+$.*

(ii) *The function u is right-continuous in the direction $\overrightarrow{e}$ on $\mathcal{O}_\alpha \cup \partial^T \mathcal{O}_\alpha \cup \partial^1 \mathcal{O}_\alpha$, i.e.*

$$u[y + h\overrightarrow{e}] \underset{h \downarrow 0^+}{\longrightarrow} u[y]\,, \qquad \text{for any } y \in \mathcal{O}_\alpha \cup \partial^T \mathcal{O}_\alpha \cup \partial^1 \mathcal{O}_\alpha \,.$$

(iii) *If Assumption 3.2 holds, then the function u is right-continuous in the direction $\overrightarrow{e}$ on $\mathcal{O}_\alpha \cup \partial^T \mathcal{O}_\alpha \cup \partial^1 \mathcal{O}_\alpha \cup \partial^0 \mathcal{O}_\alpha$.*

(iv) *If furthermore Assumptions 3.1 and 3.3 hold, then the fonction u is right-continuous in the direction $\overrightarrow{e}$ on $\overline{\mathcal{O}}_\alpha$.*

Proof. Let $y = (t, x, z) \in \overline{\mathcal{O}}_\alpha$.
(i) Fix $\nu \in [0, 1]$ and $h, h' \geq 0$. Then $(y + h\overrightarrow{e}) \in \overline{\mathcal{O}}_\alpha$ and $(y + h'\overrightarrow{e}) \in \overline{\mathcal{O}}_\alpha$. We pick any $(C, \theta) \in \mathcal{A}_\alpha(y + h\overrightarrow{e})$, $(C', \theta') \in \mathcal{A}_\alpha(y + h'\overrightarrow{e})$, and introduce the notation $(X, X') := (X^{t,x+h,C,\theta}, X^{t,x+h',C',\theta'})$ and $(X^*, (X')^*)$ for their current maxima. We then derive

$$\begin{aligned}\nu X + (1-\nu)X' &\geq \nu\{\alpha(z+h) \vee X^*\} + (1-\nu)\{\alpha(z+h') \vee (X')^*\} \\ &\geq \{\alpha(z + \nu h + (1-\nu)h')\} \vee \{\nu X + (1-\nu)X'\}^* .\end{aligned}$$

Therefore $\nu(C, \theta) + (1-\nu)(C', \theta') \in \mathcal{A}_\alpha\left(y + \{\nu h + (1-\nu)h'\}\overrightarrow{e}\right)$ and it follows from the concavity of $J(t, .)$ inherited from U, that

$$\nu J(t, C, \theta) + (1-\nu)J(t, C', \theta') \quad \leq \quad u\left(y + \left\{\nu h + (1-\nu)h'\right\}\overrightarrow{e}\right) .$$

The arbitrariness of (C, θ, C', θ') then leads to the concavity of $h \mapsto u[y + h\overrightarrow{e}]$.
(ii) Suppose $y \in \mathcal{O}_\alpha \cup \partial^T\mathcal{O}_\alpha \cup \partial^1\mathcal{O}_\alpha$. Then, there exists $h_0 > 0$ satisfying $y - h_0\overrightarrow{e} \in \overline{\mathcal{O}}_\alpha$. Recalling from (i) that the function $h \mapsto u(y + (h - h_0)\overrightarrow{e})$ is concave on $\mathbb{R}_+$, it is also continuous on $(0, \infty)$ and we deduce that u is right continuous in the direction $\overrightarrow{e}$ at point y.
(iii) Suppose now that $y \in \partial^0\mathcal{O}_\alpha$. By Lemma 3.1, $u(y) = 0$. Under Assumption 3.2, it follows from the same arguments as in the proof of Lemma 3.1 that $u(y') \leq u^*(x')$, for any $(t', x', z') \in \overline{\mathcal{O}}_\alpha$, where u^* is the value function in the classical Merton setting (i.e. $\alpha = 0$). Thus, the required regularity result is a consequence of the continuity of u^*.
(iv) Suppose finally that $y \in \partial^\alpha\mathcal{O}_\alpha$ and Assumptions 3.1, 3.2 and 3.3 hold. Recall from [7] that the value function u_∞ in the infinite time horizon is continuous on $\{(x', z'), 0 < \alpha z' \leq x' \leq z'\}$ and satisfies $u_\infty(\alpha z', z') = 0$ for any $z' > 0$. Combining (3.15) with similar arguments as above completes the proof. □

4 Numerical Approximation

4.1 Approximation scheme

In this section, we present a numerical scheme for the resolution of the Hamilton-Jacobi-Bellman equation (3.18) applying the ideas of Barles, Daher and Romano [1]. The partial differential equation is degenerate since the variable z only appears in the definition of the domain of the equation, and we decide to consider an explicit scheme.

We fix a value z_0 of interest and consider a regular discretization grid $(z_i)_{i \leq N_z}$ with step Δz of the interval $[0, 2z_0]$. For each z_i, we decompose the interval $[\alpha z_i, z_i]$ on a grid with step $\Delta^i x$ such that the number of points N_x does not depend of i, which is always possible as soon as $\alpha \in \mathbb{Q}$. We hence obtain a discretization of the set $\{(x, z) \in [0, 2z_0]^2 : \alpha z \leq x \leq z\}$ into a product of a matrix (x^i_j) of size $N_x \times N_z$ and a vector (z_j) of size N_z. Since we deal with a Neuman condition at each point (x^i_j, z_i), we also add one row to the previous

matrix by defining $x_j^{i+1} = z_i + \Delta_x^i$, whose use is detailed below. For a given horizon T, we decompose the interval $[0,T]$ with a time step Δt of order $(\Delta x)^2$.
The algorithm is constructed the following way. From an approximation $(\hat{u}(t_n, x_j^i, z_i))_{i,j}$ of the value fonction $u(t_n, ., .)$, we compute an approximation of $u(t_{n+1}, ., .)$ by

$$\hat{u}(t_{n+1}, x_j^i, z_i) = \tilde{u}(t_{n+1}, x_j^i, z_i)\mathbf{1}_{x_{j+1}^i \le z_j} + \tilde{u}(t_{n+1}, x_j^i, x_j^i)\mathbf{1}_{x_{j+1}^i > z_j}\,,$$

where $\tilde{u}$ is defined by

$$\begin{aligned}
\tilde{u}(t_{n+1}, x_0^i, z_i) &= 0\,,\\
\tilde{u}(t_{n+1}, x_j^i, z_i) &= (1-\beta\Delta t)\hat{u}(t_n, x_j^i, z_i) + \Delta t V\left(\frac{\hat{u}(t_n, x_{j+1}^i, z_i) - \hat{u}(t_n, x_j^i, z_i)}{\Delta^i x}\right)\\
&\quad -\frac{\lambda^2\Delta t}{2}\frac{[u(t_n, x_{j+1}^i, z_i) - u(t_n, x_j^i, z_i)]^2}{u(t_n, x_{j+1}^i, z_i) + 2u(t_n, x_j^i, z_i) - u(t_n, x_{j-1}^i, z_i)}\,, \quad \text{for } j > 0\,.
\end{aligned}$$

Observe that the previous relation $\tilde{u}(t_{n+1}, x_0^i, z_i) = 0$ corresponds simply to the condition $u(., \alpha z, z) = 0$ for $z \ge 0$. As for the initialization of the algorithm, we take of course $\hat{u}(0, ., .) = 0$.

Remark 4.1 The initialization of the algorithm endues a small technical problem as the previous iteration procedure can not be applied at time $t_n = 0$. This difficulty can be overcome by considering the linear form of the Hamilton-Jacobi-Bellman equation where we observe that, imposing in this time step an upper-bound c_{max} on the possible consumption strategy leads to $\hat{u}(t_1, x_0^i, z_i) = U(c_{max}(x_0^i - z_i))$. From a numerical point of view, it gives the right shape to the value function and the influence of c_{max} is still under study.

Remark 4.2 In order to ensure the convergence of the scheme, we need to check its stability, monotonicity and consistency according to the results of Barles and Souganidis [2]. The monotonicity relies classically on the construction of the scheme along the dynamic programming principle. The properties of consistency and the stability of the scheme are more intricate since the controls are not bounded and their study is left for further research.

4.2 Numerical example

Considering a power utility function $U : x \mapsto x^p/p$ and the particular set of parameters $\{\alpha, p, \sigma, \lambda, \beta\} = \{0.5, 0.2, 1, 3, 3\}$, we present in Figure 1 the numerical approximation of the value function. As the horizon T tends to infinity, we observe a fast monotone convergence of the estimated value function to the solution in infinite horizon. In particular, we observe that, as soon as the horizon T is of the order of 2 years, the use of the explicit infinite horizon value function seems reasonable.

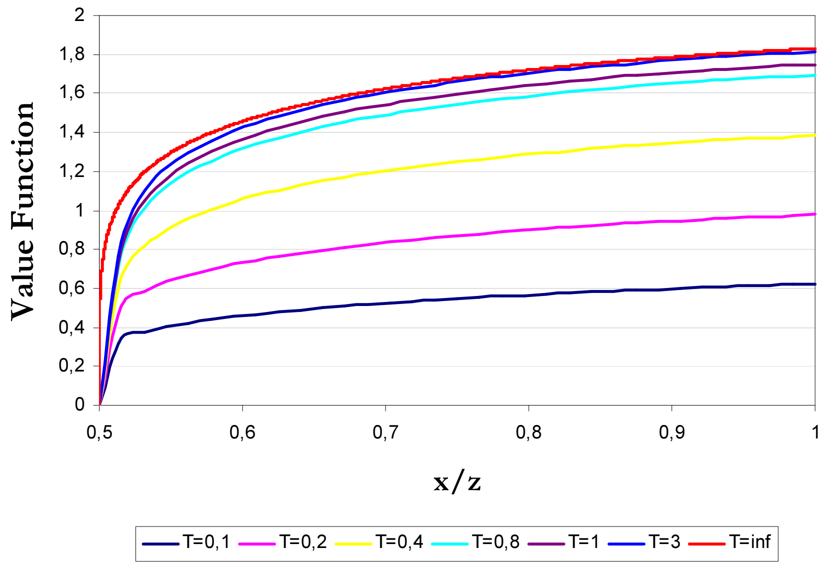

Figure 1: Value function versus the fraction of wealth x/z for different horizons T

In fact, the fast speed of convergence from u to u_∞ can be justified theoretically. Let pick (x, z) satisfying $0 \leq \alpha z \leq x \leq z$ and denote (C^*, θ^*) the optimal strategy associated to the infinite horizon problem with initial condition (x, z). This optimal strategy has been explicitly derived in [7] and, introducing (X^*, Z^*) the corresponding optimal wealth and running maximum processes, we deduce

$$0 \leq u_\infty(x, z) - u(t, x, z) \leq \mathbb{E}\left[\int_t^\infty e^{-\beta s} U(C_s^*) ds\right] = e^{-\beta t}\mathbb{E}\left[u_\infty(X_t^*, Z_t^*)\right], \quad \text{for any } t > 0.$$

Letting t going to infinity, the convergence to zero of the term on the right hand side is ensured by the transversality condition of the problem in infinite horizon. As detailed in the proof of Theorem 5.1 in [7], this term converges exponentially fast to zero. This property is indeed confirmed by our numerical results observed in Figure 1.

The PDE characterization of the value function also allows to provide a candidate for the optimal consumption-investment strategy. Indeed, the optimal controls $(\hat{C}, \hat{\theta})$ derived from the maximization in the linear form of the Hamilton-Jacobi-Bellman are given by

$$\hat{C} := -V'(u_x) \quad \text{and} \quad \hat{\theta} := -\frac{\lambda}{\sigma}\frac{u_x}{u_{xx}}.$$

Replacing the derivatives of u by their approximations as in the previous scheme leads to a computable candidate for the optimal strategy. The corresponding consumption and investment estimates are reported in Figure 2 for different maturities T. Even if there is no theoretical justification for the optimality of these strategies, we clearly observe there convergence to the optimal strategy in infinite horizon.

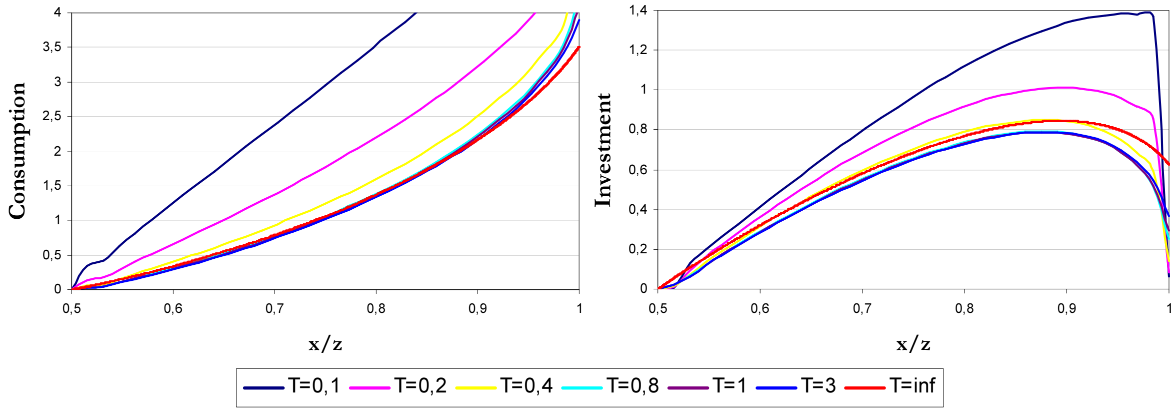

Figure 2: Consumption and Investment versus the fraction of wealth x/z for different horizons T

5 Viscosity property

This section is devoted to the proof of the following Proposition:

Proposition 5.1 *The value function u is a viscosity solution of the dynamic programming equation* (3.18).

5.1 Supersolution property

In this subsection, we prove that u is a viscosity supersolution of (3.18). We first observe from lemma 3.1 that $u_* \geq 0$ on $\partial^T \mathcal{O}_\alpha \cup \partial^0 \mathcal{O}_\alpha$. Let $y_0 := (t_0, x_0, z_0) \in \overline{\mathcal{O}}_\alpha$ and $\varphi \in C^{1,2,1}(\overline{\mathcal{O}}_\alpha)$ such that

$$0 \;=\; (u_* - \varphi)(y_0) \;=\; \inf_{\overline{\mathcal{O}}_\alpha}(u_* - \varphi)\,.$$

Without loss of generality, we can suppose that the previous infimum is indeed a strict minimum and we shall distinguish two different cases depending on the location of y_0.

1. $\underline{\mathbf{y_0} \in \mathcal{O}_\alpha}$.

Let $y_n := (t_n, x_n, z_n)_n \in \mathcal{O}_\alpha$ satisfying

$$y_n \longrightarrow y_0 \quad \text{and} \quad u(y_n) \longrightarrow u_*(y_0)\,.$$

We denote $\gamma_n := u(y_n) - \varphi(y_n) \geq 0$ and $\gamma_n^* := n^{-1} \vee \sqrt{\gamma_n}$. Since $y_0 \in \mathcal{O}_\alpha$, there exists $r > 0$ such that the open ball centered at y_0 with radius r satisfies $\mathcal{B}(y_0, r) \subset \mathcal{O}_\alpha$. We consider the constant strategy $(C, \theta) \in \mathbb{R}_+ \times \mathbb{R}$, denote $(Y^n, Z^n) := (Y^{y_n,C,\theta}, Z^{y_n,C,\theta})$ and introduce the stopping time

$$\tau_n \;:=\; \inf\left\{s \geq t_n \;:\; Y^n_s \notin \mathcal{B}(y_0, r)\right\} \wedge (t_n + \gamma_n^*)\,.$$

The dynamic programming principle implies

$$e^{-\beta t_n} u(y_n) \quad \geq \quad \mathbb{E}\left[\int_{t_n}^{\tau_n} e^{-\beta s} U(C) ds + e^{-\beta \tau_n} u\left(Y^n_{\tau_n}\right)\right].$$

Since $u \geq u_* \geq \varphi$, we deduce

$$\gamma_n + e^{\beta t_n}\mathbb{E}\left[e^{-\beta t_n}\varphi(y_n) - e^{-\beta\tau_n}\varphi\left(Y^n_{\tau_n}\right)\right] \quad \geq \quad e^{\beta t_n}\mathbb{E}\left[\int_{t_n}^{\tau_n} e^{-\beta s} U(C) ds\right].$$

Applying Itô's lemma to the regular function $e^{\beta\cdot}\varphi$, together with the previous inequality, yields

$$\begin{aligned}\gamma_n \quad \geq \quad & e^{\beta t_n}\mathbb{E}\left[\int_{t_n}^{\tau_n} e^{-\beta s}\mathcal{L}_T^{C,\theta}\varphi\left(Y^n_s\right) ds\right] \\ & + e^{\beta t_n}\mathbb{E}\left[\int_{t_n}^{\tau_n} e^{-\beta s}\varphi_z\left(Y^n_s\right) dZ^n_s + \int_{t_n}^{\tau_n} e^{-\beta s}(\sigma\lambda\theta - C)\varphi_x\left(Y^n_s\right) dW_s\right].\end{aligned}$$

Since $\varphi_x\left(Y^n\right)$ is bounded and Z^n is a constant process on the stochastic interval $[t_n, \tau_n]$, we deduce

$$\gamma_n \geq \mathbb{E}\left[\int_{t_n}^{\tau_n} e^{\beta(t_n - s)}\mathcal{L}^{C,\theta}\varphi\left(Y^n_s\right) ds\right]. \tag{5.24}$$

Dividing by γ_n^* and letting n go to infinity, since $\tau_n = t_n + \gamma_n^*$ for n large enough almost surely, the dominated convergence theorem leads to $\mathcal{L}_T^{C,\theta}\varphi(y_0) \leq 0$. From the arbitrariness of $(C,\theta) \in \mathbb{R}_+ \times \mathbb{R}$, we deduce

$$-\mathcal{L}_T\varphi(y_0) \geq 0\,.$$

2. $\underline{\mathbf{y_0} \in \partial^{\mathbf{1}}\mathcal{O}_\alpha}$.

Remark first that u_* inherits the monotony property of u derived in lemma 3.2. Thus, for any $z \geq z_0$ such that $y := (t_0, x_0, z) \in \overline{\mathcal{O}}_\alpha$, we have $\varphi(y_0) = u_*(y_0) \geq u_*(y) \geq \varphi(y)$. Since φ is a regular function, we deduce

$$-\varphi_z(y_0) \geq 0\,.$$

5.2 Subsolution property

In this subsection, we prove that u is a viscosity subsolution of (3.18). From Lemma 3.1, we have $u^* \leq 0$ on $\partial^T\mathcal{O}_\alpha \cup \partial^0\mathcal{O}_\alpha$. Let $y_0 := (t_0, x_0, z_0) \in \overline{\mathcal{O}}_\alpha$ and $\varphi \in C^{1,2,1}(\overline{\mathcal{O}}_\alpha)$ such that

$$0 \;=\; (u^* - \varphi)(y_0) \;=\; \sup_{\overline{\mathcal{O}}_\alpha}(u^* - \varphi)\,. \tag{5.25}$$

Once again, without loss of generality, we can suppose that the previous supremum is indeed a strict maximum, and we shall distinguish two different cases depending on the location of the maximum y_0.

1. $\underline{\mathbf{y_0} \in \mathcal{O}_\alpha \cup \partial^\alpha\mathcal{O}_\alpha}$.

Let introduce the function $m := -\mathcal{L}_T\varphi$, suppose that $m(y_0) > 0$ and work towards a contradiction. From (5.25) and the regularity of u^* and φ, we deduce the existence of $r > 0$ and $\eta > 0$ such that $\mathcal{B}(y_0, r) \cap \partial^T \mathcal{O}_\alpha = \mathcal{B}(y_0, r) \cap \partial^0 \mathcal{O}_\alpha = \emptyset$, and

$$\min_{\overline{\mathcal{B}}(y_0,r)\cap\overline{\mathcal{O}}_\alpha} m \ > \ 0 \quad \text{and} \quad \max_{\partial\mathcal{B}(y_0,r)\cap\overline{\mathcal{O}}_\alpha} (u^* - \varphi) \ < \ -3\eta\,. \tag{5.26}$$

Denote $\eta_r := \eta e^{-\beta r} > 0$ and take $(y_n)_n$ a sequence valued in $\mathcal{B}(y_0, r) \cap \mathcal{O}_\alpha$ satisfying

$$y_n \longrightarrow y_0\,, \quad u(y_n) \longrightarrow u^*(y_0) \quad \text{and} \quad |u(y_n) - \varphi(y_n)| \ \le \ \eta_r\,, \quad n \ge 0. \tag{5.27}$$

For any $n \ge 0$, let (C^n, θ^n) be an η_r-optimal control at point y_n and introduce the notation $(Z^n, Y^n) := (Z^{y_n,C^n,\theta^n}, Y^{y_n,C^n,\theta^n})$. We introduce the stopping time τ_n defined by

$$\tau_n \ := \ \inf\{s \ge t_n\,, Y^n_s \notin \mathcal{B}(y_0, r)\}\,.$$

By construction, Y^n is valued in $\overline{\mathcal{O}}_\alpha$, $\tau_n - t_n \le r$ and the η_r-optimality of (C^n, θ^n) leads to

$$u(y_n) \ \le \ e^{\beta t_n}\,\mathbb{E}\left[\int_{t_n}^{\tau_n} e^{-\beta s}U(C^n_s)ds + e^{-\beta\tau_n}u\left(Y^n_{\tau_n}\right)\right] + \eta_r\,. \tag{5.28}$$

Applying Ito's lemma to the regular function $e^{-\beta\cdot}\varphi$, we compute

$$\begin{aligned} e^{-\beta t_n}\varphi(y_n) \ &= \ \mathbb{E}[e^{-\beta\tau_n}\varphi(Y^n_{\tau_n})] - \mathbb{E}\left[\int_{t_n}^{\tau_n} e^{-\beta s}\left(\mathcal{L}_T^{C^n,\theta^n}\varphi\left(Y^n_s\right) - U(C^n_s)\right)ds\right] \\ &\quad - \mathbb{E}\left[\int_{t_n}^{\tau_n} e^{-\beta s}\varphi_z\left(Y^n_s\right)dZ^n_s\right]\,. \end{aligned}$$

Combining (5.27) with the negativity of $\mathcal{L}_T^{C^n,\theta^n}$ on $\overline{\mathcal{B}}(y_0, r) \cap \overline{\mathcal{O}}_\alpha$, we deduce

$$u(y_n) \ge -\eta_r + e^{\beta t_n}\mathbb{E}\left[e^{-\beta\tau_n}\varphi(Y^n_{\tau_n}) + \int_{t_n}^{\tau_n} e^{-\beta s}U(C^n_s)ds - \int_{t_n}^{\tau_n}\varphi_z\left(Y^n_s\right)dZ^n_s\right]\,. \tag{5.29}$$

Noticing that $Y^n_{\tau_n} \in \partial\mathcal{B}(y_0, r) \cap \overline{\mathcal{O}}_\alpha$ and combining (5.26) and $\tau_n - t_n \le r$, we derive

$$e^{\beta t_n}\mathbb{E}\left[e^{-\beta\tau_n}\left(\varphi\left(Y^n_s\right) - u^*\left(Y^n_s\right)\right)\right] \ \ge \ 3\eta_r\,. \tag{5.30}$$

We now compute from (5.28), (5.29), (5.30) and $u \le u^*$, that

$$\eta_r \ \le \ \mathbb{E}\left[\int_{t_n}^{\tau_n}\varphi_z\left(Y^n_s\right)dZ^n_s\right]\,. \tag{5.31}$$

Since $y_0 \in \mathcal{O}_\alpha \cup \partial^\alpha\mathcal{O}_\alpha$, we have $\mathcal{B}(y_0, r) \cap \partial^1\mathcal{O}_\alpha = \emptyset$ for r small enough. Thus Z^n is a constant process on the random interval $[t_n, \tau_n]$ and (5.31) leads to a contradiction. We therefore deduce

$$-\mathcal{L}_T\varphi(y_0) \ \le \ 0\,.$$

2. $\underline{\mathbf{y_0} \in \partial^1\mathcal{O}_\alpha}$.

Take $m := \min\{-\mathcal{L}_T\varphi, -\varphi_z\}$ and follow the lines of the proof in the previous case. This leads to (5.31) and, since $-\varphi_z(Y^n) \ge m(Y^n) > 0$ on the random interval $[t_n, \tau_n]$ according to (5.26) , we obtain a contradiction. Therefore

$$\min\{-\mathcal{L}_T\varphi, -\varphi_z\}(y_0) \ \le \ 0\,.$$

6 A comparison result

This section is devoted to the proof of a comparison result for the PDE (3.18) which ensures the uniqueness of the solution. The difficulty of the proof relies on the fact that the controls are not in a compact subset. To overcome this difficulty, we adapted the arguments of Zariphopoulou [15], in particular for the choice of the penalization function. As announced, a different version of the comparison theorem is discussed in Remark 6.1.

Theorem 6.1 *Let w and v be respectively an upper-continuous sub-solution and a lower-semicontinuous super-solution of* (3.18) *on $\overline{\mathcal{O}}_\alpha$. Suppose that the function v is right-continuous in the direction $\overrightarrow{e} = (0,1,1)$ on $\mathcal{O}_\alpha \cup \partial^1\mathcal{O}_\alpha \cup \partial^\alpha\mathcal{O}_\alpha$ and that the positive part of w and the negative part of v satisfy the following growing condition*

$$[w]^+(y) + [v]^-(y) \quad \leq \quad K(1 + x^{p'})\,, \quad y = (t,x,z) \in \overline{\mathcal{O}}_\alpha\,, \quad \text{with } p' < \frac{\gamma}{1+\gamma}\,, \tag{6.32}$$

and K a positive constant. Then, if $w \leq v$ on $\partial^0\mathcal{O}_\alpha \cup \partial^T\mathcal{O}_\alpha$, we have $w \leq v$ on $\overline{\mathcal{O}}_\alpha$.

Proof. We do not consider the case $\alpha = 0$, already covered by the literature, see Zariphopoulou [15] for example. As a consequence, observe for later use that, for any $y = (t,x,z) \in \overline{\mathcal{O}}_\alpha$, we only need to control x in order to bound y, since $\alpha z \leq x \leq z$.
We now suppose that

$$\sup_{y \in \overline{\mathcal{O}}_\alpha} [w(y) - v(y)] \quad > \quad 0 \tag{6.33}$$

and work towards a contradiction. For any $y \in \overline{\mathcal{O}}_\alpha$, we denote by (t,x,z) its components, and this convention of notation is obviously extended to elements of $\overline{\mathcal{O}}_\alpha$ of the form y_i^j with i and j any subcripts and superscripts.

1. We define the function ϕ by

$$\phi(y,y') \quad := \quad w(y) - v(y') - \delta\left(x^q + (x')^q + e^{-z} + e^{-z'}\right)\,, \quad (y,y') \in \overline{\mathcal{O}}_\alpha \times \overline{\mathcal{O}}_\alpha\,,$$

with $\delta > 0$ and $q := \gamma/(1+\gamma) < 1$. Choosing δ small enough and combining the growth condition (6.32), (6.33) and the semi-continuity properties of w and v, we deduce that the function $y \mapsto \phi(y,y)$ attains its suppremum on $\overline{\mathcal{O}}_\alpha$ and we have

$$\phi(\bar{y},\bar{y}) \;:=\; \sup_{y \in \overline{\mathcal{O}}_\alpha} \phi(y,y) \quad > \quad 0\,. \tag{6.34}$$

Since $w \leq v$ on $\partial^0\mathcal{O}_\alpha \cup \partial^T\mathcal{O}_\alpha$, (6.34) leads to $\bar{y} \in \mathcal{O}_\alpha \cup \partial^1\mathcal{O}_\alpha \cup \partial^\alpha\mathcal{O}_\alpha$. Therefore, the right-continuity of v in the direction $\overrightarrow{e}$ and the semi-continuity of w ensures that

$$\phi(\bar{y}, \bar{y} + \overrightarrow{e}/n) \underset{n\to\infty}{\longrightarrow} \phi(\bar{y},\bar{y}) \;>\; 0\,. \tag{6.35}$$

2. For any $n \geq 0$, we now define the function

$$\psi^n(y,y') \;:=\; \big[n([x-\alpha z]-[x'-\alpha z'])+1-\alpha\big]^2+\alpha(1-\alpha)\big[n(z-z')+1\big]^2 ,$$

for $(y,y') \in \overline{\mathcal{O}}_\alpha \times \overline{\mathcal{O}}_\alpha$. Since $\psi^n(\bar y, \bar y + \overrightarrow{e}/n) = 0$, we deduce from (6.35) that

$$\{\phi - \psi^n\}(\bar y, \bar y + \overrightarrow{e}/n) \;>\; 0 , \tag{6.36}$$

for n large enough. Therefore, according to (6.32), the function $\phi - \psi^n$ attains its maximum on $\overline{\mathcal{O}}_\alpha \times \overline{\mathcal{O}}_\alpha$ and we have

$$\{\phi - \psi^n\}(y_n, y_n') \;:=\; \sup_{(y,y')\in\overline{\mathcal{O}}_\alpha\times\overline{\mathcal{O}}_\alpha} \{\phi-\psi^n\}(y,y') \;>\; 0 . \tag{6.37}$$

The growing assumption (6.32) ensures the convergence along subsequences of $(y_n)_n$ and $(y_n')_n$ and, sending n to ∞, we see that $\psi^n(y_n, y_n') \to \infty$ unless $|y_n - y_n'| \to 0$. But $\phi(y_n,y_n') - \psi^n(y_n,y_n')$ is bounded from above according to (6.32) and therefore $|y_n - y_n'| \to 0$ as n goes to ∞. Denoting y_0 the common limit of $(y_n)_n$ and $(y_n')_n$, since $\{\phi-\psi^n\}(y_n,y_n') \geq \phi(\bar y, \bar y + \overrightarrow{e}/n)$, we deduce from (6.35) and the semi-properties of w and v that

$$\phi(y_0,y_0) \;\geq\; \limsup_{n\to\infty} \{\phi-\psi^n\}(y_n,y_n') \;\geq\; \phi(\bar y,\bar y) .$$

Recalling (6.34), we derive

$$\phi(y_0,y_0) > 0 \qquad \text{and} \qquad \psi^n(y_n,y_n') \underset{n\to\infty}{\longrightarrow} 0 . \tag{6.38}$$

3. We now discuss the location of (y_n, y_n') and some properties of the global penalization function given by

$$\Phi^n(y,y') \;:=\; \delta(x^q + (x')^q + e^{-z} + e^{-z'}) + \psi^n(y,y') , \quad (y,y') \in \overline{\mathcal{O}}_\alpha \times \overline{\mathcal{O}}_\alpha .$$

Since $w \leq v$ on $\partial^0\mathcal{O}_\alpha \cup \partial^T\mathcal{O}_\alpha$, we derive from (6.38) that $y_0 \in \mathcal{O}_\alpha \cup \; \partial^1\mathcal{O}_\alpha \cup \partial^\alpha\mathcal{O}_\alpha$. Furthermore, for n large enough, (6.38) implies that $x_n' - \alpha z_n' > x_n - \alpha z_n$, and we deduce that

$$y_n \in \mathcal{O}_\alpha \cup \partial^1\mathcal{O}_\alpha \cup \partial^\alpha\mathcal{O}_\alpha \quad \text{and} \quad y_n' \in \mathcal{O}_\alpha \cup \partial^1\mathcal{O}_\alpha . \tag{6.39}$$

In particular, since $x_n \neq 0$, Φ^n is regular on a neighborhood of (y_n, y_n') and we denote $D_{x,z}\Phi^n$ (resp. $D_{x',z'}\Phi^n$) its gradient with respect to (x,z) (resp. (x',z')) and $H\Phi^n$ its Hessian matrix with respect to the space variables (x,z,x',z'). Observe for later use that

$$\Phi^n_z(y_n,y_n') \;=\; -\alpha n^2(z_n' - x_n') - \delta e^{-z_n} \;<\; 0 , \qquad \text{if } y_n \in \partial^1\mathcal{O}_\alpha , \tag{6.40}$$

$$\Phi^n_{z'}(y_n,y_n') \;=\; -\alpha n^2(z_n - x_n) - \delta e^{-z_n'} \;<\; 0 , \qquad \text{if } y_n' \in \partial^1\mathcal{O}_\alpha , \tag{6.41}$$

and

$$\Phi^n_x(y_n,y_n') + \Phi^n_{x'}(y_n,y_n') \;=\; \delta q(x_n^{q-1} + (x_n')^{q-1}) \;\geq\; 0 . \tag{6.42}$$

4. For any $\epsilon > 0$, we deduce from Theorem 8.3 in [4] the existence of $b \in \mathbb{R}$ and two real symmetric matrices Λ and Λ' such that

$$\begin{aligned} &(b, D_{x,z}\Phi^n(y_n, y'_n), \Lambda) \in \overline{\mathcal{P}}^{2,+}_{\mathcal{O}_\alpha} w(y_n)\,, \\ &\big(b, -D_{x',z'}\Phi^n(y_n, y'_n), \Lambda'\big) \in \overline{\mathcal{P}}^{2,-}_{\mathcal{O}_\alpha} v(y'_n)\,, \end{aligned} \tag{6.43}$$

and

$$A \;:=\; \begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda' \end{pmatrix} - H\Phi^n(y_n, y'_n) + \epsilon\{H\Phi^n(y_n, y'_n)\}^2 \;\le\; 0\,, \tag{6.44}$$

where $\overline{\mathcal{P}}^{2,+}_{\mathcal{O}_\alpha}$ and $\overline{\mathcal{P}}^{2,-}_{\mathcal{O}_\alpha}$ denotes classically the superjet and subjet operators, see [4] for the precise definition. We compute that $H\Phi^n(y_n, y'_n)$ is explicitely given by

$$H\Phi^n(y_n, y'_n) \;=\; n^2 \begin{pmatrix} 1 & -\alpha & -1 & \alpha \\ -\alpha & \alpha & \alpha & -\alpha \\ -1 & \alpha & 1 & -\alpha \\ \alpha & -\alpha & -\alpha & \alpha \end{pmatrix} - \delta q(1-q) \begin{pmatrix} x_n^{q-2} & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & (x'_n)^{q-2} & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}.$$

Take $X := (1, 0, 1, 0)$ and observe that (6.44) implies $XAX^T \le 0$, which leads to

$$\Lambda_{1,1} - \Lambda'_{1,1} \;\le\; -\delta q(1-q)[x_n^{q-2} + (x'_n)^{q-2}] + \epsilon[q(1-q)(x_n^{q-2} + (x'_n)^{q-2})]^2 \;<\; 0\,, \tag{6.45}$$

for ϵ sufficiently small.

5. According to (6.39), (6.40) and (6.41), it follows from (6.43) and the viscosity properties of w and v that

$$\beta w(y_n) \;\le\; b + V\left[\Phi^n_x(y_n, y'_n)\right] + \sup_{\theta \in \mathbb{R}} \left\{ \sigma\lambda\theta\Phi^n_x(y_n, y'_n) + \frac{(\sigma\theta)^2}{2}\Lambda_{1,1} \right\},$$

and

$$\beta v(y'_n) \;\ge\; b + V\left[-\Phi^n_{x'}(y_n, y'_n)\right] + \sup_{\theta \in \mathbb{R}} \left\{ -\sigma\lambda\theta\Phi^n_{x'}(y_n, y'_n) + \frac{(\sigma\theta)^2}{2}\Lambda'_{1,1} \right\},$$

where V denotes the Fenchel transform of U. Combining these inequalities with the decreasing property of V and (6.42), we deduce

$$\begin{aligned} \beta\{w(y_n) - v(y'_n)\} \;&\le\; \sup_{\theta \in \mathbb{R}} \left\{ \sigma\lambda\theta\Phi^n_x(y_n, y'_n) + \frac{(\sigma\theta)^2}{2}\Lambda_{1,1} \right\} \\ &\quad - \sup_{\theta \in \mathbb{R}} \left\{ -\sigma\lambda\theta\Phi^n_{x'}(y_n, y'_n) + \frac{(\sigma\theta)^2}{2}\Lambda'_{1,1} \right\} \\ &\le\; \sup_{\theta \in \mathbb{R}} \left\{ \sigma\lambda\theta\left[\Phi^n_x + \Phi^n_{x'}\right](y_n, y'_n) + \frac{(\sigma\theta)^2}{2}(\Lambda_{1,1} - \Lambda'_{1,1}) \right\}. \end{aligned}$$

According to (6.42) and (6.45), we then deduce

$$\beta\{w(y_n) - v(y'_n)\} \;\le\; \frac{\lambda^2}{2} \frac{[\delta q(x_n^q + (x'_n)^q)]^2}{\delta q(1-q)(x_n^{q-2} + (x'_n)^{q-2}) - \epsilon[q(1-q)(x_n^{q-2} + (x'_n)^{q-2})]^2}.$$

Since this inequality holds true for any $\epsilon > 0$, it follows that

$$w(y_n) - v(y'_n) \;\; \leq \;\; \frac{\delta q (x_n^{q-1} + (x'_n)^{q-1})^2}{\gamma(1-q)(x_n^{q-2} + (x'_n)^{q-2})} \,.$$

Letting n go to infinity, we finally obtain

$$\phi(y_0, y_0) \;\leq\; w(y_0) - v(y_0) - 2\delta x_0^q \;\;\leq\;\; \left(\frac{q}{\gamma(1-q)} - 1 \right) 2\delta x_0^q \,.$$

Since $q = \gamma/(1+\gamma)$, we deduce $\phi(y_0, y_0) \leq 0$ and therefore contradict (6.38). □

Remark 6.1 The results of Theorem 6.1 hold true if we suppose that v is right-continuous in the direction $\overrightarrow{e}$ on $\mathcal{O}_\alpha \cup \partial^1 \mathcal{O}_\alpha$ instead of $\mathcal{O}_\alpha \cup \partial^1 \mathcal{O}_\alpha \cup \partial^\alpha \mathcal{O}_\alpha$, but that $w \leq v$ on $\partial^0 \mathcal{O}_\alpha \cup \partial^T \mathcal{O}_\alpha \cup \partial^\alpha \mathcal{O}_\alpha$ instead of $\partial^0 \mathcal{O}_\alpha \cup \partial^T \mathcal{O}_\alpha$. The only modification of the previous proof relies on the obtention of (6.35), which remains valid since $\bar{y} \in \mathcal{O}_\alpha \cup \partial^1 \mathcal{O}_\alpha$. Denoting furthermore that the decreasing property of V, used in part **5.** of the previous proof, relies only on the monotonicity of U, (iii) of Lemma 3.3 leads to Proposition 3.1.

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