Discrete-time Approximation of Multidimensional BSDEs with oblique reflections

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Abstract

In this paper, we study the discrete-time approximation of multi-dimensional reflected BSDEs of the type of those presented by Hu and Tang [16] and generalized by Hamadène and Zhang [15]. In comparison to the penalizing approach followed by Hamadène and Jeanblanc [14] or Elie and Kharroubi [12], we study a more natural scheme based on oblique projections. We provide a control on the error of the algorithm by introducing and studying the notion of multidimensional discretely reflected BSDE. In the particular case where the driver does not depend on the variable Z, the error on the grid points is of order $\frac{1}{2} - \epsilon$, $\epsilon > 0$.

Key words: BSDE with oblique reflections, discrete time approximation, switching problems.

MSC Classification (2000): 93E20, 65C99, 60H30.

1 Introduction

The main motivation of this paper is the discrete-time approximation of the following system of reflected Backward Stochastic Differential Equations (BSDEs)

$$\begin{cases} Y_t^i = g^i(X_T) + \int_t^T f^i(X_s, Y_s^i, Z_s^i)\mathrm{d}s - \int_t^T Z_s^i \mathrm{d}W_s + K_T^i - K_t^i, \quad 0 \le t \le T \;, \\ Y_t^i \ge \max_{j\in\mathcal{I}}\{Y_t^j - c^{ij}(X_t)\}, \quad 0 \le t \le T \;, \\ \int_0^T [Y_t^i - \max_{j\in\mathcal{I}}\{Y_t^j - c^{ij}(X_t)\}]dK_t^i = 0 \;, \quad i \in \mathcal{I} \;, \end{cases} \tag{1.1}$$

where $\mathcal{I} := \{1, \ldots, d\}$, f, g and $(c^{ij})_{i,j\in\mathcal{I}}$ are Lipschitz functions and X is the solution of a forward Stochastic Differential Equation (SDE).

These equations are linked to the solutions of *optimal switching problems*, arising for example in real option pricing. In the particular case where f does not depend on (Y, Z), a first study of these equations was made by Hamadène and Jeanblanc [14]. They derive existence and uniqueness of solution to this problem in dimension 2. The extension of this result to optimal switching problems in higher dimension is studied by Djehiche, Hamadène and Popier [9], Carmona and Ludkovski [5], Porchet, Touzi and Warin [24] or Pham, Ly Vath and Zhou [23] for an infinite time horizon consideration. In this last paper, the resolution of optimal switching problems relies mostly on their link with systems of variational inequalities.

Considering deterministic costs, Hu and Tang [16] derive existence and uniqueness of solution to this type of BSDE and relate it to optimal switching problems between one dimensional BSDEs. Extensions developed in [15] and [8] cover in particular the existence of a unique solution to the BSDE (1.1). Recently two of the authors related in [11] the solution of (1.1) to corresponding constrained BSDEs with jumps. As presented in [12], this type of BSDE can be numerically approximated combining a penalization procedure with the use of the backward scheme for BSDEs with jumps. Unfortunately, no convergence rate is available for this algorithm. We present here a more natural discretization scheme based on a geometric approach. For any $t \le T$, all the components of the Y_t process are interconnected, so that the vector Y_t lies in a random closed convex set $\mathcal{Q}(X_t)$ characterized by the cost functions $(c^{ij})_{i,j\in\mathcal{I}}$. The vector process Y is thus obliquely reflected on the boundaries of the domain $\mathcal{Q}(X)$ and we approximate these continuous reflections numerically.

As in [19, 1, 6], we first introduce a discretely reflected version of (1.1), where the reflection occurs only on a deterministic grid $\Re = \{r_0 := 0, \ldots, r_\kappa := T\}$:

$Y_T^{\Re} = \widetilde{Y}_T^{\Re} := g(X_T) \in \mathcal{Q}(X_T)$, and, for $j \leq \kappa - 1$ and $t \in [r_j, r_{j+1})$,

$$\left\{ \begin{array}{rcl} \widetilde{Y}_t^{\Re} & = & Y_{r_{j+1}}^{\Re} + \int_t^{r_{j+1}} f(X_u, \widetilde{Y}_u^{\Re}, Z_u^{\Re}) \mathrm{d}u - \int_t^{r_{j+1}} Z_u^{\Re} \mathrm{d}W_u \ , \\ Y_t^{\Re} & = & \widetilde{Y}_t^{\Re} \mathbf{1}_{\{t \notin \Re\}} + \mathcal{P}(X_t, \widetilde{Y}_t^{\Re}) \mathbf{1}_{\{t \in \Re\}}, \end{array} \right. \tag{1.2}$$

where $\mathcal{P}(X_t, .)$ is the oblique projection operator on $\mathcal{Q}(X_t)$, for $t \leq T$. Extending the approach of Hu and Tang [16], we observe that the solution to (1.2) interprets as the value process of a one-dimensional optimal BSDE switching problem with switching times belonging to $\Re$. This allows us to prove a key stability result for this equation. We control the distance between $(Y^{\Re}, Z^{\Re})$ and (Y, Z) in terms of the mesh of the reflection grid. Due to the obliqueness of the reflections, the direct argumentation of [1, 6] does not apply. Using the reinterpretation in terms of switching BSDEs, we first prove that $Y^{\Re}$ approaches Y on the grid points with a convergence rate of order $\frac{1}{2} - \varepsilon$, $\varepsilon > 0$ uniformly in $\Re$, whenever the cost function is Lipschitz and f is bounded in z, see Theorem 5.2. Imposing more regularity on the cost function, we control the convergence rate of $(Y_t^{\Re}, Z_t^{\Re})_{0 \leq t \leq T}$ to $(Y_t, Z_t)_{0 \leq t \leq T}$, see Theorem 5.3.

We then consider an Euler type approximation scheme associated to the BSDE (1.2) defined on $\pi = \{t_0, \ldots, t_n\}$ by $Y_T^{\Re,\pi} := g(X_T^{\pi})$ and, for $i \in \{n-1, \ldots, 0\}$,

$$\left\{ \begin{array}{rcl} \bar{Z}_{t_i}^{\Re,\pi} & := & (t_{i+1} - t_i)^{-1} \mathbb{E}\left[Y_{t_{i+1}}^{\Re,\pi} (W_{t_{i+1}} - W_{t_i})' \mid \mathcal{F}_{t_i} \right] \ , \\ \widetilde{Y}_{t_i}^{\Re,\pi} & := & \mathbb{E}\left[Y_{t_{i+1}}^{\Re,\pi} \mid \mathcal{F}_{t_i} \right] + (t_{i+1} - t_i) f(X_{t_i}^{\pi}, \widetilde{Y}_{t_i}^{\Re,\pi}, \bar{Z}_{t_i}^{\Re,\pi}) \ , \\ Y_{t_i}^{\Re,\pi} & := & \widetilde{Y}_{t_i}^{\Re,\pi} \mathbf{1}_{\{t_i \notin \Re\}} + \mathcal{P}(X_{t_i}^{\pi}, \widetilde{Y}_{t_i}^{\Re,\pi}) \mathbf{1}_{\{t_i \in \Re\}} \ , \end{array} \right. \tag{1.3}$$

where X^{π} is the Euler scheme associated to X. It is now well known, see e.g. [2, 25], that the convergence rate of the scheme (1.3) to the solution of (1.2) is controled by the regularity of (Y, Z) through the quantities

$$\mathbb{E}\left[\sum_{i<n} \int_{t_i}^{t_{i+1}} |Y_t^{\Re} - Y_{t_i}^{\Re}|^2 \mathrm{d}t \right] \quad \text{and} \quad \mathbb{E}\left[\sum_{i<n} \int_{t_i}^{t_{i+1}} |Z_t^{\Re} - \bar{Z}_{t_i}^{\Re}|^2 \mathrm{d}t \right] \ ,$$

with $\bar{Z}_{t_i}^{\Re} = \frac{1}{t_{i+1} - t_i} \mathbb{E}\left[\int_{t_i}^{t_{i+1}} Z_t^{\Re} \mathrm{d}t \mid \mathcal{F}_{t_i} \right]$, for $i \leq n$.

Using classical Malliavin differentiation tools, we prove a representation for $Z^{\Re}$, extending the results of [1, 6] to the system of discretely reflected BSDEs (1.2). We deduce the expected regularity results on $(Y^{\Re}, Z^{\Re})$ and, using the technics of [7], Chapter 3, we obtain in a very general setting the convergence of (1.3) to (1.2). However, due to the obliqueness of the reflections, the projection operator $\mathcal{P}(X, .)$ is only $L_{\mathcal{P}}$-lipschitz with $L_{\mathcal{P}} := \sqrt{d} > 1$, leading to a convergence rate controled by $|L_{\mathcal{P}}|^{\kappa}(|\pi|^{1/4} + \kappa^{\frac{1}{2}}|\pi|^{\frac{1}{2}})$, where we recall that κ is the number of points in the reflection grid $\Re$. The term $|L_{\mathcal{P}}|^\kappa$ can be very large even for small κ and leads to a poor logarithmic convergence rate when passing to the limit $\kappa \to \infty$ for the approximation of (1.1). In the particular case where f does not depend on z, we are able to get rid of the $|L_{\mathcal{P}}|^\kappa$ term.

Our innovative approach relies on the use of comparison results to get a control of the involved quantities:

- we interpret the solution of (1.2) as a value process of an optimization problem, which allows to get a control of the distance between the continuously and discretely reflected BSDEs,
- we introduce a convenient auxiliary process dominating both solutions (1.2) and (1.3), to get a control of the distance between these quantities.

Combining the previous estimates, we deduce the convergence of the discrete time scheme (1.3) to the solution of (1.1) with a convergence rate of order $\frac{1}{2} - \varepsilon$, $\varepsilon > 0$, on the grid points, whenever $\Re = \pi$ and f is independent of Z. Whenever the cost functions are constant, all the previous estimates hold true with $\varepsilon = 0$. We want to emphasize that all these results are obtained without any assumption on the non-degeneracy of the volatility matrix σ.

The rest of the paper is organized as follows. In Section 2, we introduce the notion of discretely obliquely reflected BSDEs, connect it with optimal switching problems and give the fundamental stability result. Section 3 focuses on the regularity of the solution to this new type of BSDE. This analysis leads to precious estimates allowing to deduce the convergence of the associated discrete time scheme, see Section 4. Afterwards, Section 5 focuses on the extension to the continuously reflected case and provides a convergence rate of the discretely reflected BSDE to the continuously one, whenever the driver f is bounded in the variable Z. The global error of the scheme is provided at the end of this section. Some *a priori* estimates are reported in the Appendix.

Notations. Throughout this paper we are given a finite time horizon T and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a d-dimensional standard Brownian motion $W = (W_t)_{t\geq 0}$. The filtration $\mathbb{F} = (\mathcal{F}_t)_{t\leq T}$ generated by the Brownian motion is supposed to satisfy the usual conditions. Here, $\mathfrak{P}$ denotes the $\sigma-$algebra on $[0,T] \times \Omega$ generated by $\mathbb{F}-$progressively measurable processes. Any element $x \in \mathbb{R}^\ell$ with $\ell \in \mathbb{N}$ will be identified to a column vector with i-th component x^i and Euclidian norm $|x|$. For $x, y \in \mathbb{R}^\ell$, $x.y$

denotes the scalar product of x and y, and x' denotes the transpose of x. We denote by $\succeq$ the component by component partial ordering relation on vectors. $\mathcal{M}^{m,d}$ denotes the set of real matrices with m lines and d columns. We denotes by C_b^k the set of functions from $\mathbb{R}^d$ to $\mathbb{R}$ with continuous and bounded derivatives up to order k. For a function $f \in C^1$, $\nabla_x f$ denotes the Jacobian matrix of f with respect to x. Finally, for ease of notation, we will sometimes write $\mathbb{E}_t[\cdot]$ instead of $\mathbb{E}[\cdot|\mathcal{F}_t]$, $t \in [0,T]$. In the following, we shall use these notations without specifying the dimension nor the dependence in $\omega \in \Omega$ when it is clearly given by the context. Finally, for any $p \geq 1$, we introduce:
• the set $\mathcal{S}^p$ of real-valued *càdlàg*[1] $\mathfrak{P}$-measurable processes $Y = (Y_t)_{0\leq t\leq T}$ satisfying $\|Y\|_{\mathcal{S}^p} := \mathbb{E}\big[\sup_{0\leq t\leq T}|Y_t|^p\big]^{\frac{1}{p}} < \infty$.
• the set $\mathcal{H}^p$ of $\mathbb{R}^d$-valued $\mathfrak{P}$-measurable processes $Z = (Z_t)_{0\leq t\leq T}$ such that $\|Z\|_{\mathcal{H}^p} := \mathbb{E}\Big[\big(\int_0^T |Z_t|^2 dt\big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} < \infty$.
• the closed subset $\mathbf{A}^p$ of $\mathcal{S}^p$ consisting of nondecreasing processes K satisfying $K_0 = 0$.
In the sequel we denote by C_L a constant whose value may change from line to line but which depends only on L. We use the notation C_L^p whenever it depends on some other parameter $p > 0$.

2 Discretely obliquely reflected BSDE

In the beginning of this section, we define and study discretely obliquely reflected BSDEs in a general setting. In particular, we show how their solutions relate to the solutions of one-dimensional optimal switching problems, where the switching times are restricted to lie in a discrete time set. This allows to prove a stability result for obliquely RBSDEs, which will be use several times in the paper.

2.1 Definition

A discretely obliquely reflected BSDE is a reflected BSDE where the reflection is only allowed on a discrete time set.
We thus consider a grid $\Re := \{r_0 = 0, \ldots, r_\kappa = T\}$ of the time interval $[0,T]$ satisfying

$$|\Re| := \max_{1\leq k\leq \kappa} |r_k - r_{k-1}| \leq \frac{L}{\kappa}. \tag{2.1}$$

We also consider a matrix valued process $C = (C^{ij})_{1\leq i,j\leq m}$ such that C^{ij} belongs to $\mathcal{S}^2$

[1]French acronym meaning right continuous with left limit.

for $i,j\in\{1,\dots,d\}$ and satisfies the following structure condition

$$\begin{cases} C_t^{ii}=0\,, & \text{for } 1\le i\le d \ \text{ and } 0\le t\le T\,;\\ \inf_{0\le t\le T} C_t^{ij} > \frac{1}{L}\,, & \text{for } 1\le i,j\le d\,,\ \text{with } i\neq j\,;\\ \inf_{0\le t\le T} C_t^{ij}+C_t^{jl}-C_t^{il}>0, & \text{for } 1\le i,j,l\le d,\ \text{with } i\neq j,\ j\neq l. \end{cases} \tag{2.2}$$

We introduce a random closed convex set family associated to C:

$$\mathcal{Q}_t := \left\{ y\in\mathbb{R}^d \mid y^i \ge \max_j (y^j - C_t^{ij})\,, \quad 1\le i\le d \right\}\,, \quad 0\le t\le T\,,$$

and the oblique projection operator onto $\mathcal{Q}_t$, denoted $\mathcal{P}_t$ and defined by

$$\mathcal{P}_t : y\in\mathbb{R}^d \ \mapsto\ \left(\max_{j\in\mathcal{I}} \left\{ y^j - C_t^{ij} \right\} \right)_{1\le i\le d}\,.$$

which is $\mathfrak{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable.

Remark 2.1. (i) It follows from the structure condition (2.2) that $\mathcal{P}$ is increasing with respect to the partial ordering relation $\succeq$, where $y\succeq y'$ means $y^i\ge (y')^i$ for all $i\in\mathcal{I}$.
(ii) An easy calculation leads to

$$|\mathcal{P}_t(y_1)-\mathcal{P}_t(y_2)| \le \sqrt{d}\,|y_1-y_2|\,, \qquad \text{for any } y_1,y_2\in\mathbb{R}^d\,.$$

We observe that the constant $\sqrt{d}$ is optimal in our setting taking for example $y_1 := (\max_{i,j} C_t^{ij}, 0,\dots,0)$ and $y_2 := (\max_{i,j} C_t^{ij}+1, 0,\dots,0)$. Thus $\mathcal{P}_t$ is $L_{\mathcal{P}}$- Lipschitz continuous with $L_{\mathcal{P}} := \sqrt{d}$.

Finally, we are also given a random variable $\xi\in[L^2(\mathcal{F}_T)]^d$ valued in $\mathcal{Q}_T$, representing the terminal value of the BSDE and a random function $F:\Omega\times[0,T]\times\mathbb{R}^d\times\mathcal{M}^{d,q}\to\mathbb{R}^d$ which is $\mathfrak{P}\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathcal{M}^{d,q})-$measurable and satisfies the Lipschitz property:

$$|F(t,y,z)-F(t,y',z')| \le L(|y-y'|+|z-z'|),$$

for all $(t,y,y',z,z')\in[0,T]\times(\mathbb{R}^d)^2\times(\mathcal{M}^{d,q})^2$, $\mathbb{P}-$a.s. We shall also assume that

($\mathbf{H}F$) The component i of $F(t,y,z)$ depends only on the component i of the vector y and on the row i of the matrix z, i.e. $F^i(t,y,z)=F^i(t,y^i,z^i)$.

Given this set of data $(\Re,C,F,\xi)$, a discretely obliquely reflected BSDE, denoted $\mathcal{D}(\Re,C,F,\xi)$, is a triplet $(\widetilde{Y}^{\Re},Y^{\Re},Z^{\Re})\in(\mathcal{S}^2\times\mathcal{S}^2\times\mathcal{H}^2)^{\mathcal{I}}$ satisfying $Y_T^{\Re}=\widetilde{Y}_T^{\Re}:=\xi\in\mathcal{Q}_T$, and defined in a backward manner, for $j\le\kappa-1$ and $t\in[r_j,r_{j+1})$, by

$$\begin{cases} \widetilde{Y}_t^{\Re} &=& Y_{r_{j+1}}^{\Re} + \int_t^{r_{j+1}} F(u,\widetilde{Y}_u^{\Re},Z_u^{\Re})\mathrm{d}u - \int_t^{r_{j+1}} Z_u^{\Re}\mathrm{d}W_u\,,\\ Y_t^{\Re} &=& \widetilde{Y}_t^{\Re}\mathbf{1}_{\{t\notin\Re\}} + \mathcal{P}_t(\widetilde{Y}_t^{\Re})\mathbf{1}_{\{t\in\Re\}}. \end{cases} \tag{2.3}$$

This rewrites equivalently for $t \in [0,T]$ as

$$\begin{cases} \widetilde{Y}_t^{\Re} &= \xi + \int_t^T F(u, \widetilde{Y}_u^{\Re}, Z_u^{\Re})\mathrm{d}u - \int_t^T Z_u^{\Re}\mathrm{d}W_u + (K_T^{\Re} - K_t^{\Re}) \, , \\ K_t^{\Re} &:= \sum_{r\in\Re\setminus\{0\}} \Delta K_r^{\Re} \mathbf{1}_{\{r\le t\}} \quad \text{with} \quad \Delta K_t^{\Re} := Y_t^{\Re} - \widetilde{Y}_t^{\Re} = -(\widetilde{Y}_t^{\Re} - \widetilde{Y}_{t-}^{\Re}) \, , \end{cases} \tag{2.4}$$

Observe that $K^{\Re} \in (\mathbf{A}^2)^{\mathcal{I}}$, since C^{ij} is non-negative and valued in $\mathcal{S}^2$, for any $i, j \in \mathcal{I}$.

We shall also use the following integrability condition for some $p \ge 2$:

$$(\mathbf{C}_p) \qquad |\xi|^p + \sup_{t\in[0,T]} |C_t|^p + \int_0^T |F(s,0,0)|^p \mathrm{d}s \;\le\; \beta \, ,$$

where β is a positive random variable satisfying $\mathbb{E}[\beta] \le C_L$. Importantly, β does not depend on $\Re$.
The proof of the following a priori estimates is postponed in Section 6.1 of the Appendix.

Proposition 2.1. *Assume that* $(\mathbf{C}_p)$ *holds for some given* $p \ge 2$*, there exists a unique solution* $(\widetilde{Y}^{\Re}, Y^{\Re}, Z^{\Re})$ *to* (2.3) *and it satisfies*

$$\|\widetilde{Y}^{\Re}\|_{\mathcal{S}^p} + \|Z^{\Re}\|_{\mathcal{H}^p} + \|K_T^{\Re}\|_{\mathbf{L}^p} \le C_L^p \, .$$

2.2 Corresponding optimal switching problem

In this subsection, we interpret the solution of the discretely obliquely RBSDE (2.4) as the value process of a corresponding optimal switching problem, where the possible switching times are restricted to belong to the grid $\Re$. Our approach relies on similar arguments as the one followed by Hu and Tang [16] in a framework with continuous reflections.

A switching strategy a is a nondecreasing sequence of stopping times $(\theta_j)_{j\in\mathbb{N}}$, combined with a sequence of random variables $(\alpha_j)_{j\in\mathbb{N}}$ valued in $\mathcal{I}$, such that α_j is $\mathcal{F}_{\theta_j}-$measurable, for any $j \in \mathbb{N}$. We denote by $\mathcal{A}$ the set of such strategies. For $a = (\theta_j, \alpha_j)_{j\in\mathbb{N}} \in \mathcal{A}$, we introduce N^a the (random) number of switches before T:

$$N^a = \#\{k \in \mathbb{N}^* \; : \; \theta_k \le T\} \, . \tag{2.5}$$

To any switching strategy $a = (\theta_j, \alpha_j)_{j\in\mathbb{N}} \in \mathcal{A}$, we associate the current state process $(a_t)_{t\in[0,T]}$ and the compound cost process $(A_t^a)_{t\in[0,T]}$ defined respectively by

$$a_t \;:=\; \alpha_0 \mathbf{1}_{\{0\le t<\theta_0\}} + \sum_{j=1}^{N^a} \alpha_{j-1} \mathbf{1}_{\{\theta_{j-1}\le t<\theta_j\}} \qquad \text{and} \qquad A_t^a \;:=\; \sum_{j=1}^{N^a} C_{\theta_j}^{\alpha_{j-1}\alpha_j} \mathbf{1}_{\{\theta_j \le t \le T\}} \, ,$$

for $0 \le t \le T$. For $(t,i) \in [0,T] \times \mathcal{I}$, the set $\mathcal{A}_{t,i}$ of admissible strategies starting from i at time t is defined by

$$\mathcal{A}_{t,i} = \{a = (\theta_j, \alpha_j)_j \in \mathcal{A} \,|\theta_0 = t,\ \alpha_0 = i,\ \mathbb{E}[|A^a_T|^2] < \infty\}\ ,$$

similarly we introduce $\mathcal{A}^{\Re}_{t,i}$ the restriction to $\Re-$admissible strategies

$$\mathcal{A}^{\Re}_{t,i} \quad := \quad \{\ a = (\theta_j, \alpha_j)_{j\in\mathbb{N}} \in \mathcal{A}_{t,i} \quad | \quad \theta_j \in \Re\ ,\ \forall j \le N^a\ \}\ ,$$

and denote $\mathcal{A}^{\Re} := \bigcup_{i \le d} \mathcal{A}^{\Re}_{0,i}$.
For $(t,i) \in [0,T] \times \mathcal{I}$, and $a \in \mathcal{A}^{\Re}_{t,i}$, we consider as in [16] the associated one dimensional switched BSDE defined by

$$U^a_u = \xi^{a_T} + \int_u^T F^{a_s}(s, U^a_s, V^a_s)\mathrm{d}s - \int_u^T V^a_s \mathrm{d}W_s - A^a_T + A^a_u\ ,\ \ t \le u \le T\ . \tag{2.6}$$

Theorem 3.1 in [16] interprets each component of the solution to the continuously reflected BSDE (1.1) as the Snell envelop associated to switched processes of the form (2.6), where the switching strategies a are not restricted to lie in the reflection grid $\Re$. The next theorem is a new version of this Snell envelop representation adapted to the context of discretely obliquely reflected BSDE (2.4).

Theorem 2.1. *Assume that* $(\mathbf{C}_2)$ *is in force. For any* $i \in \mathcal{I}$ *and* $t \in [0,T]$*, the following holds:*

(i) The process $\widetilde{Y}^{\Re}$ *dominates any* $\Re$*-switched BSDE, i.e.*

$$U^a_t \quad \le \quad (\widetilde{Y}^{\Re}_t)^i \qquad \mathbb{P}-a.s.\ , \qquad \textit{for any } a \in \mathcal{A}^{\Re}_{i,t}\ . \tag{2.7}$$

(ii) Define the strategy $a^* = (\theta^*_j, \alpha^*_j)_{j\ge0}$ *recursively by* $(\theta^*_0, \alpha^*_0) := (t,i)$ *and, for* $j \ge 1$,

$$\theta^*_j \quad := \quad \inf\left\{ s \in [\theta^*_{j-1}, T] \cap \Re \,\middle|\, (\widetilde{Y}^{\Re}_s)^{\alpha^*_{j-1}} \le \max_{k \neq \alpha^*_{j-1}} \left\{ (\widetilde{Y}^{\Re}_s)^k - C^{\alpha^*_{j-1}k}_s \right\} \right\}\ ,$$

$$\alpha^*_j \quad := \quad \min\left\{ \ell \neq \alpha^*_{j-1} \,\middle|\, (\widetilde{Y}^{\Re}_{\theta^*_j})^{\ell} - C^{\alpha^*_{j-1}\ell}_{\theta^*_j} = \max_{k \neq \alpha^*_{j-1}} \left\{ (\widetilde{Y}^{\Re}_s)^k - C^{\alpha^*_{j-1}k}_{\theta^*_j} \right\} \right\}\ .$$

Then, we have $a^* \in \mathcal{A}^{\Re}_{t,i}$ *and*

$$(\widetilde{Y}^{\Re})^i_t = U^{a^*}_t\ \ \mathbb{P}-a.s.\ . \tag{2.8}$$

(iii) The following "Snell envelop" representation holds:

$$(\widetilde{Y}^{\Re})^i_t \quad = \quad \operatorname*{ess\,sup}_{a \in \mathcal{A}^{\Re}_{t,i}} U^a_t\ , \qquad \mathbb{P}-a.s.\ . \tag{2.9}$$

Proof. Observe first that Assertion *(iii)* is a direct consequence of *(i)* and *(ii)*. Let us fix $t \in [0,T]$ and $i \in \mathcal{I}$.

Step 1. We first prove *(i)*.
Set $a = (\theta_k, \alpha_k)_{k\geq 0} \in \mathcal{A}^{\Re}_{t,i}$ and the process $(\widetilde{Y}^a, Z^a)$ defined, for $s \in [t,T]$, by

$$\widetilde{Y}^a_s := \sum_{k\geq 0} (\widetilde{Y}^{\Re}_s)^{\alpha_k} \mathbf{1}_{\{\theta_k \leq s < \theta_{k+1}\}} + \xi^{a_T} \mathbf{1}_{\{s=T\}} \quad \text{and} \quad Z^a_s := \sum_{k\geq 0} (Z^{\Re}_s)^{\alpha_k} \mathbf{1}_{\{\theta_k \leq s < \theta_{k+1}\}} . \tag{2.10}$$

Observe that these processes jump between the components of the discretely reflected BSDE (3.5) according to the strategy a, and, between two jumps, we have

$$\begin{aligned}
\widetilde{Y}^a_{\theta_k} &= (Y^{\Re}_{\theta_{k+1}})^{\alpha_k} + \int_{\theta_k}^{\theta_{k+1}} F^{\alpha_k}(s, (\widetilde{Y}^{\Re}_s)^{\alpha_k}, (Z^{\Re}_s)^{\alpha_k}) \mathrm{d}s - \int_{\theta_k}^{\theta_{k+1}} (Z^{\Re}_s)^{\alpha_k} \mathrm{d}W_s + (K^{\Re}_{\theta_{k+1}-})^{\alpha_k} - (K^{\Re}_{\theta_k})^{\alpha_k} \\
&= \widetilde{Y}^a_{\theta_{k+1}} + \int_{\theta_k}^{\theta_{k+1}} F^{a_s}(s, \widetilde{Y}^a_s, Z^a_s) \mathrm{d}s - \int_{\theta_k}^{\theta_{k+1}} Z^a_s \mathrm{d}W_s + (K^{\Re}_{\theta_{k+1}-})^{\alpha_k} - (K^{\Re}_{\theta_k})^{\alpha_k} \\
&\quad + ((Y^{\Re}_{\theta_{k+1}})^{\alpha_k} - (\widetilde{Y}^{\Re}_{\theta_{k+1}})^{\alpha_{k+1}}) , \qquad k \geq 0 .
\end{aligned} \tag{2.11}$$

Introducing

$$K^a_s := \sum_{k=0}^{N^a - 1} \left[\int_{(\theta_k \wedge s, \theta_{k+1} \wedge s)} d(K^{\Re}_u)^{\alpha_k} + \mathbf{1}_{\{\theta_{k+1} \leq s\}} \left((Y^{\Re}_{\theta_{k+1}})^{\alpha_k} - (\widetilde{Y}^{\Re}_{\theta_{k+1}})^{\alpha_{k+1}} + C^{\alpha_k \alpha_{k+1}}_{\theta_{k+1}} \right) \right] ,$$

for $s \in [t,T]$, and summing up (2.11) over k, we get, for $t \leq u \leq T$,

$$\widetilde{Y}^a_u = \xi^{a_T} + \int_u^T F^{a_s}(s, \widetilde{Y}^a_s, Z^a_s) \mathrm{d}s - \int_u^T Z^a_s \mathrm{d}W_s - A^a_T + A^a_u + K^a_T - K^a_u .$$

Using the relation $Y^{\Re}_{\theta_k} = \mathcal{P}_{\theta_k}(\widetilde{Y}^{\Re}_{\theta_k})$ for all $k \in \{0, \ldots, N^a\}$, we check that K^a is increasing. Since U^a solves (2.6), we deduce by a comparison argument (see Theorem 1.3 in [22]) that $U^a_t \leq \widetilde{Y}^a_t$. Since a is arbitrary in $\mathcal{A}^{\Re}_{t,i}$, we deduce (2.7).

Step 2. We now prove (ii).
Consider the strategy a^* given above as well as the associated process $(\widetilde{Y}^{a^*}, Z^{a^*})$ defined as in (2.10). By definition of a^*, we have

$$(Y^{\Re}_{\theta^*_{k+1}})^{\alpha^*_k} = \left(\mathcal{P}_{\theta^*_{k+1}}(\widetilde{Y}^{\Re}_{\theta^*_{k+1}}) \right)^{\alpha^*_k} = (\widetilde{Y}^{\Re}_{\theta^*_{k+1}})^{\alpha^*_{k+1}} - C^{\alpha^*_k \alpha^*_{k+1}}_{\theta^*_{k+1}} , \quad k \geq 0 ,$$

which gives

$$\int_{(\theta^*_k, \theta^*_{k+1})} d(K^{\Re}_s)^{\alpha^*_k} = 0 \quad \text{and} \quad (Y^{\Re}_{\theta^*_{k+1}})^{\alpha^*_k} - (\widetilde{Y}^{\Re}_{\theta^*_{k+1}})^{\alpha^*_k} + C^{\alpha^*_k \alpha^*_{k+1}}_{\theta^*_{k+1}} = 0 , \tag{2.12}$$

for all $k \in \{0, \ldots, N^{a^*} - 1\}$. We deduce from (2.2) that

$$\widetilde{Y}_u^{a^*} = \xi^{a^*_T} + \int_u^T F^{a^*_s}(s, \widetilde{Y}_s^{a^*}, Z_s^{a^*})ds - \int_u^T Z_s^{a^*} \mathrm{d}W_s - A_T^{a^*} + A_u^{a^*}, \quad t \leq u \leq T.$$

Hence $(\widetilde{Y}^{a^*}, Z^{a^*})$ and (U^{a^*}, V^{a^*}) are solutions of the same BSDE and $(\widetilde{Y}_t^{\Re})^i = U_t^{a^*}$. To complete the proof, we only need to check that $a^* \in \mathcal{A}^{\Re}$, i.e. $\mathbb{E}|A_T^{a^*}|^2 < \infty$. By definition of a^* on $[t, T]$ and the structure condition on the cost (2.2), we have $|A_t^{a^*}| \leq \max_{k \neq i} |C_t^{i,k}|$ which gives $\mathbb{E}[|A_t^{a^*}|^2] \leq C_L$. Combining

$$A_T^{a^*} = \widetilde{Y}_T^{a^*} - \widetilde{Y}_t^{a^*} + \int_t^T F^{a^*_s}(s, \widetilde{Y}_s^{a^*}, Z_s^{a^*})\mathrm{d}s - \int_t^T Z_s^{a^*} \mathrm{d}W_s + A_t^{a^*},$$

with the Lipschitz property of F and the fact that $(\widetilde{Y}^{\Re}, Z^{\Re}) \in (\mathcal{S}^2 \times \mathcal{H}^2)^{\mathcal{I}}$, recall Proposition 2.1, we get the square integrability of $A_T^{a^*}$ and conclude the proof. □

Remark 2.2. Although the optimal strategy a^* depends on the initial parameters t and i, we omit the script (t, i) for ease of notation.

Combining the previous representation with the a priori estimates of Proposition 2.1 and the structure condition (2.2), we deduce the following estimates, whose proof is postponed to Section 6.1 in the Appendix.

Proposition 2.2. *Assume that* $(\mathbf{C}_p)$ *holds for some given* $p \geq 2$*, then*

$$\mathbb{E}\left[\sup_{s\in[t,T]} |U_s^{a^*}|^p + \left(\int_t^T |V_u^{a^*}|^2 \mathrm{d}u\right)^{\frac{p}{2}} + |A_T^{a^*}|^p + |N^{a^*}|^p\right] \leq C_L^p,$$

for the optimal strategy $a^* \in \mathcal{A}_{t,i}^{\Re}$*,* $(t, i) \in [0, T] \times \mathcal{I}$*.*

2.3 Stability of obliquely reflected BSDEs

We now study the dependence on the solution with respect to the parameters of the BSDE. In the 'abstract' setting considered, we obtain precious estimates for the analysis of the regularity of the solution to the discretely obliquely reflected BSDE as well as the convergence of the discrete-time scheme.

We consider two discretely reflected BSDEs, with the same reflection grid $\Re$ but different parameters. For $\ell \in \{1, 2\}$, we consider an $\mathcal{F}_T$-measurable random terminal condition ${}^{\ell}\xi$, a random L-lipschitz continuous map $(y, z) \mapsto {}^{\ell}F(., y, z)$, satisfying $(\mathbf{H}F)$, and a matrix of continuous cost processes $({}^{\ell}C^{ij})_{1\leq i,j\leq d}$ satisfying the structural condition (2.2).

We suppose that the coefficients satisfy the integrablity condition $(\mathbf{C}_4)$. For $\ell \in \{1,2\}$, we denote by $({}^\ell Y^\Re, {}^\ell \widetilde{Y}^\Re, {}^\ell Z^\Re) \in (\mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{H}^2)^{\mathcal{I}}$ the solution of the obliquely discretely reflected BSDE $\mathcal{D}(\Re, {}^\ell C, {}^\ell F, {}^\ell \xi)$.
Defining $\delta Y^\Re = {}^1 Y^\Re - {}^2 Y^\Re$, $\delta \widetilde{Y}^\Re = {}^1 \widetilde{Y}^\Re - {}^2 \widetilde{Y}^\Re$, $\delta Z^\Re = {}^1 Z^\Re - {}^2 Z^\Re$, $\delta\xi := {}^1\xi - {}^2\xi$ together with

$$|\delta C_s|_\infty := \max_{i,j \in \mathcal{I}} |{}^1 C^{ij} - {}^2 C^{ij}|(s) \; , \quad |\delta F_s|_\infty := \max_{i \in \mathcal{I}} \sup_{y,z \in \mathbb{R}^d \times \mathcal{M}^{d,q}} |{}^1 F^i - {}^2 F^i|(s,y,z),$$

for $s \in [0,T]$, we prove the following stability result.

Proposition 2.3. *Assume that $(\mathbf{C}_4)$ holds. Then, we have, for any $t \in [0,T]$,*

$$\mathbb{E}\Big[|\delta Y^\Re_t|^2\Big] + \mathbb{E}\Big[|\delta \widetilde{Y}^\Re_t|^2\Big] + \frac{1}{\kappa}\mathbb{E}\Big[\int_t^T |\delta Z^\Re_s|^2 \mathrm{d}s\Big] \le C_L\Big(\mathbb{E}\Big[\int_t^T |\delta F_s|^2_\infty \mathrm{d}s + |\delta\xi|^2\Big] + \mathbb{E}\Big[\sup_{r \in \Re} |\delta C_r|^4_\infty\Big]^{\frac{1}{2}}\Big),$$

Proof. The proof divides in three steps and relies heavilly on the reinterpretation in terms of switching problems. We first introduce a convenient dominating process, and then provide successively the controls on the $\delta Y^\Re$ and $\delta Z^\Re$ terms.
Step 1. *Introduction of an auxiliary BSDE.*
Let us define $F := {}^1F \vee {}^2F$, $\xi := {}^1\xi \vee {}^2\xi$ and C by $C^{ij} := {}^1C^{ij} \wedge {}^2C^{ij}$. Observe that F satisfies $(\mathbf{H}F)$, C satisfies the structure condition (2.2) and that $(\mathbf{C}_4)$ holds for the data (C, F, ξ). We denote by $(Y^\Re, \widetilde{Y}^\Re, Z^\Re)$ the solution of the discretely obliquely reflected BSDE $\mathcal{D}(\Re, C, F, \xi)$, recalling (2.3).
Using $(\mathbf{H}F)$, the definition of F and the monotonicity property of $\mathcal{P}$, see Remark 2.1 (i), we easily obtain by a comparison argument on each interval $[r_k, r_{k+1})$, $k \in \{0, \dots, \kappa-1\}$, that

$$\widetilde{Y}^\Re \succeq {}^1\widetilde{Y}^\Re \vee {}^2\widetilde{Y}^\Re \; . \tag{2.13}$$

Recalling Theorem 2.1, we introduce the switched BSDEs associated to ${}^1Y^\Re$, ${}^2Y^\Re$ and $Y^\Re$ and denote by $\check{a} = (\check{\theta}_j, \check{a}_j)_{j \ge 0}$ the optimal strategy related to $Y^\Re$ starting from a fixed $(i,t) \in \mathcal{I} \times [0,T]$. Therefore, we have

$$(\widetilde{Y}^\Re_t)^i = U^{\check{a}}_t = \xi^{\check{a}_T} + \int_t^T F^{\check{a}_s}(s, U^{\check{a}}_s, V^{\check{a}}_s)\mathrm{d}s - \int_t^T V^{\check{a}}_s \mathrm{d}W_s - A^{\check{a}}_T + A^{\check{a}}_t \; . \tag{2.14}$$

Step 2. *Stability of the Y component.*
Since $\check{a} \in \mathcal{A}^\Re_{t,i}$, we deduce from Theorem 2.1 (iii) that

$$({}^\ell\widetilde{Y}^\Re_t)^i \;\ge\; {}^\ell U^{\check{a}_s}_t \;=\; {}^\ell\xi^{\check{a}_T} + \int_t^T {}^\ell F^{\check{a}_s}(s, {}^\ell U^{\check{a}}_s, {}^\ell V^{\check{a}}_s)\mathrm{d}s - \int_t^T {}^\ell V^{\check{a}}_s \mathrm{d}W_s - {}^\ell A^{\check{a}}_T + {}^\ell A^{\check{a}}_t \; , \; \ell \in \{1,2\} \, ,$$

where ${}^{\ell}A^{\check{a}}$ is the process of cumulated costs $({}^{\ell}C^{ij})_{i,j\in\mathcal{I}}$ associated to the strategy $\check{a}$. Combining this estimate with (2.13) and (2.14), we derive

$$|({}^1\widetilde{Y}^{\Re}_t)^i - ({}^2\widetilde{Y}^{\Re}_t)^i| \ \le \ |U^{\check{a}}_t - {}^1U^{\check{a}}_t| + |U^{\check{a}}_t - {}^2U^{\check{a}}_t| \,. \tag{2.15}$$

Since both terms on the right hand side of (2.15) are treated similarly, we focus on the first one and introduce the continuous processes $\Gamma^{\check{a}} := U^{\check{a}} + A^{\check{a}}$ and ${}^1\Gamma^{\check{a}} := {}^1U^{\check{a}} + {}^1A^{\check{a}}$. Applying Ito's formula, we compute, for all $t \le u \le T$,

$$\begin{aligned}\mathbb{E}_t\Big[|\Gamma^{\check{a}}_u - {}^1\Gamma^{\check{a}}_u|^2 + \int_u^T |V^{\check{a}}_s - {}^1V^{\check{a}}_s|^2 \mathrm{d}s\Big] &\le \\ \mathbb{E}_t\Big[|\Gamma^{\check{a}}_T - {}^1\Gamma^{\check{a}}_T|^2 + 2\int_u^T (\Gamma^{\check{a}}_s - {}^1\Gamma^{\check{a}}_s)[F^{\check{a}_s}(s, U^{\check{a}}_s, {}^1V^{\check{a}}_s) &- {}^1F^{\check{a}_s}(s, {}^1U^{\check{a}}_s, {}^1V^{\check{a}}_s)]\mathrm{d}s\Big] \,.\end{aligned} \tag{2.16}$$

Since $F = {}^1F \vee {}^2F$ and ${}^1F$ is Lipschitz continuous, we also get

$$\begin{aligned}&|F^{\check{a}_s}(s, U^{\check{a}}_s, {}^1V^{\check{a}}_s) - {}^1F^{\check{a}_s}(s, {}^1U^{\check{a}}_s, {}^1V^{\check{a}}_s)| \le \\ &\qquad |\delta F_s|_\infty + L(|\Gamma^{\check{a}}_s - {}^1\Gamma^{\check{a}}_s| + |A^{\check{a}}_s - {}^1A^{\check{a}}_s| + |V^{\check{a}}_s - {}^1V^{\check{a}}_s|)\,, \quad 0 \le s \le T\,.\end{aligned}$$

Using classical arguments, we then deduce from the last inequality and (2.16) that

$$|\Gamma^{\check{a}}_t - {}^1\Gamma^{\check{a}}_t|^2 \ \le C_L \left(\mathbb{E}_t\Big[|\delta\xi^{\check{a}_T}|^2 \int_t^T |\delta F_s|^2_\infty \mathrm{d}s\Big] + \sup_{t \le s \le T} \mathbb{E}_t\big[|A^{\check{a}}_s - {}^1A^{\check{a}}_s|^2\big] \right) \tag{2.17}$$

Moreover, using the inequality $|x \vee y - y| \le |x - y|$ for $x, y \in \mathbb{R}$ and the convexity of the function $x \mapsto x^2$, we compute

$$\begin{aligned}\mathbb{E}_t\big[|A^{\check{a}}_s - {}^1A^{\check{a}}_s|^2\big] &= \mathbb{E}_t\left[\left|\sum_{k=1}^{N^{\check{a}}} \big[{}^2C^{\check{\alpha}_{k-1}\check{\alpha}_k} \wedge^1 C^{\check{\alpha}_{k-1}\check{\alpha}_k} -^1 C^{\check{\alpha}_{k-1}\check{\alpha}_k}\big](\check{\theta}_k)\mathbf{1}_{\{\check{\theta}_k \le s\}}\right|^2\right] \\ &\le \mathbb{E}_t\Big[|N^{\check{a}}| \sup_{r\in\Re} |\delta C_r|^2_\infty\Big]\,, \qquad t \le s \le T\,.\end{aligned} \tag{2.18}$$

Plugging in (2.17) and recalling the definition of $\Gamma^{\check{a}}$ and ${}^1\Gamma^{\check{a}}$, we get

$$|U^{\check{a}}_t - {}^1U^{\check{a}}_t|^2 \ \le \ C_L\mathbb{E}_t\Big[|N^{\check{a}}| \sup_{r\in\Re} |\delta C_r|^2_\infty + \int_t^T |\delta F_s|^2_\infty \mathrm{d}s + |\delta\xi|^2\Big] \,.$$

The exact same reasoning leads to the same estimate for $|U^{\check{a}}_t - {}^2U^{\check{a}}_t|^2$. Therefore, we deduce from (2.15) and Cauchy Schwartz inequality that

$$\mathbb{E}\Big[|({}^2\widetilde{Y}^{\Re}_t)^i - ({}^1\widetilde{Y}^{\Re}_t)^i|^2\Big] \ \le \ C_L\Big(\mathbb{E}\big[|N^{\check{a}}|^2\big]^{\frac{1}{2}}\, \mathbb{E}\Big[\sup_{r\in\Re} |\delta C_r|^4_\infty\Big]^{\frac{1}{2}} + \mathbb{E}\Big[\int_t^T |\delta F_s|^2_\infty \mathrm{d}s + |\delta\xi|^2\Big]\Big). \tag{2.19}$$

Using Proposition 2.2, we compute, since i is arbitrary,

$$\mathbb{E}\Big[|{}^2\widetilde{Y}_t^{\Re} - {}^1\widetilde{Y}_t^{\Re}|^2\Big] \le C_L\Big(\mathbb{E}\Big[\int_t^T |\delta F_s|_\infty^2 \mathrm{d}s + |\delta\xi|^2\Big] + \mathbb{E}\Big[\sup_{r\in\Re}|\delta C_r|_\infty^4\Big]^{\frac{1}{2}}\Big). \qquad (2.20)$$

Step 3. *Stability of the Z component.*
Applying Ito's formula to the *càdlàg* process $|\delta\widetilde{Y}^{\Re}|^2$ and noting $\delta\tilde{K} = {}^1K^{\Re} - {}^2K^{\Re}$, we obtain

$$\mathbb{E}\left[|\delta\widetilde{Y}_t^{\Re}|^2 + \int_t^T |\delta Z_s^{\Re}|^2\mathrm{d}s + \sum_{t<r\le T}|\Delta\delta\tilde{K}_r^{\Re}|^2\right] = \mathbb{E}\Big[|\delta\widetilde{Y}_T^{\Re}|^2 + 2\int_t^T \delta Y_s^{\Re}\delta F_s\mathrm{d}s + 2\int_t^T \delta Y_r^{\Re}\mathrm{d}\delta\tilde{K}_r^{\Re}\Big],$$

where we used the fact that $|\delta\widetilde{Y}^{\Re}|^2 - |\delta Y^{\Re}|^2 - 2\delta Y^{\Re}(\delta\widetilde{Y}^{\Re} - \delta Y^{\Re}) = |\Delta\delta\tilde{K}^{\Re}|^2$. Since δK is a pure jump process, we compute

$$\mathbb{E}\Big[\int_t^T \delta Y_r^{\Re}\mathrm{d}\delta\tilde{K}_r^{\Re}\Big] \le \mathbb{E}\left[\alpha\sum_{t<r\le T, r\in\Re}|\delta Y_r^{\Re}|^2 + \frac{1}{\alpha}\sum_{t<r\le T}|\Delta\delta\tilde{K}_r^{\Re}|^2\right], \quad \alpha > 0,$$

which, for α large enough and using standard arguments, leads to

$$\mathbb{E}\left[\int_t^T |\delta Z_s^{\Re}|^2\mathrm{d}s + \sum_{t<r\le T}|\Delta\delta\tilde{K}_r^{\Re}|^2\right] \le C_L\left(\mathbb{E}\big[|\delta\xi|^2\big] + \mathbb{E}\left[\int_t^T |\delta F_s|_\infty^2\mathrm{d}s + \sum_{t<r\le T, r\in\Re}|\delta Y_r^{\Re}|^2\right]\right).$$

Since (2.20) holds true for any $t \in [0, T]$, we deduce

$$\mathbb{E}\left[\int_t^T |\delta Z_s^{\Re}|^2\mathrm{d}s + \sum_{t<r\le T}|\Delta\delta\tilde{K}_r^{\Re}|^2\right] \le C_L\kappa\left(\mathbb{E}\big[|\delta\xi|^2\big] + \mathbb{E}\Big[\int_t^T |\delta F_s|_\infty^2\mathrm{d}s\Big] + \mathbb{E}\Big[\sup_{r\in\Re}|\delta C_r|_\infty^4\Big]^{\frac{1}{2}}\right),$$

which concludes the proof of the proposition. □

3 Regularity of discretely obliquely reflected BSDEs

This section is dedicated to the derivation of regularity properties for the solution of discretely reflected BSDEs. These results are obtained in a Markovian diffusion setting. This means that the randomness of the parameter (C, F, ξ), is due to a state process X, which is the solution of a Stochastic Differential Equation (SDE). In this framework, we focus on the $\mathcal{H}^2$-regularity of the $Z^{\Re}$ component of the solution of the BSDEs. The main results are retrieved by means of kernel regularization and Malliavin differentiation arguments. Finally, we extend this result to the case where the diffusion X is replaced by its Euler scheme.

3.1 A diffusion setting for discretely RBSDEs

Let X be the solution on $[0,T]$ to the following SDE:

$$X_t = X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}W_s\;, \quad 0 \le t \le T\;, \tag{3.1}$$

where $X_0 \in \mathbb{R}^m$ and $(b,\sigma) : \mathbb{R}^m \to \mathbb{R}^m \times \mathcal{M}^{m,q}(\mathbb{R})$ are L-Lipschitz functions.

Under the above assumption, the following estimates are well known (see e.g. [18])

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |X_t|^p\Big] \le C_L^p \text{ and } \sup_{s\in[0,T]}\left(\mathbb{E}\Big[\sup_{u\in[0,T],|u-s|\le h} |X_s - X_u|^p\Big]\right)^{\frac{1}{p}} \le C_L^p\,\sqrt{h}\;, \tag{3.2}$$

for any $p > 0$. In the sequel, we shall denote by β^X a positive random variable, which may change from line to line, but which depends only on $\sup_{t\in[0,T]}|X_t|$ and which satisfies $\mathbb{E}\big[|\beta^X|^p\big] \le C_L^p$ for all $p > 0$. Importantly, β^X does not depend on $\Re$.

Remark 3.1. Observe that, as in [1, 7] and contrary to [19], we make no uniform ellipticity condition on σ. This allows us to treat the case of non homogenous diffusion by setting e.g. $X_t^1 = t$, $t \in [0,T]$.

In this context, we are given a matrix valued maps $c := (c^{ij})$ where $c^{ij} : \mathbb{R}^m \to \mathbb{R}^+$, are L-Lipschitz continuous and satisfy:

$$\begin{cases} c^{ii}(.) = 0\,, & \text{for } 1 \le i \le d\;; \\ \inf_{x\in\mathbb{R}^m} c^{ij}(x) > 0\,, & \text{for } 1 \le i,j \le d\,, \text{ with } i \ne j\;; \\ \inf_{x\in\mathbb{R}^m}\{c^{ij}(x)+c^{jl}(x)-c^{il}(x)\} > 0, & \text{for } 1 \le i,j,l \le d, \text{ with } i \ne j,\, j \ne l. \end{cases} \tag{3.3}$$

We then introduce a family $(\mathcal{Q}(x))_{x\in\mathbb{R}^m}$ of closed convex domains:

$$\mathcal{Q}(x) := \left\{ y \in \mathbb{R}^d \mid y^i \ge \max_{j\in\mathcal{I}}(y^j - c^{ij}(x))\,, \quad \forall i \in \mathcal{I} \right\}\;, \quad \text{where} \quad \mathcal{I} := \{1,\dots,d\}\,. \tag{3.4}$$

We introduce the oblique projection operator $\mathcal{P}(x,.)$ onto $\mathcal{Q}(x)$ defined by

$$\mathcal{P} : (x,y) \in \mathbb{R}^m \times \mathbb{R}^d \;\mapsto\; \left(\max_{j\in\mathcal{I}}\left\{y^j - c^{ij}(x)\right\}\right)_{1\le i\le d}\,.$$

Finally, we are given
- an L-Lipschitz function $g : \mathbb{R}^m \to \mathbb{R}^d$ such that $g(x) \in \mathcal{Q}(x)$ for all $x \in \mathbb{R}^m$,
- a generator function, i.e. an L-lipschitz map $f : \mathbb{R}^m \times \mathbb{R}^d \times \mathcal{M}^{d,q} \to \mathbb{R}^d$.

From now on, we shall appeal to the following assumption:

($\mathbf{H}f$) the component i of $f(.,y,z)$ depends only on the component i of the vector y and on the column i of the matrix z i.e. $f^i(.,y,z) = f^i(.,y^i,z^i)$.

We denote by $(Y^{\Re}, \widetilde{Y}^{\Re}, Z^{\Re})$ the solution of the discretely reflected BSDE $\mathcal{D}(\Re, c(X), f(X,.,.), g(X))$ which reads on each interval $[r_j, r_{j+1})$, for $j < \kappa$:

$$\left\{ \begin{array}{rcl} \widetilde{Y}_t^{\Re} & = & Y_{r_{j+1}}^{\Re} + \int_t^{r_{j+1}} f(X_u, \widetilde{Y}_u^{\Re}, Z_u^{\Re}) \mathrm{d}u - \int_t^{r_{j+1}} Z_u^{\Re} \mathrm{d}W_u \ , \\ Y_t^{\Re} & = & \widetilde{Y}_t^{\Re} \mathbf{1}_{\{t \notin \Re\}} + \mathcal{P}(X_t, \widetilde{Y}_t^{\Re}) \mathbf{1}_{\{t \in \Re\}}. \end{array} \right. \tag{3.5}$$

Or equivalently on $[0,T]$ as

$$\left\{ \begin{array}{rcl} \widetilde{Y}_t^{\Re} & = & g(X_T) + \int_t^T f(X_u, \widetilde{Y}_u^{\Re}, Z_u^{\Re}) \mathrm{d}u - \int_t^T Z_u^{\Re} \mathrm{d}W_u + (K_T^{\Re} - K_t^{\Re}) \ , \quad 0 \le t \le T \ , \\ K_t^{\Re} & := & \sum_{r \in \Re \setminus \{0\}} \Delta K_r^{\Re} \mathbf{1}_{\{r \le t\}} \ \text{ and } \ \Delta K_t^{\Re} = Y_t^{\Re} - \widetilde{Y}_t^{\Re} = -(\widetilde{Y}_t^{\Re} - \widetilde{Y}_{t-}^{\Re}) \ , \quad 0 \le t \le T. \end{array} \right. \tag{3.6}$$

From (3.2), it follows that the data $(c(X), f(X,.,.), g(X))$ satisfies the integrability condition $(\mathbf{C}_p)$ for all $p \ge 2$. We thus deduce from the proof of Proposition 2.1 and Proposition 2.2, the following estimate on $(Y^{\Re}, \widetilde{Y}^{\Re}, Z^{\Re})$ and their associated optimal switched BSDEs, recalling Theorem 2.1.

Proposition 3.1. *There exists a unique solution* $(\widetilde{Y}^{\Re}, Y^{\Re}, Z^{\Re})$ *to* (3.5) *and it satisfies*

$$\mathbb{E}_t \left[\sup_{s \in [t,T]} |\widetilde{Y}_s^{\Re}|^p + \left(\int_t^T |Z_s^{\Re}|^2 \mathrm{d}s \right)^{\frac{p}{2}} + |K_T^{\Re} - K_t^{\Re}|^p \right] \le \beta^X \ , \ \forall t \le T \ . \tag{3.7}$$

Moreover, for all $(t,i) \in [0,T] \times \mathcal{I}$*, the optimal strategy* $a^* \in \mathcal{A}_{t,i}^{\Re}$ *satisfies*

$$\mathbb{E}_t \left[\sup_{s \in [t,T]} |U_s^{a^*}|^p + \left(\int_t^T |V_s^{a^*}|^2 ds \right)^{\frac{p}{2}} + |A_T^{a^*}|^p + |N^{a^*}|^p \right] \le \beta^X \ . \tag{3.8}$$

3.2 Malliavin differentiability of $(X, Y^{\Re}, \widetilde{Y}^{\Re}, Z^{\Re})$

We shall sometimes use the following regularity assumption on the coefficients:
($\mathbf{H}r$) The coefficients b, σ, g f, and $(c^{ij})_{i,j}$ are $C^{1,b}$ in all their variables, with the Lipschitz constants dominated by L.

We denote by $\mathbb{D}^{1,2}$ the set of random variables G which are differentiable in the Malliavin sense and such that $\|G\|_{\mathbb{D}^{1,2}}^2 := \|G\|_{\mathbf{L}^2}^2 + \int_0^T \|D_t G\|_{\mathbf{L}^2}^2 dt < \infty$, where $D_t G$ denotes the Malliavin derivative of G at time $t \le T$. After possibly passing to a suitable version, an adapted process belongs to the subspace $\mathcal{L}_a^{1,2}$ of $\mathcal{H}^2$ whenever $V_s \in \mathbb{D}^{1,2}$ for all $s \le T$ and $\|V\|_{\mathcal{L}_a^{1,2}}^2 := \|V\|_{\mathcal{H}^2}^2 + \int_0^T \|D_t V\|_{\mathcal{H}^2}^2 dt < \infty$. For a general presentation on Malliavin calculus for stochastic differential equations, the reader may refer to [20].

Remark 3.2. Under $(\mathbf{H}r)$, the solution of (3.1) is Malliavin differentiable and its derivative satisfies

$$\| \sup_{s \le T} |D_s X| \|_{\mathcal{S}^p} < \infty \,, \tag{3.9}$$

and we have

$$\sup_{s \le u} \|D_s X_t - D_s X_u\|_{\mathbf{L}^p} + \| \sup_{t \le s \le T} |D_t X_s - D_u X_s| \,\|_{\mathbf{L}^p} \quad \le \quad C_L^p |t-u|^{1/2} \,, \tag{3.10}$$

for any $0 \le u \le t \le T$. Let $G \in \mathbb{D}^{1,2}(\mathbb{R}^d)$. Since X belongs to $\mathcal{L}_a^{1,2}$ under $(\mathbf{H}r)$, and $\mathcal{P}$ is $L_{\mathcal{P}}$-lipschitz continuous, we deduce that $\mathcal{P}(X_t, G) \in \mathbb{D}^{1,2}(\mathbb{R}^d)$. Using Lemma 5.1 in [1], we compute

$$D_s(\mathcal{P}(X_t,G))^i = \tag{3.11}$$
$$\sum_{j=1}^d (D_s G^j - D_s c_{ij}(X_t)) \mathbf{1}_{\{G^j - c^{ij}(X_t) > \max_{\ell<j}(G^\ell - c^{i\ell}(X_t))\}} \mathbf{1}_{\{G^j - c^{ij}(X_t) \ge \max_{\ell>j}(G^\ell - c^{i\ell}(X_t))\}} .$$

Combining (3.11), Proposition 5.3 in [10] and an induction argument, we obtain that $(Y^{\Re}, \widetilde{Y}^{\Re}, Z^{\Re})$ is Malliavin differentiable and that a version of $(D_u \widetilde{Y}^{\Re}, D_u Z^{\Re})$ is given by

$$\begin{aligned} D_u(\widetilde{Y}^{\Re}_t)^i =& D_u(Y^{\Re}_{r_{j+1}})^i - \sum_{k=1}^d \int_t^{r_{j+1}} D_u(Z^{\Re}_s)^{ik} \mathrm{d}W^k_s + \int_t^{r_{j+1}} \nabla_x f^i(X_s, (\widetilde{Y}^{\Re}_s)^i, (Z^{\Re}_s)^{i\cdot}) D_u X_s \mathrm{d}s \\ &+ \int_t^{r_{j+1}} \nabla_{y^i} f^i(X_s, (\widetilde{Y}^{\Re}_s)^i, (Z^{\Re}_s)^{i\cdot}) D_u(\widetilde{Y}^{\Re}_s)^i \mathrm{d}s + \int_t^{r_{j+1}} \nabla_z f^i(X_s, (\widetilde{Y}^{\Re}_s)^i, (Z^{\Re}_s)^{i\cdot}) D_u(Z^{\Re}_s)^{i\cdot} \mathrm{d}s \end{aligned} \tag{3.12}$$

for $0 \le u \le t \le r_{j+1}$ and $j < \kappa$. Here, $\nabla_z f^i$ denotes $\sum_{\ell=1}^d \nabla_{z^{\ell\cdot}} f^i$, recalling $(\mathbf{H}f)$.

3.3 Representation of Z

For $a \in \mathcal{A}^{\Re}$, we introduce the process Λ^a defined by

$$\begin{aligned} \Lambda^a_{t,s} := \exp\Big\{ & \int_t^s \nabla_z f^{a_r}(X_r, \widetilde{Y}^{\Re}_r, Z^{\Re}_r) \mathrm{d}W_r \\ & - \int_t^s \Big(\frac{1}{2} |\nabla_z f^{a_r}(X_r, \widetilde{Y}^{\Re}_r, Z^{\Re}_r)|^2 - \nabla_y f^{a_r}(X_r, \widetilde{Y}^{\Re}_r, Z^{\Re}_r) \Big) \mathrm{d}r \Big\} \,, \end{aligned} \tag{3.13}$$

for $0 \le t \le s \le T$.

For later use, we remark

$$\sup_{a \in \mathcal{A}^{\Re}} \| \sup_{t \le s \le T} \Lambda^a_{t,s} \|_{\mathcal{L}^p} \le C_L^p \,, \quad 0 \le t \le T, \quad p \ge 2 \,, \tag{3.14}$$

and deduce from the dynamics of Λ that

$$\sup_{a\in\mathcal{A}^{\Re}}\left(\|\Lambda^{a}_{t,t}-\Lambda^{a}_{t,u}\|_{\mathcal{L}^p}+\|\sup_{t\le s\le T}|\Lambda^{a}_{u,s}-\Lambda^{a}_{t,s}|\,\|_{\mathcal{L}^p}\right)\le C^p_L\sqrt{t-u}\,,\quad u\le t\le T\,,\quad p\ge 2\,. \tag{3.15}$$

Proposition 3.2. *Under* ($\mathbf{H}r$)*, there is a version of* $Z^{\Re}$ *such that,*

$$(Z^{\Re}_t)^i=\mathbb{E}_t\Big[\nabla_x g^{a^*_T}(X_T)\Lambda^{a^*}_{t,T}D_tX_T+\int_t^T\nabla_x f^{a^*_s}(X_s,\widetilde{Y}^{\Re}_s,Z^{\Re}_s)\Lambda^{a^*}_{t,s}D_tX_s\mathrm{d}s\\ -\sum_{j=1}^{N^{a^*}}\nabla_x c^{\alpha^*_{j-1}\alpha^*_j}(X_{\theta^*_j})\Lambda^{a^*}_{t,\theta^*_j}(D_tX)_{\theta^*_j}\Big]\,, \tag{3.16}$$

for $(t,i)\in[0,T]$*, with* $a^*=(\theta^*_j,\alpha^*_j)_{j\ge0}\in\mathcal{A}^{\Re}_{t,i}$ *the optimal strategy given in Theorem 2.1 and recalling* (2.5)*.*

Proof. We fix $j<\kappa$ and, observing that the process a^* is constant on the interval $[\theta^*_j,\theta^*_{j+1})$, we deduce from (3.12) and Ito's formula that

$$\Lambda^{a^*}_{t,t}D_u(\widetilde{Y}^{\Re}_t)^{\alpha^*_j}=\mathbb{E}_t\left[\Lambda^{a^*}_{t,\theta^*_{j+1}}(D_u(Y^{\Re})^{\alpha^*_j})_{\theta^*_{j+1}}+\int_t^{\theta^*_{j+1}}\nabla_x f^{\alpha^*_j}(X_s,\widetilde{Y}^{\Re}_s,Z^{\Re}_s)\Lambda^{a^*}_{t,s}D_uX_s\mathrm{d}s\right],$$

for $\theta^*_j\le u\le t<\theta^*_{j+1}$. Combining (3.11) and the definition of a^* given in Theorem 2.1 (ii), we compute

$$\Lambda^{a^*}_{t,\theta^*_{j+1}}(D_u(Y^{\Re})^{\alpha^*_j})_{\theta^*_{j+1}}=\Lambda^{a^*}_{t,\theta^*_{j+1}}(D_u(\widetilde{Y}^{\Re})^{\alpha^*_{j+1}})_{\theta^*_{j+1}}-\nabla_x c^{\alpha^*_j\alpha^*_{j+1}}(X_{\theta^*_{j+1}})\Lambda^{a^*}_{t,\theta^*_{j+1}}(D_tX)_{\theta^*_{j+1}},$$

for $j<\kappa$. Plugging the second equality into the first one and summing up over j concludes the proof. □

We conclude this section by providing a 'weak' regularity property of $Z^{\Re}$ in the general Lipschitz setting. In order to get rid of the previous Assumption ($\mathbf{H}r$), we make use of kernel regularization arguments. Since this procedure is very classical, we do not detail it here precisely, see e.g. the proofs of Proposition 4.2 in [7] or Proposition 3.3 in [1].

Proposition 3.3. *There is a version of* $Z^{\Re}$ *satisfying*

$$\mathbb{E}\Big[\int_s^t|Z^{\Re}_u|^2\mathrm{d}u\Big]\le C_L|t-s|\,,\quad s\le t\le T\,. \tag{3.17}$$

Proof. Combining (3.9), with (3.14), (3.16) and Doob's inequality, we observe that

$$\sup_{t\in[0,T]}\|Z^{\Re}_t\|_{\mathbf{L}^p}\le C^p_L\,,\quad p\ge2\,,$$

holds under $(\mathbf{H}r)$. Therefore (3.17) is satisfied under $(\mathbf{H}r)$. As in the proof of Proposition 4.2 in [7], the stability results of Proposition 2.3 allow us to use classical Kernel regularization arguments. Since the previous estimate holds uniformly for the sequence of approximating regularized BSDE, the proof is complete. □

3.4 Regularity results

We consider a grid $\pi := \{t_0 = 0, \dots, t_n = T\}$ on the time interval $[0, T]$, with modulus $|\pi| := \max_{0 \le i \le n-1} |t_{i+1} - t_i|$, such that $\Re \subset \pi$.

We want to control the following quantities, representing the $\mathcal{H}^2$-regularity of $(\widetilde{Y}, Z)$:

$$\mathbb{E}\Big[\int_0^T |\widetilde{Y}_t^{\Re} - \widetilde{Y}_{\pi(t)}^{\Re}|^2 \mathrm{d}t\Big] \quad \text{and} \quad \mathbb{E}\Big[\int_0^T |Z_t^{\Re} - \bar{Z}_{\pi(t)}^{\Re}|^2 \mathrm{d}t\Big] , \tag{3.18}$$

where $\pi(t) := \sup\{t_i \in \pi\, ;\ t_i \le t\}$ is defined on $[0, T]$ as the projection to the closest previous grid point of π and

$$\bar{Z}_{t_i}^{\Re} := \frac{1}{t_{i+1} - t_i} \mathbb{E}\left[\int_{t_i}^{t_{i+1}} Z_s^{\Re} \mathrm{d}s \mid \mathcal{F}_{t_i}\right] , \quad i \in \{0, \dots, n-1\}. \tag{3.19}$$

Remark 3.3. Observe that $(\bar{Z}_s^{\Re})_{s \le T} := (\bar{Z}_{\pi(s)}^{\Re})_{s \le T}$ interprets as the best $\mathcal{H}^2$-approximation of the process $Z^{\Re}$ by adapted processes which are constant on each interval $[t_i, t_{i+1})$, for all $i < n$.

Proposition 3.4. *The following holds*

$$\frac{1}{T} \mathbb{E}\Big[\int_0^T |\widetilde{Y}_t^{\Re} - \widetilde{Y}_{\pi(t)}^{\Re}|^2 \mathrm{d}t\Big] \le \sup_{t \in [0,T]} \mathbb{E}\Big[|\widetilde{Y}_t^{\Re} - \widetilde{Y}_{\pi(t)}^{\Re}|^2\Big] \le C_L |\pi|.$$

Proof. Observe first that

$$\mathbb{E}\Big[|\widetilde{Y}_t^{\Re} - \widetilde{Y}_{\pi(t)}^{\Re}|^2\Big] \le \mathbb{E}\left[\left|\int_{\pi(t)}^t f(X_s, \widetilde{Y}_s^{\Re}, Z_s^{\Re}) \mathrm{d}s + \int_{\pi(t)}^t Z_s^{\Re} \mathrm{d}W_s\right|^2\right] , \quad 0 \le t \le T .$$

The proof is concluded combining this estimate with (3.2), Proposition 3.1 and Proposition 3.3. □

We now turn to the study of the regularity of the process $Z^{\Re}$.

Theorem 3.1. *The process $Z^{\Re}$ satisfies*

$$\mathbb{E}\Big[\int_0^T |Z_s^{\Re} - \bar{Z}_s^{\Re}|^2 \mathrm{d}s\Big] \quad \le \quad C_L(|\pi|^{\frac{1}{2}} + \kappa |\pi|) . \tag{3.20}$$

Proof. A regularization argument as in proof of Proposition 3.3 allows us to work under $(\mathbf{H}r)$. From Remark 3.3, it is clear that

$$\mathbb{E}\Big[\int_0^T |Z^{\Re}_s - \bar{Z}^{\Re}_s|^2 \mathrm{d}s\Big] \;\leq\; \mathbb{E}\Big[\int_0^T |Z^{\Re}_s - Z^{\Re}_{\pi(s)}|^2 \mathrm{d}s\Big]\;. \tag{3.21}$$

For $s \leq T$ and $a = (\alpha_k, \theta_k)_{k\geq 0} \in \mathcal{A}^{\Re}_{s,\ell}$, $\ell \in \mathcal{I}$, we define $(V^a_{s,t})_{s\leq t\leq T}$ by

$$V^a_{s,t} := \mathbb{E}_t\Big[\nabla_x g^{a_T}(X_T)\Lambda^a_{s,T} D_s X_T + \int_s^T \nabla_x f^{a_u}(X_u, \widetilde{Y}^{\Re}_u, Z^{\Re}_u)\Lambda^a_{s,u} D_s X_u \mathrm{d}u - \sum_{k=1}^{N^a} \nabla_x c_{\alpha_{j-1},\alpha_j}(X_{\theta_k})\Lambda^a_{s,\theta_k}(D_s X)_{\theta_k}\Big]\;.$$

We now fix $\ell \in \mathcal{I}$ and denote, for $u \leq T$, by $a^u \in \mathcal{A}^{\Re}_{u,\ell}$ the optimal strategy associated to the representation of $(\widetilde{Y}^{\Re}_u)^\ell$, recalling (ii) in Theorem 2.1.
Observe that, by definition, we have

$$N^{a^t} = N^{a^u} \text{ and } a^t = a^u \;\;,\quad r_j \leq t \leq u < r_{j+1}\,,\quad j < \kappa\,. \tag{3.22}$$

Fix $i < n$, and deduce from Proposition 3.2 and (3.22) that

$$\mathbb{E}\Big[|(Z^{\Re}_t)^\ell - (Z^{\Re}_{t_i})^\ell|^2\Big] = \mathbb{E}\Big[|V^{a^t}_{t,t} - V^{a^{t_i}}_{t_i,t_i}|^2\Big] \;\leq\; 2\left(\mathbb{E}\Big[|V^{a^{t_i}}_{t,t} - V^{a^{t_i}}_{t_i,t}|^2\Big] + \mathbb{E}\Big[|V^{a^{t_i}}_{t_i,t} - V^{a^{t_i}}_{t_i,t_i}|^2\Big]\right)\,, \tag{3.23}$$

for $t \in [t_i, t_{i+1})$. Combining $(\mathbf{H}r)$, (3.9), (3.10), (3.14), (3.15) and Cauchy-Schwartz inequality with the definition of V^a, we deduce

$$\mathbb{E}\Big[|V^{a^{t_i}}_{t,t} - V^{a^{t_i}}_{t_i,t}|^2\Big] \leq C_L|\pi|^{\frac{1}{2}}\,,\quad t_i \leq t \leq t_{i+1}\,,\quad i \leq n\,. \tag{3.24}$$

Since $V^{a^{t_i}}_{t_i,\cdot}$ is a martingale on $[t_i, t_{i+1}]$, we obtain

$$\begin{aligned}
\mathbb{E}\Big[|V^{a^{t_i}}_{t_i,t} - V^{a^{t_i}}_{t_i,t_i}|^2\Big] &\leq \mathbb{E}\Big[|V^{a^{t_i}}_{t_i,t_{i+1}} - V^{a^{t_i}}_{t_i,t_i}|^2\Big]\\
&\leq \mathbb{E}\Big[|V^{a^{t_i}}_{t_{i+1},t_{i+1}}|^2 - |V^{a^{t_i}}_{t_i,t_i}|^2\Big] + \mathbb{E}\Big[|V^{a^{t_i}}_{t_i,t_{i+1}}|^2 - |V^{a^{t_i}}_{t_{i+1},t_{i+1}}|^2\Big]\\
&\leq \mathbb{E}\Big[|V^{a^{t_i}}_{t_{i+1},t_{i+1}}|^2 - |V^{a^{t_i}}_{t_i,t_i}|^2\Big] + C_L|\pi|^{\frac{1}{2}}\,,\qquad t_i \leq t \leq t_{i+1}\,,
\end{aligned} \tag{3.25}$$

where the last inequality follows from (3.24). Combining (3.23), (3.24), (3.25) and summing up over i, we obtain

$$\mathbb{E}\Big[\int_0^T |(Z^{\Re}_t)^\ell - (Z^{\Re}_{\pi(t)})^\ell|^2 \mathrm{d}t\Big] \leq C_L|\pi|^{\frac{1}{2}} + |\pi|\left(\mathbb{E}\Big[|V^{a^{r_{\kappa-1}}}_{T,T}|^2 - |V^{a^0}_{0,0}|^2\Big] + \sum_{j=1}^{\kappa-1}(|V^{a^{r_{j-1}}}_{r_j,r_j}|^2 - |V^{a^{r_j}}_{r_j,r_j}|^2)\right).$$

Combined with (3.9) and (3.14), this concludes the proof since ℓ is arbitrary. □

3.5 Extension

We shall approximate the process X by its Euler scheme X^π, with dynamics

$$X_t^\pi \;=\; X_0 + \int_0^t b(X_{\pi(s)}^\pi)\mathrm{d}s + \int_0^t \sigma(X_{\pi(s)}^\pi)\mathrm{d}W_s \;, \quad 0 \le t \le T \;. \tag{3.26}$$

Classically, we have the following upper-bound, uniformly in π:

$$\mathbb{E}\Big[\sup_{0\le t\le T} |X_t^\pi|^p\Big]^{1/p} \;\le\; C_L^p \;\;,\;\; p \ge 2 \;. \tag{3.27}$$

The control of the error between X and its Euler scheme X^π is well understood, see e.g. [17], and we have

$$\mathbb{E}\Big[\sup_{0\le t\le T} |X_t - X_t^\pi|^p\Big]^{1/p} \;\le\; C_L^p\,|\pi|^{\frac{1}{2}} \;\;,\;\; p \ge 2 \;. \tag{3.28}$$

In this context, we denote by $(Y^{eu}, \widetilde{Y}^{eu}, Z^{eu})$ the unique solution of the reflected BSDE $\mathcal{D}(\Re, c(X^\pi), f(X^\pi, .), g(X^\pi))$. Our main result here is the counterpart of Proposition 3.4 and Theorem 3.1 when X is replaced by X^π.

Proposition 3.5. *The following holds*

$$\mathbb{E}\Big[\int_0^T |\widetilde{Y}_t^{eu} - \widetilde{Y}_{\pi(t)}^{eu}|^2\mathrm{d}t\Big] \le C_L|\pi| \quad \textit{and} \quad \mathbb{E}\Big[\int_0^T |Z_s^{eu} - \bar{Z}_s^{eu}|^2\mathrm{d}s\Big] \;\le C_L(|\pi|^{\frac{1}{2}} + \kappa|\pi|) \;.$$

Proof. We only sketch the main step of the proof since it follows formally exactly the same arguments as the one used to obtain Proposition 3.4 and Theorem 3.1.

Step 1. We use a kernel regularization argument which allows us to work under $(\mathbf{H}r)$. In this case, we observe that X^π belongs to $\mathcal{L}_a^{1,2}$ and satisfies

$$D_s X_t^\pi = \sigma(X_{\pi(s)}^\pi) + \int_s^t \nabla_x b(X_{\pi(r)}^\pi) D_s X_{\pi(r)}^\pi \mathrm{d}r + \int_s^t \sum_{j=1}^q \nabla_x \sigma^j(X_{\pi(r)}^\pi) D_s X_{\pi(r)}^\pi \mathrm{d}W_r^j \;,$$

for $s \le t$. One then checks, see Remark 5.2 in [1] for details, that

$$\|\sup_{s\le T} |D_s X^\pi|\|_{\mathcal{S}^p} < \infty \;, \tag{3.29}$$

$$\sup_{s\le u} \|D_s X_t^\pi - D_s X_u^\pi\|_{\mathbf{L}^p} + \|\sup_{t\le s\le T} |D_t X_s^\pi - D_u X_s^\pi|\;\|_{\mathbf{L}^p} \le C_L^p |t-u|^{1/2} \;,\;\; 0 \le u \le t \le T \;.$$

It is also straightforward that $(Y^{eu}, \widetilde{Y}^{eu}, Z^{eu})$ is Malliavin differentiable and satisfies (3.12) with X^π instead of X.

Step 2. In order to retrieve the results of the Proposition, one then follows exactly the same steps and arguments as the one used in the previous Section 3.3 and Section 3.4. □

4 A discrete-time approximation for discretely reflected BSDEs

We present here a discrete time scheme for the approximation of the solution of the discretely obliquely reflected BSDE (3.5).
Recall that $\pi := \{t_0 = 0, \dots, t_n = T\}$ is a grid on the time interval $[0, T]$, such that $\Re \subset \pi$ and $|\pi|n \leq L$. In the sequel, the process X is approximated by its Euler scheme X^π, see Section 3.5 for details.

4.1 An Euler scheme for discretely obliquely reflected BSDEs

We introduce an Euler-type approximation scheme for the discretely reflected BSDEs. Starting from the terminal condition

$$Y_T^{\Re,\pi} = \widetilde{Y}_T^{\Re,\pi} := g(X_T^\pi) \in \mathcal{C}(X_T^\pi) ,$$

we compute recursively, for $i \leq n-1$,

$$\left\{ \begin{array}{rcl} \bar{Z}_{t_i}^{\Re,\pi} & = & (t_{i+1} - t_i)^{-1} \mathbb{E}\left[Y_{t_{i+1}}^{\Re,\pi} (W_{t_{i+1}} - W_{t_i})' \mid \mathcal{F}_{t_i} \right] , \\ \widetilde{Y}_{t_i}^{\Re,\pi} & = & \mathbb{E}\left[Y_{t_{i+1}}^{\Re,\pi} \mid \mathcal{F}_{t_i} \right] + (t_{i+1} - t_i) f(X_{t_i}^\pi, \widetilde{Y}_{t_i}^{\Re,\pi}, \bar{Z}_{t_i}^{\Re,\pi}) , \\ Y_{t_i}^{\Re,\pi} & = & \widetilde{Y}_{t_i}^{\Re,\pi} \mathbf{1}_{\{t_i \notin \Re\}} + \mathcal{P}(X_{t_i}^\pi, \widetilde{Y}_{t_i}^{\Re,\pi}) \mathbf{1}_{\{t_i \in \Re\}} . \end{array} \right. \qquad (4.1)$$

This kind of backward scheme has been already considered when no reflection occurs, see e.g. [2], and in the reflected case, see e.g. [1, 19, 7]. See also [4] for a recent survey on the subject.

Combining an induction argument with the Lispchitz-continuity of f, g and the projection operator, one easily checks that the above processes are square integrable and that the conditional expectations are well defined at each step of the algorithm.

Remark 4.1. (i) This so-called "moonwalk" algorithm is given by an implicit formulation, and one should use a fixed point argument to compute explicitly $\widetilde{Y}^{\Re,\pi}$ at each grid point.
(ii) In the two dimensional case, Hamadene and Jeanblanc [14] interpret $Y^1 - Y^2$ as the solution of a doubly reflected BSDE. It is worth noticing that the solution of the

corresponding discrete time scheme developed by [7] for the approximation of doubly reflected BSDE exactly coincides with $(Y^{\Re,\pi})^1 - (Y^{\Re,\pi})^2$ derived here.

For later use, we introduce the piecewise continuous time scheme associated to the triplet $(Y^{\Re,\pi}, \widetilde{Y}^{\Re,\pi}, \bar{Z}^{\Re,\pi})$. By the martingale representation theorem, there exists $Z^{\Re,\pi} \in \mathcal{H}^2$ such that

$$Y^{\Re,\pi}_{t_{i+1}} = \mathbb{E}_{t_i}\Big[Y^{\Re,\pi}_{t_{i+1}}\Big] + \int_{t_i}^{t_{i+1}} Z^{\Re,\pi}_u \mathrm{d}W_u \ \ , \ \ i \le n-1 \ ,$$

and by the ItÃŹ isometry, for $i \le n-1$,

$$\bar{Z}^{\Re,\pi}_{t_i} = \frac{1}{t_{i+1}-t_i}\mathbb{E}\left[\int_{t_i}^{t_{i+1}} Z^{\Re,\pi}_s \mathrm{d}s \mid \mathcal{F}_{t_i}\right] \ . \tag{4.2}$$

We set $\bar{Z}^{\Re,\pi}_t := \bar{Z}^{\Re,\pi}_{\pi(t)}$ for $t \in [0,T]$, define $\widetilde{Y}^{\Re,\pi}$ by

$$\widetilde{Y}^{\Re,\pi}_t = Y^{\Re,\pi}_{t_{i+1}} + (t_{i+1}-t) f(X^\pi_{t_i}, \widetilde{Y}^{\Re,\pi}_{t_i}, \bar{Z}^{\Re,\pi}_{t_i}) - \int_t^{t_{i+1}} Z^{\Re,\pi}_u \mathrm{d}W_u \ , \quad t_i \le t \le t_{i+1}, \ \ i \in \mathcal{I}, \tag{4.3}$$

and introduce $Y^{\Re,\pi}$ on $[0,T]$ by $Y^{\Re,\pi}_t := \widetilde{Y}^{\Re,\pi}_t \mathbf{1}_{\{t \notin \Re\}} + \mathcal{P}(X^\pi_t, \widetilde{Y}^{\Re,\pi}_t)\mathbf{1}_{\{t \in \Re\}}$.

This can be rewritten as

$$\begin{cases} \widetilde{Y}^{\Re,\pi}_t &= g(X^\pi_T) + \int_t^T f(X^\pi_{\pi(u)}, \widetilde{Y}^{\Re,\pi}_{\pi(u)}, \bar{Z}^{\Re,\pi}_u)\mathrm{d}u - \int_t^T Z^{\Re,\pi}_u \mathrm{d}W_u + (K^{\Re,\pi}_T - K^{\Re,\pi}_t) \ , \\ K^{\Re,\pi}_t &:= \sum_{r \in \Re \setminus \{0\}} \Delta K^{\Re,\pi}_r \mathbf{1}_{\{r \le t\}} \ \text{ and } \ \Delta K^{\Re,\pi}_t := Y^{\Re,\pi}_t - \widetilde{Y}^{\Re,\pi}_t = -(\widetilde{Y}^{\Re,\pi}_t - \widetilde{Y}^{\Re,\pi}_{t-}) \ , \\ Y^{\Re,\pi}_t &= \widetilde{Y}^{\Re,\pi}_t \mathbf{1}_{\{t \notin \Re\}} + \mathcal{P}(X^\pi_t, \widetilde{Y}^{\Re,\pi}_t)\mathbf{1}_{\{t \in \Re\}} \ , \qquad 0 \le t \le T. \end{cases} \tag{4.4}$$

We finally provide a useful a priori estimate for the solution of the discrete time scheme whenever f does not depend on z, whose proof is postponed Section 6.2 of the Appendix.

Proposition 4.1. *If f does not depend on z and $|\pi|L < 1$, the following bound holds*

$$\mathbb{E}\left[\sup_{0 \le i \le n} |\widetilde{Y}^{\Re,\pi}_{t_i}|^p\right] \ \le \ C^p_L, \qquad p \ge 2, \tag{4.5}$$

recall that C^p_L neither depends on $\Re$ nor on π.

4.2 Convergence Results

The next proposition provides a control on the error between the discrete-time scheme (4.1) and the solution of the discretely reflected BSDE (3.5).

Proposition 4.2. *The following holds*

$$\sup_{t\in[0,T]} \mathbb{E}\Big[|\widetilde{Y}^{\Re}_t - \widetilde{Y}^{\Re,\pi}_t|^2 + |Y^{\Re}_t - Y^{\Re,\pi}_t|^2\Big] + \mathbb{E}\Big[\int_0^T |Z^{\Re}_s - \bar{Z}^{\Re,\pi}_s|^2 \mathrm{d}s\Big] \le C_L |L_{\mathcal{P}}|^{2\kappa}(|\pi|^{\frac{1}{2}} + \kappa|\pi|)\,, \tag{4.6}$$

where we recall that $L_{\mathcal{P}} = \sqrt{d}$ *is the Lipschitz constant of the projection operator* $\mathcal{P}$.

Proof. As in Section 3.5, we consider $(Y^{eu}, \widetilde{Y}^{eu}, Z^{eu})$ the unique solution of the reflected BSDE $\mathcal{D}(\Re, c(X^\pi), f(X^\pi, .), g(X^\pi))$. Using Proposition 2.3, the Lipschitz property of f, g, c and (3.28), we obtain

$$\sup_{t\in[0,T]} \mathbb{E}\Big[|\widetilde{Y}^{\Re}_t - \widetilde{Y}^{eu}_t|^2 + |Y^{\Re}_t - Y^{eu}_t|^2\Big] + \frac{1}{\kappa}\mathbb{E}\Big[\int_0^T |Z^{\Re}_s - Z^{eu}_s|^2 \mathrm{d}s\Big] \le C_L|\pi|\,. \tag{4.7}$$

Using the same arguments as in the proof of Proposition 3.4.1 Step 1.a in [6] e.g., we get the following inequality:

$$\begin{aligned}\sup_{t\in[t_i,t_{i+1})} \mathbb{E}\Big[|\widetilde{Y}^{eu}_t - \widetilde{Y}^{\Re,\pi}_t|^2 + |Y^{eu}_t - Y^{\Re,\pi}_t|^2\Big] + \mathbb{E}\Big[\int_{t_i}^{t_{i+1}} |Z^{eu}_s - \bar{Z}^{\Re,\pi}_s|^2 \mathrm{d}s\Big] \le \\ C_L\left(\mathbb{E}\Big[|Y^{eu}_{t_{i+1}} - Y^{\Re,\pi}_{t_{i+1}}|^2 + \int_{t_i}^{t_{i+1}} (|\widetilde{Y}^{eu}_s - \widetilde{Y}^{\Re}_{\pi(s)}|^2 + |Z^{eu}_s - \bar{Z}^{\Re}_{\pi(s)}|^2)\mathrm{d}s\Big]\right)\,.\end{aligned} \tag{4.8}$$

There are two differences with the proof of Proposition 3.4.1 in [6]. First, $\mathcal{P}$ here depends both on x and y: but this is not a problem since $(Y^{eu}, \widetilde{Y}^{eu}, Z^{eu})$ and $(Y^{\Re,\pi}, \widetilde{Y}^{\Re,\pi}, Z^{\Re,\pi})$ are parametrized by the same forward process X^π.
Second, $\mathcal{P}$ is not 1-Lipschitz but only $L_{\mathcal{P}}$-Lipschitz, with $L_{\mathcal{P}} > 1$, in its y component. This explain the term $|L_{\mathcal{P}}|^{2\kappa}$ in (4.6). Indeed, we have, for $i < n$,

$$|Y^{eu}_{t_{i+1}} - Y^{\Re,\pi}_{t_{i+1}}|^2 = |\mathcal{P}(X^\pi_{t_{i+1}}, \widetilde{Y}^{eu}_{t_{i+1}}) - \mathcal{P}(X^\pi_{t_{i+1}}, \widetilde{Y}^{\Re,\pi}_{t_{i+1}})|^2 \le |L_{\mathcal{P}}|^2 |\widetilde{Y}^{eu}_{t_{i+1}} - \widetilde{Y}^{\Re,\pi}_{t_{i+1}}|^2.$$

This leads, using an induction argument (see e.g. Step 1.b in the proof of Proposition 3.4.1 in [6]), to

$$\begin{aligned}\sup_{t\in[0,T]} \mathbb{E}\Big[|\widetilde{Y}^{eu}_t - \widetilde{Y}^{\Re,\pi}_t|^2 + |Y^{eu}_t - Y^{\Re,\pi}_t|^2\Big] + \mathbb{E}\Big[\int_0^T |Z^{eu}_s - \bar{Z}^{\Re,\pi}_s|^2 \mathrm{d}s\Big] \le \\ C_L |L_{\mathcal{P}}|^{2\kappa}\left(|\pi| + \int_0^T (|\widetilde{Y}^{eu}_s - \widetilde{Y}^{\Re}_{\pi(s)}|^2 + |Z^{eu}_s - \bar{Z}^{\Re}_{\pi(s)}|^2)\mathrm{d}s\right).\end{aligned}$$

Combining the last inequality with Proposition 3.5 and (4.7) completes the proof. □

The term $|L_{\mathcal{P}}|^{2\kappa}$, even when κ is small can be very large. Moreover, we shall see in the next section that it yields to a poor convergence rate for continuously reflected BSDEs.

This term is due to the "geometric" approach, used in the proof of Proposition 4.2, and the fact that $\mathcal{P}$ is only $L_{\mathcal{P}}$-Lipschitz with $L_{\mathcal{P}} > 1$. We obtain below a better control, using the stability results proved at the end of Section 2 but unfortunately under the assumption that f does not depend on z. The optimal choice for κ in terms of $|\pi|$ is discussed in Section 5.3 below.

Theorem 4.1. *If f does not depend on z, the following holds*

$$\sup_{t\in[0,T]} \mathbb{E}\Big[|\widetilde{Y}_t^{\Re} - \widetilde{Y}_t^{\Re,\pi}|^2 + |Y_t^{\Re} - Y_t^{\Re,\pi}|^2\Big] \;\le\; C_L|\pi|\;,$$

$$\mathbb{E}\Big[\int_0^T |Z_t^{\Re} - \bar{Z}_t^{\Re,\pi}|^2 \mathrm{d}t\Big] \;\le\; C_L(\kappa|\pi| + |\pi|^{\frac{1}{2}})\;,$$

for $|\pi|$ small enough.

Proof. We use here the stability results of Proposition 2.3 setting $({}^1Y^{\Re}, {}^1\widetilde{Y}^{\Re}, {}^1Z^{\Re}) = (Y^{\Re}, \widetilde{Y}^{\Re}, Z^{\Re})$ with ${}^1F : (s,y,z) \mapsto f(X_s, \widetilde{Y}_s^{\Re})$ and $({}^2Y^{\Re}, {}^2\widetilde{Y}^{\Re}, {}^2Z^{\Re}) = (Y^{\Re,\pi}, \widetilde{Y}^{\Re,\pi}, Z^{\Re,\pi})$, with ${}^2F : (s,y,z) \mapsto f(X^{\pi}_{\pi(s)}, \widetilde{Y}^{\Re,\pi}_{\pi(s)})$. Combining (4.5) and Proposition 3.1 with the Lipschitz property of f, it is clear that $(\mathbf{C}_4)$ holds. Applying Proposition 2.3 and (3.28), we derive, for $t \in [0,T]$,

$$\mathbb{E}|\widetilde{Y}_t^{\Re} - \widetilde{Y}_t^{\Re,\pi}|^2 + \frac{1}{\kappa}\int_t^T \mathbb{E}|Z_s^{\Re} - Z_s^{\Re,\pi}|^2 ds \le$$
$$C_L\Big(|\pi| + \int_t^T \mathbb{E}|\widetilde{Y}_s^{\Re} - \widetilde{Y}_{\pi(s)}^{\Re}|^2\, ds + \textstyle\int_t^T \mathbb{E}|\widetilde{Y}_{\pi(s)}^{\Re,\pi} - \widetilde{Y}_{\pi(s)}^{\Re}|^2 \mathrm{d}s\Big)\,. \qquad (4.9)$$

Applying the discrete version of Gronwall's lemma to estimate (4.9) rewritten at time $t = t_j \in \pi$, we deduce

$$\mathbb{E}|\widetilde{Y}_{t_j}^{\Re} - \widetilde{Y}_{t_j}^{\Re,\pi}|^2 \le C_L\Big(|\pi| + \int_t^T \mathbb{E}|\widetilde{Y}_s^{\Re} - \widetilde{Y}_{\pi(s)}^{\Re}|^2 ds\Big)\;, \quad 0 \le t \le t_j \le T, \quad t_j \in \pi. \quad (4.10)$$

Plugging this estimate into (4.9), we compute

$$\mathbb{E}|\widetilde{Y}_t^{\Re} - \widetilde{Y}_t^{\Re,\pi}|^2 + \frac{1}{\kappa}\int_t^T \mathbb{E}|Z_s^{\Re} - Z_s^{\Re,\pi}|^2 ds \le C_L\Big(|\pi| + \int_t^T \mathbb{E}|\widetilde{Y}_s^{\Re} - \widetilde{Y}_{\pi(s)}^{\Re}|^2 ds\Big)\;,\; 0 \le t \le T\;,$$

which combined with Proposition 3.4 leads to the first claim of the theorem.

Observe from the representations (3.19) and (4.2) that

$$\mathbb{E}\Big[\int_0^T |Z_t^{\Re} - \bar{Z}_t^{\Re,\pi}|^2 \mathrm{d}t\Big] \le C_L\left(\mathbb{E}\Big[\int_0^T |Z_t^{\Re} - \bar{Z}_t^{\Re}|^2 \mathrm{d}t\Big] + \mathbb{E}\Big[\int_0^T |Z_t^{\Re} - Z_t^{\Re,\pi}|^2 \mathrm{d}t\Big]\right)\;.$$

Plugging (3.20), estimate (4.9) written at time $t = 0$ and the first claim of this Theorem into this expression concludes the proof. □

5 Extension to the continuously reflected case

In this section, we extend the convergence results of the scheme (4.1) to the case of continuously reflected BSDEs. To this end, we show that the error between discretely and continuously obliquely reflected BSDEs is controled in a convenient way.

5.1 Continuously obliquely reflected BSDEs

In the sequel, we shall use the following assumption on f:
- (**H**z) The function f is bounded in its last variable : $\sup_{z\in\mathcal{M}^{d,q}}|f(0,0,z)|\le C_L$.
and the following assumption on the cost c:
- (**H**c) For $i,j\in\mathcal{I}$, the function c^{ij} is equal to ${}^1c^{ij} - {}^2c^{ij}$, with ${}^1c^{ij}$ is C^2 with bounded first and second derivatives and ${}^2c^{ij}$ is a convex function with bounded first derivative. This last assumption is needed to retrieve some regularity on the reflecting process K (see Lemma 5.1 below).
We denote by $(Y,Z,K)\in(\mathcal{S}^2\times\mathcal{H}^2\times\mathbf{A}^2)^{\mathcal{I}}$ the solution of the continuously obliquely reflected BSDE $\mathcal{C}([0,T],c(X),f(X,.),g(X_T))$ defined by

$$\begin{cases} Y^i_t = g^i(X_T) + \int_t^T f^i(X_s,Y^i_s,Z^i_s)\mathrm{d}s - \int_t^T Z^i_s\mathrm{d}W_s + K^i_T - K^i_t\ , \\ Y^i_t \ge \max_{j\in\mathcal{I}}\{Y^j_t - c^{ij}(X_t)\}\ , \quad 0\le t\le T\ , \\ \int_0^T[Y^i_t - \max_{j\in\mathcal{I}}\{Y^j_t - c^{ij}(X_t)\}]\mathrm{d}K^i_t = 0\ , \quad i\in\mathcal{I}\ . \end{cases} \tag{5.1}$$

Under the assumption on f, g and c, the existence and uniqueness of such a solution is given in [15, 16].

The solution of (5.1) has also a representation property in term of switched BSDEs, recalling (2.6). Here of course the switching times of the strategy are not restricted to take their values in $\Re$. We refer to [8] for more details.

Theorem 5.1. *There exists, for any fixed initial condition $(t,i)\in[0,T]\times\mathcal{I}$, an optimal switching strategy $\dot{a} := (\dot{\theta}_k,\dot{\alpha}_k)_{k\ge 0}\in\mathcal{A}_{t,i}$, such that*

$$Y^i_t = U^{\dot{a}}_t = \operatorname*{ess\,sup}_{a\in\mathcal{A}_{t,i}} U^a_t\ , \qquad \mathbb{P}-a.s.\ . \tag{5.2}$$

We deduce from (5.2), Theorem 2.1 (iii), the monotonicity property of $\mathcal{P}$ and (5.1):

$$Y \succeq Y^{\Re} \succeq \widetilde{Y}^{\Re}\ , \qquad \text{for any grid } \Re\ . \tag{5.3}$$

Moreover, most of the estimates presented in Section 2 for discretely reflected BSDEs hold true for continuously reflected BSDEs. For reader's convenience, we collect them in the following proposition. The proof itself is postponed to Section 6.3 of the Appendix.

Proposition 5.1. *The following a priori estimates holds. For any $p \geq 2$,*

$$|Y_t|^p + \mathbb{E}_t\left[\left(\int_t^T |Z_s|^2 \mathrm{d}s\right)^{\frac{p}{2}}\right] + \mathbb{E}_t[|K_T - K_t|^p] \leq \mathbb{E}_t[\beta^X] \ , \quad 0 \leq t \leq T \ , \tag{5.4}$$

and, for all $(t,i) \in [0,T] \times \mathcal{I}$, the optimal strategy $\dot{a} \in \mathcal{A}_{t,i}$ satisfies

$$\mathbb{E}_t\left[\sup_{s\in[t,T]} |U_s^{\dot{a}}|^p\right] + \mathbb{E}_t[|N^{\dot{a}}|^p] \leq \mathbb{E}_t[\beta^X] \,. \tag{5.5}$$

5.2 Error between discretely and continuously reflected BSDEs

We first provide a control of the error on the grid points of $\Re$ between the solutions of the obliquely discretely and continuously reflected BSDEs (3.6) and (5.1).

Theorem 5.2. *Under* $(\mathbf{H}z)$*, the following holds*

$$\mathbb{E}\left[\sup_{r\in\Re}\left\{|Y_r - \widetilde{Y}_r^{\Re}|^2 + |Y_r - Y_r^{\Re}|^2\right\}\right] \ \leq \ C_L^{\varepsilon}|\Re|^{1-\varepsilon} \ , \quad \varepsilon > 0 \,. \tag{5.6}$$

Moreover, if the cost functions are constant, the last inequality holds true with $\epsilon = 0$.

Proof. The proof of this result relies mainly on the interpretation in terms of switched BSDEs provided in Section 2.2. For a fixed $(t,i) \in [0,T] \times \mathcal{I}$, we associate to the optimal strategy $\dot{a} = (\dot{\theta}_k, \dot{\alpha}_k)_k \in \mathcal{A}_{t,i}$ not restricted to lie in the grid $\Re$, the corresponding 'discretized' strategy $a := (\theta_k, \alpha_k)_{k\geq 0} \in \mathcal{A}_{t,i}^{\Re}$ defined by

$$\theta_k \ := \ \inf\left\{r \geq \dot{\theta}_k \ ; \ \ r \in \Re\right\} \qquad \text{and} \qquad \alpha_k := \dot{\alpha}_k \ , \qquad k \geq 0 \ . \tag{5.7}$$

Step 1. We first derive two key controls on the distance between $A^{\dot{a}}$ and A^a.
We fix $p \geq 2$ and, since $\dot{\theta}_k \leq \theta_k$, $k \geq 1$, we compute

$$\begin{aligned}
\left(\int_t^T |A_s^{\dot{a}} - A_s^a|^2 \mathrm{d}s\right)^{\frac{p}{2}} \ &= \ \left(\int_t^T \left|\sum_{k=1}^{N^{\dot{a}}} c^{\dot{\alpha}_{k-1}\dot{\alpha}_k}(X_{\dot{\theta}_k})\mathbf{1}_{\dot{\theta}_k\leq s} - c^{\dot{\alpha}_{k-1}\dot{\alpha}_k}(X_{\theta_k})\mathbf{1}_{\theta_k\leq s}\right|^2 \mathrm{d}s\right)^{\frac{p}{2}} \\
&\leq \ C_L^p \int_t^T \left|\sum_{k=1}^{N^{\dot{a}}} \left[c^{\dot{\alpha}_{k-1}\dot{\alpha}_k}(X_{\theta_k}) - c^{\dot{\alpha}_{k-1}\dot{\alpha}_k}(X_{\dot{\theta}_k})\right] \mathbf{1}_{\theta_k\leq s}\right|^p \mathrm{d}s \\
&\quad + C_L^p \left(\int_t^T \left|\sum_{k=1}^{N^{\dot{a}}} c^{\dot{\alpha}_{k-1}\dot{\alpha}_k}(X_{\dot{\theta}_k})\mathbf{1}_{\dot{\theta}_k\leq s<\theta_k}\right|^2 \mathrm{d}s\right)^{\frac{p}{2}} \ .
\end{aligned} \tag{5.8}$$

Using the convexity inequality $(\sum_{k=1}^n |x_k|)^p \leq n^{p-1}\sum_{k=1}^n |x_k|^p$, we obtain

$$\left(\int_t^T \left|\sum_{k=1}^{N^{\dot a}} c^{\dot\alpha_{k-1}\dot\alpha_k}(X_{\dot\theta_k})\mathbf{1}_{\dot\theta_k\leq s<\theta_k}\right|^2 \mathrm{d}s\right)^{\frac{p}{2}} \leq C_L^p(1+\sup_{t\in[0,T]}|X_t|^p)|N^{\dot a}|^p|\Re|^{\frac{p}{2}} . \quad (5.9)$$

Using once again the same convexity inequality with $p=2$, the Lipschitz property of the maps $(c^{ij})_{i,j\in\mathcal{I}}$ and the definition of $\dot\theta_k$ and θ_k, we get

$$\begin{aligned}\int_t^T \left|\sum_{k=1}^{N^{\dot a}} \left[c^{\dot\alpha_{k-1}\dot\alpha_k}(X_{\theta_k}) - c^{\dot\alpha_{k-1}\dot\alpha_k}(X_{\dot\theta_k})\right]\mathbf{1}_{\theta_k\leq s}\right|^p \mathrm{d}s &\leq C_L^p|N^{\dot a}|^{p-1}\sum_{k=1}^{N^{\dot a}}|X_{\theta_k}-X_{\dot\theta_k}|^p \\ &\leq C_L^p|N^{\dot a}|^p\chi^{|\Re|,p} ,\end{aligned}$$

where $\chi^{|\Re|,p} := \sum_{k=1}^{\kappa}\sup_{r\in[r_{k-1},r_k]}|X_r - X_{r_k}|^p$.
Plugging this estimate and (5.9) in (5.8), we deduce

$$\left(\int_t^T \left|A_s^{\dot a} - A_s^a\right|^2 \mathrm{d}s\right)^{\frac{p}{2}} \leq C_L^p|N^{\dot a}|^p\left(\left(1+\sup_{s\in[0,T]}|X_s|^p\right)|\Re|^{\frac{p}{2}} + \chi^{|\Re|,p}\right) . \quad (5.10)$$

Observe also that, for $r\in\Re$, we have $\mathbf{1}_{\dot\theta_k\leq r} = \mathbf{1}_{\theta_k\leq r}$ which gives

$$|A_r^{\dot a} - A_r^a|^p \leq \left(\sum_{k=1}^{N^{\dot a}}\left|c^{\dot\alpha_{k-1}\dot\alpha_k}(X_{\dot\theta_k}) - c^{\dot\alpha_{k-1}\dot\alpha_k}(X_{\theta_k})\right|\mathbf{1}_{\theta_k\leq r}\right)^p \leq C_L|N^{\dot a}|^p\chi^{|\Re|,p} . \quad (5.11)$$

Step 2. We now prove the main result of the theorem.
We introduce the processes $\Gamma := U^a - A^a$ and $\dot\Gamma := U^{\dot a} - A^{\dot a}$, so that

$$|U^a - U^{\dot a}| \leq |\Gamma - \dot\Gamma| + |A^a - A^{\dot a}| . \quad (5.12)$$

Applying Ito's formula to the continuous process $|\dot\Gamma - \Gamma|^2$ on $[t,T]$, using Gronwall Lemma and the Lipschitz property of f, we obtain

$$|\dot\Gamma_t - \Gamma_t|^2 \leq C_L\,\mathbb{E}_t\Big[\int_t^T \left|[f^{\dot a_s} - f^{a_s}](X_s,U_s^{\dot a},V_s^{\dot a})\right|^2\mathrm{d}s + \int_t^T\left|A_s^{\dot a} - A_s^a\right|^2\mathrm{d}s\Big] . \quad (5.13)$$

Elevating this expression to the power $\frac{p}{2}$, we deduce

$$|\dot\Gamma_t - \Gamma_t|^p \leq C_L^p\,\mathbb{E}_t\Big[\left(\int_t^T\left|[f^{\dot a_s} - f^{a_s}](X_s,U_s^{\dot a},V_s^{\dot a})\right|^2\mathrm{d}s\right)^{\frac{p}{2}} + \left(\int_t^T\left|A_s^{\dot a} - A_s^a\right|^2\mathrm{d}s\right)^{\frac{p}{2}}\Big] . \quad (5.14)$$

Combining the definition of θ with the Lipschitz property of f and $(\mathbf{H}z)$, we compute

$$\int_t^T \left|[f^{\dot a_s} - f^{a_s}](X_s, U_s^{\dot a}, V_s^{\dot a})\right|^2 \mathrm{d}s = \int_t^T \left| \sum_{k=1}^{N^{\dot a}} f^{\alpha_{k-1}}(X_s, U_s^{\dot a}, V_s^{\dot a})(\mathbf{1}_{\dot\theta_{k-1} \le s < \dot\theta_k} - \mathbf{1}_{\theta_{k-1} \le s < \theta_k}) \right|^2 \mathrm{d}s$$
$$\le C_L |N^{\dot a}|^2 \sup_{s\in[0,T]} (1 + |X_s|^2 + |U_s^{\dot a}|^2)|\Re| .$$

Plugging the last inequality and (5.10) in (5.14), we deduce

$$|\dot\Gamma_t - \Gamma_t|^p \le C_L^p \mathbb{E}_t \left[|N^{\dot a}|^p \left(\sup_{s\in[0,T]} (1 + |X_s|^p + |U_s^{\dot a}|^p)|\Re|^{\frac{p}{2}} + \chi^{|\Re|,p} \right) \right] .$$

Restricting to the case where $t \in \Re$, we deduce from (5.11) and (5.12) that

$$|Y_t^i - (\widetilde Y_t^{\Re})^i|^2 \le C_L^p \left(\mathbb{E}_t \left[|N^{\dot a}|^p \sup_{s\in[0,T]} (1 + |X_s|^p + |Y_s|^p) \right]^{\frac{2}{p}} |\Re| + \mathbb{E}_t \left[|N^{\dot a}|^p |\chi^{|\Re|,p} \right]^{\frac{2}{p}} \right) .$$

Using Cauchy-Schwartz inequality and Proposition 5.1 with the last inequality, we obtain

$$|Y_t^i - (\widetilde Y_t^{\Re})^i|^2 \le C_L^p \left(\beta^X |\Re| + \beta^X \mathbb{E}_t \left[|\chi^{|\Re|,p}|^2 \right]^{\frac{1}{p}} \right) .$$

Using again Cauchy Schwartz inequality and defining $M_t := \mathbb{E}_t \left[|\chi^{|\Re|,p}|^2 \right]$, we get

$$\mathbb{E}\left[\sup_{t\in\Re} |Y_t^i - (\widetilde Y_t^{\Re})^i|^2 \right] \le C_L^p \left(|\Re| + \mathbb{E}\left[\sup_{t\in[0,T]} |M_t|^{\frac{2}{p}} \right]^{\frac{1}{2}} \right) . \tag{5.15}$$

Combining Burkholder-Davis-Gundy and convexity inequalities with (3.2), we compute

$$\mathbb{E}\left[\sup_{t\in[0,T]} |M_t|^{\frac{2}{p}} \right] \le C_L^p \left(|M_0|^{\frac{2}{p}} + \mathbb{E}\left[|M_T|^2 \right]^{\frac{1}{p}} \right) \le C_L^p \mathbb{E}\left[|\chi^{|\Re|,p}|^4 \right]^{\frac{1}{p}} \le C_L^p |\kappa|^{\frac{4}{p}} |\Re|^2 .$$

Plugging this expression in (5.15), we deduce (5.6) from the condition $\kappa|\Re| \le L$ and the arbitrariness of i.

Step 3. We finally consider the particular case where the cost functions are constant. Following the same arguments as in Step 1., we observe that (5.10) turns into

$$\left(\int_t^T \left| A_s^{\dot a} - A_s^a \right|^2 \mathrm{d}s \right)^{\frac{p}{2}} \le C_L^p |N^{\dot a}|^p (1 + \sup_{s\in[0,T]} |X_s|^p) |\Re|^{\frac{p}{2}} ,$$

and that $A_r^{\dot a} - A_r^a = 0$ for $r \in \Re$. The same reasoning as in Step 2. then leads to

$$|Y_t^i - \widetilde Y_t^i|^2 \le C_L^2 \mathbb{E}_t \left[|N^{\dot a}|^p \sup_{s\in[0,T]} (1 + |X_s|^p + |\dot Y_s|^p) \right]^{\frac{2}{p}} |\Re| .$$

Using Cauchy Schwartz and Proposition 5.1 concludes the proof. □

We now present the main result of this section, which allows to control the error between the solutions of the continuously and the discretely obliquely reflected BSDE at any time between 0 and T.

Theorem 5.3. *Under* $(\mathbf{H}z)$-$(\mathbf{H}c)$*, the following holds*

$$\sup_{t\in[0,T]} \mathbb{E}\Big[|Y_t - \widetilde{Y}^{\Re}_t|^2 + |Y_t - Y^{\Re}_t|^2\Big] + \mathbb{E}\Big[\int_0^T |Z_s - Z^{\Re}_s|^2 \mathrm{d}s\Big] \quad \leq \quad C^{\epsilon}_L |\Re|^{\frac{1}{2}-\epsilon} \;, \quad \epsilon > 0 \;.$$

If furthermore the cost functions are constant, the previous estimate holds true for $\epsilon = 0$.

In order to prove this theorem, we first state the following lemma discussing the regularity of K.

Lemma 5.1. *Under* $(\mathbf{H}z)$-$(\mathbf{H}c)$*, there exists some positive process* η *satisfying* $||\eta||_{\mathcal{H}^2} \leq C_L$ *and such that, for all* $i \in \mathcal{I}$, $\mathrm{d}K^i_s \leq \eta_s \mathrm{d}s$ *in the sense of random measure.*

Proof. We follow here the main idea of the proof of Proposition 4.2 in [10] and divide the proof in three steps.

Step 1. Fix $i, j \in \mathcal{I}$. We first observe using Itô-Tanaka Formula, that, under $(\mathbf{H}c)$,

$$c^{ij}(X_t) = c^{ij}(X_0) + \int_0^t b^{ij}_s \mathrm{d}s + \int_0^t \nu^{ij}_s \mathrm{d}W_s - \int_0^t \mathrm{d}\Delta^{ij}_s \;, \quad 0 \leq t \leq T \;,$$

where Δ^{ij} is an increasing process and

$$||b^{ij}||_{\mathcal{H}^2} + ||\nu^{ij}||_{\mathcal{H}^2} \leq C_L. \tag{5.16}$$

We then introduce $\Gamma^{ij} := Y^i - Y^j + c^{ij}(X) \geq 0$. Using once again Itô-Tanaka Formula, we compute

$$\begin{aligned}[\Gamma^{ij}_t]^+ = [\Gamma^{ij}_0]^+ &+ \int_0^t (-f^i(X_s, Y^i_s, Z^i_s) + f^j(X_s, Y^j_s, Z^j_s) + b^{ij}_s)\mathbf{1}_{\{\Gamma^{ij}_s>0\}} \mathrm{d}s \\ &+ \int_0^t (\nu^{ij}_s + Z^i_s - Z^j_s)\mathbf{1}_{\{\Gamma^{ij}_s>0\}} \mathrm{d}W_s + \int_0^t \mathbf{1}_{\{\Gamma^{ij}_s>0\}}(-\mathrm{d}K^i_s + \mathrm{d}K^j_s - \mathrm{d}\Delta^{ij}_s) + \frac{1}{2}\int_0^t \mathrm{d}L^{ij}_s \;,\end{aligned}$$

for $0 \leq t \leq T$, where L^{ij} is the local time at 0 of the continuous semi-martingale Γ^{ij}. Since $\Gamma^{ij} \geq 0$ and Δ^{ij}, L^{ij} are increasing processes, we compute

$$\begin{aligned}\mathbf{1}_{\{\Gamma^{ij}_s=0\}} \mathrm{d}K^i_s &\leq (-f^i(X_s, Y^i_s, Z^i_s) + f^j(X_s, Y^j_s, Z^j_s) + b^{ij}_s)\mathbf{1}_{\{\Gamma^{ij}_s=0\}} \mathrm{d}s + \mathbf{1}_{\{\Gamma^{ij}_s=0\}} \mathrm{d}K^j_s \\ &\leq C_L(1 + |X_s| + \sup_{\ell\in\mathcal{I}} |Y^\ell_s| + \sup_{\ell,k\in\mathcal{I}} |b^{\ell k}_s|)\mathrm{d}s + \mathbf{1}_{\{\Gamma^{ij}_s=0\}} \mathrm{d}K^j_s \;, \end{aligned} \tag{5.17}$$

for $0 \le s \le T$, where we used $(\mathbf{H}z)$ in order to obtain the last inequality.

Step 2. We now prove that

$$\mathbf{1}_{\{\Gamma_s^{ij}=0\}} \mathrm{d}K_s^j = 0\,, \tag{5.18}$$

in the sense of random measure. We first observe that $\mathbf{1}_{\{\Gamma_s^{ij}=0\}} \mathrm{d}K_s^j = \gamma_s^{ij} \mathrm{d}K_s^j$ with $\gamma_s^{ij} := \mathbf{1}_{\{\Gamma_s^{ij}=0\}} \mathbf{1}_{\{Y_s^j - \mathcal{P}^j(X_s,Y_s)=0\}}$. Indeed, if $\mathbf{1}_{\{Y_s^j - \mathcal{P}^j(X_s,Y_s)>0\}} \mathrm{d}K_s^j$ were a positive random measure on $[0,T]$, this would contradict the minimality condition (5.1) for K.

Suppose the existence of a stopping time τ smaller than T, such that

$$\Gamma_\tau^{ij} = 0 \text{ and } Y_\tau^j - \mathcal{P}^j(X_\tau, Y_\tau) = 0. \tag{5.19}$$

By definition of the projection $\mathcal{P}$, we have

$$Y_\tau^j - \mathcal{P}^j(X_\tau, Y_\tau) = Y_\tau^j - Y_\tau^{k_\tau} + c^{jk_\tau}(X_\tau)\,, \tag{5.20}$$

where k_τ takes value in $\mathcal{I}$. Moreover $Y_\tau^i - Y_\tau^{k_\tau} + c^{ik_\tau}(X_\tau) \ge 0$, which leads, combined with (5.19) and (5.20), to $c^{ij}(X_\tau) + c^{jk_\tau}(X_\tau) - c^{ik_\tau}(X_\tau) \le 0$ and contradicts then (3.3).

Thus $\gamma_\tau^{ij} = 0$ for any stopping time τ smaller than T and we deduce that γ^{ij} is undistinguishable from 0, which proves (5.18).

Step 3. To conclude, using once again the minimality condition for K in (5.1), observe that $\mathrm{d}K_s^i = \sum_j \mathbf{1}_{\{\Gamma_s^{ij}=0\}} \mathrm{d}K_s^i \le \eta_s \mathrm{d}s$, with $\eta := C_L(1+|X|+\sup_{\ell\in\mathcal{I}}|Y^\ell|+\sup_{\ell,k\in\mathcal{I}}|b^{\ell k}|)$ which satisfies $\|\eta\|_{\mathcal{H}^2} \le C_L$, recalling (3.2), (5.4) and (5.16). □

Proof of Theorem 5.3.
Fix $t \in [0,T]$ and introduce $\delta\widetilde{Y} := Y - \widetilde{Y}^{\Re}$, $\delta Y := Y - Y^{\Re}$, $\delta Z := Z - Z^{\Re}$ and $\delta f := f(X,Y,Z) - f(X,\widetilde{Y}^{\Re}, Z^{\Re})$. Applying Ito's formula to the *càdlàg* process $|\delta\widetilde{Y}|^2$, we get

$$|\delta\widetilde{Y}_t|^2 + \int_t^T |\delta Z_s|^2 \mathrm{d}s \;=\; |\delta\widetilde{Y}_T|^2 - 2\int_{(t,T]} \delta\widetilde{Y}_{s-} \mathrm{d}\delta\widetilde{Y}_s - \sum_{t<s\le T} |\delta\widetilde{Y}_s - \delta Y_s|^2\,. \tag{5.21}$$

Recalling that $\delta\widetilde{Y}_{s-} = \delta Y_s$, $\int_{(t,T]} \delta Y_s \mathrm{d}K_s^{\Re} \ge 0$ and the Lipschitz property of f, standard arguments lead to

$$\mathbb{E}\Big[|\delta\widetilde{Y}_t|^2 + \int_t^T |\delta Z_s|^2 \mathrm{d}s\Big] \;\le\; C_L \mathbb{E}\Big[\int_t^T \delta Y_s \mathrm{d}K_s\Big] \;\le\; C_L \sum_{j<\kappa} \mathbb{E}\Big[\int_{r_j}^{r_{j+1}} \delta Y_s \mathrm{d}K_s\Big]\,. \tag{5.22}$$

Using the expression of δY and Lemma 5.1, we obtain

$$\delta Y_s \le \delta Y_{r_{j+1}} + \int_s^{r_{j+1}} (\delta f_u + \eta_u)\mathrm{d}u - \int_s^{r_{j+1}} \delta Z_u \mathrm{d}W_u \ , \quad r_j \le s < r_{j+1} \ , \quad j < \kappa \, .$$

Combining $(\mathbf{H}z)$, (3.2), (3.7), (5.4) and the fact that $||\eta||_{\mathcal{H}^2} \le C_L$, we deduce

$$\begin{aligned} \sum_{j<\kappa} \mathbb{E}\Big[\int_{r_j}^{r_{j+1}} \delta Y_s \mathrm{d}K_s\Big] &\le \mathbb{E}\Big[\sum_{j<\kappa} \int_{r_j}^{r_{j+1}}\int_s^{r_{j+1}} (\delta f_u + \eta_u)\mathrm{d}u\mathrm{d}K_s\Big] + \mathbb{E}\Big[\sum_{j<\kappa} \int_{r_j}^{r_{j+1}} \delta Y_{r_{j+1}} \mathrm{d}K_s\Big] \\ &\le C_L|\Re| + \mathbb{E}\Big[K_T \sup_{r\in\Re} |\delta Y_r|\Big] \ . \end{aligned}$$

Plugging this expression in (5.22) and using Cauchy Schwartz inequality together with (5.6) and Proposition 2.1 concludes the proof. □

5.3 Convergence of the discrete-time scheme

Combining the previous results with the control of the error between the discrete-time scheme and the discretely obliquely reflected BSDE derived in Section 4, we obtain the convergence of the discrete time scheme to the solution of the continuously obliquely reflected BSDE. In the next theorem, we detail the corresponding approximation error for different optimal choices of reflection time step $|\Re|$ with respect to the discrete time step $|\pi|$.

Theorem 5.4. *The following holds.*
(i) If $(\mathbf{H}f)$*-*$(\mathbf{H}c)$ *holds, taking* $|\Re| \sim \frac{\log L_{\mathcal{P}}}{-\varepsilon \log|\pi|}$ *for* $\varepsilon > 0$*, we have*

$$\sup_{t\in[0,T]} \mathbb{E}\Big[|Y_t - \widetilde{Y}_t^{\Re,\pi}|^2 + |Y_t - Y_t^{\Re,\pi}|^2\Big] + \mathbb{E}\Big[\int_0^T |Z_s - \bar{Z}_s^{\Re,\pi}|^2 \mathrm{d}s\Big] \le \frac{C_L^\varepsilon}{[-\log(|\pi|)]^{\frac{1}{2}-\varepsilon}} \ .$$

(ii) If f *does not depend on* z *and* $|\pi|L < 1$*, taking similar grids* $\Re = \pi$*, we have*

$$\sup_{i\le n} \mathbb{E}\Big[|Y_{t_i} - Y_{t_i}^{\Re,\pi}|^2 + |Y_{t_i} - \widetilde{Y}_{t_i}^{\Re,\pi}|^2\Big] \quad \le \quad C_L^\varepsilon |\pi|^{1-\varepsilon} \ , \quad \varepsilon > 0 \ ,$$

Moreover under $(\mathbf{H}c)$*,*

$$\sup_{t\in[0,T]} \mathbb{E}\Big[|Y_t - Y_t^{\Re,\pi}|^2 + |Y_t - \widetilde{Y}_t^{\Re,\pi}|^2\Big] \quad \le \quad C_L^\epsilon |\pi|^{\frac{1}{2}-\varepsilon} \ , \quad \varepsilon > 0 \ .$$

(iii) Under $(\mathbf{H}c)$*, if* f *does not depend on* z *and* $|\pi|L < 1$*, taking* $|\Re| \sim |\pi|^{2/3}$*, we get*

$$\mathbb{E}\Big[\int_0^T |Z_s - \bar{Z}_s^{\Re,\pi}|^2 \mathrm{d}s\Big] \le C_L^\varepsilon |\pi|^{\frac{1}{3}-\varepsilon} \ , \quad \varepsilon > 0 \ .$$

(iv) Furthermore, for constant cost functions, (ii) and (iii) hold true with $\varepsilon = 0$*.*

Proof. For $\varepsilon > 0$, setting $\Re$ such that $|\Re| \sim \frac{\log L_{\mathcal{P}}}{-\varepsilon \log |\pi|}$, we obtain combining Proposition 4.2 and Theorem 5.3 that

$$\sup_{t\in[0,T]} \mathbb{E}\Big[|Y_t - \widetilde{Y}_t^{\Re,\pi}|^2 + |Y_t - Y_t^{\Re,\pi}|^2\Big] + \mathbb{E}\Big[\int^T |Z_s - \bar{Z}_s^{\Re,\pi}|^2 \mathrm{d}s\Big] \leq C_L^{\varepsilon}\Big[\Big(\frac{-1}{\log(|\pi|)}\Big)^{\frac{1}{2}-\varepsilon} \vee |\pi|^{\frac{1}{2}-\varepsilon}\Big] .$$

Therefore (i) is proved. Furthermore (ii), (iii) and (iv) are direct consequences of Theorem 4.1 and Theorem 5.2 or Theorem 5.3. □

6 Appendix

6.1 A priori estimates for discretely RBSDEs

We collect here the proofs for a priori estimates given in Proposition 2.1 and Proposition 2.2.

Proof of Proposition 2.1
Observing that on each interval $[r_j, r_{j+1})$, $(Y^{\Re}, \widetilde{Y}^{\Re}, Z^{\Re})$ solves a standard BSDE, existence and uniqueness follow from a concatenation procedure and [21]. The rest of the proof divides in two steps controlling separately $\widetilde{Y}^{\Re}$ and $(Z^{\Re}, K^{\Re})$.

Step 1. *Control of* $\widetilde{Y}^{\Re}$
As in proof of Theorem 2.4 in [15], we consider two non-reflected BSDEs bounding $\widetilde{Y}^{\Re}$. Define the $\mathbb{R}^d$-valued random variable $\breve{\xi}$ and the random map $\breve{F}$ by $(\breve{\xi})^j := \sum_{i=1}^d |\xi|^i$ and $(\breve{F})^j := \sum_{i=1}^d |(F)^i|$ for $1 \leq j \leq d$.
We then denote by $(\breve{Y}, \breve{Z}) \in (\mathcal{S}^2 \times \mathcal{H}^2)^{\mathcal{I}}$ the solution to the following non-reflected BSDE

$$\breve{Y}_t \;=\; \breve{\xi} + \int_t^T \breve{F}(s, \breve{Y}_s, \breve{Z}_s)\mathrm{d}s - \int_t^T \breve{Z}_s \mathrm{d}W_s\ , \quad 0 \leq t \leq T. \tag{6.1}$$

Since all the components of $\breve{Y}$ are similar, $\breve{Y} \in \mathcal{C}$.
We also introduce $({}^0Y, {}^0Z)$ the solution to the BSDE

$${}^0Y_t \;=\; \xi + \int_t^T F(s, {}^0Y_s, {}^0Z_s)\mathrm{d}s - \int_t^T {}^0Z_s \mathrm{d}W_s\ , \quad 0 \leq t \leq T\ .$$

Using a comparison argument on each interval $[r_j, r_{j+1})$ and the monotony property of $\mathcal{P}$, we straightforwardly deduce ${}^0Y \preceq Y^{\Re} \preceq \breve{Y}$.
Since $({}^0Y, \breve{Y})$ are solutions to standard non-reflected BSDEs, usual arguments lead to

$$\sup_{0\leq s\leq T} |\widetilde{Y}_s^{\Re}|^p \;\leq\; \sup_{0\leq s\leq T} |{}^0Y_s|^p + \sup_{0\leq s\leq T} |\breve{Y}_s|^p =: \bar{\beta}\ , \tag{6.2}$$

where the positive random variable $\bar{\beta}$ satisfies classically $\mathbb{E}[\bar{\beta}] \le C_L$, under condition $(\mathbf{C}_p)$ for a given $p \ge 2$.

Step 2. *Control of* $(Z^{\Re}, K^{\Re})$
We fix $t \le T$ and applying Ito's formula to the *càdlàg* process $|\widetilde{Y}^{\Re}|^2$ on $[0,t]$ to derive

$$|\widetilde{Y}^{\Re}_t|^2 = \ |\widetilde{Y}^{\Re}_0|^2 + 2\int_{(0,t]} \widetilde{Y}^{\Re}_{s-} \mathrm{d}\widetilde{Y}^{\Re}_s + \int_{(0,t]} |Z^{\Re}_s|^2 \mathrm{d}s + \sum_{s \le t} (|\widetilde{Y}^{\Re}_s|^2 - |\widetilde{Y}^{\Re}_{s-}|^2 - 2\widetilde{Y}^{\Re}_{s-}\Delta Y^{\Re}_s)\ .$$

Since the last term on the right-hand side is non negative we deduce that

$$\begin{aligned}|\widetilde{Y}^{\Re}_t|^2 + \int_t^T |Z^{\Re}_s|^2 \mathrm{d}s \le |\widetilde{Y}^{\Re}_T|^2 \ &+2\int_t^T \widetilde{Y}^{\Re}_{s-} F(s, \widetilde{Y}^{\Re}_s, Z^{\Re}_s)\mathrm{d}s \\ &+2\int_{(t,T]} \widetilde{Y}^{\Re}_{s-}\mathrm{d}K^{\Re}_s + 2\int_t^T (Z^{\Re}_s \widetilde{Y}^{\Re}_s)\mathrm{d}W_s.\end{aligned}$$

Using standard arguments, together with (6.2) and $(\mathbf{C}_p)$ for a fixed $p \ge 2$, we compute

$$\int_t^T |Z^{\Re}_s|^2 \mathrm{d}s \ \le \ C_L \left(\bar{\beta}^{\frac{2}{p}} + \bar{\beta}^{\frac{1}{p}}(K^{\Re}_T - K^{\Re}_t) + \int_t^T (Z^{\Re}_s \widetilde{Y}^{\Re}_s)\mathrm{d}W_s \right) . \tag{6.3}$$

Moreover, we get from (2.4) and $(\mathbf{C}_p)$ that

$$|K^{\Re}_T - K^{\Re}_t|^2 \ \le \ C_L \left[\bar{\beta}^{\frac{2}{p}} + \int_t^T |Z^{\Re}_s|^2 \mathrm{d}s + \left(\int_t^T Z^{\Re}_s \mathrm{d}W_s \right)^2 \right] . \tag{6.4}$$

Combining (6.3) and (6.4) we obtain

$$\int_t^T |Z^{\Re}_s|^2 \mathrm{d}s \ \le \ \frac{C_L}{\varepsilon}\bar{\beta}^{\frac{2}{p}} + \varepsilon \int_t^T |Z^{\Re}_s|^2 \mathrm{d}s + \varepsilon \left(\int_t^T Z^{\Re}_s \mathrm{d}W_s \right)^2 + C_L \int_t^T (Z^{\Re}_s \widetilde{Y}^{\Re}_s)\mathrm{d}W_s, \tag{6.5}$$

for any $\varepsilon > 0$. Elevating the previous estimate to the power $p/2$, it follows from Burkholder-Davis-Gundy inequality that

$$\begin{aligned}\mathbb{E}_t\left[\left(\int_t^T |Z^{\Re}_s|^2 \mathrm{d}s \right)^{\frac{p}{2}} \right] \ &\le C_L^p \left(\varepsilon^{-\frac{p}{2}} \mathbb{E}_t[\bar{\beta}] + \varepsilon^{\frac{p}{2}} \mathbb{E}_t\Big[(\textstyle\int_t^T |Z^{\Re}_s|^2 \mathrm{d}s)^{\frac{p}{2}} \Big] + \mathbb{E}_t\Big[(\textstyle\int_t^T |Z^{\Re}_s \widetilde{Y}^{\Re}_s|^2 \mathrm{d}s)^{\frac{p}{4}} \Big] \right), \\ &\le C_L^p \left(\varepsilon^{-\frac{p}{2}} \mathbb{E}_t[\bar{\beta}] + \varepsilon^{-\frac{p}{2}} \mathbb{E}_t\Big[\sup_{s \in [t,T]} |\widetilde{Y}^{\Re}_s|^p \Big] + \varepsilon^{\frac{p}{2}} \mathbb{E}_t\Big[(\textstyle\int_t^T |Z^{\Re}_s|^2 \mathrm{d}s)^{\frac{p}{2}} \Big] \right)\end{aligned}$$

Using (6.2) and $(\mathbf{C}_p)$, we deduce, for ε small enough,

$$\mathbb{E}_t\left[(\int_t^T |Z^{\Re}_s|^2 \mathrm{d}s)^{\frac{p}{2}} \right] \le C_L^p \mathbb{E}_t[\bar{\beta}] \ . \tag{6.6}$$

Taking (6.4) up to the power $\frac{p}{2}$, and combining Burkholder-Davis-Gundy inequality with (6.6) yields $\mathbb{E}_t\big[|K^{\Re}_T - K^{\Re}_t|^p\big] \le C_L^p \mathbb{E}_t[\bar{\beta}]$, which concludes the proof of the Proposition, recalling $(\mathbf{C}_p)$. □

Proof of Proposition 2.2
Fix $(t,i) \in [0,T] \times \mathcal{I}$ and $p \geq 2$. According to the identification of (U^{a^*}, V^{a^*}) with $(\widetilde{Y}^{a^*}, Z^{a^*})$, obtained in the proof of Theorem 2.1, we deduce from Proposition 2.1 the expected controls on U^{a^*} and V^{a^*}. Writing the equation satisfied by (U^{a^*}, V^{a^*}) and using standard arguments for BSDEs, we observe that

$$\mathbb{E}_t\Big[|A_T^{a^*}|^p\Big] \leq C_L^p \left(\mathbb{E}_t\left[\sup_{s\in[t,T]} |U_s^{a^*}|^p + \left(\int_t^T |V_s^{a^*}|^2 ds \right)^{\frac{p}{2}} \right] + |A_t^{a^*}|^p \right) .$$

By definition of a^* and (2.2), we have $|A_t^{a^*}| \leq \max_{k\neq i} |C_t^{i,k}|$, which plugged in the previous inequality leads to $\mathbb{E}_t\big[|A_T^{a^*}|^p\big] \leq C_L^p \mathbb{E}_t\big[\bar{\beta}\big]$, recalling $(\mathbf{C}_p)$.
We finally complete the proof, noticing from (2.2) that $\mathbb{E}_t\big[|N^{a^*}|^p\big] \leq C_L^p \mathbb{E}_t\big[|A_T^{a^*}|^p\big]$. □

6.2 A priori estimates for the Euler scheme

This paragraph provides the proof of Proposition 4.1, concerning a-priori estimates for the Euler scheme associated to RBSDEs.

Proof of Proposition 4.1
The proof follows exactly the same arguments as in Step 1 of the proof of Proposition 2.1 above. The only difficulty is the use of a comparison argument for Euler Scheme that we provide right below in Lemma 6.1. □

We detail here a comparison theorem for discrete-time schemes of BSDEs in the case where the driver does not depend on the variable z.
For $k = 1, 2$, let ξ_k be a square integrable random variable and $\psi_k : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^d$ a $L-$Lipschitz generator function. We suppose that $\xi_1 \geq \xi_2$ and $\psi_1 \geq \psi_2$ on $\mathbb{R}^m \times \mathbb{R}^d$. For a time grid π, we denote by $Y^{\pi,k}$ the discrete-time scheme starting from the terminal condition $Y_T^{\pi,k} := \xi_k$ and computing recursively, for $i = n-1, \ldots, 0$,

$$Y_{t_i}^{\pi,k} \quad = \quad \mathbb{E}\left[Y_{t_{i+1}}^{\pi,k} \mid \mathcal{F}_{t_i} \right] + (t_{i+1} - t_i)\psi_k(X_{t_i}^{\pi}, Y_{t_i}^{\pi,k}) \, . \tag{6.7}$$

Lemma 6.1. *For any π such that $|\pi|L < 1$, we have $Y_{t_i}^{\pi,1} \geq Y_{t_i}^{\pi,2}$, $i \leq n$.*

Proof. Since the results holds true on the grid point $t_n = T$ and follows from a backward induction on π, we just prove $Y_{t_{n-1}}^{\pi,1} \geq Y_{t_{n-1}}^{\pi,2}$. Using (6.7), we compute

$$Y_{t_{n-1}}^{\pi,1} - Y_{t_{n-1}}^{\pi,2} = \mathbb{E}_{t_{n-1}}\left[\xi_1 - \xi_2 \Big| \mathcal{F}_{t_{n-1}} \right] + (T - t_{n-1})\Lambda_{n-1}\left(Y_{t_{n-1}}^{\pi,1} - Y_{t_{n-1}}^{\pi,2} \right) + \Delta_{n-1} \, , \tag{6.8}$$

where $\Delta_{n-1} := \psi_1(X_{t_{n-1}}^{\pi} Y_{t_{n-1}}^{\pi,2}) - \psi_2(X_{t_{n-1}}^{\pi} Y_{t_{n-1}}^{\pi,2}) \geq 0$ and

$$\Lambda_{n-1} \;:=\; \begin{cases} \frac{\psi_1(X_{t_{n-1}}^{\pi} Y_{t_{n-1}}^{\pi,1}) - \psi_1(X_{t_{n-1}}^{\pi} Y_{t_{n-1}}^{\pi,2})}{Y_{t_{n-1}}^{\pi,1} - Y_{t_{n-1}}^{\pi,2}} & \text{if} \quad Y_{t_{n-1}}^{\pi,1} - Y_{t_{n-1}}^{\pi,2} \neq 0, \\ 0 & \text{else} \, . \end{cases} \tag{6.9}$$

Since ψ_1 is $L-$Lipschitz, the condition $|\pi|L < 1$, implies $(T - t_{n-1})\Lambda_{n-1} < 1$. Plugging this estimate, $\Delta_{n-1} \geq 0$ and $\xi_1 \geq \xi_2$ and ψ_1 in (6.8), we complete the proof. □

6.3 A priori estimates for continuously RBSDEs

This last paragraph is dedicated to the proof of Proposition 5.1.

Proof of Proposition 5.1

The proof of (5.4) is a direct adaptation of the proof of Proposition 2.1. The only difference is in Step 1: we approximate (Y, Z, K) by a sequence of penalized BSDEs (see the proof of Theorem 2.4 in [15] or Step 3 in the proof of Theorem 5.3) which are bounded by ${}^0Y$ and $\check{Y}$. Estimate (5.5) follows from the exact same arguments as the one used in the proof of Proposition 2.2. □

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