Probabilistic Representation and Approximation for Coupled Systems of Variational Inequalities

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Abstract

Our study is dedicated to the probabilistic representation and numerical approximation of solutions to coupled systems of variational inequalities. The dynamics of each component of the solution is driven by a different linear parabolic operator and suffers a non-linear dependence in all the components of the solution. This dynamics is combined with a global structural constraint between all the components of the solution including the practical example of optimal switching problems. In this paper, we interpret the unique viscosity solution to this type of coupled systems of variational inequalities as the solution to one-dimensional constrained BSDEs with jumps introduced recently in [6]. In the spirit of [3], this new representation allows for the introduction of a natural entirely probabilistic numerical scheme for the resolution of these systems.

Key words: BSDE with jumps, variational inequalities, viscosity solutions, Monte Carlo simulations, Switching problems.
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1 Introduction

Pardoux and Peng [12] developed the theory of Backward Stochastic Differential Equations, providing a probabilistic representation of solution to quasilinear parabolic PDEs. Coupling the diffusion process with a pure jump process, Pardoux Pradeilles and Rao [13] extend this representation to systems of coupled semilinear PDEs with different linear differential operators on each line. Introducing restrictions on the domain of the backward process, El Karoui et al [5] cover the class of variational inequalities. Constraining instead the jump part of the solution, Kharroubi, Ma, Pham and Zhang [11] allow to consider quasilinear variational inequalities.

The focus of this note is to extend this type of Feynman-Kac representation to the more general class of coupled systems of quasilinear variational inequalities, arising for example in optimal impulse or switching problems. We will typically consider systems of PDE of the form

$$\Big[-\frac{\partial v_i}{\partial t} - \mathcal{L}^i v_i - f_i(., (v_k)_{1\le k\le m}, \sigma_i^\top D_x v_i)\Big] \wedge \min_{1\le j\le m} h_{i,j}(., v_i, v_j, \sigma_i^\top D_x v_i) = 0, \tag{1.1}$$

$$\text{on } \mathcal{I}\times[0,T)\times\mathbb{R}^d, \qquad \text{with terminal condition} \quad v_i(T,.) = g_i \text{ on } \mathcal{I}\times\mathbb{R}^d, \tag{1.2}$$

where, for any $i \in \mathcal{I} := \{1, \ldots, m\}$, $\mathcal{L}^i$ is a linear second order local operator

$$\mathcal{L}^i v_i(t,x) \quad := \quad b_i(x) \cdot D_x v_i(t,x) + \frac{1}{2}\mathrm{tr}(\sigma_i \sigma_i^\top(x) D_x^2 v_i(t,x))\,, \tag{1.3}$$

and b, σ, f, h and g are Lipschitz continuous functions. As observed by [2], this PDE appears in the resolution of optimal switching problems as well as stochastic target problems with jumps. The major difficulty relies in the coupling between all the components $(v_i)_{i \leq m}$ of the solution and the use of different linear operators at each line. When m is high, the numerical resolution of (1.1)-(1.2) by classical PDE approximation methods is very tricky and highly computational, and we intend to provide here a probabilistic representation to (1.1)-(1.2) leading to an efficient probabilistic numerical scheme.Whenever b and σ are independent of the regime $i \in \mathcal{I}$ and the constraint functions are of the form $h_{i,j} : (., y_i, y_j, .) \mapsto y_i - y_j - c_{i,j}$, Hu and Tang [10] interpret the vector solution to (1.1)-(1.2) as a multi-dimensional BSDE with terminal condition and oblique reflections. The challenging derivation of a convergent numerical approximation for this type of BSDE is of great interest and is currently under study. The approach of this paper relies instead on a recent reinterpretation of obliquely multi-dimensional reflected BSDEs in terms of one-dimensional constrained BSDE with jumps, as introduced in [6]. The idea is to consider, as in [13], a random regime driven by a pure jump transmutation process, allowing to retrieve simultaneously some information concerning all the components of the solution.

Being given a d-dimensional Brownian motion W and an independent Poisson measure μ on $\mathbb{R}_+ \times \mathcal{I}$, we consider, for any initial condition $e := (t, i, x) \in [0, T] \times \mathcal{I} \times \mathbb{R}^d =: E$, the unique $\mathcal{I} \times \mathbb{R}^d$-valued solution (I^e_s, X^e_s) of the SDE:

$$\left\{ \begin{array}{lcl} I_s & = & i + \int_t^s \int_{\mathcal{I}} (j - I_{r^-}) \mu(dr, dj) \\ X_s & = & x + \int_t^s b(I_r, X_r) dr + \int_t^s \sigma(I_r, X_r) \cdot dW_r \end{array} \right. , \qquad t \leq s \leq T\,. \tag{1.4}$$

Formally, given a smooth solution v to (1.1)-(1.2), the process $Y := v_{I^e}(., X^e_.)$ satisfies

$$Y_t = g_{I^e_T}(X^e_T) + \int_t^T f_{I^e_s}(X^e_s, Y_s + U_s, Z_s) ds + K_T - K_t - \int_t^T Z_s \cdot dW_s - \int_t^T \int_{\mathcal{I}} U_s(j) \mu(ds, dj) \tag{1.5}$$

on $[0,T]$, where we denote $Z_s := \sigma^\top_{I^e_{s^-}}(X^e_s) D_x v_{I^e_{s^-}}(s, X^e_s)$, $U_s(.) := v_.(s, X^e_s) - v_{I^e_{s^-}}(s, X^e_s)$, and $K_s := \int_0^s [-\frac{\partial v_{I^e_u}}{\partial t} - \mathcal{L}^{I^e_u} v_{I^e_u} - f_{I^e_u}(., (v_k)_{1 \leq k \leq m}, \sigma^\top_{I^e_u} D_x v_{I^e_u})(u, X^e_u)] du$. Since v satisfies (1.1), we expect the following constraint to be satisfied:

$$h_{I^e_{s-}, j}(X^e_s, Y_{s-}, Y_{s-} + U_s(j), Z_s) \quad \geq \quad 0\,, \qquad j \in \mathcal{I},\ \ t \leq s \leq T\,. \tag{1.6}$$

The BSDE (1.5) combined with constraint (1.6) enters into the class of constrained BSDEs with jumps and admits a unique minimal solution under mild conditions on the coefficients. We reinterpret the Y-component of the solution as the unique viscosity solution to the coupled system of variational inequalities (1.1)-(1.2). This new Feynman-Kac representation is meaningful to the BSDE literature since:

- It extends the results of [11] to more general constraints and driver functions depending on U, allowing a strong coupling between the dynamics of the value function components and gives a minimality condition in some particular cases.

- It enlarges the conclusions of Peng and Xu [14] derived in the no-jump case.
- It offers a PDE representation to reflected BSDEs with interconnected obstacles introduced in [9], since they relate directly to constrained BDSE with jumps, see [6].
- It generalizes the use of diffusion-transmutation process in [13] to systems of variational inequalities.

This representation leads to a natural probabilistic algorithm for the resolution of (1.1)-(1.2). The constrained BSDE with jumps is replaced by a penalized BSDEs with jumps, which is approximated by the dicrete time scheme studied in [3] and [7]. This leads to a convergent numerical scheme based on time discretization, Monte Carlo simulations and projections.

The rest of the paper is organized as follows: In Section 2, we discuss existence, uniqueness, penalization and give a minimality condition for constrained BSDE with jumps (1.5)-(1.6). Section 3 presents the viscosity properties and the last section details the numerical approximation.

Notations. Throughout this paper, we are given a finite horizon T and a probability space $(\Omega, \mathcal{G}, \mathbf{P})$ endowed with a d-dimensional standard Brownian motion $W = (W_t)_{t\geq 0}$, and an independent Poisson random measure μ on $\mathbb{R}_+ \times \mathcal{I}$, with intensity measure $\lambda(di)dt$ for some positive finite measure λ on $\mathcal{I} := \{1, \dots, m\}$. We denote $E := [0,T] \times \mathcal{I} \times \mathbb{R}^d$. For a smooth function $\varphi :\ [0,T] \times \mathbb{R}^d \times \mathcal{I} \to \mathbb{R}$, $\frac{\partial \varphi}{\partial t}$, $D_x\varphi$ and $D_x^2\varphi$ denote resp. the derivative of φ w.r.t. t, the gradient and the Hessian matrix of φ w.r.t. x. The dependence in $\omega \in \Omega$ is omitted whenever explicit.

2 Constrained Forward Backward SDEs with jumps

We present in this section the constrained Forward Backward SDEs with jumps and recall the existence and uniqueness results of [6]. We discuss the correspondence between the value function associated to Y and the U component of the solution. Under additional regularity of the value function, we provide a Skorohod type minimality condition for the considered BSDE.

2.1 Existence and uniqueness of a minimal solution

As discussed above, the forward process is a transmutation-diffusion process composed by a pure jump process I and a diffusion without jump X whose dynamics depends on I. For any initial condition $e := (t, i, x) \in E$, (I^e, X^e) is the unique solution to (1.4), starting from (i, x) at time t.

For any initial condition $e \in E$, a solution to the constrained BSDE with jumps is a quadruplet $(Y^e, Z^e, U^e, K^e) \in \mathcal{S}^2 \times \mathbf{L}^2_W \times \mathbf{L}^2_{\tilde{\mu}} \times \mathbf{A}^2$ satisfying (1.5)-(1.6), where,

- $\mathcal{S}^2$ is the set of real valued $\mathcal{G}$-adapted *càdlàg* processes Y on $[0,T]$ such that $\|Y\|_{\mathcal{S}^2} := \mathbb{E}\left[\sup_{0\leq r\leq T} |Y_r|^2\right]^{\frac{1}{2}} < \infty$,
- $\mathbf{L}^2_W$ is the set of predictable $\mathbb{R}^d$-valued processes Z s. t. $\|Z\|_{\mathbf{L}^2_W} := \mathbb{E}\left[\left(\int_0^T |Z_r|^2 dr\right)\right]^{\frac{1}{2}} < \infty$,
- $\mathbf{L}^2_{\tilde{\mu}}$ is the set of $\mathcal{P} \otimes \sigma(\mathcal{I})$ measurable maps $U : \Omega \times [0,T] \times \mathcal{I} \to \mathbb{R}$ such that $\|U\|_{\mathbf{L}^2_{\tilde{\mu}}} := \mathbb{E}\left[\int_0^T \int_{\mathcal{I}} |U_s(j)|^2 \lambda(dj) ds\right]^{\frac{1}{2}} < \infty$,
- $\mathbf{A}^2$ is the closed subset of $\mathcal{S}^2$ composed by nondecreasing processes K with $K_0 = 0$.

Furthermore, (Y,Z,U,K) is referred to as the minimal solution to (1.5)-(1.6) whenever, for any other solution (Y',Z',U',K'), we have $Y \le Y'$ a.s.. In order to ensure existence and uniqueness of a minimal solution to (1.5)-(1.6) for any initial condition, we impose the following assumptions.

(H0) The following holds:

(i) For any $(i,j) \in \mathcal{I}^2$, f_i, g_i and $h_{i,j}$ are Lipschitz functions with linear growth.

(ii) The function $h_{i,j}(x,y,.,z)$ is non-increasing for all $(i,x,y,z,j) \in \mathcal{I} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{I}$.

(iii) There exist two constants $C_1 \ge C_2 > -1$ and a measurable map $\gamma : \mathcal{I} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times [\mathbb{R}^{\mathcal{I}}]^2 \times \mathcal{I} \to [C_2, C_1]$ such that, for any $(i,x,y,z,u,u') \in \mathcal{I} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times [\mathbb{R}^{\mathcal{I}}]^2$,

$$f_i(x,y+u,z) - f_i(x,y+u',z') \quad \le \quad \int_{\mathcal{I}} (u_j - u'_j)\gamma(i,x,y,z,u,u',j)\lambda(dj)\,.$$

(H1) For any $e = (t,i,x) \in E$, there exists a quadruple $(\tilde{Y}^e, \tilde{Z}^e, \tilde{U}^e, \tilde{K}^e) \in \mathcal{S}^{\mathbf{2}} \times \mathbf{L}^2_W \times \mathbf{L}^2_{\tilde{\mu}} \times \mathbf{A}^2$ solution to (1.5)-(1.6), with $\tilde{Y}^e_t = \tilde{v}_{I^e_t}(t, X^e_t)$, for some deterministic function $\tilde{v}$ satisfying $|\tilde{v}_i(t,x)| \le C(1+|x|)$ on E.

We provide in Remark 3.2 a more tractable sufficient condition under which **(H1)** holds.

Theorem 2.1. *Suppose* **(H0)**-**(H1)** *holds. For any $e := (t,i,x) \in E$, there exists a unique quadruple $(Y^e, Z^e, U^e, K^e) \in \mathcal{S}^{\mathbf{2}} \times \mathbf{L}^2_W \times \mathbf{L}^2_{\tilde{\mu}} \times \mathbf{A}^2$ minimal solution to* (1.5)-(1.6), *and $v_i : (t,x) \mapsto Y^{t,i,x}_t$ defines a deterministic map from E into $\mathbb{R}$.*

Proof. This result is a direct application of Theorem 2.1 in [6]. □

2.2 Related penalized BSDE

For any initial condition $e \in E$ and $n \in \mathbb{N}$, we denote by $(Y^{e,n}, Z^{e,n}, U^{e,n})$ the solution to the following penalized BSDE with jump

$$\begin{aligned} Y_t &= g_{I^e_T}(X^e_T) + \int_t^T f_{I^e_s}(X^e_s, Y_s + U_s, Z_s)ds - \int_t^T \int_{\mathcal{I}} U_s(j)\mu(ds,dj) - \int_t^T Z_s \cdot dW_s \\ &+ n\int_t^T \int_{\mathcal{I}} [h_{I^e_{s^-},j}(X^e_s, Y_{s^-}, Y_{s^-} + U_s(j), Z_s)]^- \lambda(dj)ds\,, \qquad 0 \le t \le T\,. \end{aligned} \tag{2.1}$$

Under **(H0)**, following the arguments of [1], we verify that there exists a unique solution to (2.1) and introduce $K^{e,n} := \int_0^{\cdot} \int_{\mathcal{I}} [h_{I^e_{s^-},j}(X^e_s, Y^{e,n}_{s^-}, Y^{e,n}_{s^-} + U^{e,n}_s(j), Z^{e,n}_s)]^- \lambda(dj)ds$, for any $e \in E$ and $n \in \mathbb{N}$. Furthermore, it converges to the solution of (1.5)-(1.6).

Proposition 2.1. *If* **(H0)**-**(H1)** *holds, $(Y^{e,n})_{n\in\mathbb{N}}$ converges increasingly to Y^e, for any $e \in E$. Additionally, If the process Y^e is quasi-left continuous in time, we have*

$$\|Y^e - Y^{e,n}\|_{\mathcal{S}^2} + \|Z^e - Z^{e,n}\|_{\mathbf{L}^2_W} + \|U^e - U^{e,n}\|_{\mathbf{L}^2_{\tilde{\mu}}} + \|K^e - K^{e,n}\|_{\mathcal{S}^2} \underset{n\to\infty}{\longrightarrow} 0\,, \quad e \in E \tag{2.2}$$

Proof. Fix $e \in E$ and observe from Proposition 2.1 in [6] that $Y^{e,n}$ converges increasingly to Y^e. Since μ is a Poisson measure, the process $Y^{e,n}$ is quasi-left continuous. If Y^e has the same regularity, the predictable projections of Y^e and $Y^{e,n}$ are simply given by $(Y^e_{t^-})_t$ and $(Y^{e,n}_{t^-})_t$ leading to $Y^e_{t^-} = \lim_{n\to\infty} Y^{e,n}_{t^-}$. We deduce from the weak version of Dini's theorem, see [4] p.

202, that $Y^{e,n}$ converges uniformly to Y^e on $[0,T]$, and the dominated convergence theorem leads to $\|Y^e - Y^{e,n}\|_{\mathcal{S}^2} \underset{n\to\infty}{\longrightarrow} 0$. Combined with standard estimates of the form

$$\|Z^{e,n+p} - Z^{e,n}\|^2_{\mathbf{L}^2_W} + \|U^{e,n+p} - U^{e,n}\|^2_{\mathbf{L}^2_{\tilde{\mu}}} + \|K^{e,n+p} - K^{e,n}\|^2_{\mathcal{S}^2} \quad \leq \quad C\|Y^{e,n+p} - Y^{e,n}\|^2_{\mathcal{S}^2},$$

this implies that the sequences (Z^n), (U^n) and (K^n) are Cauchy and hence convergent. □

Remark 2.1. Under the additional Assumption **(H2)** below, $(v_i)_{i\in\mathcal{I}}$ interprets as the unique viscosity solution to (1.1)-(1.2), see Theorem 3.2. In this case, $(v_i)_{i\in\mathcal{I}}$ is continuous, $Y_t = v_{I_t}(t, X_t)$ is quasi left continuous and Proposition 2.1 holds.

We denote by $(v^n)_{n\in\mathbb{N}}$ the sequence of deterministic functions defined by $v^n : e \in E \mapsto Y^{e,n}_t$ and we shall use indifferently the notation $v^n(t,i,x)$ or $v^n_i(t,x)$, for $(t,i,x) \in E$. Under **(H0)-(H1)**, we know from Proposition 2.1 that v is the pointwise limit of $(v^n)_{n\in\mathbb{N}}$.

2.3 Representation of U and minimality condition

Proposition 2.2. *Let* **(H0)-(H1)** *hold. For any $e \in E$ and stopping time θ valued in $[t,T]$, we have $Y^e_\theta = v_{I^e_\theta}(\theta, X^e_\theta)$, and the process U represents as*

$$U^e_s(j) \quad = \quad v_j(s, X^e_s) - v_{I^e_{s^-}}(s, X^e_s)\,, \qquad j \in \mathcal{I}, \qquad t \leq s \leq T\,. \tag{2.3}$$

Proof. According to Proposition 2.1, we simply need to provide similar representations for the penalized BSDE (2.1). Fix $e \in E$. For any stopping time θ valued in $[t,T]$, uniqueness of solution to (2.1) and the Markov property of (I^e, X^e) directly lead to $Y^{e,n}_\theta = v^n_{I^e_\theta}(\theta, X^e_\theta)$. Denoting $\tilde{U}^{e,n}_s(j) := v^n_j(s, X^e_s) - v^n_{I^e_{s^-}}(s, X^e_s)$, for $j \in \mathcal{I}$ and $0 \leq s \leq T$, we deduce from (2.1) that

$$\int_{\mathcal{I}} \tilde{U}^{e,n}_s(j)\mu(ds,dj) \quad = \quad Y^{e,n}_s - Y^{e,n}_{s-} \quad = \quad \int_{\mathcal{I}} \tilde{U}^{e,n}_s(j)\mu(ds,dj)\,, \qquad 0 \leq s \leq T\,.$$

Therefore $\mathbf{E}\left[\int_0^T \int_{\mathcal{I}} (U^{e,n}_s(j) - U^{e,n}_s(j))^2 \lambda(dj)ds\right] = 0$ and the proof is complete. □

Under an extra regularity assumption on the function v satisfied under Assumption **(H2)** below, the previous representation leads to a Skorohod type minimality condition for (1.5)-(1.6).

Corollary 2.1. *Let* **(H0)-(H1)** *hold. Suppose $(v_i)_{i\in\mathcal{I}}$ is continuous and the function h does not depend on z. Then, for any $e \in E$, the minimal solution (Y^e, Z^e, U^e, K^e) satisfies*

$$\int_t^T \min_{j\in\mathcal{I}} \left[h_{I^e_{s^-},j}(X^e_s, Y^e_{s^-}, Y^e_{s^-} + U^e_s(j))\right] dK^e_s \quad = \quad 0. \tag{2.4}$$

Proof. Fix $e \in E$. Since $(v_i)_{i\in\mathcal{I}}$ is continuous, the process Y^e inherits the quasi-left continuity of (I^e, X^e). Combining (2.3) and Proposition 2.1 leads to $\max_{j\in\mathcal{I}} \|U^e(j) - U^{e,n}(j)\|_{\mathcal{S}^2} \underset{n\to\infty}{\longrightarrow} 0$.We deduce from (2.2) and Lemma 5.8 in [8], which also holds for càglàd functions, that

$$\int_t^T \min_{j\in\mathcal{I}} \left[h_{I^e_{s^-},j}(X^e_s, Y^{e,n}_{s^-}, Y^{e,n}_{s^-} + U^{e,n}_s(j))\right] dK^{e,n}_s \underset{n\to\infty}{\longrightarrow} \int_t^T \min_{j\in\mathcal{I}} \left[h_{I^e_{s^-},j}(X^e_s, Y^e_{s^-}, Y^e_{s^-} + U^e_s(j))\right] dK^e_s\,.$$

Since $\int_t^T \min_{j\in\mathcal{I}} \left[h_{I^e_{s^-},j}(X^e_s, Y^{e,n}_{s^-}, Y^{e,n}_{s^-} + U^{e,n}_s(j))\right] dK^{e,n}_s \leq 0$ and (1.6) holds, we get (2.4). □

3 Link with coupled systems of variational inequalities

In this section, we interpret the minimal solution to (1.5)-(1.6) as the unique viscosity solution to the PDE (1.1)-(1.2), thus generalizing the representation derived in [11], [13] and [14].

3.1 Viscosity properties of the penalized BSDE

The penalized parabolic integral partial differential equation (IPDE) associated to (2.1) is naturally defined, for each $n \in \mathbb{N}$, by

$$\begin{cases} -\dfrac{\partial \varphi_i}{\partial t} - \mathcal{L}^i\varphi_i - f_i(.,(\varphi_j)_{j\in\mathcal{I}}, \sigma_i^\top D_x\varphi_i) - n\displaystyle\int_{\mathcal{I}} \left[h_{i,j}\big(.,\varphi_i,\varphi_j,\sigma_i^\top D_x\varphi_i\big)\right]^-\lambda(dj) = 0 \\ \text{on } [0,T)\times\mathbb{R}^d\times\mathcal{I}, \qquad \text{and} \qquad v_i(T,x) = g_i \text{ on } \mathcal{I}\times\mathbb{R}^d\,, \end{cases} \tag{3.1}$$

where $\mathcal{L}$ is the m-dimensional Dynkin operator associated to X and defined in (1.3). Since the penalized BSDE enters into the class of BSDE with jumps studied by Pardoux, Pradeilles and Rao [13], we deduce the following Feynman-Kac representation result.

Proposition 3.1. *Under* **(H0)-(H1)***, the functions* $(v^n)_n$ *are continuous viscosity solutions to* (3.1)*. Indeed, for any* $n \in \mathbb{N}$*,* $v^n(T,.) = g$ *and, for any* $(i,t,x) \in \mathcal{I}\times[0,T)\times\mathbb{R}^d$ *and* $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^d)$ *such that* (t,x) *is a null global minimum (resp. max.imum) of* $(v_i^n - \varphi)$*, we have*

$$\left[-\frac{\partial\varphi}{\partial t} - \mathcal{L}^i\varphi - f_i(.,(v_j^n)_{j\in\mathcal{I}},\sigma_i^\top D_x\varphi) - n\int_{\mathcal{I}}[h_{i,j}(.,v_i^n,v_j^n,\sigma_i^\top D_x\varphi)]^-\lambda(dj)\right](t,x) \;\geq\; (\textit{resp.} \;\leq)\; 0.$$

Proof. Fix $n \in \mathbb{N}$. The continuity of v^n follows from similar arguments as in the proof of Lemma 2.1 in [13]. According to the representations detailed in the proof of Proposition 2.2, the viscosity property of v^n fits in the framework of Theorem 4.1 in [13], up to the comparison theorem for BSDE, which is replaced by Theorem 2.5 in Royer [15]. □

3.2 Viscosity properties of the constrained BSDE with jumps

Formally, passing to the limit in (3.1) when n goes to infinity, we expect v to be solution of (1.1) on $[0,T)\times\mathbb{R}^d\times\mathcal{I}$. As for the boundary condition, we can not expect to have $v(T^-,.) = g$, and we shall consider the relaxed boundary condition given by

$$\min\left[v_i - g_i\;,\;\min_{j\in\mathcal{I}} h_{i,j}\left(.,v_i,v_j,\sigma^\top D_x v_i\right)\right](T^-,x) \quad=\quad 0 \text{ on } \mathcal{I}\times\mathbb{R}^d. \tag{3.2}$$

Remark 3.1. In the particular case where the driver function f is independent of (y,z,u) and the constraint function is given by $\tilde{h}_{i,j} : (x,y,y+v,z) \mapsto -c_{i,j} - v$ with c a given cost function, we retrieve the system of variational inequalities associated to switching problems

$$\min\Big[-\frac{\partial v_i}{\partial t} - \mathcal{L}^i v_i - f_i,\;\min_{j\in\mathcal{I}}[v_i - v_j - c_{i,j}]\Big] \quad= 0\,,\;\text{on } [0,T)\times\mathbb{R}^d\times\mathcal{I}\,, \tag{3.3}$$

$$\min\left[v_i - g_i,\;\min_{j\in\mathcal{I}}\left[v_i - v_j - c_{i,j}\right]\right](T^-,.) \quad= 0\,,\;\text{on } \mathbb{R}^d\times\mathcal{I}\,. \tag{3.4}$$

Thus, if (3.4) satisfies a comparison theorem, $v(T^-,.)$ interprets as the smallest function grower to g satisfying (3.4). In particular, we retrieve the terminal condition $v(T^-,.) = g$ proposed by [10] when the terminal condition g satisfies the cost constraint.

In order to define viscosity solution of (1.1)-(3.2), we introduce, for any locally bounded vector function $(u_i)_{i\in\mathcal{I}}$ on $[0,T]\times\mathbb{R}^d$ its lower semicontinuous and upper semicontinuous (lsc and usc in short) envelopes u_* and u^* defined, for $(t,x)\in[0,T]\times\mathbb{R}^d$, by

$$u_*(t,x) = \liminf_{(t',x')\to(t,x),t'<T} u(t',x'), \quad \text{and} \quad u^*(t,x) = \limsup_{(t',x')\to(t,x),t'<T} u(t',x')\,.$$

Definition 3.1. *A vector function $(u_i)_{i\in\mathcal{I}}$, lsc (resp. usc) on $[0,T)\times\mathbb{R}^d$, is called a viscosity supersolution (resp. subsolution) to* (1.1)-(3.2) *if, for each $(i,t,x)\in\mathcal{I}\times[0,T]\times\mathbb{R}^d$ and $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^d)$ such that (t,x) is a null global minimum (resp. maximum) of $(u_i-\varphi)$, we have,*

$$\text{if } t<T,\ \min\Big[-\frac{\partial\varphi}{\partial t}-\mathcal{L}^i\varphi-f_i(.,(u_j)_{j\in\mathcal{I}},\sigma_i^\top D_x\varphi),\min_{j\in\mathcal{I}}h_{i,j}(.,u_i,u_j,\sigma_i^\top D_x\varphi)\Big](t,x)\geq(\text{resp.}\leq)\,0,$$
$$\text{if } t=T,\ \min\Big[u_i-g_i,\ \min_{j\in\mathcal{I}}h_{i,j}(.,u_i,u_j,\sigma_i^\top D_x\varphi)\Big](T,x)\ \geq(\ \text{resp.}\ \ \leq)\ 0\,.$$

A locally bounded vector function $(u_i)_{i\in\mathcal{I}}$ on $[0,T)\times\mathbb{R}^d$ is called a viscosity solution to (1.1)-(3.2) *if u_* and u^* are respectively viscosity supersolution and subsolution to* (1.1)-(3.2).

Theorem 3.1. *Under* **(H0)-(H1)***, the function v is a (discontinuous) viscosity solution to* (1.1)-(3.2).

Proof. First, following the lines of the proof of Lemma 3.3 and Remark 3.2 in [11], standard estimates on the penalized BSDE (2.1) lead to

$$\mathbf{E}\Big[\sup_{t\in[0,T]}|Y_t^{e,n}|^2\Big]\leq C\Big(1+\mathbf{E}\Big[|g_{I_T^e}(X_T^e)|^2+\int_t^T|X_s^e|^2ds+\sup_{s\in[0,T]}|\tilde{v}_{I_s^e}(s,X_s^e)|^2\Big]\Big),\quad e\in E\,.$$

Combining Fatou's lemma with standard estimates on X and linear growth conditions on g and $\tilde{v}$, see **(H1)**, we get that $\sup_{t\in[0,T]}|v_i(t,x)|^2\leq C(1+|x|^2)$ with $C>0$. Thus, v is locally bounded.

We observe that the viscosity property of v in the interior of the domain is based on the same arguments as the one presented in Theorem 4.1 of [11]. The only difference relies on the more general form of the coefficients f and h, which is not a relevant issue since they are continuous. In order to alleviate the presentation of the paper, we choose to omit it here and detail only the viscosity property (3.2) on the maturity boundary.

(i) Let first consider the supersolution property of v_* to (3.2). Let $(i,x_0)\in\mathcal{I}\times\mathbb{R}^d$ and $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^d)$ such that (T,x_0) is a null global minimum of $([v_*]_i-\varphi)$. Passing to the limit the viscosity properties of the penalized BSDE, we get

$$\min_{j\in\mathcal{I}}h_{i,j}(x,[v_*]_i,[v_*]_j,\sigma^\top D_x\varphi)](T,x_0)\quad\geq\quad 0\,.$$

Furthermore $v^n(T,.)=g$, $n\in\mathbb{N}$, so that the monotonic property of the sequence of continuous functions $(v^n)_{n\in\mathbb{N}}$ leads to $v_*(T,.)\geq g$. Therefore v_* is a viscosity supersolution to (3.2).

(ii) We now turn to the subsolution property of v^*. Let reason by contradiction and suppose the existence of $(i,x_0)\in\mathcal{I}\times\mathbb{R}^d$ and $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^d)$ such that

$$0\ =\ (v_i^*-\varphi)(T,x_0)\quad=\quad\max_{[0,T]\times\mathbb{R}^d}(v_i^*-\varphi)\,, \tag{3.5}$$

and $\min\Big[\varphi - g_i\ ,\ \min_{j\in\mathcal{I}} h_{i,j}(.,\varphi,v_j^*,\sigma_i^\top D_x\varphi)\Big](T,x_0) =: 2\varepsilon\ >\ 0$. The regularity of v^*, φ and $D_x\varphi$ as well as the monotonic property of h lead to the existence of an open neighborhood $\mathcal{O}$ of $(T,x_0)\in[0,T]\times\mathbb{R}^d$, and $\Upsilon, r>0$ such that for all $(t,x,\eta,\eta')\in\mathcal{O}\times(-\Upsilon,\Upsilon)\times B(0,r)$, we get

$$\min\Big[\varphi-\eta-g_i\ ,\ \min_{j\in\mathcal{I}} h_{i,j}(x,\varphi-\eta,v_j^*,\sigma_i^\top[D_x\varphi+\eta'])\Big](t,x)\quad\geq\quad\varepsilon\,. \tag{3.6}$$

Let introduce classically $(t_k,x_k)_k$ a sequence valued in $[0,T)\times\mathbb{R}^d$ satisfying $(t_k,x_k)\to(T,x_0)$ and $v_i(t_k,x_k)\to v_i^*(T,x_0)$. Pick $\delta>0$ such that $[t_k,T]\times B(x_k,\delta)\subset\mathcal{O}$ for k large enough, and introduce the modified test function φ^k given by

$$\varphi^k(t,x)\quad:=\quad\varphi(t,x)+\left(\zeta\frac{|x-x_k|^2}{\delta^2}+C_k\phi\left(\frac{x-x_k}{\delta}\right)+\sqrt{T-t}\right),$$

where $0<\zeta<\Upsilon\wedge\delta r$, ϕ is a regular function in $C^2(\mathbb{R}^d)$ such that $\phi|_{\bar B(0,1)}\equiv 0$, $\phi|_{\bar B(0,1)^c}>0$, $\lim_{|x|\to\infty}\frac{\phi(x)}{1+|x|}=\infty$, and $C_k>0$ is a constant to be determined precisely later on. We deduce from (3.5) that $(v^*-\varphi^k)(t,x)\leq-\zeta$, for $(t,x)\in[t_k,T]\times\partial B(x_k,\delta)$. Choosing C_k large enough, the particular form of the function ϕ leads to

$$(v_i^*-\varphi^k)(t,x)\quad\leq\quad-\frac{\zeta}{2}\,,\qquad\text{for }(t,x)\in B(x_k,\delta)^c\times[t_k,T]\,. \tag{3.7}$$

Thanks to the $\sqrt{T-t}$ term in the modified test function φ^k, we deduce that

$$\Big[-\frac{\partial\varphi^k}{\partial t}-\mathcal{L}^i\varphi^k-f_i\Big(.,(v_j^*+[\varphi^k-\eta-v_i^*]\mathbf{1}_{j=i})_{j\in\mathcal{I}},\sigma_i^\top D_x\varphi^k\Big)\Big](t,x)\quad\geq\quad 0\,, \tag{3.8}$$

for any $(t,x,\eta)\in[t_k,T)\times B(x_k,\delta)\times(-\Upsilon+\zeta,\Upsilon)$ and k large enough. Choose now $\eta<\Upsilon\wedge\frac{\zeta}{2}\wedge\varepsilon$ and introduce the stopping time $\theta_k:=\inf\big\{s\geq t_k\ ;\ X_s^{e_k}\notin B(x_k,\delta)\quad\text{or}\quad I_s^{e_k}\neq I_{s^-}^{e_k}\big\}\wedge T$, where $e_k:=(t_k,i,x_k)$. Let finally consider the process (Y^k,Z^k,U^k,K^k) given on $[t_k,\theta_k]$ by

$$\left\{\begin{array}{lll}
Y_s^k & := & \Big[\varphi^k(s,X_s^{e_k})-\eta\Big]\mathbf{1}_{s\in[t_k,\theta_k)}+v_{I_s^{e_k}}(\theta_k,X_{\theta_k}^{e_k})\mathbf{1}_{s=\theta_k}\,,\quad Z_s^k\ :=\ \sigma_{I_{s-}^{e_k}}^\top(X_s^{e_k})D_x\varphi^k(s,X_s^{e_k}),\\
U_s^k & := & \Big(\Big[v_j^*(s,X_s^{e_k})-[\varphi^k(s,X_s^{e_k})-\eta]\Big]\mathbf{1}_{j\neq I_{s-}^{e_k}}\Big)_{j\in\mathcal{I}}\,,\\
K_s^k & := & -\int_{t_k}^s\Big[\Big(\frac{\partial\varphi^k}{\partial t}+\mathcal{L}^{I_r^{e_k}}\varphi^k\Big)+f_{I_r^{e_k}}(.,(v_j^*+[\varphi^k-\eta-v_{I_r^{e_k}}^*]\mathbf{1}_{j=I_r^{e_k}})_{j\in\mathcal{I}},Z_r^k)\Big](r,X_r^{e_k})dr\\
 & & -\ \int_{t_k}^s\int_{\mathcal{I}}(\varphi^k-\eta-v_j^*)(r,X_r^{e_k})\mu(dr,dj)+\ \Big[\varphi^k-\eta-v_{I_{\theta_k}^{e_k}}\Big](\theta_k,X_{\theta_k}^{e_k})\mathbf{1}_{s=\theta_k}.
\end{array}\right.$$

One easily checks from (3.6)-(3.7)-(3.8) that (Y^k,Z^k,U^k,K^k) is solution to

$$Y_s=v_{I_{\theta_k}^{e_k}}(\theta_k,X_{\theta_k}^{e_k})+\int_s^{\theta_k}f_{I_r^{e_k}}(X_r^{e_k},Y_r+U_r,Z_r)dr-\int_s^{\theta_k}Z_r\cdot dW_r-\int_s^{\theta_k}\int_{\mathcal{I}}U_r(j)\mu(dr,dj)+K_{\theta_k}-K_r$$

on $[t_k,\theta_k]$, together with the constraint $h_{I_{r-}^{e_k},j}(X_r^{e_k},Y_{r^-},Y_{r^-}+U_r(j),Z_r)\geq 0$ a.e., $j\in\mathcal{I}$. Since $(Y^{e_k},Z^{e_k},U^{e_k},K^{e_k})$ is a minimal solution to this constrained BSDE with jumps and we deduce

$$\varphi^k(t_k,x_k)-\eta\ =\ \varphi(t_k,x_k)+\sqrt{T-t_k}-\eta\ \geq\ v_i(t_k,x_k)\,,\qquad\text{for all }k\text{ large enough}\ .$$

Letting k go to infinity, this contradicts (3.5) and concludes the proof. □

Remark 3.2. The main drawback of this representation is the necessity of Assumption **(H1)**. Following similar arguments as Proposition 6.3 in [11], observe that it is satisfied whenever there exist a Lipschitz vector function $(w_i)_{i\in\mathcal{I}}\in[C^2(\mathbb{R}^d)]^{\mathcal{I}}$ supersolution to (3.2) satisfying a linear growth condition, and a constant $C>0$ such that $\mathcal{L}^iw_i+f(.,(w_j)_{j\in\mathcal{I}},\sigma_i^\top Dw_i)\leq C$ on $\mathbb{R}^d$, $i\in\mathcal{I}$.

3.3 A comparison argument

In this section, we provide sufficient conditions characterizing the value function v as the unique viscosity solution to (1.1)-(3.2). This gives in particular the continuity of v, leading to the strong convergence by penalization and the minimality condition, presented in Section 2. The proof relies as usual on a comparison argument, which holds under the following additional assumptions.

(H2) The following holds:

(i) For any $i \in \mathcal{I}$, f_i is convex in $((y_j)_{j\in\mathcal{I}}, z)$ and increasing in u_i.

(ii) For any $i, j \in \mathcal{I}$, $h_{i,j}$ is concave in (y_i, y_j, z) and decreasing in y_i.

(iii) There exists a nonnegative vector function $(\Lambda_i)_{i\in\mathcal{I}} \in [C^2(\mathbb{R}^d)]^{\mathcal{I}}$ and a positive constant ρ such that, for all $i \in \mathcal{I}$, $\Lambda_i \geq g_i$, $\lim_{|x|\to\infty} \frac{\Lambda_i(x)}{1+|x|} = \infty$ and we have :

$$\mathcal{L}^i \Lambda_i + f_i(., (\Lambda_j)_{j\in\mathcal{I}}, \sigma_i^\top D_x \Lambda_i) \leq \rho \Lambda_i \qquad \text{and} \qquad \min_{j\in\mathcal{I}} h_{i,j}(., \Lambda_i, \Lambda_j, \sigma_i^\top D_x \Lambda_i) > 0 \; .$$

An example where Assumption **(H2)** holds is given for the case of optimal switching in [2].

Remark 3.3. As in Bouchard [2], (iii) allows to construct a nice strict supersolution to (1.1) allowing to control solutions to (1.1)-(3.2) by convex perturbations. Following the approach of [11], the general form of f and h forces us to add the extra convexity assumptions (i) and (ii).

Theorem 3.2. *Let* **(H0)**-**(H1)**-**(H2)** *hold. Then, for any U lsc (resp. V usc) viscosity supersolution (resp. subsolution) to* (1.1)-(3.2) *satisfying* $[|U| + |V|](t, x) \leq C(1 + |x|)$ *on* $[0, T] \times \mathbb{R}^d$, *we have* $U_i \geq V_i$ *on* $[0, T] \times \mathbb{R}^d$, $i \in \mathcal{I}$. *In particular, v is continuous and the unique viscosity solution to* (1.1)-(3.2) *satisfying a linear growth condition.*

We omit the proof of this comparison theorem which is a natural extension of Theorem 4.1 in [11]. Following the arguments of the proof of Proposition 3.3 in [14], v still interprets as the minimal viscosity solution to (1.1)-(3.2) in the class of functions with linear growth, whenever only a comparison theorem for the IPDE (3.1) holds.

4 Numerical issues

The numerical resolution of systems of variational inequalities of the form (1.1)-(1.2) usually relies on the use of iterated free boundary. We first solve the system without boundary condition and consider recursively the system constrained by the boundary condition coming from the previous iteration. In a switching problem, we constrain the solution associated to $n+1$ possible switches by the obstacle built with the solution where only n switches are allowed. Such a numerical approach is computationally demanding.We detail here a natural convergent algorithm based on the approximation of the solution to the corresponding constrained BSDE with jump (1.5)-(1.6). We combine a penalization procedure with the discrete time scheme studied by [3] and the statistical estimation projection presented in [7]. Thanks to the previous Feynman-Kac representation, this gives rise to a convergent probabilistic algorithm solving coupled systems of variational inequalities.

Let fix an initial condition $e \in E$ and omit it in the notations for ease of presentation. Suppose that **(H0)-(H1)-(H2)** holds. The algorithm divides in three steps.

Step 1. Approximation by penalization. We first approach the constrained BSDE with jump (1.5)-(1.6) by its penalized version (2.1) characterized by a driver $f^n := f - n[h]^-$ as in Section 2.2. We deduce from Proposition 2.1 that the penalization error converges to 0 as n goes to infinity, see (2.2).

Step 2. Time discretization. Observe that the pure Jump process I can be simulated perfectly and denote by (τ_l) its jump times on $[0,T]$. Let introduce the Euler time scheme approximation X^h of the forward process X defined on the concatenation $(s_l)_l$ of the regular time grid $\{t_k := kh,\ k = 1,\ldots,T/h\}$ with the jumps (τ_l) of I:

$$X_0^h = X_0 \quad \text{and} \quad X_{s_{l+1}}^h := X_{s_l}^h + b_{I_{s_l}}(X_{s_l}^h)(s_{l+1} - s_l) + \sigma_{I_{s_l}}(X_{s_l}^h)[W_{s_{l+1}} - W_{s_l}].$$

We deduce an approximation $Y_T^{n,h}$ of Y_T^n at maturity given by $g_{I_T}(X_T^h)$. The penalized BSDE (2.1) can now be discretized by an extension of the scheme exposed in [3]. An approximation of Y^n at time 0 is computed recursively following the backward scheme for $k = T/h - 1, \cdots, 0$:

$$\left\{ \begin{array}{lcl} Z_{t_k}^{n,h} & := & \frac{1}{h}\mathbf{E}_{t_k}\left[Y_{t_{k+1}}^{n,h}(W_{t_{k+1}} - W_{t_k})\right] \\ U_{t_k}^{n,h}(i) & := & \frac{1}{h}\mathbf{E}_{t_k}\left[Y_{t_{k+1}}^{n,h}\frac{\tilde{\mu}((t_k,t_{k+1}]\times\{i\})}{\lambda(i)}\right], \quad i \in \mathcal{I} \\ Y_{t_k}^{n,h} & := & \mathbf{E}_{t_k}\left[Y_{t_{k+1}}^{n,h} + \int_{t_k}^{t_{k+1}} f_{I_s}^n(X_{t_k}^h, Y_{t_{k+1}}^{n,h}, Z_{t_k}^{n,h}, U_{t_k}^{n,h})ds\right] \end{array} \right. \tag{4.1}$$

where $\mathbf{E}_{t_k}$ denotes the conditional expectation with respect to $\mathcal{G}_{t_k}$. Following the arguments of Section 2.5 in [3] and identifying $(Y^{n,h}, Z^{n,h}, U^{n,h})$ as a constant by part process on each interval $(t_k, t_{k+1}]$, we verify the convergence of this time-discretization approximation :

$$\|Y^n - Y^{n,h}\|_{\mathcal{S}^2} + \|Z^n - Z^{n,h}\|_{\mathbf{L^2_W}} + \|U^n - U^{n,h}\|_{\mathbf{L^2_{\tilde{\mu}}}} \underset{h\to\infty}{\longrightarrow} 0, \quad n \in \mathbb{N}. \tag{4.2}$$

Step 3. Approximation of the conditional expectations. The last step consists in estimating the conditional expectation operators $\mathbf{E}_{t_k}$ arising in (4.1). We adopt here the approach of Longstaff-Schwarz generalized by [7] relying on least square regressions.

Fix $N \in \mathbb{N}$ and simulate N independent copies of the Brownian increments $(W_{t_{k+1}}^j - W_{t_k}^j)_{0\le k\le T/h}$ and the poisson measure $(\tilde{\mu}^j((t_k, t_{k+1}] \times \mathcal{I})_{0\le k\le T/h}$. For each simulation $j \le N$, define I_j^N and $X_j^{h,N}$ the trajectories of I and X^h. By induction, one can easily verify the Markov property of the process $(Y^{n,h}, Z^{n,h}, U^{n,h})$ defined in (4.1):

$$Y_{t_i}^{n,h} = c_k^{n,h}(I_{t_i}, X_{t_i}^h), \quad Z_{t_i}^{n,h} = a_k^{n,h}(I_{t_i}, X_{t_i}^h), \quad U_{t_i}^{n,h} = b_k^{n,h}(I_{t_i}, X_{t_i}^h),$$

for some deterministic functions $(a_k^{n,h}, b_k^{n,h}, c_k^{n,h})_{k\le n}$. The idea is to approximate these functions using Ordinary Least Square (OLS) estimators. Given $L \in \mathbb{N}$, we introduce a collection of basis functions $(a_l^L, b_l^L, c_l^L)_{1\le l\le L}$ of $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$. For each trajectory $j \le N$, define the associated terminal value given by $Y_{j,t_n}^{n,h,L,N} := g_{I_{k,t_n}^N}(X_{j,t_n}^{h,N})$. Now we define recursively $(Z_{j,t_k}^{n,h,L,N}, U_{j,t_k}^{n,h,L,N})$, backward in time for $k = T/h - 1, \cdots, 0$, by computing OLS approximations as follows:

$$(\hat{\alpha}_1, \cdots, \hat{\alpha}_L) := \arg \min_{\alpha_1, \cdots, \alpha_L} \frac{1}{N} \sum_{j=1}^{N} \Big| \frac{1}{h} Y^{n,h,L,N}_{j,t_{k+1}} [W^j_{t_{k+1}} - W^j_{t_k}] - \sum_{l=1}^{L} \alpha_l a^L_l (I^N_{j,t_k}, X^{h,N}_{j,t_k}) \Big|^2,$$

$$(\hat{\beta}, \cdots, \hat{\beta}_L)(i) := \arg \min_{\beta_1, \cdots, \beta_L} \frac{1}{N} \sum_{j=1}^{N} \Big| \frac{1}{h} Y^{n,h,L,N}_{t_{k+1}} \frac{\tilde{\mu}^j((t_k, t_{k+1}] \times \{i\})}{\lambda(i)} - \sum_{l=1}^{L} \beta_l b^L_l (I^N_{j,t_k}, X^{h,N}_{j,t_k}) \Big|^2,$$

for $i \in \mathcal{I}$, leading to the approximation

$$Z^{n,h,L,N}_{j,t_k} := \sum_{l=1}^{L} \hat{\alpha}_l a^L_l (I^N_{j,t_k}, X^{h,N}_{j,t_k}) \quad \text{and} \quad U^{n,h,L,N}_{j,t_k}(i) := \sum_{l=1}^{L} \hat{\beta}_l(j) b^L_l (I^N_{j,t_k}, X^{h,N}_{j,t_k}), \quad i \in \mathcal{I}.$$

It remains to introduce $(\hat{\gamma}_1, \cdots, \hat{\gamma}_L)$ the minimizer of the mean square error

$$\frac{1}{N} \sum_{j=1}^{N} \Big| Y^{n,h,L,N}_{j,t_{k+1}} + \int_{t_k}^{t_{k+1}} f^n_{I^N_{j,s}} (X^{h,N}_{j,t_k}, Y^{n,h,L,N}_{t_{k+1}}, Z^{n,h,L,N}_{t_k}, U^{n,h,L,N}_{t_k}) ds - \sum_{l=1}^{L} \gamma_l c^L_l (I^N_{j,t_k}, X^{h,N}_{j,t_k}) \Big|^2$$

in order to deduce the OLS approximation $Y^{n,h,L,N}_{j,t_k} := \sum_{l=1}^{L} \hat{\gamma}_l c^L_l (I^N_{j,t_k}, X^{h,N}_{j,t_k})$.

We refer to [7] for the control of the statistical error due to the approximation of the conditional expectation operators by OLS projections, and, by extension,

$$\|Y^{n,h} - Y^{n,hL,N}\|_{\mathcal{S}^2} + \|Z^{n,h} - Z^{n,h,L,N}\|_{\mathbf{L^2_W}} + \|U^{n,h} - U^{n,h,L,N}\|_{\mathbf{L^2_{\tilde{\mu}}}} \underset{N,L \to \infty}{\longrightarrow} 0, \; n \in \mathbb{N}, \; h > 0. \quad (4.3)$$

The convergence of the algorithm follows from (2.2), (4.2) and (4.3). The derivation of a convergence rate requires precisions on the influence of n on the discretization and statistical errors, as well as a control of the penalization error. This challenging point is left for further research.

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