Thin-shell concentration for convex measures

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Abstract

We prove that for $s < 0$, $s$-concave measures on $\mathbb{R}^n$ satisfy a thin-shell concentration similar to the log-concave case. It leads to a Berry-Esseen type estimate for most of their one dimensional marginal distributions. We also establish sharp reverse Hölder inequalities for $s$-concave measures.

1 Introduction

For any subsets $A, B \subset \mathbb{R}^n$, the Minkowski sum is defined by

$$A + B = \{a + b : a \in A, b \in B\}.$$

Let $s \in [-\infty, 1]$. A measure $\mu$ on $\mathbb{R}^n$ is called $s$-concave whenever

$$\mu((1 - \lambda)A + \lambda B) \geq ((1 - \lambda)\mu(A)^s + \lambda\mu(B)^s)^{1/s},$$

for every $\lambda \in [0, 1]$ and every compact subsets $A, B \subset \mathbb{R}^n$ such that $\mu(A)\mu(B) > 0$. When $s = 0$, this inequality should be read as

$$\mu((1 - \lambda)A + \lambda B) \geq \mu(A)^{1-\lambda}\mu(B)^\lambda$$

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and it defines $\mu$ as a log-concave measure. When $s = -\infty$, the measure is said to be convex and the inequality is replaced by

$$
\mu((1 - \lambda)A + \lambda B) \geq \min(\mu(A), \mu(B)).
$$

Notice that the class of $s$-concave measures on $\mathbb{R}^n$ is decreasing in $s$ so that any $s$-concave measure is a convex measure. Any $s$-concave measure with $s \geq 0$ is log-concave and the thin-shell concentration for log-concave measures has been studied in [16, 17, 18, 21, 22]. The purpose of this paper is to prove a thin-shell concentration for $s$-concave measures in the case $s < 0$, which we consider from now on. By measure, we always mean probability measure.

The class of $s$-concave measures was introduced and studied in [10, 11], where a complete characterization was established. An $s$-concave measure is supported on some convex subset of an affine subspace where it has a density (see Section 2 for more details). When the support of an $s$-concave measure $\mu$ generates the whole space, we say that $\mu$ is full-dimensional.

A random vector with an $s$-concave distribution is called $s$-concave. The linear image of an $s$-concave random vector is also $s$-concave. We say that a random vector is full-dimensional if its distribution is full-dimensional. It is known that any semi-norm of an $s$-concave random vector with $s < 0$ has moments of all order $p \in (0, -1/s)$ (see [10] and [1]). The Euclidean norm of an $s$-concave random vector $X$ has a finite moment of order 2 if and only if $s > -1/2$. Since we are interested in comparison of moments of the Euclidean norm with the moment of order 2, we will always assume that $-1/2 < s < 0$.

Let $n \geq 1$ be an integer. The Euclidean space $\mathbb{R}^n$ is equipped with its Euclidean norm $|\cdot|_2$ and its scalar product $\langle \cdot, \cdot \rangle$. Its unit sphere is denoted by $S^{n-1}$ and its unit ball by $B^n_2$. We say that a random vector $X$ is isotropic if $\mathbb{E}X = 0$ and for every $\theta \in S^{n-1}$, $\mathbb{E}\langle X, \theta \rangle^2 = 1$. Observe that if $X$ is an $s$-concave full-dimensional random vector and $-1/2 < s$, we can always find an affine transformation $A$ such that $AX$ is isotropic.

Let $p \in \mathbb{R}$ and $X \in \mathbb{R}^n$ be a random vector. Assume that $|X|_2$ has finite moments of order 2 and $p$ with the convention that $(\mathbb{E}|X|_2^p)^{1/p} = \exp(\mathbb{E}\ln|X|_2)$ for $p = 0$. We define

$$
\alpha_p(X) := \left| \frac{(\mathbb{E}|X|_2^p)^{1/p}}{(\mathbb{E}|X|_2^2)^{1/2}} - 1 \right|.
$$

Our main result is the following

**Theorem 1.** Let $r > 2$. Let $X \in \mathbb{R}^n$ be a full-dimensional $(-1/r)$-concave random vector.

If $X$ is isotropic, then for any $p$ such that $|p| \leq c \min(r, n^{1/3})$, we have

$$
\alpha_p(X) \leq \frac{C|p - 2|}{r} + \left( \frac{C|p - 2|}{n^{1/3}} \right)^{3/5},
$$
where $C$ and $c$ are universal constants.

In the general case (when $X$ is not isotropic), let $A$ be an affine transformation such that $AX$ is full-dimensional and isotropic. Then for any $p \in \mathbb{R}$ such that $|p| \leq c \min \left( r, \frac{n^{1/3}}{\|A\|^{2/3} \|A^{-1}\|^{2/3}} \right)$, we have

$$
\alpha_p(X) := \left| \frac{(E|X|^p)^{1/p}}{(E|X|^2)^{1/2}} - 1 \right| \leq \frac{C |p - 2|}{r} + \left( \frac{C |p - 2| (\|A\| \|A^{-1}\|)^{2/3}}{n^{1/3}} \right)^{3/5},
$$

where $C$ and $c$ are universal constants.

We also show (see Remark 15) that for $r > n + \sqrt{n}$, the estimate of $\alpha_p(X)$ in Theorem 1 can be improved and recovers the estimate of the log-concave case from [18].

To present the connections between moment inequalities, concentration in a thin-shell property and the Berry-Esseen theorem for one dimensional marginals, let us introduce some notations.

Let $X \in \mathbb{R}^n$ be an isotropic random vector. Thus $E|X|^2 = n$. Define $\varepsilon(X)$ to be the smallest number $\varepsilon > 0$ such that

$$
\mathbb{P} \left( \left| \frac{|X|^2}{\sqrt{n}} - 1 \right| \geq \varepsilon \right) \leq \varepsilon.
$$

If $\varepsilon(X) = o(1)$ with respect to the dimension $n$, we say that $X$ is concentrated in a thin-shell. This is the usual jargon of the subject. More rigorously, it suggests that we are considering a sequence of random vectors $(X_n)$ with $X_n \in \mathbb{R}^n$ and that $\varepsilon(X_n) = o(1)$ as $n$ goes to $\infty$. It was shown in [2] (see also [13, 14]) that if an isotropic random vector $X$ uniformly distributed on a convex body in $\mathbb{R}^n$ is such that $\varepsilon(X) = o(1)$, then almost all one dimensional marginal distributions of $X$ satisfy a Berry-Esseen theorem. More generally, let $X \in \mathbb{R}^n$ be an isotropic random vector, it was proved in [7] that

$$
\sigma_{n-1} \left( \theta \in S^{n-1} : \sup_{t \in \mathbb{R}} |\mathbb{P}(\langle X, \theta \rangle \leq t) - \Phi(t)| \geq 4\varepsilon(X) + \delta \right) \leq 4n^{3/8} e^{-cn^4},
$$

where $\sigma_{n-1}$ denotes the rotation invariant probability measure on the unit sphere $S^{n-1}$, $\Phi$ is the standard normal distribution function and $c > 0$ is a universal constant. It is worth noticing that the result from [7] does not assume log-concavity. Assuming only that $X$ is isotropic, we get that if $\varepsilon(X)$ is $o(1)$ then almost all the one dimension marginal distributions of $X$ are approximately Gaussian. The fact that indeed for all log-concave random vector $\varepsilon(X) = o(1)$ was proved later in [21, 17] and the best estimate at this date [18] is that

$$
\varepsilon(X) = O(n^{-1/6} \log n).
$$

Now let $p > 2$ and assume that $X$ is isotropic and that $|X|^2$ has a finite moment of order $p$. Then $\varepsilon(X)$ is $o(1)$ if and only if $\alpha_p(X)$ is $o(1)$, see
Remark 4 below. Hence Theorem 1 ensures that if \( r \to +\infty \) with the dimension \( n \) then any isotropic \((-1/r\)-concave random vector satisfies a thin-shell concentration and therefore almost all its one dimensional marginals verify a Berry-Esseen theorem. As a matter of fact, this condition on \( r \) is necessary. If \( r \) is fixed and does not depend on the dimension \( n \), Proposition 5 gives an example of an isotropic \((-1/r\)-concave random vector \( X \in \mathbb{R}^n \) which does not satisfy a thin-shell property. Remark 6 also shows the asymptotic sharpness of Theorem 1, since for this example, for a fixed \( p > 2 \), \( \alpha_p(X) \geq C(p - 2)/r \) for \( r \) and \( n \) large enough, where \( C > 0 \) is a universal constant.

To build the proof of Theorem 1, we need to extend to the case of \( s \)-concave measures several tools coming from the study of log-concave measures. This is the purpose of Section 2. Some of them were already achieved by Bobkov [8], like analog of the Ball’s bodies [5] in the \( s \)-concave setting. Some others were also noticed previously (see e.g. [8], [1]) but not with the most accurate point of view. These new ingredients are analog to the results of [12] in the log-concave setting and are at the heart of our proof. As in the approach of [16] or [18], an important ingredient is the log-Sobolev inequality on \( SO(n) \). It follows e.g. from the work of Bakry and Émery [4] and the calculus of the Ricci curvature of \( SO(n) \) (see [20, Formula (F6)] for example) that for any Lipschitz function \( f : SO(n) \to \mathbb{R}^+ \) (see sections 3 and 4 for definitions)

\[
(2) \quad \mathbb{E}(f(U) \log f(U)) - \mathbb{E} f(U) \log(\mathbb{E} f(U)) \leq \frac{c}{n} \mathbb{E} (|\nabla \log f(U)|^2 f(U)),
\]

where \( U \) is uniformly distributed on \( SO(n) \). It allows to get reverse Hölder inequalities (see inequality (15) in [16]): for every \( f : SO(n) \to \mathbb{R} \), let \( L \) be the log-Lipschitz constant of \( f \) (that is the Lipschitz constant of \( \log f \)), then for every \( q > r > 0 \),

\[
(3) \quad (\mathbb{E}|f(U)|^q)^{1/q} \leq \exp \left( \frac{c L^2}{n} (q - r) \right) (\mathbb{E}|f(U)|^r)^{1/r},
\]

where \( U \) is uniformly distributed on \( SO(n) \).

Let \( X \) be a \((-1/r\)-concave random vector in \( \mathbb{R}^n \) with full-dimensional support and distributed according to a measure with a density function \( w : \mathbb{R}^n \to \mathbb{R}_+ \). For any linear subspace \( E \), denote by \( P_E \) the orthogonal projection onto \( E \) and for any \( x \in E \) denote by

\[
\pi_{E^\perp} w(x) = \int_{x + E^\perp} w(y) dy
\]

the marginal of \( w \) on \( E \). Given an integer \( k \) between 1 and \( n \), a real number \( p \in (-k, r) \), a linear subspace \( E_0 \) of \( \mathbb{R}^n \) of dimension \( k \) and \( \theta_0 \in S(E_0) \), where \( S(E_0) \) denotes the unit sphere of \( E_0 \), we define the function \( h_{k,p} : SO(n) \to \mathbb{R}_+ \) by

\[
(4) \quad h_{k,p}(u) := |S^{k-1}| \int_0^{\infty} t^{p+k-1} \pi_{u(E_0)} w(tu(\theta_0)) dt,
\]
for every $u \in SO(n)$, where $|S^{k-1}|$ denotes the area of the sphere.

Following the approach of [22, 16], we observe that for any $p \in (-k, r)$

\[
E|X|^p = \frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)}E_{h_{k,p}}(U),
\]

where $U$ is uniformly distributed on $SO(n)$. In view of (5) and the definition of $h_{k,p}$, we notice that it is of importance to work with family of measures which are stable after taking the marginals and it is clear from the definition that for any subspace $E$, if $X$ is $(-1/r)$-concave, then $P_{E}X$ is also $(-1/r)$-concave.

In the next section 2, we first introduce more notation and recall important facts concerning convex measures. Then we give an example of an isotropic $(-1/r)$-concave random vector $X \in \mathbb{R}^n$ that does not satisfy a thin-shell property, when $r$ is fixed with respect to the dimension. Finally, we extend to the case of $s$-concave measures several tools coming from the study of log-concave measures that will be essential in the proof of Theorem 1. Section 3 is devoted to the proof of Theorem 1. Some of the results of these two sections are either classical or variation of known results; their proofs are shifted to the appendix.

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2 Preliminary results for $s$-concave measures

We first recall some properties of $s$-concave measures and their relation to $\beta$-concave functions.

The class of $s$-concave measures was introduced and studied in [10, 11], where the following complete characterization was established. An $s$-concave measure $\mu$ on $\mathbb{R}^n$ is supported on some convex subset of an affine subspace where it has a density. When this subspace is the whole space, we say that $\mu$ is full-dimensional. In this case, its density $w$ is $\beta$-concave with $\beta = s/(1-ns)$. Recall that a function $f : \mathbb{R}^n \to \mathbb{R}_+$ is called $\beta$-concave whenever

\[
f((1-\lambda)x + \lambda y) \geq ((1-\lambda)f(x)^\beta + \lambda f(y)^\beta)^{1/\beta}
\]

for every $\lambda \in [0, 1]$ and every $x, y \in \mathbb{R}^n$ such that $f(x)f(y) > 0$, where the right hand side is replaced by $f(x)^{1-\lambda}f(y)^{\lambda}$ for $\beta = 0$. Note that when $\beta < 0$ which will be the case below, $\beta$-concavity means that $f^\beta$ is convex on its convex support $\{f > 0\}$.

We will use a similar language for probability measure, random vector and function which are related here as distribution, law of a random vector and density of probability. It is important to remember that when $X \in \mathbb{R}^n$ is
(-1/r)-concave full-dimensional, then the result recalled above states that its distribution has a support that generates \( \mathbb{R}^n \) and has a density which is \((-1/(n + r))\)-concave.

Recall that for every \( x > 0 \), \( \Gamma(x) = \int_0^\infty u^{x-1}e^{-u} \, du \) and for every \( x, y > 0 \), \( B(x, y) = \int_0^1 u^{x-1}(1-u)^{y-1} \, du = \int_0^\infty u^{x-1}(u + 1)^{-y} \, du \).

The following inequality of Paley-Zygmund type is well known.

**Lemma 2.** Let \( 2 < p < s \). Let \( Y \) be a non-negative random variable with finite \( s \)-moment. Then for every \( 0 \leq t \leq (\mathbb{E}Y^p)^{1/p} \) we have

\[
\mathbb{P}(Y \geq t) \geq \left( \frac{\mathbb{E}Y^p - t^p}{(\mathbb{E}Y^s)^{p/s}} \right)^{s/(s-p)}.
\]

**Proof.** Using Hölder inequality, we have

\[
\mathbb{E}Y^p = \mathbb{E}Y^p 1_{Y < t} + \mathbb{E}Y^p 1_{Y \geq t} \leq t^p + (\mathbb{E}Y^s)^{p/s} \mathbb{P}(Y \geq t)^{1-p/s}.
\]

Thus

\[
\mathbb{P}(Y \geq t) \geq \left( \frac{\mathbb{E}Y^p - t^p}{(\mathbb{E}Y^s)^{p/s}} \right)^{s/(s-p)}.
\]

\[\square\]

**Proposition 3.** Let \( 2 < p < s \). Let \( X \in \mathbb{R}^n \) be an isotropic random vector such that \( |X|_2 \) has a finite \( s \)-moment. Then

\[
\min \left( \frac{\alpha_p(X)}{2}, \left( \frac{p\alpha_p(X)/2}{\alpha_s(X) + 1} \right)^{s/(s-p)} \right) \leq \varepsilon(X) \leq ((\alpha_p(X) + 1)^p - 1)^{1/3}.
\]

**Proof.** Let \( \varepsilon > 0 \). Applying Lemma 2 to \( Y = |X|_2/(\mathbb{E}|X|_2^{1/2})^2, t = \varepsilon + 1 \) and noticing that \( \mathbb{E}Y^p = (\alpha_p(X) + 1)^p, \mathbb{E}Y^s = (\alpha_s(X) + 1)^s \), we get that

\[
\mathbb{P} \left( \frac{|X|_2}{(\mathbb{E}|X|_2^{1/2})^2} \geq 1 + \varepsilon \right) \geq \left( \frac{(\alpha_p(X) + 1)^p - (\varepsilon + 1)^p}{(\alpha_s(X) + 1)^p} \right)^{s/(s-p)}
\]

whenever \( 0 < \varepsilon \leq \alpha_p(X) \). Since for \( p \geq 1 \) and \( x \geq y \geq 1 \), \( x^p - y^p \geq p(x-y) \), we have

\[
\mathbb{P} \left( \frac{|X|_2}{(\mathbb{E}|X|_2^{1/2})^2} \geq 1 + \varepsilon \right) \geq \left( \frac{p(\alpha_p(X) - \varepsilon)}{(\alpha_s(X) + 1)^p} \right)^{s/(s-p)}.
\]

Therefore

\[
\mathbb{P} \left( \frac{|X|_2}{(\mathbb{E}|X|_2^{1/2})^2} \geq 1 + \varepsilon \right) \geq \left( \frac{p\alpha_p(X)/2}{(\alpha_s(X) + 1)^p} \right)^{s/(s-p)}
\]

whenever \( 0 < \varepsilon \leq \alpha_p(X)/2 \). The left-hand side inequality follows.
Since for \( q \geq 1 \), \(|x - 1| \leq |x^q - 1|\) for every \( x \geq 0 \), Markov inequality gives

\[
P\left( \frac{|X|}{(E|X|^2)^{1/2}} - 1 \geq \varepsilon \right) \leq P\left( \frac{|X|^q}{(E|X|^2)^{q/2}} - 1 \geq \varepsilon \right) \leq \frac{E\left( \frac{|X|^2}{(E|X|^2)^{1/2}} - 1 \right)^2}{\varepsilon^2}.
\]

To conclude the right-hand side inequality, take \( q = p/2 \) and observe that

\[
E\left( \frac{|X|^2}{(E|X|^2)^{q/2}} - 1 \right)^2 = (\alpha_2 \varepsilon + 1)^{2q} + 1 - 2(\alpha_q \varepsilon + 1)^q \leq (\alpha_2 \varepsilon + 1)^{2q} - 1.
\]

\[\square\]

**Remark 4.** Let \( 2 < p < s \). Let \( X \in \mathbb{R}^n \) be an isotropic random vector such that \( |X|^2 \) has a finite \( s \)-moment. Proposition 3 shows that \( \varepsilon(X) \) is \( o(1) \) if and only if \( \alpha_p(X) \) is \( o(1) \) when \( n \to \infty \).

Now we estimate \( \varepsilon(X) \) for an example which shows that an isotropic \((-1/r\))-concave random vector \( X \in \mathbb{R}^n \) may not satisfy a thin-shell property. In the proposition below, the notation \( \lim\inf \) refers to the limit inferior.

**Proposition 5.** Let \( r > 2 \). There exists a sequence \( (X_n)_n \) of isotropic \((-1/r\))-concave random vectors \( X_n \in \mathbb{R}^n \) such that

\[\lim_{n \to \infty} \varepsilon(X_n) \geq c(r) > 0,\]

where \( c(r) > 0 \) depends only on \( r \).

**Proof.** Let \( r > 2 \) and \( 2 < p < r \) and let \( X_n \in \mathbb{R}^n \) be an isotropic random vector with density

\[f_{n,r}(x) = \frac{c_1}{(1 + c_2|x|^r)^{r+n}},\]

where \( c_1 \) and \( c_2 \) are normalization factors. From [10, 11], such a random vector is \((-1/r\))-concave. An immediate computation gives that

\[
\frac{(E|X_n|^p)^{1/p}}{(E|X_n|^2)^{1/2}} = \left( \frac{B(n+p,r-p)}{B(n,r)} \right)^{1/p} \left( \frac{B(n+2,r-2)}{B(n,r)} \right)^{-1/2}.
\]

For fixed \( r \) and \( 2 < p < r \), we have

\[\lim_{n \to +\infty} \frac{(E|X_n|^p)^{1/p}}{(E|X_n|^2)^{1/2}} = \left( \frac{\Gamma(r-p)}{\Gamma(r)} \right)^{1/p} \left( \frac{\Gamma(r-2)}{\Gamma(r)} \right)^{-1/2},\]

and by the strict log-convexity of the Gamma function, we have

\[\lim_{n \to +\infty} (\alpha_p(X_n) + 1) = \lim_{n \to +\infty} \frac{(E|X_n|^p)^{1/p}}{(E|X_n|^2)^{1/2}} > 1.\]
As a consequence for any $2 < p < r$, $\lim_{n \to +\infty} \alpha_p(X_n) > 0$.

Now let $2 < p < s < r$. From Proposition 3, we get

\[
\lim_{n \to +\infty} \varepsilon(X_n) \geq \lim_{n \to +\infty} \min \left( \frac{\alpha_p(X_n)}{2}, \left( \frac{p\alpha_p(X_n)/2}{(\alpha_s(X_n) + 1)^p} \right)^{s/(s-p)} \right) > 0.
\]

Choose $p = (2 + r)/2$ and $s = (p + r)/2$ for which $2 < p < s < r$ and note that the right hand side term in equation (7) depends only on $r$. This concludes the proof.

\begin{remark}
Let $2 < p < r$ and let $r \to \infty$. Applying Stirling formula in (6) when $r \to \infty$, a calculation gives that

\[
\lim_{r \to \infty} r \lim_{n \to \infty} \alpha_p(X_n) = (p - 2)/2.
\]

This asymptotic estimate shows that for a fixed $p > 2$ and $r$ and $n$ large enough, then $\alpha_p(X_n) \geq C(p - 2)/r$ where $C > 0$ is a universal constant. This proves the sharpness of Theorem 1 under these conditions.
\end{remark}

We now prove some inequalities for $s$-concave measures that will be useful tools in the next section.

\begin{theorem}
(1) Let $f : [0, \infty) \to [0, \infty)$ be a measurable function such that $\|f\|_\infty > 0$. Then

\[
p \mapsto \left( \int_0^{\infty} pt^{p-1} f(t) \, dt / \|f\|_\infty \right)^{1/p}
\]

is non-decreasing on its domain of definition.

(2) Let $\alpha > 0$ and $f : [0, \infty) \to [0, \infty)$ be $(-1/\alpha)$-concave, continuous and integrable. Define $H_f : [0, \alpha) \to \mathbb{R}_+$ by

\[
H_f(p) = \begin{cases} 
\frac{1}{B(p, \alpha - p)} \int_0^{+\infty} t^{p-1} f(t) \, dt & \text{for } 0 < p < \alpha \\
f(0) & \text{for } p = 0.
\end{cases}
\]

Then $H_f$ is log-concave on $[0, \alpha)$.
\end{theorem}

The proof of the first part may be treated as in Lemma 2.1 in [24] and the proof of the second part is identical to the well known $(1/n)$-concave case [12]. We postpone the proof of Theorem 7 to the appendix.

We present several consequences of this result such as some reverse Hölder inequalities with sharp constants in the spirit of Borell’s [12] and Berwald’s [6] inequalities.
Corollary 8. Let \( r > 0 \) and \( \mu \) be a \((-1/r)\)-concave measure on \( \mathbb{R}^n \). Let \( \phi : \mathbb{R}^n \to \mathbb{R}_+ \) such that \( \{\phi > 0\} \) is convex and \( \phi \) is concave on \( \{\phi > 0\} \). Then the function

\[
p \mapsto \begin{cases} \frac{1}{pB(p, r-p)} \int \phi(x)^p d\mu(x) & \text{for } 0 < p < r \\ \mu(\{\phi > 0\}) & \text{for } p = 0 \end{cases}
\]

is log-concave on \([0, r)\).

Moreover, if \( \mu(\{\phi > 0\}) > 0 \) then for any \( 0 < p \leq q < r \),

\[
\left( \int_{\mathbb{R}^n} \phi(x)^q \frac{d\mu(x)}{\mu(\{\phi > 0\})} \right)^{1/q} \leq \left( \frac{qB(q, r-q)}{(pB(p, r-p))^{1/p}} \int_{\mathbb{R}^n} \phi(x)^p \frac{d\mu(x)}{\mu(\{\phi > 0\})} \right)^{1/p}.
\]

Proof. By the concavity of \( \phi \), for every \( u, v \geq 0 \) and every \( \lambda \in [0, 1] \)

\[
(1 - \lambda)\{\phi > u\} + \lambda\{\phi > v\} \subset \{\phi > (1 - \lambda)u + \lambda v\}.
\]

By the \((-1/r)\)-concavity of \( \mu \), the function \( f(t) = \mu(\{\phi > t\}) \) is \((-1/r)\)-concave and it is clearly continuous on \( \mathbb{R}_+ \). Observe by Fubini that for any \( p > 0 \),

\[
\int_{\mathbb{R}^n} \phi(x)^p d\mu(x) = \int_0^{+\infty} pt^{p-1} f(t) dt.
\]

The result follows from the part (2) of Theorem 7. The moreover part follows from the log-concavity since then \( p \mapsto (H_f(p)/f(0))^{1/p} \) is a non-increasing function. \( \square \)

The second corollary concerns the function \( h_{k,p} \) defined in (4).

Corollary 9. Let \( r > 0 \) and \( u \in SO(n) \). For any \((-1/(r+n))\)-concave function \( w : \mathbb{R}^n \to \mathbb{R}_+ \) and any subspace \( E_0 \) of dimension \( k \leq n \), the function

\[
p \mapsto \begin{cases} \frac{h_{k,p}(u)}{B(p + k, r-p)} & \text{for } p > -k + 1 \\ |S^{k-1}| \pi_{u(E_0)} w(0) & \text{for } p = -k + 1 \end{cases}
\]

is log-concave on \([-k + 1, r)\).

Proof. Since \( w \) is \((-1/(r+n))\)-concave, we note that \( t \mapsto \pi_{t(E_0)} w(tu(\theta_0)) \) is \((-1/(r+k))\)-concave and it is clearly continuous on \( \mathbb{R}_+ \). Theorem 7 proves the result. \( \square \)

We finish with some geometric properties of a family of bodies introduced by K. Ball in [5] in the log-concave case.
Corollary 10. Let $\alpha > 0$. Let $w : \mathbb{R}^n \to \mathbb{R}_+$ be a $(-1/\alpha)$-concave function such that $w(0) > 0$. For $0 < a < \alpha$ let

$$K_a(w) = \left\{ x \in \mathbb{R}^n; \; a \int_0^{+\infty} t^{a-1} w(tx) dt \geq w(0) \right\}.$$  

Then for any $0 < a \leq b < \alpha$

$$\left( \frac{w(0)}{\|w\|_\infty} \right)^{\frac{1}{b}-\frac{1}{a}} K_a(w) \subset K_b(w) \subset \left( \frac{bB(b, \alpha - b)}{aB(a, \alpha - a)} \right)^{1/b} K_a(w).$$

Proof. Notice that the sets $K_a$ are star-shaped with respect to the origin, that is for every $x \in K_a$ and every $\lambda \in [0,1]$, $\lambda x \in K_a$. The radial function of $K_a$ is

$$\rho_{K_a}(x) := \sup \{ r: rx \in K_a \} = \left( a \int_0^{+\infty} t^{a-1} \frac{w(tx)}{w(0)} dt \right)^{\frac{1}{a}}.$$ 

For any $x \in \mathbb{R}^n$, let $f$ be the continuous $(-1/\alpha)$-concave function defined on $\mathbb{R}^+$ by $f(t) = w(tx)/w(0)$. By (1) of Theorem 7, the function $a \mapsto \left( \int_0^{+\infty} t^{a-1} \frac{f(t)}{\|f\|_\infty} dt \right)^{\frac{1}{a}}$ is non-decreasing. The left hand side inclusion follows. Moreover, from (2) of Theorem 7, the function $H_f : [0,\alpha) \to \mathbb{R}_+$ is log-concave on $[0,\alpha)$ with $H_f(0) = 1$. For $0 < a \leq b < \alpha$, we have thus $H_f(b)^{1/b} \leq H_f(a)^{1/a}$. The right hand side inclusion follows. \qed

3 Thin shell for convex measures

The purpose of this section is to prove Theorem 1. We follow the strategy of the log-concave case initiated in [21, 17, 22] and further developed in [16, 18].

The support function $h_K$ of a non-empty compact set $K \subset \mathbb{R}^n$ is defined by

$$\forall \theta \in \mathbb{R}^n, \; h_K(\theta) = \sup_{x \in K} \langle x, \theta \rangle.$$ 

To any random vector $X$ in $\mathbb{R}^n$ and any $p \geq 1$, we associate its $Z^+_p$-body defined by its support function

$$\forall \theta \in \mathbb{R}^n, \; h_{Z^+_p(X)}(\theta) = (\mathbb{E} \langle X, \theta \rangle^p_+)^{1/p}.$$ 

When the distribution of $X$ has a density $g$, we write $Z^+_p(g) = Z^+_p(X)$. Extending a theorem of Ball [5] for log-concave functions, Bobkov proved in [8, Remark 2.6] (see also [15, Theorem 3.1]) that if $w$ is $(-1/(r+n))$-concave on $\mathbb{R}^n$ such that $w(0) > 0$, then

(8) $K_a(w)$ is convex and compact for any $0 < a \leq r + n - 1$.  

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In the case of log-concave measures [25, 26, 18, 19], several relations between the $Z_p^+$ bodies and the convex sets $K_a$ are known. We need their analogue in the setting of $s$-concave measures for negative $s$. We start with two technical lemmas. We postpone their proofs to the appendix.

**Lemma 11.** Let $x, y \geq 1$, then
\begin{equation}
(9) \quad c \frac{x}{x + y} \leq (xB(x, y))^{1/x} \leq C \frac{x}{x + y},
\end{equation}
where $c, C$ are positive universal constants. Moreover, for $k, r > 1$, the extension by continuity at 0 of the function $p \mapsto \frac{1}{p} \log \frac{B(k + p, r - p)}{B(k, r)}$ is differentiable on $[-(k - 1), \frac{r - 1}{2}]$ and satisfies
\begin{equation}
(10) \quad 0 \leq \frac{d}{dp} \left( \frac{1}{p} \log \frac{B(k + p, r - p)}{B(k, r)} \right) \leq \frac{1}{r - 1} + \frac{1}{k - 1}
\end{equation}
for $p \in [-\frac{k - 1}{2}, \frac{r - 1}{2}]$.

In this paper, we use the notion of geometric distance between sets, defined for every compact subsets $K, L \subset \mathbb{R}^n$ containing 0 in their interior by
\[
d(K, L) = \inf \{t_2/t_1 : t_1 L \subset K < t_2 L, t_1, t_2 > 0\}.
\]
Let $n \geq 1, r \geq 2$ and $w$ be the $(-1/(r + n))$-concave density of a probability measure $\mu$ on $\mathbb{R}^n$. Then by Corollary 8 and Lemma 11, for $1 \leq p \leq q \leq r - 1$, one has
\[
Z_p^+(w) \subset Z_q^+(w) \subset C \frac{q}{p} \left( \inf_{\theta \in S^{n-1}} \mu \{x : \langle x, \theta \rangle > 0\} \right)^{\frac{1}{q} - \frac{1}{p}} Z_p^+(w).
\]
Fix $\theta \in S^{n-1}$ and define $F(t) = \mu(\{x : \langle x, \theta \rangle \leq t\})$, for $t \in \mathbb{R}$. One has $\int_{\mathbb{R}} t F'(t) dt = \int_{\mathbb{R}^n} (x, \theta) w(x) dx = 0$ and $F$ is $(-1/r)$-concave. Using Jensen’s inequality, we get
\[
F(0)^{-\frac{1}{r}} = F \left( \int_{\mathbb{R}} t F'(t) dt \right)^{-\frac{1}{r}} \leq \int_{\mathbb{R}} F(t)^{-\frac{1}{r}} F'(t) dt = \left[ F(t)^{1 - \frac{1}{r}} \right]_{-\infty}^{+\infty} = \frac{1}{1 - \frac{1}{r}}.
\]
Hence $\mu(\{x : \langle x, \theta \rangle > 0\}) \geq (1 - \frac{1}{r})^{-1} \geq 1/4$ for $r \geq 2$. We have recovered here in a simple way a Grünbaum’s type inequality for convex measures due to Bobkov [8, Theorem 5.2]. We deduce that, for $1 \leq p \leq q \leq r - 1$,
\begin{equation}
(11) \quad Z_p^+(w) \subset Z_q^+(w) \subset C \frac{q}{p} Z_p^+(w) \quad \text{and} \quad d(Z_p^+(w), Z_q^+(w)) \leq C \frac{q}{p},
\end{equation}

**Lemma 12.** Let $r, m$ and $p$ be such that $m$ is a positive integer, $r \geq m + 1$ and $-m \leq p \leq r - 1$. Let $F$ be a subspace of $\mathbb{R}^n$ of dimension $m$ and let $g$ be a $(-1/(r + m))$-concave density of a probability measure on $F$ such that $\int_{F} x g(x) dx = 0$. Then we have
\[
d(K_{m+p}(g), Z_{\max(m,p)}^+(g)) \leq c,
\]
where $c$ is a universal constant.
As in [18], an important ingredient in the proof of the thin-shell concentration inequality is an estimate from above of the log-Lipschitz constant of the map on $SO(n) : u \mapsto h_{k,p}(u)$. Let $\mathcal{M}_n(\mathbb{R})$ be the set of square $n \times n$ matrices. We equip

$$SO(n) = \{ u \in \mathcal{M}_n(\mathbb{R}) : u^t u = Id, \det(u) = 1 \}$$

with its standard invariant Riemannian metric, which we specify for concreteness on $T_{Id}SO(n)$, the tangent space at the identity element $Id \in SO(n)$. Since $u^t u = Id$, this tangent space may be identified with the set of anti-symmetric matrices $\{ B \in \mathcal{M}_n(\mathbb{R}) : B^t + B = 0 \}$. We define the scalar product $\langle B, \tilde{B} \rangle = \frac{1}{2} \text{tr}(B^t \tilde{B})$ on $T_{Id}SO(n)$.

**Proposition 13.** Let $n \geq 1$, $r > 10$ and $w$ be the $(-1/(r + n))$-concave density of a probability measure on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} xw(x)dx = 0$. Let $k$ be an integer such that $k \geq 2$, $2k - 1 \leq n$ and $2k \leq r$. Let $p$ such that $-\frac{k}{2} \leq p \leq r - 1$. Denote by $L_{k,p}$ the log-Lipschitz constant of the map on $SO(n) : u \mapsto h_{k,p}(u)$. Then

$$L_{k,p} \leq C \max(k, p) d(Z_{\max(k,p)}^+(w), B_n^2),$$

where $C$ is a universal constant.

**Proof.** For any subspace $F$ of dimension $m$, the marginal $\pi_F(w)$ is a $(-1/(r + m))$-concave function on $F$ and from (8), for any $a \in [0, r + m - 1]$, we associate the convex body $K_a(\pi_F(w))$ in $F$. Then the proof of Theorem 2.1 in [18, section 2.2] gives the upper bound:

$$L_{k,p} \leq \max_F \{(m + p) d(K_{m+p}(\pi_F(w)), B_2(F)) \}$$

over all subspaces $F$ of dimension $m = k, k + 1, 2k - 1$, where $B_2(F)$ is the Euclidean unit ball in $F$. By assumptions on $k$, we get that for these values of $m, m \leq 2k - 1 \leq n$ and $r \geq 2k \geq m + 1$ and $p \geq -k/2 \geq -m/2$. Hence from Lemma 12, we have

$$d(K_{m+p}(\pi_F(w)), B_2(F)) \leq c d(Z_{\max(m,p)}^+(\pi_F(w)), B_2(F)).$$

By definition, if $X$ is the random vector with density $w$ on $\mathbb{R}^n$, the marginal $\pi_F(w)$ is the density of the projection of $X$ onto $F$, namely $P_F X$. By identification of the support functions, we have that, for any $\theta \in F$,

$$h_{Z_{\theta}^+}^p(\pi_F(w))(\theta) = \mathbb{E}(P_F X, \theta)^p_+ = \mathbb{E}(X, \theta)^p_+.$$

This means that $Z_{\theta}^p(\pi_F(w)) = P_F(Z_{\theta}^+(w))$. Since the distance to the Euclidean ball cannot increase after projections, we conclude that

$$d(K_{m+p}(\pi_F(w)), B_2(F)) \leq c d(Z_{\max(m,p)}^+(w), B_n^2).$$

By equation (11), for $m = k, k + 1, 2k - 1$, one has

$$d(Z_{\max(m,p)}^+(w), Z_{\max(k,p)}^+(w)) \leq c.$$

This finishes the proof. \qed
We define the $q$-condition number of a random vector $X$ to be
\[
\rho_q(X) = \frac{\sup_{|\theta|_2=1} (\mathbb{E} \langle X, \theta \rangle_q^q)^{1/q}}{\inf_{|\theta|_2=1} (\mathbb{E} \langle X, \theta \rangle_q^q)^{1/q}}.
\]
Obviously, if $w$ is the density of a full-dimensional random vector $X$ in $\mathbb{R}^n$ then $\rho_q(X) = d(Z^+_q(w), B^n_2)$.

**Proposition 14.** With the same assumptions as in Proposition 13, if a random vector $X$ with density $w$ is isotropic then
\[
L_{k,p} \leq C \max(k,p)^2.
\]
More generally if $A$ is such that $AX$ is isotropic then
\[
(12) \quad L_{k,p} \leq C \max(k,p)^2 \|A\|\|A^{-1}\|.
\]

**Proof.** Let $q = \max(k,p)$, then one has $1 \leq q \leq r - 1$. Using the triangular inequality we get
\[
\rho_q(X) = d(Z^+_q(w), B^n_2) \leq d(Z^+_q(w), Z^+_2(w)) d(Z^+_2(w), B^n_2).
\]
From equation (11) we deduce that $d(Z^+_q(w), Z^+_2(w)) \leq cq$. For any $\theta \in S^{n-1}$, $\mathbb{E}\langle X, \theta \rangle = 0$, hence $\mathbb{E}\langle X, \theta \rangle_+ = \mathbb{E}\langle -X, \theta \rangle_+$. Using this equality and equation (11) we deduce that
\[
(\mathbb{E}\langle -X, \theta \rangle_+^2)^{1/2} \leq c\mathbb{E}\langle -X, \theta \rangle_+ = c\mathbb{E}\langle X, \theta \rangle_+ \leq c (\mathbb{E}\langle X, \theta \rangle_+^2)^{1/2}.
\]
Thus
\[
\mathbb{E}\langle X, \theta \rangle_+^2 \leq \mathbb{E}\langle X, \theta \rangle^2 = \mathbb{E}\langle X, \theta \rangle_+^2 + \mathbb{E}\langle -X, \theta \rangle_+^2 \leq C \mathbb{E}\langle X, \theta \rangle_+^2.
\]
Hence if $X$ is isotropic we deduce that $d(Z^+_2(w), B^n_2) \leq c'$. We conclude that
\[
\rho_q(X) = d(Z^+_q(w), B^n_2) \leq C'q.
\]
The conclusion follows from Proposition 13. In the general case, notice that $Z^+_q(AX) = AZ^+_q(X)$ and $d(AB^n_2, B^n_2) = \|A\|\|A^{-1}\|$, thus
\[
\rho_q(X) \leq \rho_q(AX)\|A\|\|A^{-1}\|.
\]

**Proof of Theorem 1.** Without loss of generality, we can assume that $r > 32$. Indeed, if $r \leq 32$ then the statement in Theorem 1 is valid for $|p| \leq cr$ and it gives only a comparison of $(\mathbb{E}|X|^p_2)^{1/p}$ with $(\mathbb{E}|X|^2_2)^{1/2}$ up to a constant factor. The result is a consequence of Theorem 5.2 in [1].

From now on, we assume that $r > 32$ and that $|p| \leq \frac{r}{\tilde{r}}$. We start by presenting a complete argument following [16]. This will give a complete
proof of Theorem 1 with a slightly weaker result. In the second part, we just indicate the needed modifications of the argument of [18] to get the complete conclusion.

In this first part, we will prove that for any \( p \in \left[ \frac{1}{\sqrt{n}}, \min(cn^{1/8}, \frac{r}{8}) \right] \)

\[
(13) \quad \left( \mathbb{E}|X|^p \mathbb{E}|X|^p \right)^{1/p} \leq 1 + \frac{Cp}{r} + \left( \frac{Cp}{n^{1/3}} \right)^{3/5}. 
\]

Assuming (13), few elementary steps are needed to prove that for any \( p \) such that \( |p| \leq \min(cn^{1/8}, \frac{r}{8}) \),

\[
(14) \quad \left| \left( \frac{\mathbb{E}|X|^p}{\mathbb{E}|X|^p} \right)^{1/p} - 1 \right| \leq \frac{C(1 + |p|)}{r} + \left( \frac{C(1 + |p|)}{n^{1/3}} \right)^{3/5},
\]

which is already enough to get a thin-shell concentration. Indeed, for \( p \geq 2 \), by Hölder inequality, we have

\[
0 \leq \frac{\mathbb{E}|X|^p}{\mathbb{E}|X|^p} - 1 \leq \frac{\mathbb{E}|X|^p}{\mathbb{E}|X|^p} - 1
\]

and we conclude by (13). For \( p \leq -2 \), we have \( |p| = -p \geq 2 \) and from Hölder inequality and (13),

\[
0 \leq \frac{\mathbb{E}|X|^p}{\mathbb{E}|X|^p} - 1 \leq \frac{\mathbb{E}|X|^p}{\mathbb{E}|X|^p} - 1 \leq \frac{C|p|}{r} + \left( \frac{C|p|}{n^{1/3}} \right)^{3/5}.
\]

An elementary computation shows that

\[
\left| \left( \frac{\mathbb{E}|X|^p}{\mathbb{E}|X|^p} \right)^{1/p} - 1 \right| \leq \frac{C|p|}{r} + \left( \frac{C|p|}{n^{1/3}} \right)^{3/5}.
\]

For \( p \in [-2, 2] \), by Hölder inequality, we have

\[
0 \leq 1 - \frac{(\mathbb{E}|X|^p)^{1/p}}{(\mathbb{E}|X|^p)^{1/p}} \leq 1 - \frac{(\mathbb{E}|X|^p)^{1/p}}{(\mathbb{E}|X|^p)^{1/p}}
\]

and we conclude by the previous estimate for \( p = -2 \). This concludes the proof of (14).

Let us start the proof of (13). Let \( p \in \left[ \frac{1}{\sqrt{n}}, \min(cn^{1/8}, \frac{r}{8}) \right] \) and \( k \) be an integer greater or equal than 2 such that \( p < k \leq n \). We will optimize the choice of \( k \) at the end of the proof. Recall that by (5),

\[
\mathbb{E}|X|^p = \frac{\Gamma((p + n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p + k)/2)} \mathbb{E} h_{k,p}(U),
\]

where \( U \) is uniformly distributed on \( SO(n) \). Using that the function \( \frac{d}{dp} \log \Gamma(p) \) is concave (see for example the proof of Lemma 11 in the appendix), we deduce that

\[
(15) \quad \frac{d}{dp} \left( \frac{1}{p} \log \frac{\Gamma((p + n)/2)\Gamma(k/2)}{\Gamma((p + k)/2)\Gamma(n/2)} \right) \leq 0.
\]
It follows that for any $0 < p < k$,
\[
\frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)\Gamma((-p+n)/2)\Gamma(k/2)} \leq 1.
\]
Then for all $0 < p < r$ and $n \geq k > p$ we have
\[
E[X]^p_{\mathcal{E}} X \leq E h_{k,p}(U) E h_{k,-p}(U).
\]
Applying log-Sobolev inequality (3) to $h_{k,p}$ and $h_{k,-p}$ we get
\[
E h_{k,p}(U)^2 \leq e^{\frac{c l^2_{k,p}}{n}} (E h_{k,p}(U))^2, \quad E h_{k,-p}(U)^2 \leq e^{\frac{c l^2_{k,-p}}{n}} (E h_{k,-p}(U))^2.
\]
Since $\text{Var} f = E f^2 - (E f)^2$ we deduce that
\[
\begin{cases}
\text{Var} h_{k,p}(U) \leq \left( e^{\frac{c l^2_{k,p}}{n}} - 1 \right) (E h_{k,p}(U))^2, \\
\text{Var} h_{k,-p}(U) \leq \left( e^{\frac{c l^2_{k,-p}}{n}} - 1 \right) (E h_{k,-p}(U))^2.
\end{cases}
\]
By Corollary 9, we know that $p \mapsto h_{k,p}(u) \Gamma(k+p, r-p)$ is log-concave on $[-k+1, r)$ hence
\[
h_{k,p}(u) h_{k,-p}(u) \leq \left( \frac{B(k+p, r-p) B(k-p, r+p)}{B(k, r)} \right) h_{k,0}^2(u).
\]
Taking the expectation with respect to $SO(n)$, we get that
\[
E h_{k,p}(U) h_{k,-p}(U) \leq \left( \frac{B(k+p, r-p) B(k-p, r+p)}{B(k, r)} \right) E h_{k,0}^2(U).
\]
Since $E h_{k,0}(U) = 1$ we deduce from (17) that
\[
E h_{k,0}^2(U) \leq e^{\frac{c l^2_{k,0}}{n}}.
\]
Assume that $k$ is such that $k \leq r$ then by (10), we know that for $p \leq (k-1)/2$,
\[
\left( \frac{B(k+p, r-p) B(k-p, r+p)}{B(k, r)} \right)^{1/p} \leq e^{2p(\frac{1}{k-1} + \frac{1}{r-1})} \leq e^{4p(\frac{1}{k} + \frac{1}{r})}
\]
since $k, r \geq 2$. Hence
\[
E h_{k,p}(U) h_{k,-p}(U) \leq e^{\frac{c l^2_{k,0}}{n} + 4p(\frac{1}{k} + \frac{1}{r})}.
\]
Moreover
\[
E h_{k,p}(U) h_{k,-p}(U) = E h_{k,p}(U) E h_{k,-p}(U) + \text{Cov}(h_{k,p}(U), h_{k,-p}(U)) \\
\geq E h_{k,p}(U) E h_{k,-p}(U) - \sqrt{\text{Var} h_{k,p}(U) \text{Var} h_{k,-p}(U)}
\]
\[
\geq E h_{k,p}(U) E h_{k,-p}(U) \left( 1 - \sqrt{\left( e^{\frac{c l^2_{k,p}}{n}} - 1 \right) \left( e^{\frac{c l^2_{k,-p}}{n}} - 1 \right)} \right)
\]
\[
(20) \geq E h_{k,p}(U) E h_{k,-p}(U) \left( 1 - \sqrt{\left( e^{\frac{c l^2_{k,p}}{n}} - 1 \right) \left( e^{\frac{c l^2_{k,-p}}{n}} - 1 \right)} \right)
\]
where the last inequality follows from (18). Assume moreover that \( k \) is such that \( 2k - 1 \leq n \) and \( 2k \leq r \) then for \( p \leq (k - 1)/2 \), we can evaluate \( L_{k,p} \), \( L_{k,-p} \) and \( L_{k,0} \) from Proposition 14 since the assumptions are fulfilled. We get that if \( X \) is isotropic then \( \max(L_{k,p}, L_{k,-p}, L_{k,0}) \leq Ck^2 \). If \( k \leq c_0 n^{1/4} \) for a small enough numerical constant \( c_0 \), we have

\[
\sqrt{\left(e^{L_{k,p}^2/n} - 1\right) \left(e^{L_{k,-p}^2/n} - 1\right)} \leq c' \frac{k^4}{n} \leq \frac{1}{10}.
\]

Combining this estimate with (20) and (19), we have proved that if \( k \) is an integer such that \( k \geq 2, 2k - 1 \leq n, 2k \leq r, k \leq c_0 n^{1/4} \) and \( 2p + 1 \leq k \) (this set of integers is not empty since \( r > 32 \) and \( p \leq r/8 \)) then

\[
\mathbb{E}h_{k,p}(U) \mathbb{E}h_{k,-p}(U) \leq e^{4p^2\left(\frac{k}{2} + \frac{1}{n}\right) + C\frac{k^4}{pn}} \leq e^{4p^2\left(\frac{k}{2} + \frac{1}{n}\right) + C\frac{k^4}{pn}}.
\]

For \( p \leq 1 \), we also force \( k \) to satisfy \( k \leq C_0 p^{1/4} n^{1/4} \). Hence taking the power \( 1/p \) in the last expression, we conclude from (16) that

\[
(\mathbb{E}|X|^p_2 \mathbb{E}|X|^{-p}_2)^{1/p} \leq e^{4p^2\left(\frac{k}{2} + \frac{1}{n}\right) + C\frac{k^4}{pn}} \leq 1 + cp \left(\frac{1}{k} + \frac{1}{r}\right) + c \frac{k^4}{pn},
\]

since \( p/k, p/r \) and \( k^4/pn \) are bounded by universal constants. It remains to optimize the choice of \( k \). Let \( p_0 = n^{-1/2} \). In this case we choose \( k = 2 \) and get

\[
(\mathbb{E}|X|^p_2 \mathbb{E}|X|^{-p}_2)^{1/p_0} \leq 1 + \frac{C}{\sqrt{n}}.
\]

If \( p \geq n^{-1/2} \) we choose \( k \) to be an integer such that \( \min(r/4, (p^2n)^{1/5}) \leq k \leq 2 \min(r/4, (p^2n)^{1/5}) \) with the restriction \( 2p + 1 \leq k \leq cn^{1/4} \) and that \( k \leq cp^{1/4} n^{1/4} \). For any \( p \) such that \( p_0 \leq p \leq \min(cn^{1/8}, r/8) \), the integer \( k \) satisfies \( k \geq 2, 2k - 1 \leq n, 2k \leq r, k \leq c_0 n^{1/4} \) and \( 2p + 1 \leq k \) and we get that

\[
(\mathbb{E}|X|^p_2 \mathbb{E}|X|^{-p}_2)^{1/p} \leq 1 + \frac{Cp}{r} + \left(\frac{Cp}{n^{1/3}}\right)^{3/5}.
\]

This ends the proof of (13).

In the second part, we follow the argument developed in [18] to get a better estimate. We deal now with the case of \( p \) being positive or negative and, as already said, we can assume without loss of generality that \( r > 34 \) and \( |p| \leq r/8 \). As in [18], our goal is to estimate

\[
\frac{d}{dp} \log((\mathbb{E}|X|^p_2)^{1/2}) = \frac{d}{dp} \log((\mathbb{E}h_{k,p}(U))^{1/2}) + \frac{d}{dp} \left(\frac{1}{p} \log \frac{\Gamma((p + n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p + k)/2)}\right).
\]
Most of the computation of section 3.2 in [18] is identical. All the ingredients needed for the proof have been established and, adapting the argument done in section 3.2 in [18], we get

\[
\frac{d}{dp} \log((E|X|_2^p)^{1/p}) \leq \frac{c}{p^2 n} (2L_{k,p}^2 + 3L_{k,0}^2) + \frac{C}{k - 1} + \frac{C}{r - 1}.
\]

For convenience of the reader, we will shortly reproduce the proof of (22) in the appendix.

Assume that \(X\) is isotropic. For any \(2|p| \leq k \leq r/2\) (this set of integers is not empty since \(r > 32\) and \(|p| \leq r/8\)), we know by Proposition 14, that \(L_{k,p}\) and \(L_{k,0}\) are smaller than \(Ck^2\). We get that

\[
\frac{d}{dp} \log((E|X|_2^p)^{1/p}) \leq C \left( \frac{k^4}{p^2 n} + \frac{1}{k} + \frac{1}{r} \right).
\]

We have to minimize this expression for \(k\) being an integer greater or equal than 2 and \(k \in [2|p|, r/2]\). For \(|p| \in [n^{-1/2}, cn^{1/3}]\), we set \(k\) being an integer such that \(\min(r/4, 2(p^2 n)^{1/5}) \leq k \leq 2 \min(r/4, 2(p^2 n)^{1/5})\). Therefore \(k\) satisfies the restrictions and we get for any \(p\) such that \(|p| \in [n^{-1/2}, cn^{1/3}]\),

\[
\frac{d}{dp} \log((E|X|_2^p)^{1/p}) \leq C \left( \frac{1}{(p^2 n)^{1/5}} + \frac{1}{r} \right).
\]

After integration over \(p\), we get that for all \(p \in [n^{-1/2}, c \min(r, n^{1/3})]\)

\[
\left| \log \left( \frac{(E|X|_2^p)^{1/p}}{(E|X|_2^0)^{1/2}} \right) \right| \leq C \left| p - 2 \right| + \frac{C |p^{3/5} - 2^{3/5}|}{n^{1/5}}.
\]

Since \(|p^{3/5} - 2^{3/5}| \leq |p - 2|^{3/5}\) and all terms in the right hand side of the inequality are bounded by a universal constant, we conclude by adjusting that

\[
\left| \frac{(E|X|_2^p)^{1/p}}{(E|X|_2^0)^{1/2}} - 1 \right| \leq C \left| p - 2 \right| + \frac{C |p - 2|^{3/5}}{n^{1/3}}, \quad \forall p \in [n^{-1/2}, c \min(r, n^{1/3})].
\]

Since (23) holds only for \(|p| \geq n^{-1/2}\), we use (21) to bridge the gap between \(-n^{-1/2}\) and \(n^{-1/2}\). Indeed, from (21), the previous inequality for \(p_0 = n^{-1/2}\) and using that \(|p_0 - 2| = 2 - p_0 \leq 2\), we get that for \(p \in [-p_0, p_0]\),

\[
(E|X|_2^p)^{1/p} \geq (E|X|_2^{-p_0})^{-1/p_0} \geq \frac{1}{1 + \frac{C}{\sqrt{n}} (E|X|_2^{p_0})^{1/p_0}} \geq 1 - \frac{2C}{r} - \frac{(2C/n^{1/3})^{3/5}}{1 + \frac{C}{n^{1/3}}} (E|X|_2^0)^{1/2}.
\]

An easy adaptation of the constants leads to the conclusion of Theorem 1 for all \(p \in [-n^{-1/2}, n^{-1/2}]\).

Integrating again (23), we get, for \(p \in [-c \min(r, n^{1/3}), -n^{-1/2}]\),

\[
\frac{(E|X|_2^p)^{1/p}}{(E|X|_2^{-p_0})^{-1/p_0}} \geq 1 - \frac{C |p + p_0|}{r} - \left( \frac{C |p + p_0|}{n^{1/3}} \right)^{3/5}.
\]
Using that $|p + p_0| \leq |p - 2|$ and the previous comparison of the moment of order $-p_0$ with the moment of order 2 and adjusting the constants, this proves that for all $p \in [-c \min(r, n^{1/3}), -n^{-1/2}]$,

$$\left| \left( \frac{\mathbb{E}|X|^p}{\mathbb{E}|X|^2} \right)^{1/p} - 1 \right| \leq \frac{C|p - 2|}{r} + \left( \frac{C|p - 2|}{n^{1/3}} \right)^{3/5}.$$  

This concludes the proof of the first part of Theorem 1.

If $X$ is such that $AX$ is isotropic, we know from Proposition 14 that for any integer $k$ such that $2|p| \leq k \leq r/2$,

$$\max(L_{k,p}, L_{k,0}) \leq Ck^2 \|A\|\|A^{-1}\|.$$  

The proof is identical to the previous one replacing $n$ by $\frac{n}{\|A\|^2\|A^{-1}\|^2}$. \hfill \Box

**Remark 15.** In [18], a preprocessing step consisted in adding a Gaussian isotropic vector to the random vector $X$ in order to start at the very beginning with a better information on the $Z_p^+$ bodies associated to the measure. In [22, 16], this convolution argument played a role of regularization. It is natural to ask if such a process could be done in the situation of $s$-concave measure. Nothing is doable by adding a Gaussian vector because for $s < 0$, the new vector does not belong to any class of $s$-concave vectors. However, for $r > n$, we can build a similar argument, adding to $X$ a random vector $Z$ uniformly distributed on the Euclidean ball, see also [9]. Since $Z$ is $(1/n)$-concave and $X$ is $(-1/r)$-concave, the new vector $Y = \frac{X + Z}{\sqrt{2}}$ will be $(-1/(r - n))$-concave. For any $p \geq 1$, we have (see inequality (4.7) in [18])

$$\alpha_p(X) \leq \alpha_{2p}(Y)(2 + \alpha_{2p}(Y))$$

so that it remains to bound $\alpha_{2p}(Y)$. It is easy to see that $Y$ is such that for every $q \geq 2$ and every $\theta \in S_n$, $(\mathbb{E} \langle Y, \theta \rangle^q)^{1/q} \geq c\sqrt{q}$. Adapting the proof of Proposition 14, we get $L_{k,p} \leq C \max(k, p)^{3/2}$. As in [18], this improvement leads to the following estimate: if $r - n > 2$, then for any $p$ such that $1 \leq p \leq c \min(r - n, \sqrt{n})$

$$\alpha_{2p}(Y) \leq \frac{C(2p - 2)}{r - n} + \left( \frac{C(2p - 2)}{\sqrt{n}} \right)^{1/2}.$$  

For $r > n + \sqrt{n}$, we recover the same thin-shell thin-shell concentration as in the log-concave case. It would be interesting to understand in which precise sense the $s$-concave measures are close to the log-concave measures for $s \in (-1/n, 1/n)$. Another question is to know what kind of preprocessing argument like in [23] would enable to recover the small ball estimates from [1].
4 Appendix

Proof of Theorem 7. (1) This result is classical. In the symmetric case, it follows from Lemma 2.1 in [24]. The general case is similar. We provide its proof for completeness. We may assume, without loss of generality, that \(\|f\|_{\infty} = 1\). Denote \(I_p(f) = \int_0^{+\infty} t^{p-1} f(t) dt\). From Hölder inequality, the function \(p \mapsto \log(I_p(f))\) is convex on its convex support, thus the domain of definition of \(I_p(f)\) is an interval. Let \(0 < p < q\) be fixed such that \(I_p(f) < +\infty\) and \(I_q(f) < +\infty\). Let \(a = (pI_p(f))^{1/p}\) and \(\varphi(t) = t^{q-1}(f(t) - 1_{[0,a]}(t))\). Notice that \(\varphi \leq 0\) on \([0,a]\), \(\varphi \geq 0\) on \([a, +\infty)\) and \(\int_0^{+\infty} \varphi(t) dt = 0\). Thus

\[
I_q(f) - I_q(1_{[0,a]}) = \int_0^{+\infty} t^{q-p} \varphi(t) dt = \int_0^{+\infty} (t^{q-p} - a^{q-p}) \varphi(t) dt \geq 0,
\]

since the integrand is non negative on \(\mathbb{R}_+\). We conclude that

\[
I_q(f) \geq I_q(1_{[0,a]}) = \frac{a^q}{q} = \frac{1}{q} (pI_p(f))^{\frac{q}{p}}.
\]

(2) Since \(f\) is \((-1/\alpha)\)-concave, there exists a convex function \(\varphi : [0, \infty) \to (0, \infty)\) such that \(f = \varphi^{-\alpha}\). Since \(f\) is integrable it follows that \(\varphi\) tends to +\(\infty\) at +\(\infty\). From the convexity of \(\varphi\), one deduces that for some constant \(c > 0\), \(\varphi(t) \geq c(1 + t)\). Thus \(f(t) \leq (c + ct)^{-\alpha}\), for every \(t \geq 0\). Therefore, \(t^{p-1}f\) is integrable for every \(p < \alpha\), which means that \(H_f(p) < +\infty\) for every \(0 < p < \alpha\). Let \(p \in (0, \alpha)\) and \(m, M > 0\). Define \(g : \mathbb{R}_+ \to \mathbb{R}_+\) by \(g(t) = m (1 + \frac{t}{M})^{-\alpha}\). Then

\[
\int_0^{+\infty} t^{p-1} g(t) dt = mM^p \int_0^{+\infty} v^{p-1}(1 + v)^{-\alpha} dv = mM^p B(p, \alpha - p).
\]

Thus \(H_g(p) = mM^p\), which implies that \(\log(H_g)\) is affine on \((0, \alpha)\). Take \(0 < a < b < c < \alpha\). Let \(\lambda \in [0,1]\) be such that \(b = (1 - \lambda)a + \lambda c\). Choose \(m\) and \(M\) such that \(mM^a = H_f(a)\) and \(mM^b = H_f(b)\) so that \(H_g(a) = H_f(a)\) and \(H_g(b) = H_f(b)\). If we prove that

\[
(24) \quad \int_0^{+\infty} t^{\alpha-1} (g - f)(t) dt \geq 0,
\]

that is \(H_g(c) \geq H_f(c)\), then using that \(\log(H_g)\) is affine, we will deduce that

\[
H_f(b) = H_g(b) = H_g(a)^{1-\lambda} H_g(c)^{\lambda} \geq H_f(a)^{1-\lambda} H_f(c)^{\lambda}
\]

and this will prove the log-concavity of \(H\) on \((0, \alpha)\). If \(f = g\) then \((24)\) is satisfied so that in the following we assume that the function \(h := g - f \neq 0\). Let

\[
H_1(t) = \int_t^{+\infty} s^{a-1} h(s) ds \quad \text{and} \quad H_2(t) = \int_t^{+\infty} s^{b-a-1} H_1(s) ds.
\]
Since $h(t) = O(t^{-\alpha})$ at infinity, we deduce that $H_1(t) = O(t^{a-\alpha})$ and $H_2(t) = O(t^{b-\alpha})$. We have $\int_0^{+\infty} t^{a-1} h(t) dt = 0$ thus $H_1(\infty) = H_1(0) = 0$. Obviously $H_2(\infty) = 0$. We also observe

$$0 = \int_0^{+\infty} t^{b-1} h(t) dt = \int_0^{+\infty} t^{b-a} t^{a-1} h(t) dt = - \int_0^{+\infty} t^{b-a} H'_1(t) dt$$

$$= [t^{b-a} H_1(t)]_0^{+\infty} + (b-a) \int_0^{+\infty} t^{b-a-1} H_1(t) dt = (b-a)H_2(0),$$

whence $H_2(\infty) = H_2(0) = 0$. Since $\int_0^{+\infty} t^{b-a-1} H_1(t) dt = 0$ and $H_1 \neq 0$, the function $H_1$ has at least one change of sign. Moreover, using that $H_1(0) = H_1(\infty) = 0$, we deduce that $H'_1$ and therefore $h$ has at least two sign changes. Since $h = g - f$ has the same sign as $f^{-\alpha} - g^{-\alpha}$ which is convex, it cannot have more than two sign changes. Thus it has exactly two sign changes at some $0 < t_1 < t_2$. Moreover, from the convexity of $f^{-\alpha} - g^{-\alpha}$, the sign of $h$ has to be negative on $(t_1, t_2)$ and positive on $(0, t_1)$ and $(t_2, +\infty)$. From an easy study of the function $H_2$, we deduce that $H_2 \geq 0$. Therefore, using that $H_1(0) = H_1(\infty) = H_2(0) = H_2(\infty) = 0$, we get

$$\int_0^{+\infty} t^{a-1} h(t) dt = \int_0^{+\infty} t^{c-a} t^{a-1} h(t) dt = - \int_0^{+\infty} t^{c-a} H'_1(t) dt$$

$$= [-t^{c-a} H_1(t)]_0^{+\infty} + (c-a) \int_0^{+\infty} t^{c-a-1} H_1(t) dt$$

$$= (c-a) \int_0^{+\infty} t^{c-b} t^{b-a-1} H_1(t) dt$$

$$= (c-a)[-t^{c-b} H_2(t)]_0^{+\infty} + (c-a) (c-b) \int_0^{+\infty} t^{c-b-1} H_2(t) dt$$

$$= (c-a)(c-b) \int_0^{+\infty} t^{c-b-1} H_2(t) dt \geq 0.$$

This proves (24) and establish the log-concavity of $H_f$ on $(0, \alpha)$. To get it on $[0, \alpha)$, it is enough to prove that $H_f$ is continuous at 0. This follows from the observation that

$$B(p, \alpha - p) \sim \frac{\Gamma(p)}{p^{\alpha-1}} \sim \frac{1}{p} \quad \text{thus} \quad H_f(p) \sim p \int_0^{+\infty} t^{p-1} f(t) dt.$$

And it is classical that, for a continuous function $f$, the right hand side term tends to $f(0)$ when $p \to 0$. \(\square\)

**Proof of Lemma 11.** Equation (9) follows easily from the classical bounds for the Gamma function (see [3]), valid for $x \geq 1$:

$$\sqrt{2\pi} x^{-\frac{1}{2}} e^{-x} \leq \Gamma(x) \leq \sqrt{2\pi} x^{-\frac{1}{2}} e^{-x + \frac{1}{2}}.$$

For equation (10), we write that

$$\frac{B(k+p,r-p)}{B(k,r)} = \frac{\Gamma(k+p)\Gamma(r-p)}{\Gamma(k)\Gamma(r)}.$$
Denote \( G(p) = \log \Gamma(p) \), for \( p > 0 \). We know that \( G''(p) = \sum_{i \geq 0} 1/(p + i)^2 \) hence \( G'' \) is non-increasing and \( 0 \leq G''(p) \leq 1/(p - 1) \), for \( p > 1 \). Denote \( F_k(p) = \frac{G(k + p) - G(k)}{p} \), for \( k > 0 \) and \( p > -k \). We have \( F_k(p) = \int_0^1 G'(k + up)du \). Using that \( G'' \) is non-increasing, we get that for \( k > 1 \) and \( p \geq -(k - 1)/2 \),

\[
F_k'(p) = \int_0^1 G''(k + up)udu \leq \frac{G''(k + 1/2)}{2} \int_0^1 udu = \frac{1}{2} G''(k + 1/2) \leq \frac{1}{k - 1}
\]

and \( F_k'(p) \geq 0 \). Therefore, if \( k > 1, r > 1 \) and \( -\frac{k - 1}{2} \leq p \leq -\frac{r - 1}{2} \) then

\[
0 \leq \frac{d}{dp} \left( \frac{1}{p} \log \frac{B(k + p, r - p)}{B(k, r)} \right) = \frac{d}{dp} (F_k(p) - F_r(-p)) = F_k'(p) + F_r'(-p) \leq \frac{1}{k - 1} + \frac{1}{r - 1}.
\]

**Proof of Lemma 12.** We present here a similar but simpler proof than in the appendix of [18]. By integration in polar coordinates, it is well known [25] (see also [19]) that we have the following relation between the \( Z^+_q \)-bodies associated with \( g \) and the \( Z^+_q \)-bodies associated with one of the convex bodies \( K_a(g) \): for any \( 0 < q < r \)

\[
(25) \quad Z^+_q(g) = g(0)^{1/q} Z^+_q(K_{m+q}(g)),
\]

where for any body \( K \), \( Z^+_q(K) \) denotes the convex body whose support function is defined by

\[
\forall \theta \in \mathbb{R}^n, \quad h_{Z^+_q(K)}(\theta) = \left( \int_K \langle x, \theta \rangle^q_+ dx \right)^{\frac{1}{q}}.
\]

Let \( \theta \in \mathbb{R}^n \) and \( K \) be a convex body containing 0. From Berwald’s inequalities [6] applied to \( K \cap \{ \langle x, \theta \rangle \geq 0 \} \) and the function \( x \mapsto \langle x, \theta \rangle_+ \) which is concave on \( K \cap \{ \langle x, \theta \rangle \geq 0 \} \), the function

\[
p \mapsto \left( \frac{\int_K \langle x, \theta \rangle^p_+ dx}{mB(p + 1, m) Vol(K \cap \{ \langle x, \theta \rangle \geq 0 \})} \right)^\frac{1}{p}
\]

is decreasing. Observe that for every \( \theta \in \mathbb{R}^n \), \( \lim_{p \to \infty} \left( \int_K \langle x, \theta \rangle^p_+ dx \right)^{\frac{1}{p}} = h_K(\theta) \) and that

\[
(mB(p + 1, m))^{\frac{1}{p}} = \left( m \int_0^1 u^p(1 - u)^{m-1}du \right)^{\frac{1}{p}} \to 1 \quad \text{as} \quad p \to +\infty.
\]

We deduce that

\[
\left( \frac{\int_K \langle x, \theta \rangle^q_+ dx}{mB(q + 1, m) Vol(K \cap \{ \langle x, \theta \rangle \geq 0 \})} \right)^\frac{1}{q} \geq h_K(\theta).
\]
Now we establish that
\[ h_K(\theta) \geq \frac{h_{Z_+^+(K)}(\theta)}{\text{Vol}(K \cap \{(x, \theta) \geq 0\})^{1/q}} \geq (qB(q, m + 1))^{1/q} h_K(\theta). \]

(26) \[ h_K(\theta) \geq \frac{h_{Z_+^+(K)}(\theta)}{\text{Vol}(K \cap \{(x, \theta) \geq 0\})^{1/q}} \geq (qB(q, m + 1))^{1/q} h_K(\theta). \]

Now we establish that
\[ d(K_{m+p}(g), B_2(F)) \leq c \ d(Z_{\text{max}(m,p)}^+(g), B_2(F)). \]

First assume that \( p \geq m \). Take \( q = \max(p, m) = p \) and \( K = K_{m+p}(g) \). By Lemma 11, for any \( p \geq m \geq 1 \), \( (pB(p, m + 1))^{1/p} \geq cp/(m + p + 1) \geq c/3 \) and we deduce from (26) that
\[ c K_{m+p}(g) \subset \frac{1}{\text{Vol}(K_{m+p}(g) \cap \{(x, \theta) \geq 0\})^{1/p}} Z_+^+(K_{m+p}(g)) \subset K_{m+p}(g), \]
where \( c \) is a universal constant. Together with (25), we conclude that
\[ d(K_{m+p}(g), Z_+^+(g)) = d(K_{m+p}(g), Z_+^+(K_{m+p}(g))) \leq c \]
for a universal constant \( c \). This gives the conclusion.

In the case when \( -\frac{m}{2} < p < m \), Corollary 10 with \( a = m + p \), \( b = 2m \) and \( \alpha = r + m \) gives
\[ \left( \frac{g(0)}{\|g\|_{\infty}} \right)^{\frac{1}{m+p}-\frac{1}{2m}} K_{m+p}(g) \subset K_{2m}(g) \subset \frac{(2m)B(2m, r - m))^{1/2m}}{((m + p)B(m + p, r - p))^{1/m+p}} K_{m+p}(g). \]

From Lemma 11, we have
\[ \frac{(2m)B(2m, r - m))^{1/2m}}{((m + p)B(m + p, r - p))^{1/m+p}} \leq c \frac{2m}{m + p} \leq 4c \]
since \( p \geq -\frac{m}{2} \). Thus \( K_{2m}(g) \subset 4cK_{m+p}(g) \).

For the other inclusion, we use the fact that \( \int xg(x)dx = 0 \) and the sharp lower bound of the ratio \( g(0)/\|g\|_{\infty} \), proved in Lemma 7.2 of [1],
\[ \frac{g(0)}{\|g\|_{\infty}} \geq \left( \frac{r - 1}{r + m - 1} \right)^{r+m} \geq e^{-2m} \]
for \( m \leq r - 1 \). Since \( p \geq -\frac{m}{2} \), we have \( (m - p)/(m + p) \leq 3 \) hence
\[ \left( \frac{g(0)}{\|g\|_{\infty}} \right)^{\frac{1}{m+p}-\frac{1}{2m}} \geq e^{-3}. \]

Therefore \( d(K_{m+p}(g), K_{2m}(g)) \leq c' \) where \( c' \) is a universal constant. We conclude that \( d(K_{m+p}(g), B_2(F)) \leq c'd(K_{2m}(g), B_2(F)) \) and using (27) we have \( d(K_{2m}(g), B_2(F)) \leq c' d(Z_{m}^+(g), B_2(F)). \) This finishes the proof. \( \square \)
**Proof of inequality** (22). Our goal is to estimate

\[
\frac{d}{dp} \log \left( \left( \mathbb{E}[|X|^2] \right)^{\frac{1}{p}} \right) = \frac{d}{dp} \log \left( \left( \mathbb{E}h_{k,p}(U) \right)^{\frac{1}{p}} \right) + \frac{d}{dp} \left( \frac{1}{p} \log \frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)} \right).
\]

As already mentioned in (15), by concavity of \( p \mapsto \frac{d}{dp} \log \Gamma \left( \frac{p+n}{2} \right) \Gamma \left( \frac{k}{2} \right) \Gamma \left( \frac{n}{2} \right) \Gamma \left( \frac{p+k}{2} \right) \), we have

\[
\frac{d}{dp} \left( \frac{1}{p} \log \frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)} \right) \leq 0.
\]

We use the following convention: let \((\Omega, \mu)\) be a measurable space, for any measurable function \( f : \Omega \rightarrow \mathbb{R}^+ \), we set

\[
\mathbb{E}_\mu(f) = \int f \, d\mu \quad \text{and} \quad \text{Ent}_\mu(f) = \mathbb{E}_\mu(f \log f) - \mathbb{E}_\mu(f) \log(\mathbb{E}_\mu(f)).
\]

Let \( w \) be the density of the distribution of \( X \) on \( \mathbb{R}^n \). Since \( X \) is \((-1/r)-concave\), \( w \) is \((-1/(r+n))-concave\) on \( \mathbb{R}^n \). To any fixed \( u \in SO(n) \), we associate the measure \( \mu_u \) on \( \mathbb{R}^+ \) with density

\[
t \mapsto |S^{k-1}| t^{k-1} \pi_u(E_0) w(tu(\theta_0))
\]

so that

\[
h_{k,p}(u) = |S^{k-1}| \int_0^\infty t^{p+k-1} \pi_u(E_0) w(tu(\theta_0)) \, dt.
\]

Define also \( \mu_{k,p} \) the measure on \( \mathbb{R}^+ \) with density

\[
t \mapsto |S^{k-1}| t^{k-1} \pi U(E_0) w(tU(\theta_0)).
\]

Then \( \mathbb{E}h_{k,p}(U) = \mathbb{E}_U \mathbb{E}_{\mu_U}(t^p) = \mathbb{E}_{\mu_{k,p}}(t^p) \). Since \( w \) is a density of probability, \( \mu_{k,p} \) is a probability measure on \( \mathbb{R}^+ \). A classical fact, verified by direct computation, is that

\[
\frac{d}{dp} \log \left( \left( \mathbb{E}_{\mu}(f^p) \right)^{1/p} \right) = \frac{1}{p^2} \frac{\text{Ent}_{\mu}(f^p)}{\mathbb{E}_{\mu}(f^p)}.
\]

Therefore

\[
\frac{d}{dp} \log \left( \left( \mathbb{E}h_{k,p}(U) \right)^{\frac{1}{p}} \right) = \frac{d}{dp} \log \left( \left( \mathbb{E}_{\mu_{k,p}}(t^p) \right)^{\frac{1}{p}} \right)
\]

\[
= \frac{1}{p^2} \frac{\text{Ent}_{\mu_{k,p}}(t^p)}{\mathbb{E}_{\mu_{k,p}}(t^p)} = \frac{1}{p^2} \frac{\text{Ent}_{\mu_{k,p}}(t^p)}{\mathbb{E}h_{k,p}(U)}.
\]

(28)

The numerator can be decomposed into two terms:

\[
\text{Ent}_{\mu_{k,p}}(t^p) = \mathbb{E}_U \text{Ent}_{\mu_U}(t^p) + \mathbb{E}_U \mathbb{E}_{\mu_U}(t^p) \mathbb{E}_U \text{Ent}_{\mu_U}(t^p) + \mathbb{E}_U h_{k,p}(U).
\]
To control the second term, we use the log-Sobolev inequality (2):

\[
\frac{1}{p^2} \mathbb{E} \left[ \text{Ent}_{\mu^p(U)} \right] \leq \frac{c}{p^2 n} \mathbb{E} \left[ (\nabla \log h_{k,p}(U) h_{k,p}(U)) \right) \leq \frac{cL_{k,p}^2}{p^2 n}.
\]

To control the first term, we start by observing that for a fixed \( u \in SO(n) \),

\[
\mathbb{E} \left[ \text{Ent}_{\mu^p(U)} \right] = \frac{1}{dp} \frac{dp}{p} \left( \log \frac{h_{k,p}(u)}{B(p+k,r-p)} - \log \frac{h_{k,0}(u)}{B(k,r)} + \log \frac{B(p+k,r-p)}{B(k,r)} + \log h_{k,0}(u) \right).
\]

By Corollary 9, the map \( p \mapsto \frac{h_{k,p}(u)}{B(p+k,r-p)} \) is log-concave on \((-k + 1, r)\). This implies that

\[
\frac{d}{dp} \left( \log \frac{h_{k,p}(u)}{B(p+k,r-p)} - \log \frac{h_{k,0}(u)}{B(k,r)} \right) \leq 0.
\]

We know from Lemma 11 that, for all \( p \in [-\frac{k-1}{2}, \frac{r-1}{2}] \),

\[
\frac{d}{dp} \left( \frac{1}{p} \log \frac{B(k+p,r-p)}{B(k,r)} \right) \leq C \left( \frac{1}{k-1} + \frac{1}{r-1} \right).
\]

Therefore, for any fixed \( u \in SO(n) \),

\[
\frac{1}{p^2} \mathbb{E} \left[ \text{Ent}_{\mu^p(U)} \right] \leq Ch_{k,p}(u) \left( \frac{1}{k-1} + \frac{1}{r-1} \right) - \frac{1}{p^2} h_{k,p}(u) \log h_{k,0}(u).
\]

Integrating over \( u \in SO(n) \), we deduce that

\[
\frac{1}{p^2} \mathbb{E} \left[ \text{Ent}_{\mu^p(U)} \right] \leq C \left( \frac{1}{k-1} + \frac{1}{r-1} \right) + \frac{1}{p^2} \mathbb{E} h_{k,p}(U) \log(h_{k,0}(U)^{-1}) \mathbb{E} h_{k,p}(U).
\]

From Jensen and Hölder inequalities,

\[
\frac{\mathbb{E}(h_{k,p}(U) \log h_{k,0}(U)^{-1})}{\mathbb{E} h_{k,p}(U)} \leq \log \left( \frac{\mathbb{E}(h_{k,p}(U) h_{k,0}(U)^{-1})}{\mathbb{E} h_{k,p}(U)} \right) - \log \left( \frac{(\mathbb{E}(h_{k,0}(U)^{-2}))^{1/2}}{\mathbb{E} h_{k,p}(U)} \right) + \log (\mathbb{E}(h_{k,0}(U)^{-2}))^{1/2}.
\]

From (3), the first term is upper bounded by \( \frac{c}{n} L_{k,p}^2 \). For the second term, we first use (3) with \( f = h_{k,0}^{-1}, q = 2 \) and \( r = 0 \), then we use (3) again with \( f = h_{k,0}, q = 1 \) and \( r = 0 \). Since \( \mathbb{E} h_{k,0}(U) = \mathbb{E} \mu_{k,0}(1) = 1 \), we deduce that this term is bounded by \( \frac{c}{n} L_{k,0}^2 \). Combining this last inequality with (30), (29) and (28), we conclude that

\[
\frac{d}{dp} \log((\mathbb{E}[X]_2^2)) \leq \frac{c}{p^2 n} (2L_{k,p}^2 + 3L_{k,0}^2) + \frac{C}{k-1} + \frac{C}{r-1}.
\]
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References


