Thin-shell concentration for convex measures

by

MATTHIEU FRADELIZI, OLIVIER GUÉDON and
ALAIN PAJOR (Paris-Est)

Abstract. We prove that for $s < 0$, $s$-concave measures on $\mathbb{R}^n$ exhibit thin-shell concentration similar to the log-concave case. This leads to a Berry–Esseen type estimate for most of their one-dimensional marginal distributions. We also establish sharp reverse Hölder inequalities for $s$-concave measures.

1. Introduction. For any subsets $A, B \subseteq \mathbb{R}^n$, their Minkowski sum is defined by

$$A + B = \{a + b : a \in A, b \in B\}.$$ 

Let $s \in [-\infty, 1]$. A measure $\mu$ on $\mathbb{R}^n$ is called $s$-concave whenever

$$\mu((1 - \lambda)A + \lambda B) \geq ((1 - \lambda)^s \mu(A)^s + \lambda \mu(B)^s)^{1/s}$$

for every $\lambda \in [0, 1]$ and any compact subsets $A, B \subseteq \mathbb{R}^n$ such that $\mu(A) \mu(B) > 0$. When $s = 0$, this inequality should be read as

$$\mu((1 - \lambda)A + \lambda B) \geq \mu(A)^1 - \lambda \mu(B)^\lambda$$

and it defines $\mu$ as a log-concave measure. When $s = -\infty$, the measure is said to be convex and the inequality is replaced by

$$\mu((1 - \lambda)A + \lambda B) \geq \min(\mu(A), \mu(B)).$$

Notice that the class of $s$-concave measures on $\mathbb{R}^n$ is decreasing in $s$ so that any $s$-concave measure is a convex measure. Any $s$-concave measure with $s \geq 0$ is log-concave, and thin-shell concentration for log-concave measures has been studied in [16, 17, 19, 22, 23]. The purpose of this paper is to prove thin-shell concentration for $s$-concave measures in the case $s < 0$, which we consider from now on. By measure, we always mean probability measure.

The class of $s$-concave measures was introduced and studied in [10, 11], where a complete characterization was established. An $s$-concave measure is

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supported on some convex subset of an affine subspace where it has a density (see Section 2 for more details). When the support of an \( s \)-concave measure \( \mu \) generates the whole space, we say that \( \mu \) is \textit{full-dimensional}.

A random vector with an \( s \)-concave distribution is called \textit{\( s \)-concave}. The linear image of an \( s \)-concave random vector is also \( s \)-concave. We say that a random vector is full-dimensional if its distribution is \textit{full-dimensional}. It is known that any seminorm of an \( s \)-concave random vector with \( s < 0 \) has moments of all order \( p \in (0, -1/s) \) (see [10] and [1]). The Euclidean norm of an \( s \)-concave random vector \( X \) has a finite moment of order 2 if and only if \( s > -1/2 \). Since we are interested in comparison of moments of the Euclidean norm with the moment of order 2, we will always assume that \(-1/2 < s < 0\).

Let \( n \geq 1 \) be an integer. The Euclidean space \( \mathbb{R}^n \) is equipped with its Euclidean norm \( |\cdot|_2 \) and scalar product \( \langle \cdot, \cdot \rangle \). Its unit sphere is denoted by \( S^{n-1} \) and its unit ball by \( B_2^n \). We say that a random vector \( X \) is \textit{isotropic} if \( \mathbb{E} X = 0 \) and for every \( \theta \in S^{n-1} \), \( \mathbb{E} \langle X, \theta \rangle^2 = 1 \). Observe that if \( X \) is an \( s \)-concave full-dimensional random vector and \( s > -1/2 \), we can always find an affine transformation \( A \) such that \( AX \) is isotropic.

Let \( p \in \mathbb{R} \) and \( X \in \mathbb{R}^n \) be a random vector. Assume that \( |X|_2 \) has finite moments of order 2 and \( p \) with the convention that \( (\mathbb{E} |X|_2^p)^{1/p} = \exp(\mathbb{E} \ln |X|_2) \) for \( p = 0 \). We define

\[
\alpha_p(X) := \left| \frac{\mathbb{E} |X|_2^p}{(\mathbb{E} |X|_2^2)^{1/2}} - 1 \right|.
\]

Our main result is the following

\textbf{Theorem 1.} Let \( r > 2 \). Let \( X \in \mathbb{R}^n \) be a full-dimensional \((-1/r)\)-concave random vector. If \( X \) is isotropic, then for any \( p \) such that \( |p| \leq c \min(r, n^{1/3}) \), we have

\[
\alpha_p(X) \leq \frac{C|p - 2|}{r} + \left( \frac{C|p - 2|}{n^{1/3}} \right)^{3/5},
\]

where \( C \) and \( c \) are universal constants.

In the general case (when \( X \) is not isotropic), let \( A \) be an affine transformation such that \( AX \) is full-dimensional and isotropic. Then for any \( p \in \mathbb{R} \) such that \( |p| \leq c \min(r, \|A\|^{2/3}\|A^{-1}\|^{2/3}) \), we have

\[
\alpha_p(X) \leq \frac{C|p - 2|}{r} + \left( \frac{C|p - 2|\|A\|\|A^{-1}\|^{2/3}}{n^{1/3}} \right)^{3/5},
\]

where \( C \) and \( c \) are universal constants.

We also show (see Remark 15) that for \( r > n + \sqrt{n} \), the estimate of \( \alpha_p(X) \) in Theorem 1 can be improved and recovers the estimate of the log-concave case from [19].
To present connections between moment inequalities, thin-shell concentration and the Berry–Esseen theorem for one-dimensional marginals, let us introduce some notation.

Let \( X \in \mathbb{R}^n \) be an isotropic random vector. Thus \( \mathbb{E}|X|^2 = n \). Define \( \varepsilon(X) \) to be the smallest number \( \varepsilon > 0 \) such that

\[
\mathbb{P}\left( \left| \frac{|X|^2}{\sqrt{n}} - 1 \right| \geq \varepsilon \right) \leq \varepsilon.
\]

If \( \varepsilon(X) = o(1) \) with respect to the dimension \( n \), we say that \( X \) is concentrated in a thin shell. This is the usual jargon of the subject. More rigorously, it suggests that we are considering a sequence \( (X_n) \) of random vectors with \( X_n \in \mathbb{R}^n \) and that \( \varepsilon(X_n) = o(1) \) as \( n \) goes to \( \infty \). It was shown in [2] (see also [14, 13]) that if an isotropic random vector \( X \) uniformly distributed on a convex body in \( \mathbb{R}^n \) is such that \( \varepsilon(X) = o(1) \), then almost all one-dimensional marginal distributions of \( X \) satisfy a Berry–Esseen theorem. More generally, let \( X \in \mathbb{R}^n \) be an isotropic random vector; it was proved in [7] that

\[
\sigma_{n-1}\left( \theta \in S^{n-1} : \sup_{t \in \mathbb{R}} |\mathbb{P}(\langle X, \theta \rangle \leq t) - \Phi(t)| \geq 4\varepsilon(X) + \delta \right) \leq 4n^{3/8} e^{-cn\delta^4},
\]

where \( \sigma_{n-1} \) denotes the rotation invariant probability measure on the unit sphere \( S^{n-1} \), \( \Phi \) is the standard normal distribution function and \( c > 0 \) is a universal constant. It is worth noticing that the result from [7] does not assume log-concavity. Assuming only that \( X \) is isotropic, we find that if \( \varepsilon(X) \) is \( o(1) \) then almost all one-dimensional marginal distributions of \( X \) are approximately Gaussian. Later in [22, 17] it was proved that indeed \( \varepsilon(X) = o(1) \) for all log-concave random vectors, and the best estimate to date [19] is

\[
\varepsilon(X) = O(n^{-1/6} \log n).
\]

Now let \( p > 2 \) and assume that \( X \) is isotropic and that \( |X|^2 \) has a finite moment of order \( p \). Then \( \varepsilon(X) \) is \( o(1) \) if and only if \( \alpha_p(X) \) is \( o(1) \) (see Remark 4 below). Hence Theorem 1 ensures that if \( r \to \infty \) with the dimension \( n \) then any isotropic \((-1/r)\)-concave random vector exhibits thin-shell concentration and therefore almost all of its one-dimensional marginals satisfy a Berry–Esseen theorem. As a matter of fact, this condition on \( r \) is necessary. If \( r \) is fixed and does not depend on the dimension \( n \), Proposition 5 gives an example of an isotropic \((-1/r)\)-concave random vector \( X \in \mathbb{R}^n \) which does not have thin-shell concentration. Remark 6 also shows the asymptotic sharpness of Theorem 1 since for this example, for a fixed \( p > 2 \), \( \alpha_p(X) \geq C(p-2)/r \) for \( r \) and \( n \) large enough, where \( C > 0 \) is a universal constant.

To prove Theorem 1 we need to extend to the case of \( s \)-concave measures several tools coming from the study of log-concave measures. This is the
purpose of Section 2. Some of them were already established by Bobkov [8], like an analog of Ball’s bodies [5] in the s-concave setting. Some others were also noticed previously (see e.g. [8], [1]) but not with the most accurate point of view. These new ingredients are analogous to the results of [12] in the log-concave setting and are at the heart of our proof. As in the approach of [16] or [19], an important ingredient is the log-Sobolev inequality on SO(n). It follows e.g. from the work of Bakry and Émery [4] and the calculation of the Ricci curvature of SO(n) (see [21, formula (F6)] for example) that for any Lipschitz function $f : \text{SO}(n) \to \mathbb{R}^+$ (see Sections 3 and 4 for definitions)

$$
\mathbb{E}(f(U) \log f(U)) - \mathbb{E} f(U) \log(\mathbb{E} f(U)) \leq \frac{c}{n} \mathbb{E}(|\nabla \log f(U)|^2 f(U)),
$$

where $U$ is uniformly distributed on SO(n). This allows one to get reverse Hölder inequalities (see [16, (15)]): for every $f : \text{SO}(n) \to \mathbb{R}$, let $L$ be the log-Lipschitz constant of $f$ (that is, the Lipschitz constant of $\log f$); then for every $q > r > 0$,

$$
(\mathbb{E} |f(U)|^q)^{1/q} \leq \exp \left( \frac{cL^2}{n} (q - r) \right) (\mathbb{E} |f(U)|^r)^{1/r},
$$

where $U$ is uniformly distributed on SO(n).

Let $X$ be a $(-1/r)$-concave random vector in $\mathbb{R}^n$ with full-dimensional support and distributed according to a measure with a density function $w : \mathbb{R}^n \to \mathbb{R}^+$. For any linear subspace $E$, denote by $P_E$ the orthogonal projection onto $E$ and for any $x \in E$ denote by

$$
\pi_E w(x) = \int_{x+E^\perp} w(y) \, dy
$$

the marginal of $w$ on $E$. Given an integer $k$ between 1 and $n$, a real number $p \in (-k, r)$, a linear subspace $E_0$ of $\mathbb{R}^n$ of dimension $k$, and $\theta_0 \in S(E_0)$, where $S(E_0)$ denotes the unit sphere of $E_0$, we define the function $h_{k,p} : \text{SO}(n) \to \mathbb{R}^+$ by

$$
h_{k,p}(u) := |S^{k-1}| \int_0^\infty t^{p+k-1} \pi_{u(E_0)} w(tu(\theta_0)) \, dt
$$

for every $u \in \text{SO}(n)$, where $|S^{k-1}|$ denotes the area of the sphere.

Following the approach of [23, 16], we observe that for any $p \in (-k, r),$

$$
\mathbb{E} |X|^p = \frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)} \mathbb{E} h_{k,p}(U),
$$

where $U$ is uniformly distributed on SO(n). In view of (5) and the definition of $h_{k,p}$, we notice that it is of importance to work with families of measures which are stable under taking marginals, and it is clear from the definition that for any subspace $E$, if $X$ is $(-1/r)$-concave, then so is $P_E X$. 
In Section 2, we first introduce more notation and recall important facts concerning convex measures. Then we give an example of an isotropic \((-1/r)\)-concave random vector \(X \in \mathbb{R}^n\) that does not have thin-shell concentration, when \(r\) is fixed with respect to the dimension. Finally, we extend to the case of \(s\)-concave measures several tools coming from the study of log-concave measures that will be essential in the proof of Theorem 1. Section 3 is devoted to the proof of Theorem 1. Some of the results of these two sections are either classical or variations of known results; their proofs are shifted to the appendix.

2. Preliminary results for \(s\)-concave measures. We first recall some properties of \(s\)-concave measures and their relation to \(\beta\)-concave functions.

The class of \(s\)-concave measures was introduced and studied in \([10, 11]\), where the following complete characterization was established. An \(s\)-concave measure \(\mu\) on \(\mathbb{R}^n\) is supported on some convex subset of an affine subspace where it has a density. When this subspace is the whole space, we say that \(\mu\) is full-dimensional. In this case, its density \(w\) is \(\beta\)-concave with \(\beta = s/(1-ns)\).

Recall that a function \(f : \mathbb{R}^n \to \mathbb{R}_+\) is called \(\beta\)-concave whenever

\[
    f((1-\lambda)x + \lambda y) \geq ((1-\lambda)f(x)^\beta + \lambda f(y)^\beta)^{1/\beta}
\]

for every \(\lambda \in [0, 1]\) and all \(x, y \in \mathbb{R}^n\) such that \(f(x)f(y) > 0\), where the right hand side is replaced by \(f(x)^{1-\lambda}f(y)^{\lambda}\) for \(\beta = 0\). Note that when \(\beta < 0\), which will be the case below, \(\beta\)-concavity means that \(f^\beta\) is convex on its convex support \(\{f > 0\}\).

We will use a similar language for probability measure, random vector and function which are related here as distribution, law of a random vector and density of probability. It is important to remember that when \(X \in \mathbb{R}^n\) is \((-1/r)\)-concave full-dimensional, then the result recalled above states that its distribution has a support that generates \(\mathbb{R}^n\) and has a density which is \((-1/(n+r))\)-concave.

Recall that for every \(x > 0\), \(\Gamma(x) = \int_0^\infty u^{x-1}e^{-u} du\), and for every \(x, y > 0\),

\[
    B(x, y) = \int_0^1 u^{x-1}(1-u)^{y-1} du = \int_0^\infty u^{x-1}(u+1)^{-(x+y)} du.
\]

The following inequality of Paley–Zygmund type is well known.

**Lemma 2.** Let \(2 < p < s\). Let \(Y\) be a non-negative random variable with finite \(s\)-moment. Then for every \(0 \leq t \leq (\mathbb{E}Y^p)^{1/p}\) we have

\[
    \mathbb{P}(Y \geq t) \geq \left( \frac{\mathbb{E}Y^p - t^p}{(\mathbb{E}Y^s)^{p/s}} \right)^{s/(s-p)}.
\]

**Proof.** Using the Hölder inequality, we have

\[
    \mathbb{E}Y^p = \mathbb{E}Y^p 1_{Y < t} + \mathbb{E}Y^p 1_{Y \geq t} \leq t^p + (\mathbb{E}Y^s)^{p/s}\mathbb{P}(Y \geq t)^{1-p/s}.
\]
Thus
\[ P(Y \geq t) \geq \left( \frac{\mathbb{E} Y^p - t^p}{(\mathbb{E} Y^s)^{p/s}} \right)^{s/(s-p)} \]

**Proposition 3.** Let \( 2 < p < s \). Let \( X \in \mathbb{R}^n \) be an isotropic random vector such that \(|X|_2\) has a finite s-moment. Then
\[
\min\left( \frac{\alpha_p(X)}{2}, \left( \frac{p\alpha_p(X)/2}{(\alpha_s(X) + 1)^p} \right)^{s/(s-p)} \right) \leq \varepsilon(X) \leq (\alpha_p(X) + 1)^p - 1)^{1/3}.
\]

**Proof.** Let \( \varepsilon > 0 \). Applying Lemma 2 to \( Y = |X|_2/(\mathbb{E} |X|_2^2)^{1/2} \), \( t = \varepsilon + 1 \) and noticing that \( \mathbb{E} Y^p = (\alpha_p(X) + 1)^p \), \( \mathbb{E} Y^s = (\alpha_s(X) + 1)^s \), we get
\[
P\left( \frac{|X|_2}{(\mathbb{E} |X|_2^2)^{1/2}} \geq 1 + \varepsilon \right) \geq \left( \frac{(\alpha_p(X) + 1)^p - (\varepsilon + 1)^p}{(\alpha_s(X) + 1)^p} \right)^{s/(s-p)}
\]
whenever \( 0 < \varepsilon \leq \alpha_p(X) \). Since \( x^p - y^p \geq p(x - y) \) for \( p \geq 1 \) and \( x \geq y \geq 1 \), we have
\[
P\left( \frac{|X|_2}{(\mathbb{E} |X|_2^2)^{1/2}} \geq 1 + \varepsilon \right) \geq \left( \frac{p(\alpha_p(X) - \varepsilon)}{(\alpha_s(X) + 1)^p} \right)^{s/(s-p)}
\]
Therefore
\[
P\left( \frac{|X|_2}{(\mathbb{E} |X|_2^2)^{1/2}} \geq 1 + \varepsilon \right) \geq \left( \frac{p(\alpha_p(X) - \varepsilon)}{(\alpha_s(X) + 1)^p} \right)^{s/(s-p)}
\]
whenever \( 0 < \varepsilon \leq \alpha_p(X)/2 \), and the left-hand inequality follows.

Since \(|x - 1| \leq |x^q - 1|\) for \( q \geq 1 \) and every \( x \geq 0 \), the Markov inequality gives
\[
P\left( \frac{|X|_2}{(\mathbb{E} |X|_2^2)^{1/2}} - 1 \right) \geq \varepsilon \leq \mathbb{E}\left( \frac{|X|_2^q}{(\mathbb{E} |X|_2^2)^{q/2}} - 1 \right) \geq \varepsilon
\]
\[
\leq \frac{\mathbb{E}\left| \frac{|X|_2^q}{(\mathbb{E} |X|_2^2)^{q/2}} - 1 \right|^2}{\varepsilon^2}.
\]
To deduce the right-hand inequality of the statement, take \( q = p/2 \) and observe that
\[
\mathbb{E}\left| \frac{|X|_2^q}{(\mathbb{E} |X|_2^2)^{q/2}} - 1 \right|^2 = (\alpha_{2q}(X) + 1)^{2q} + 1 - 2(\alpha_q(X) + 1)^q
\]
\[
\leq (\alpha_{2q}(X) + 1)^{2q} - 1.
\]

**Remark 4.** Let \( 2 < p < s \). Let \( X \in \mathbb{R}^n \) be an isotropic random vector such that \(|X|_2\) has a finite s-moment. Proposition 3 shows that \( \varepsilon(X) \) is \( o(1) \) if and only if \( \alpha_p(X) \) is \( o(1) \) when \( n \to \infty \).

Now we estimate \( \varepsilon(X) \) for an example which shows that an isotropic \((-1/r)\)-concave random vector \( X \in \mathbb{R}^n \) may fail to have thin-shell concentration.
Proposition 5. Let $r > 2$. There exists a sequence $(X_n)_n$ of isotropic $(-1/r)$-concave random vectors $X_n \in \mathbb{R}^n$ such that

$$\liminf_{n \to \infty} \varepsilon(X_n) \geq c(r) > 0,$$

where $c(r) > 0$ depends only on $r$.

Proof. Let $r > 2$ and $2 < p < r$ and let $X_n \in \mathbb{R}^n$ be an isotropic random vector with density

$$f_{n,r}(x) = \frac{c_1}{(1 + c_2|x|^2)^{r+n}},$$

where $c_1$ and $c_2$ are normalization factors. From [10, 11], such a random vector is $(-1/r)$-concave. An immediate computation gives

$$\left(\mathbb{E}|X_n|^p\right)^{1/p} = \left(\frac{B(n+p,r-p)}{B(n,r)}\right)^{1/p} \left(\frac{B(n+2,r-2)}{B(n,r)}\right)^{-1/2}.$$ 

For fixed $r$ and $2 < p < r$, we have

$$\lim_{n \to \infty} \left(\mathbb{E}|X_n|^p\right)^{1/p} = \left(\frac{\Gamma(r-p)}{\Gamma(r)}\right)^{1/p} \left(\frac{\Gamma(r-2)}{\Gamma(r)}\right)^{-1/2}.$$

and by the strict log-convexity of the Gamma function, we have

$$\lim_{n \to \infty} (\alpha_p(X_n) + 1) = \lim_{n \to \infty} \frac{\left(\mathbb{E}|X_n|^p\right)^{1/p}}{\left(\mathbb{E}|X_n|^2\right)^{1/2}} > 1.$$

As a consequence for any $2 < p < r$, $\lim_{n \to \infty} \alpha_p(X_n) > 0$.

Now let $2 < p < s < r$. From Proposition 3 we get

$$\liminf_{n \to \infty} \varepsilon(X_n) \geq \lim_{n \to \infty} \min\left(\frac{\alpha_p(X_n)}{2}, \left(\frac{p\alpha_p(X_n)}{\alpha_s(X_n) + 1}^p\right)^{s/(s-p)}\right) > 0.$$

Choose $p = (2 + r)/2$ and $s = (p + r)/2$ for which $2 < p < s < r$ and note that the middle term in (7) depends only on $r$. This concludes the proof. ■

Remark 6. Let $2 < p < r$ and let $r \to \infty$. A calculation applying the stirling formula in (6) when $r \to \infty$ gives

$$\lim_{r \to \infty} \lim_{n \to \infty} \alpha_p(X_n) = (p-2)/2.$$ 

This asymptotic estimate shows that for a fixed $p > 2$ and $r$ and $n$ large enough, then $\alpha_p(X_n) \geq C(p-2)/r$ where $C > 0$ is a universal constant. This proves the sharpness of Theorem 1 under these conditions.

We now prove some inequalities for $s$-concave measures that will be useful tools in the next section.
Theorem 7.

(i) Let $f : [0, \infty) \to [0, \infty)$ be a measurable function such that $\|f\|_\infty > 0$. Then

$$p \mapsto \left( \int_0^\infty t^{p-1} f(t) \frac{dt}{\|f\|_\infty} \right)^{1/p}$$

is non-decreasing on its domain of definition.

(ii) Let $\alpha > 0$ and $f : [0, \infty) \to [0, \infty)$ be $(-1/\alpha)$-concave, continuous and integrable. Define $H_f : [0, \alpha) \to \mathbb{R}_+$ by

$$H_f(p) = \begin{cases} \frac{1}{B(p, \alpha - p)} \int_0^{\infty} t^{p-1} f(t) \; dt & \text{for } 0 < p < \alpha, \\ f(0) & \text{for } p = 0. \end{cases}$$

Then $H_f$ is log-concave on $[0, \alpha)$.

The proof of (i) may be obtained as in [25, Lemma 2.1] and the proof of (ii) is identical to the well known $(1/n)$-concave case [12]. We postpone the proof of Theorem 7 to the appendix.

We present several consequences of this result such as some reverse Hölder inequalities with sharp constants in the spirit of Borell’s [12] and Berwald’s [6] inequalities.

Corollary 8. Let $r > 0$ and $\mu$ be a $(-1/r)$-concave measure on $\mathbb{R}^n$. Let $\phi : \mathbb{R}^n \to \mathbb{R}_+ = [0, \infty]$ be such that $\{\phi > 0\}$ is convex and $\phi$ is concave on $\{\phi > 0\}$. Then the function

$$p \mapsto \begin{cases} \frac{1}{pB(p, r - p)} \int_{\mathbb{R}^n} \phi(x)^p \; d\mu(x) & \text{for } 0 < p < r, \\ \mu(\{\phi > 0\}) & \text{for } p = 0, \end{cases}$$

is log-concave on $[0, r)$.

Moreover, if $\mu(\{\phi > 0\}) > 0$ then for any $0 < p \leq q < r$,

$$\left( \int_{\mathbb{R}^n} \phi(x)^q \frac{d\mu(x)}{\mu(\{\phi > 0\})} \right)^{1/q} \leq \left( \frac{qB(q, r - q)}{pB(p, r - p)} \right)^{1/q} \left( \int_{\mathbb{R}^n} \phi(x)^p \frac{d\mu(x)}{\mu(\{\phi > 0\})} \right)^{1/p}. \leq \int_{\mathbb{R}^n} \phi(x)^p \; d\mu(x) = \int_0^{\infty} \phi(x)^p \; d\mu(x) \right).$$

Proof. By the concavity of $\phi$, for all $u, v \geq 0$ and $\lambda \in [0, 1]$

$$(1 - \lambda)\{\phi > u\} + \lambda\{\phi > v\} \subset \{\phi > (1 - \lambda)u + \lambda v\}.$$ By the $(-1/r)$-concavity of $\mu$, the function $f(t) = \mu(\{\phi > t\})$ is $(-1/r)$-concave and it is clearly continuous on $\mathbb{R}_+$. Observe that for any $p > 0$, by Fubini’s theorem,

$$\int_{\mathbb{R}^n} \phi(x)^p \; d\mu(x) = \int_0^{\infty} pt^{p-1} f(t) \; dt.$$
The first part of the result follows from Theorem 7(ii). The “moreover” part follows from log-concavity since $p \mapsto (H_f(p)/f(0))^{1/p}$ is then non-increasing.

The second corollary concerns the function $h_{k,p}$ defined in (4).

**Corollary 9.** Let $r > 0$ and $u \in \text{SO}(n)$. For any $(-1/(r+n))$-concave function $w : \mathbb{R}^n \to \mathbb{R}_+$ and any subspace $E_0$ of dimension $k \leq n$, the function

$$p \mapsto \begin{cases} h_{k,p}(u) & \text{for } p > -k + 1, \\ \frac{B(p + k, r - p)}{|S^{k-1}| \pi w(E_0) w(0)} & \text{for } p = -k + 1, \end{cases}$$

is log-concave on $[-k + 1, r)$.

**Proof.** Since $w$ is $(-1/(r+n))$-concave, we note that $t \mapsto \pi U(E_0) w(tu(\theta_0))$ is $(-1/(r+k))$-concave and it is clearly continuous on $\mathbb{R}_+$. Theorem 7 yields the result.

We finish with some geometric properties of a family of bodies introduced by K. Ball [5] in the log-concave case.

**Corollary 10.** Let $\alpha > 0$. Let $w : \mathbb{R}^n \to \mathbb{R}_+$ be a $(-1/\alpha)$-concave function such that $w(0) > 0$. For $0 < a < \alpha$ let

$$K_a(w) = \left\{ x \in \mathbb{R}^n : a \int_0^\infty t^{a-1} w(tx) \, dt \geq w(0) \right\}.$$

Then for any $0 < a \leq b < \alpha$,

$$\left( \frac{w(0)}{\|w\|_\infty} \right)^{1/a - 1/b} K_a(w) \subset K_b(w) \subset \left( \frac{bB(b, \alpha - b)}{B(a, \alpha - a)} \right)^{1/a} K_a(w).$$

**Proof.** Notice that the sets $K_a$ are star-shaped with respect to the origin, that is, $\lambda x \in K_a$ for all $x \in K_a$ and $\lambda \in [0, 1]$. The radial function of $K_a$ is

$$\rho_{K_a}(x) := \sup \{ r : rx \in K_a \} = \left( a \int_0^\infty t^{a-1} \frac{w(tx)}{w(0)} \, dt \right)^{1/a}.$$

For any $x \in \mathbb{R}^n$, let $f$ be the continuous $(-1/\alpha)$-concave function defined on $\mathbb{R}^+$ by $f(t) = w(tx)/w(0)$. By Theorem 7(i), the function

$$a \mapsto \left( \int_0^\infty t^{a-1} \frac{f(t)}{\|f\|_\infty} \, dt \right)^{1/a}$$

is non-decreasing. Hence the left-hand inclusion follows. Moreover, from Theorem 7(ii), the function $H_f : [0, \alpha) \to \mathbb{R}_+$ is log-concave on $[0, \alpha)$ with $H_f(0) = 1$. For $0 < a \leq b < \alpha$, we thus have $H_f(b)^{1/b} \leq H_f(a)^{1/a}$, implying the right-hand inclusion.
3. Thin-shell concentration for convex measures. The purpose of this section is to prove Theorem 1. We follow the strategy of the log-concave case initiated in [22, 17, 23] and further developed in [16, 19].

The support function $h_K$ of a non-empty compact set $K \subset \mathbb{R}^n$ is defined by
\[
\forall \theta \in \mathbb{R}^n, \quad h_K(\theta) = \sup_{x \in K} \langle x, \theta \rangle.
\]

To any random vector $X$ in $\mathbb{R}^n$ and any $p \geq 1$, we associate its $Z_p^+$-body defined by its support function
\[
\forall \theta \in \mathbb{R}^n, \quad h_{Z_p^+(X)}(\theta) = (\mathbb{E} \langle X, \theta \rangle_p^p)^{1/p}.
\]

When the distribution of $X$ has a density $g$, we write $Z_p^+(g) = Z_p^+(X)$.

Extending a theorem of Ball [5] for log-concave functions, Bobkov [8, Remark 2.6] (see also [15, Theorem 3.1]) proved that if $w$ is $\left(-\frac{1}{r+n}\right)$-concave on $\mathbb{R}^n$ and $w(0) > 0$, then
\[
K_a(w) \text{ is convex and compact for any } 0 < a \leq r + n - 1.
\]

In the case of log-concave measures [26, 27, 19, 20], several relations between the $Z_p^+$-bodies and the convex sets $K_a$ are known. We need their analogue in the setting of $s$-concave measures for negative $s$. We start with two technical lemmas. We postpone their proofs to the appendix.

**Lemma 11.** Let $x, y \geq 1$. Then
\[
c \frac{x}{x+y} \leq \frac{1}{p} \log \frac{B(k+p,r-p)}{B(k,r)} \leq C \frac{x}{x+y},
\]
where $c, C$ are universal positive constants. Moreover, for $k, r > 1$, the extension by continuity at 0 of the function
\[
p \mapsto \frac{1}{p} \log \frac{B(k+p,r-p)}{B(k,r)}
\]
is differentiable on $[-\frac{k-1}{2}, \frac{r-1}{2}]$ and satisfies
\[
0 \leq \frac{d}{dp} \left( \frac{1}{p} \log \frac{B(k+p,r-p)}{B(k,r)} \right) \leq \frac{1}{r-1} + \frac{1}{k-1}
\]
for $p \in \left[-\frac{k-1}{2}, \frac{r-1}{2}\right]$.

In this paper, we use the notion of geometric distance between sets, defined for any compact subsets $K, L \subset \mathbb{R}^n$ containing 0 in their interior by
\[
d(K, L) = \inf \{t_2/t_1 : t_1 L \subset K \subset t_2 L, \ t_1, t_2 > 0\}.
\]

Let $n \geq 1$, $r \geq 2$ and $w$ be the $\left(-\frac{1}{r+n}\right)$-concave density of a probability measure $\mu$ on $\mathbb{R}^n$. Then by Corollary 8 and Lemma 11, for $1 \leq p \leq q \leq r-1$,
\[
Z_p^+(w) \subset Z_q^+(w) \subset c \left( \inf_{\theta \in S^{n-1}} \mu(\{x : \langle x, \theta \rangle > 0\}) \right)^{1/q-1/p} Z_p^+(w).
\]
Fix \( \theta \in S^{n-1} \) and define \( F(t) = \mu(\{ x : \langle x, \theta \rangle \leq t \}) \) for \( t \in \mathbb{R} \). Then
\[
\int_{\mathbb{R}} tF'(t) dt = \int_{\mathbb{R}^n} \langle x, \theta \rangle w(x) dx = 0
\]
and \( F \) is \((-1/r)\)-concave. Using Jensen’s inequality, we get
\[
F(0)^{-1/r} = F\left( \int_{\mathbb{R}} tF'(t) dt \right)^{-1/r} \leq \int_{\mathbb{R}} F(t)^{-1/r} F'(t) dt = \left[ \frac{F(t)^{1-1/r}}{1-1/r} \right]_{-\infty}^{\infty} = \frac{1}{1-1/r}.
\]
Hence \( \mu(\{ x : \langle x, \theta \rangle > 0 \}) \geq (1 - 1/r)^r \geq 1/4 \) for \( r \geq 2 \). We have recovered here in a simple way a Grünbaum type inequality for convex measures due to Bobkov [8, Theorem 5.2]. We deduce that, for \( 1 \leq p \leq q \leq r - 1 \),
\[
(11) \quad Z_p^+(w) \subset Z_q^+(w) \subset C_{p/q}^q Z_p^+(w) \quad \text{and} \quad d(Z_p^+(w), Z_q^+(w)) \leq C_{p/q}.
\]

**Lemma 12.** Let \( r, m \) and \( p \) be such that \( m \) is a positive integer, \( r \geq m + 1 \) and \( -m/2 \leq p \leq r - 1 \). Let \( F \) be a subspace of \( \mathbb{R}^n \) of dimension \( m \) and let \( g \) be a \((-1/(r + m))\)-concave density of a probability measure on \( F \) such that \( \int_F xg(x) dx = 0 \). Then
\[
d(K_{m+p}(g), Z_{\max(m,p)}^+(g)) \leq c,
\]
where \( c \) is a universal constant.

As in [19], an important ingredient in the proof of the thin-shell concentration inequality is an estimate from above of the log-Lipschitz constant of the map \( u \mapsto h_{k,p}(u) \) on \( \text{SO}(n) \). Let \( \mathcal{M}_n(\mathbb{R}) \) be the set of square \( n \times n \) matrices. We equip \( \text{SO}(n) = \{ u \in \mathcal{M}_n(\mathbb{R}) : u^t u = \text{Id}, \det(u) = 1 \} \) with its standard invariant Riemannian metric, which we specify for concreteness on \( T_{\text{Id}} \text{SO}(n) \), the tangent space at the identity element \( \text{Id} \in \text{SO}(n) \). Since \( u^t u = \text{Id} \), this tangent space may be identified with the set of anti-symmetric matrices \( \{ B \in \mathcal{M}_n(\mathbb{R}) : B^t + B = 0 \} \). We define the scalar product \( \langle B, B \rangle = \frac{1}{2} \text{tr}(B^t B) \) on \( T_{\text{Id}} \text{SO}(n) \).

**Proposition 13.** Let \( n \geq 1, r > 10 \) and \( w \) be the \((-1/(r + n))\)-concave density of a probability measure on \( \mathbb{R}^n \) such that \( \int_{\mathbb{R}^n} xw(x) dx = 0 \). Let \( k \) be an integer such that \( k \geq 2, 2k - 1 \leq n \) and \( 2k \leq r \). Let \( p \) be such that \( -k/2 \leq p \leq r - 1 \). Denote by \( L_{k,p} \) the log-Lipschitz constant of the map \( u \mapsto h_{k,p}(u) \) on \( \text{SO}(n) \). Then
\[
L_{k,p} \leq C \max(k, p)d(Z_{\max(k,p)}^+(w), B_2^n),
\]
where \( C \) is a universal constant.
Proof. For any subspace $F$ of dimension $m$, the marginal $\pi_F(w)$ is a $(-1/(r+m))$-concave function on $F$ and from [8], to any $a \in [0, r+m-1]$, we associate the convex body $K_a(\pi_F(w))$ in $F$. Then the proof of Theorem 2.1 in [19, Section 2.2] gives the upper bound:

$$L_{k,p} \leq \max_F \{(m+p) d(K_{m+p}(\pi_F(w)), B_2(F))\}$$

over all subspaces $F$ of dimension $m = k, k+1, 2k-1$, where $B_2(F)$ is the Euclidean unit ball in $F$. By the assumptions on $k$, for these values of $m$, we have $m \leq 2k-1 \leq n$ and $r \geq 2k \geq m+1$ and $p \geq -k/2 \geq -m/2$. Hence from Lemma 12 we have

$$d(K_{m+p}(\pi_F(w)), B_2(F)) \leq cd(Z_{\max(m,p)}^+(\pi_F(w)), B_2(F)).$$

By definition, if $X$ is a random vector with density $w$ on $\mathbb{R}^n$, the marginal $\pi_F(w)$ is the density of the projection $P_F X$ of $X$ onto $F$. By identification of the support functions, we see that, for any $\theta \in F$,

$$h^p_{Z^+_p(\pi_F(w))}(\theta) = \mathbb{E} \langle P_F X, \theta \rangle^p = \mathbb{E} \langle X, \theta \rangle^p.$$ 

This means that $Z^+_p(\pi_F(w)) = P_F Z^+_p(w)$. Since the distance to the Euclidean ball cannot increase after projections, we conclude that

$$d(K_{m+p}(\pi_F(w)), B_2(F)) \leq cd(Z_{\max(m,p)}^+(w), B_2^*).$$

By [11], for $m = k, k+1, 2k-1$, one has

$$d(Z_{\max(m,p)}^+(w), Z_{\max(k,p)}^+(w)) \leq c.$$ 

This finishes the proof. $\blacksquare$

We define the $q$-condition number of a random vector $X$ to be

$$\rho_q(X) = \sup_{\|\theta\|_2=1} \left( \frac{\mathbb{E} \langle X, \theta \rangle^q}{\inf_{\|\theta\|_2=1} \mathbb{E} \langle X, \theta \rangle^q} \right)^{1/q}.$$ 

Obviously, if $w$ is the density of a full-dimensional random vector $X$ in $\mathbb{R}^n$ then $\rho_q(X) = d(Z^+_q(w), B_2^n)$.

**Proposition 14.** With the same assumptions as in Proposition 13, if a random vector $X$ with density $w$ is isotropic then

$$L_{k,p} \leq C \max(k,p)^2.$$ 

More generally if $A$ is such that $AX$ is isotropic then

$$L_{k,p} \leq C \max(k,p)^2 \|A\|\|A^{-1}\|.$$ 

(12)

**Proof.** Let $q = \max(k,p)$. Then $1 \leq q \leq r-1$. Using the triangular inequality we get

$$\rho_q(X) = d(Z^+_q(w), B_2^n) \leq d(Z^+_q(w), Z^+_q(w))d(Z^+_q(w), B_2^n).$$ 

From \([11]\) we deduce that \(d(Z^q_+(w), Z^q_+(w)) \leq cq\). For any \(\theta \in S^{n-1}\), \(\mathbb{E} \langle X, \theta \rangle = 0\), hence \(\mathbb{E} \langle X, \theta \rangle_+ = \mathbb{E} \langle -X, \theta \rangle_+\). Using this equality and \([11]\) we deduce that

\[
(\mathbb{E} \langle -X, \theta \rangle^2_+)^{1/2} \leq c \mathbb{E} \langle -X, \theta \rangle_+ = c \mathbb{E} \langle X, \theta \rangle_+ \leq c(\mathbb{E} \langle X, \theta \rangle^2_+)^{1/2}.
\]

Thus

\[
\mathbb{E} \langle X, \theta \rangle^2_+ \leq \mathbb{E} \langle X, \theta \rangle^2 = \mathbb{E} \langle X, \theta \rangle^2_+ + \mathbb{E} \langle -X, \theta \rangle^2_+ \leq C \mathbb{E} \langle X, \theta \rangle^2_+.
\]

Hence if \(X\) is isotropic we deduce that \(d(Z^q_+(w), B^n_2) \leq c\). We conclude that

\[
\rho_q(X) = d(Z^q_+(w), B^n_2) \leq C' q.
\]

The first conclusion follows from Proposition \([13]\). In the general case, notice that \(Z^q_+(AX) = AZ^q_+(X)\) and \(d(AB^n_2, B^n_2) = \|A\|\|A^{-1}\|\), thus

\[
\rho_q(X) \leq \rho_q(AX)\|A\|\|A^{-1}\|. \quad \blacksquare
\]

**Proof of Theorem \([4]\)** Without loss of generality, we can assume \(r > 32\). Indeed, if \(r \leq 32\) then the statement in Theorem 1 is valid for \(|p| \leq cr\) and it gives only a comparison of \((\mathbb{E} |X|^p_2)^{1/p}\) with \((\mathbb{E} |X|^2_2)^{1/2}\) up to a constant factor. The result is a consequence of Theorem 5.2 in \([1]\).

From now on, we assume that \(r > 32\) and \(|p| \leq r/8\). We start by presenting a complete argument following \([16]\). This will give a complete proof of a slightly weaker result. In the second part, we just indicate the needed modifications of the argument of \([19]\) to get the complete conclusion.

In this first part, we will prove that for any \(p \in [1/\sqrt{n}, \min(cn^{1/8}, r/8)]\),

\[
(\mathbb{E} |X|^p_2 \mathbb{E} |X|^{-p}_2)^{1/p} \leq 1 + \frac{Cp}{r} + \left( \frac{Cp}{n^{1/3}} \right)^{3/5}.
\]

Assuming \((13)\), few elementary steps are needed to prove that for any \(p\) such that \(|p| \leq \min(cn^{1/8}, r/8)\),

\[
\left| \frac{(\mathbb{E} |X|^p_2)^{1/p}}{(\mathbb{E} |X|^2_2)^{1/2}} - 1 \right| \leq \frac{C(1 + |p|)}{r} + \left( \frac{C(1 + |p|)}{n^{1/3}} \right)^{3/5},
\]

which is already enough to get thin-shell concentration. Indeed, for \(p \geq 2\), by the Hölder inequality, we have

\[
0 \leq \frac{(\mathbb{E} |X|^p_2)^{1/p}}{(\mathbb{E} |X|^2_2)^{1/2}} - 1 \leq \frac{(\mathbb{E} |X|^p_2)^{1/p}}{(\mathbb{E} |X|^{-p}_2)^{-1/p}} - 1
\]

and we conclude by \((13)\). For \(p \leq -2\), we have \(|p| = -p \geq 2\) and from the Hölder inequality and \((13)\),

\[
0 \leq \frac{(\mathbb{E} |X|^2_2)^{1/2}}{(\mathbb{E} |X|^p_2)^{1/p}} - 1 \leq \frac{(\mathbb{E} |X|^{-p}_2)^{1/|p|}}{(\mathbb{E} |X|^{-p}_2)^{-1/|p|}} - 1 \leq \frac{C|p|}{r} + \left( \frac{C|p|}{n^{1/3}} \right)^{3/5}.
\]
An elementary computation shows that

\[
\left( \frac{\mathbb{E} |X|^p}{\mathbb{E} |X|^2} \right)^{1/2} - 1 \leq \frac{C|p|}{r} \left( \frac{C|p|}{n^{1/3}} \right)^{3/5}.
\]

For \( p \in [-2, 2] \), by the Hölder inequality,

\[
0 \leq 1 - \frac{(\mathbb{E} |X|^p)^{1/p}}{(\mathbb{E} |X|^2)^{1/2}} \leq 1 - \frac{(\mathbb{E} |X|^2)^{-1/2}}{(\mathbb{E} |X|^2)^{1/2}}
\]

and we conclude by the previous estimate for \( p = -2 \). This concludes the proof of (14).

Let us start the proof of (13). Let \( p \in [1/\sqrt{n}, \min(cn^{1/8}, r/8)] \) and \( k \) be an integer greater or equal than 2 such that \( p < k \leq n \). We will optimize the choice of \( k \) at the end of the proof. Recall that by (5),

\[
\mathbb{E} |X|^p = \frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)} \mathbb{E} h_{k,p}(U),
\]

where \( U \) is uniformly distributed on SO(n). Using the fact that the function \( \frac{d}{dp} \log \Gamma(p) \) is concave (see for example the proof of Lemma 11 in the appendix), we deduce that

\[
\frac{d}{dp} \left( \frac{1}{p} \log \frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma((p+k)/2)\Gamma(n/2)} \right) \leq 0.
\]

It follows that for any \( 0 < p < k , \)

\[
\frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)} \frac{\Gamma((-p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((-p+k)/2)} \leq 1.
\]

Then for all \( 0 < p < r \) and \( n \geq k > p \) we have

\[
\mathbb{E} |X|^p \mathbb{E} |X|^q \leq \mathbb{E} h_{k,p}(U) \mathbb{E} h_{k,-p}(U).
\]

Applying the log-Sobolev inequality (3) to \( h_{k,p} \) and \( h_{k,-p} \) we get

\[
\mathbb{E} h_{k,p}(U)^2 \leq e^{cL_{k,p}^2/n} (\mathbb{E} h_{k,p}(U))^2, \quad \mathbb{E} h_{k,-p}(U)^2 \leq e^{cL_{k,-p}^2/n} (\mathbb{E} h_{k,-p}(U))^2.
\]

Since \( \text{Var} f = \mathbb{E} f^2 - (\mathbb{E} f)^2 \) we deduce that

\[
\text{Var} h_{k,p}(U) \leq (e^{cL_{k,p}^2/n} - 1) (\mathbb{E} h_{k,p}(U))^2, \quad \text{Var} h_{k,-p}(U) \leq (e^{cL_{k,-p}^2/n} - 1) (\mathbb{E} h_{k,-p}(U))^2.
\]

By Corollary 3, we know that \( p \mapsto h_{k,p}(u)/B(k+p,r-p) \) is log-concave on \([-k+1, r)\) hence

\[
h_{k,p}(u)h_{k,-p}(u) \leq \left( \frac{B(k+p,r-p)}{B(k,r)} \frac{B(k-p,r+p)}{B(k,r)} \right) h_{k,0}(u).
\]
Taking the expectation with respect to SO(n), we get
\[ \mathbb{E} h_{k,p}(U) h_{k,-p}(U) \leq \left( \frac{B(k + p, r - p)}{B(k, r)} \frac{B(k - p, r + p)}{B(k, r)} \right) \mathbb{E} h_{k,0}^2(U). \]

Since \( \mathbb{E} h_{k,0}(U) = 1 \) we deduce from (17) that
\[ \mathbb{E} h_{k,0}^2(U) \leq e^{cL_{k,0}^2/n}. \]

Assume that \( k \leq r \). Then by (10), we know that for \( p \leq (k - 1)/2 \),
\[ \left( \frac{B(k + p, r - p)}{B(k, r)} \frac{B(k - p, r + p)}{B(k, r)} \right)^{1/p} \leq e^{2p(1/k + 1/r)} \leq e^{4p(1/k + 1/r)} \]
since \( k, r \geq 2 \). Hence
\[ \mathbb{E} h_{k,p}(U) h_{k,-p}(U) \leq e^{cL_{k,0}^2/n + 4p^2(1/k + 1/r)}. \]

Moreover
\[ \mathbb{E} h_{k,p}(U) h_{k,-p}(U) = \mathbb{E} h_{k,p}(U) \mathbb{E} h_{k,-p}(U) + \text{Cov}(h_{k,p}(U), h_{k,-p}(U)) \]
\[ \geq \mathbb{E} h_{k,p}(U) \mathbb{E} h_{k,-p}(U) - \sqrt{\text{Var } h_{k,p}(U) \text{Var } h_{k,-p}(U)} \]
\[ \geq \mathbb{E} h_{k,p}(U) \mathbb{E} h_{k,-p}(U)(1 - \sqrt{(e^{cL_{k,p}^2/n - 1})(e^{cL_{k,-p}^2/n - 1})}), \]
where the last inequality follows from (18). Assume moreover that \( 2k - 1 \leq n \) and \( 2k \leq r \). Then for \( p \leq (k - 1)/2 \), we can evaluate \( L_{k,p}, L_{k,-p} \) and \( L_{k,0} \) from Proposition 14 since the assumptions are fulfilled. We find that if \( X \) is isotropic then \( \max(L_{k,p}, L_{k,-p}, L_{k,0}) \leq Ck^2 \). If \( k \leq c_0 n^{1/4} \) for a small enough numerical constant \( c_0 \), we have
\[ \sqrt{(e^{cL_{k,p}^2/n - 1})(e^{cL_{k,-p}^2/n - 1})} \leq c \frac{k^4}{n} \leq \frac{1}{10}. \]
Combining this estimate with (20) and (19), we have proved that if \( k \) is an integer such that \( k \geq 2, 2k - 1 \leq n, 2k \leq r, k \leq c_0 n^{1/4} \) and \( 2p + 1 \leq k \) (this set of integers is not empty since \( r > 32 \) and \( p \leq r/8 \)) then
\[ \mathbb{E} h_{k,p}(U) \mathbb{E} h_{k,-p}(U) \leq e^{4p^2(1/k + 1/r) + ck^4/n} \leq e^{4p^2(1/k + 1/r) + Ck^4/n}. \]

For \( p \leq 1 \), we also force \( k \) to satisfy \( k \leq C_0 p^{1/4} n^{1/4} \). Hence taking the power \( 1/p \) in the last expression, we conclude from (16) that
\[ (\mathbb{E} |X|^p_2 \mathbb{E} |X|^{-p}_2)^{1/p} \leq e^{4p(1/k + 1/r) + Ck^4/(pn)} \leq 1 + cp \left( \frac{1}{k} + \frac{1}{r} \right) + c \frac{k^4}{pn}, \]
since \( p/k, p/r \) and \( k^4/(pn) \) are bounded by universal constants.

It remains to optimize the choice of \( k \). Let \( p_0 = n^{-1/2} \). In this case we choose \( k = 2 \) and get
\[ (\mathbb{E} |X|^{p_0}_2 \mathbb{E} |X|^{-p_0}_2)^{1/p_0} \leq 1 + C/\sqrt{n}. \]
If \( p \geq n^{-1/2} \) we choose \( k \) to be an integer such that \( \text{min}(r/4, (p^2 n)^{1/5}) \leq k \leq 2 \min(r/4, (p^2 n)^{1/5}) \) with the restriction \( 2p + 1 \leq k \leq cp^{1/4}n^{1/4} \). For any \( p \) such that \( p_0 \leq p \leq \min(cn^{1/8}, r/8) \), the integer \( k \) satisfies \( k \geq 2, 2k - 1 \leq n, 2k \leq r, k \leq c_0 n^{1/4} \) and \( 2p + 1 \leq k \) and we get

\[
(\mathbb{E} |X|_2^p \mathbb{E} |X|_2^{-p})^{1/p} \leq 1 + \frac{Cp}{r} + \left( \frac{Cp}{n^{1/3}} \right)^{3/5}.
\]

This ends the proof of (13).

In the second part, we follow the argument developed in [19] to get a better estimate. We deal now with the case of \( p \) being positive or negative and, as already said, we can assume without loss of generality that \( r > 34 \) and \( |p| \leq r/8 \). As in [19], our goal is to estimate

\[
\frac{d}{dp} \log((\mathbb{E} |X|_2^p)^{1/p}) = \frac{d}{dp} \log((\mathbb{E} h_{k,p}(U))^{1/p})
\]

\[
+ \frac{d}{dp} \left( \frac{1}{p} \log \frac{\Gamma((p + n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p + k)/2)} \right).
\]

Most of the computation of Section 3.2 in [19] can be repeated. All the ingredients needed for the proof have been established and, adapting the argument in [19], we get

\[
\frac{d}{dp} \log((\mathbb{E} |X|_2^p)^{1/p}) \leq \frac{c}{p^2 n} (2L^2_{k,p} + 3L^2_{k,0}) + \frac{C}{k - 1} + \frac{C}{r - 1}.
\]

For convenience of the reader, we will briefly reproduce the proof of (22) in the appendix.

Assume that \( X \) is isotropic. For any \( 2|p| \leq k \leq r/2 \) (this set of integers is not empty since \( r > 32 \) and \( |p| \leq r/8 \)), we know by Proposition 14 that \( L_{k,p} \) and \( L_{k,0} \) are smaller than \( Ck^2 \). We get

\[
\frac{d}{dp} \log((\mathbb{E} |X|_2^p)^{1/p}) \leq C \left( \frac{k^4}{p^2 n} + \frac{1}{k} + \frac{1}{r} \right).
\]

We have to minimize this expression for \( k \) being an integer \( \geq 2 \) in the interval \([2|p|, r/2]\). For \(|p| \in [n^{-1/2}, cn^{1/3}]\), we set \( k \) to be an integer such that \( \min(r/4, 2(p^2 n)^{1/5}) \leq k \leq 2 \min(r/4, 2(p^2 n)^{1/5}) \). Therefore \( k \) satisfies the restrictions, and for any \( p \) such that \(|p| \in [n^{-1/2}, cn^{1/3}]\), we get

\[
\frac{d}{dp} \log((\mathbb{E} |X|_2^p)^{1/p}) \leq C \left( \frac{1}{(p^2 n)^{1/5}} + \frac{1}{r} \right).
\]

After integration over \( p \), we find that for all \( p \in [n^{-1/2}, c \min(r, n^{1/3})] \),

\[
\left| \log \left( \frac{\mathbb{E} |X|^p_2}{\mathbb{E} |X|_2^{1/2}} \right)^{1/p} \right| \leq \frac{C|p - 2|}{r} + \frac{C|p^{3/5} - 2^{3/5}|}{n^{1/5}}.
\]

Since \(|p^{3/5} - 2^{3/5}| \leq |p - 2|^{3/5}\) and other terms on the right hand side of the
Inequality are bounded by a universal constant, we conclude that
\[
\left| \frac{\mathbb{E}|X|^p_2}{\mathbb{E}|X|_2^{2p}} - 1 \right| \leq \frac{C|p - 2|}{r} + \left( \frac{C|p - 2|}{n^{1/3}} \right)^{3/5}, \quad \forall p \in [n^{-1/2}, c \min(r, n^{1/3})].
\]
Since (23) holds only for \(|p| \geq n^{-1/2}\), we use (21) to bridge the gap between 
\(-n^{-1/2}\) and \(n^{-1/2}\). Indeed, from (21), the previous inequality for \(p_0 = n^{-1/2}\) and \(|p_0 - 2| = 2 - p_0 \leq 2\), we deduce that for \(p \in [-p_0, p_0]\),
\[
\left| \frac{\mathbb{E}|X|^p_2}{\mathbb{E}|X|_2^{2p}} - 1 \right| \leq \frac{1}{1 + C/\sqrt{n}} \left( \frac{\mathbb{E}|X|_2^{p_0}}{\mathbb{E}|X|_2^{2p_0}} \right)^{1/p_0}
\]
\[
\geq \frac{1 - 2C/r - (2C/n^{1/3})^{3/5}}{1 + C/n^{1/5}} (\mathbb{E}|X|_2^{2})^{1/2}.
\]
An easy adaptation of the constants leads to the conclusion of Theorem 1 for all \(p \in [-n^{-1/2}, n^{-1/2}]\).

Integrating (23) again, we get, for \(p \in [-c \min(r, n^{1/3}), -n^{-1/2}]\),
\[
\frac{\mathbb{E}|X|^p_2}{\mathbb{E}|X|_2^{2p}} \geq 1 - \frac{C|p + p_0|}{r} - \left( \frac{C|p + p_0|}{n^{1/3}} \right)^{3/5}.
\]
Using \(|p + p_0| \leq |p - 2|\) and the previous comparison of the moment of order 
\(-p_0\) with the moment of order 2 and adjusting the constants proves that for all \(p \in [-c \min(r, n^{1/3}), -n^{-1/2}]\),
\[
\left| \frac{\mathbb{E}|X|^p_2}{\mathbb{E}|X|_2^{2p}} - 1 \right| \leq \frac{C|p - 2|}{r} + \left( \frac{C|p - 2|}{n^{1/3}} \right)^{3/5}.
\]
This concludes the proof of the first part of Theorem 1

If \(X\) is such that \(AX\) is isotropic, we know from Proposition 14 that for any integer \(k\) such that \(2|p| \leq k \leq r/2\),
\[
\max(L_{k,p}, L_{k,0}) \leq C k^2 \|A\| \|A^{-1}\|.
\]
The proof is identical to the previous one after replacing \(n\) by \(\|A\|^2 \|A^{-1}\|^2\).

**Remark 15.** In [19], a preprocessing step consisted in adding a Gaussian isotropic vector to the random vector \(X\) in order to start at the very beginning with a better information on the \(Z^+_p\)-bodies associated to the measure. In [23, 16], this convolution argument played a role of regularization. It is natural to ask if such a process could be done in the situation of \(s\)-concave measure. Adding a Gaussian vector does not help because for \(s < 0\), the new vector does not belong to any class of \(s\)-concave vectors. However, for \(r > n\), we can give a similar argument, adding to \(X\) a random vector \(Z\) uniformly distributed on the Euclidean ball (see also [9]). Since \(Z\) is \((1/n)\)-concave and \(X\) is \((-1/r)\)-concave, the new vector \(Y = (X + Z)/\sqrt{2}\) will be \((-1/(r - n))\)-concave. For any \(p \geq 1\), we have (see [19, inequality (4.7)])
\[
\alpha_p(X) \leq \alpha_{2p}(Y)(2 + \alpha_{2p}(Y)),
\]
so that it remains to bound $\alpha_{2p}(Y)$. It is easy to see that $(\mathbb{E} (Y, \theta)^q)_{+}^{1/q} \geq c\sqrt{q}$ for all $q \geq 2$ and $\theta \in S^{n-1}$. Adapting the proof of Proposition 14 we get $L_{k,p} \leq C \max(k, p)^{3/2}$. As in [19], this improvement leads to the following estimate: if $r-n > 0$ for any $p$ such that $1 \leq p \leq c \min(r-n, \sqrt{n})$,

$$\alpha_{2p}(Y) \leq \frac{C(2p-2)}{r-n} + \left(\frac{C(2p-2)}{\sqrt{n}}\right)^{1/2}.$$ 

For $r > n + \sqrt{n}$, we recover the same thin-shell concentration as in the log-concave case. It would be interesting to understand in which precise sense $s$-concave measures are close to log-concave measures for $s \in (-1/n, 1/n)$. Another question is to know what kind of preprocessing argument as in [24] would enable one to recover the small ball estimates from [1].

4. Appendix

Proof of Theorem 7. (i) This result is classical. In the symmetric case, it follows from Lemma 2.1 in [25]. The general case is similar. We provide the proof for completeness. We may assume, without loss of generality, that $\|f\|_{\infty} = 1$. Denote $I_p(f) = \int_{0}^{\infty} t^{p-1} f(t) \, dt$. From the Hölder inequality, the function $p \mapsto \log(I_p(f))$ is convex on its convex support, thus the domain of definition of $I_p(f)$ is an interval. Let $0 < p < q$ be fixed such that $I_p(f) < \infty$ and $I_q(f) < \infty$. Let $a = (pI_p(f))^{1/p}$ and $\varphi(t) = t^{p-1}(f(t) - 1_{[0,a]}(t))$. Notice that $\varphi \leq 0$ on $[0, a]$, $\varphi \geq 0$ on $[a, \infty)$ and $\int_{0}^{\infty} \varphi(t) \, dt = 0$. Thus

$$I_q(f) - I_q(1_{[0,a]}) = \int_{0}^{\infty} t^{q-p} \varphi(t) \, dt = \int_{0}^{\infty} (t^{q-p} - a^{q-p}) \varphi(t) \, dt \geq 0,$$

since the integrand is non-negative on $\mathbb{R}_+$. We conclude that

$$I_q(f) \geq I_q(1_{[0,a]}) = \frac{a^q}{q} = \frac{1}{q} (pI_p(f))^{q/p}.$$ 

(ii) Since $f$ is $(-1/\alpha)$-concave, there exists a convex function $\varphi : [0, \infty) \to (0, \infty]$ such that $f = \varphi^{-\alpha}$. Since $f$ is integrable it follows that $\varphi$ tends to $\infty$ at $\infty$. From the convexity of $\varphi$, one deduces that $\varphi(t) \geq c(1+t)$ for some constant $c > 0$. Thus $f(t) \leq (c + ct)^{-\alpha}$ for every $t \geq 0$. Therefore, $t^{p-1} f$ is integrable for every $p < \alpha$, which means that $H_f(p) < \infty$ for every $0 < p < \alpha$. Let $p \in (0, \alpha)$ and $m, M > 0$. Define $g : \mathbb{R}_+ \to \mathbb{R}_+$ by $g(t) = m(1 + t/M)^{-\alpha}$. Then

$$\int_{0}^{\infty} t^{p-1} g(t) \, dt = m M^p \int_{0}^{\infty} v^{p-1} (1 + v)^{-\alpha} \, dv = m M^p B(p, \alpha - p).$$

Thus $H_g(p) = m M^p$, which implies that $\log(H_g)$ is affine on $(0, \alpha)$. Take $0 < a < b < c < \alpha$. Let $\lambda \in [0, 1]$ be such that $b = (1-\lambda)a + \lambda c$. Choose $m$ and $M$ such that $m M^a = H_f(a)$ and $m M^b = H_f(b)$ so that $H_g(a) = H_f(a)$.
and \( H_g(b) = H_f(b) \). If we prove that

\[
\int_{0}^{\infty} t^{c-1}(g - f)(t) \, dt \geq 0,
\]

that is, \( H_g(c) \geq H_f(c) \), then using that \( \log(H_g) \) is affine, we will deduce that

\[
H_f(b) = H_g(b) = H_g(a)^{1-\lambda} H_g(c)^{\lambda} \geq H_f(a)^{1-\lambda} H_f(c)^{\lambda},
\]

and this will prove the log-concavity of \( H \) on \((0, \alpha)\). If \( f = g \) then (24) is satisfied so that in the following we assume that \( h := g - f \neq 0 \). Let

\[
H_1(t) = \int_{t}^{\infty} s^{a-1} h(s) \, ds \quad \text{and} \quad H_2(t) = \int_{t}^{\infty} s^{b-a-1} H_1(s) \, ds.
\]

Since \( h(t) = O(t^{-\alpha}) \) at infinity, we deduce that \( H_1(t) = O(t^{a-\alpha}) \) and \( H_2(t) = O(t^{b-\alpha}) \). We have \( \int_{0}^{\infty} t^{a-1} h(t) \, dt = 0 \), thus \( H_1(\infty) = H_1(0) = 0 \). Obviously \( H_2(\infty) = 0 \). We also observe that

\[
0 = \int_{0}^{\infty} t^{b-1} h(t) \, dt = \int_{0}^{\infty} t^{b-a} t^{a-1} h(t) \, dt = - \int_{0}^{\infty} t^{b-a} H_1'(t) \, dt
\]

\[
= [t^{b-a} H_1(t)]_{0}^{\infty} + (b - a) \int_{0}^{\infty} t^{b-a-1} H_1(t) \, dt = (b - a) H_2(0),
\]

whence \( H_2(\infty) = H_2(0) = 0 \). Since \( \int_{0}^{\infty} t^{b-a-1} H_1(t) \, dt = 0 \) and \( H_1 \neq 0 \), the function \( H_1 \) has at least one change of sign. Moreover, using that \( H_1(0) = H_1(\infty) = 0 \), we deduce that \( H_1' \) and therefore \( h \) has at least two sign changes. Since \( h = g - f \) has the same sign as \( f^{-\alpha} - g^{-\alpha} \) which is convex, it cannot have more than two sign changes. Thus it has exactly two sign changes at some \( 0 < t_1 < t_2 \). Moreover, from the convexity of \( f^{-\alpha} - g^{-\alpha} \), \( h \) has to be negative on \((t_1, t_2)\) and positive on \((0, t_1)\) and \((t_2, \infty)\). From an easy study of the function \( H_2 \), we deduce that \( H_2 \geq 0 \). Therefore, using \( H_1(0) = H_1(\infty) = H_2(0) = H_2(\infty) = 0 \), we get

\[
\int_{0}^{\infty} t^{c-1} h(t) \, dt = \int_{0}^{\infty} t^{c-a} t^{a-1} h(t) \, dt = - \int_{0}^{\infty} t^{c-a} H_1'(t) \, dt
\]

\[
= [-t^{c-a} H_1(t)]_{0}^{\infty} + (c - a) \int_{0}^{\infty} t^{c-a-1} H_1(t) \, dt
\]

\[
= (c - a) \int_{0}^{\infty} t^{c-b} t^{b-a-1} H_1(t) \, dt
\]

\[
= (c - a)[-t^{c-b} H_2(t)]_{0}^{\infty} + (c - a)(c - b) \int_{0}^{\infty} t^{c-b-1} H_2(t) \, dt
\]

\[
= (c - a)(c - b) \int_{0}^{\infty} t^{c-b-1} H_2(t) \, dt \geq 0.
\]
This proves (24) and establishes the log-concavity of $H_f$ on $(0, \alpha)$. To get it on $[0, \alpha)$, it is enough to prove that $H_f$ is continuous at 0. This follows from the observation that

$$B(p, \alpha - p) \underset{p \to 0}{\sim} \Gamma(p) \underset{p \to 0}{\sim} \frac{1}{p}, \quad \text{thus} \quad H_f(p) \underset{p \to 0}{\sim} p \int_0^\infty t^{p-1} f(t) \, dt.$$ 

And it is classical that, for a continuous function $f$, the right-hand side term tends to $f(0)$ when $p \to 0$. 

**Proof of Lemma 11** Estimates (9) follow easily from the classical bounds for the Gamma function (see [3]), valid for $x \geq 1$:

$$\sqrt{2\pi x^{x-1/2}} e^{-x} \leq \Gamma(x) \leq \sqrt{2\pi x^{x-1/2}} e^{-x+1/12}.$$ 

For (10), we write

$$\frac{B(k+p,r-p)}{B(k,r)} = \frac{\Gamma(k+p)\Gamma(r-p)}{\Gamma(k)\Gamma(r)}.$$ 

Denote $G(p) = \log \Gamma(p)$ for $p > 0$. We know that $G''(p) = \sum_{i \geq 0} 1/(p+i)^2$, hence $G''$ is non-increasing and $0 \leq G''(p) \leq 1/(p-1)$ for $p > 1$. Denote

$$F_k(p) = \frac{G(k+p) - G(k)}{p} \quad \text{for} \quad k > 0 \quad \text{and} \quad p > -k.$$ 

We have $F_k(p) = \int_0^1 G''(k+up) \, du$. Using that $G''$ is non-increasing, we deduce that for $k > 1$ and $p \geq -(k-1)/2$,

$$F_k'(p) = \int_0^1 G''(k + up) u \, du \leq G''(k + 1/2) \int_0^1 u \, du = \frac{1}{2} G''(k + 1/2) \leq \frac{1}{k-1},$$

and $F_k'(p) \geq 0$. Therefore, if $k > 1$, $r > 1$ and $-\frac{k-1}{2} \leq p \leq \frac{r-1}{2}$ then

$$0 \leq \frac{d}{dp} \left( \frac{1}{p} \log \frac{B(k+p,r-p)}{B(k,r)} \right) = \frac{d}{dp} (F_k(p) - F_r(-p)) = F_k'(p) + F_r'(-p) \leq \frac{1}{k-1} + \frac{1}{r-1}.$$ 

**Proof of Lemma 12** We present here a similar proof to one in the appendix of [19]. Applying Corollary 10 to $w = g$, $n = m$, $\alpha = r + m$, we deduce that, for $m/2 \leq a \leq b \leq r + m - 1$,

$$\left( \frac{g(0)}{\|g\|_\infty} \right)^{1/a-1/b} K_a(g) \subset K_b(g) \subset \left( \frac{bB(b,r+m-b)}{(aB(a,r+m-a))^{1/a}} \right)^{1/b} K_a(g).$$
From Lemma 11 we have
\[
\frac{(bB(b, r + m - b))^{1/b}}{(aB(a, r + m - a))^{1/a}} \leq \frac{b}{a}.
\]
Moreover since \( \int xg(x) \, dx = 0 \), from Lemma 7.2 of [1], one has
\[
\frac{g(0)}{\|g\|_{\infty}} \geq \left( \frac{r - 1}{r + m - 1} \right)^{r+m} \geq e^{-2m}.
\]
Since \( 1/a - 1/b \leq 1/a \leq 2/m \), we deduce that
\[
\frac{g(0)}{\|g\|_{\infty}} \geq e^{-4/m}.
\]
We conclude that for \( m/2 \leq a \leq b \leq r + m - 1 \),
\[
(25) \quad e^{-4/m} K_a(g) \subset K_b(g) \subset c \frac{b}{a} K_a(g).
\]
By integration in polar coordinates, it is well known [26] (see also [20]) that we have the following relation between the \( Z^{+}_q \)-bodies associated with \( g \) and the \( Z^{+}_q \)-bodies associated with one of the convex bodies \( K_a(g) \): for any \( 0 < q < r \),
\[
(26) \quad Z^{+}_q(g) = g(0)^{1/q} Z^{+}_q(K_{m+q}(g)),
\]
where for any body \( K \), \( Z^{+}_q(K) \) denotes the convex body whose support function is defined by
\[
\forall \theta \in \mathbb{R}^m, \quad h_{Z^{+}_q(K)}(\theta) = \left( \int_{K} \langle x, \theta \rangle_+^q \, dx \right)^{1/q}.
\]
Let \( \theta \in \mathbb{R}^m \) and \( K \) be a convex body containing 0. From Berwald’s inequalities [6] applied to \( K \cap \{ \langle x, \theta \rangle \geq 0 \} \) and the function \( x \mapsto \langle x, \theta \rangle_+ \) which is concave on \( K \cap \{ \langle x, \theta \rangle \geq 0 \} \), the function
\[
p \mapsto \left( \frac{\int_K \langle x, \theta \rangle_+^p \, dx}{mB(p + 1, m) \text{Vol}(K \cap \{ \langle x, \theta \rangle \geq 0 \})} \right)^{1/p}
\]
is decreasing. Observe that \( \lim_{p \to \infty} (\int_K \langle x, \theta \rangle_+^p \, dx)^{1/p} = h_K(\theta) \) for all \( \theta \in \mathbb{R}^m \), and
\[
(mB(p + 1, m))^{1/p} = \left( m \int_0^1 u^p (1 - u)^{m-1} \, du \right)^{1/p} \xrightarrow{p \to \infty} 1.
\]
We deduce that
\[
\left( \frac{\int_K \langle x, \theta \rangle_+^q \, dx}{mB(q + 1, m) \text{Vol}(K \cap \{ \langle x, \theta \rangle \geq 0 \})} \right)^{1/q} \geq h_K(\theta).
\]
Note also that \( \int_K \langle x, \theta \rangle_+^q \, dx \leq h_K(\theta)^q \text{Vol}(K \cap \{ \langle x, \theta \rangle \geq 0 \}) \) and that \( mB(q + 1, m) = qB(m+1) \). Therefore
\[
(27) \quad h_K(\theta) \geq \frac{h_{Z^{+}_q(K)}(\theta)}{\text{Vol}(K \cap \{ \langle x, \theta \rangle \geq 0 \})^{1/q}} \geq (qB(q, m+1))^{1/q} h_K(\theta).
\]
Now we establish that for $q = \max(p, m)$, 
\begin{equation}
\label{eq:28}
d(K_{m+q}(g), Z^+_q(g)) \leq c.
\end{equation}

By Lemma \[11\] for any $q \geq m \geq 1$, $(qB(q, m+1))^{1/q} \geq cq/(m+q+1) \geq c/3$ and we deduce from (27) that for every $\theta \in \mathbb{R}^n$, 
\begin{align*}
h_{K_{m+q}}(g)(\theta) &\geq \frac{h_{Z^+_q(K_{m+q}(g))}(\theta)}{\operatorname{Vol}(K_{m+q}(g) \cap \{(x, \theta) \geq 0\})^{1/q}} \geq \frac{c}{3h_{K_{m+q}}(g)}(\theta),
\end{align*}
where $c$ is a universal constant.

Thus (28) is proved. Together with (26), we conclude that 
\begin{equation}
\label{eq:29}
d(K_{m+q}(g), Z^+_q(g)) = d(K_{m+q}(g), Z^+_q(K_{m+q}(g)))
\end{equation}
for a universal constant $c$. Applying (25) for $a = m+1$ and $b = m+q$, we get 
\[e^{-4}K_{m+1}(g) \subset K_{m+q}(g) \subset c\frac{m+q}{m+1}K_{m+1}(g).
\]

Since $q \geq m$ and $(\frac{m+q}{m+1})^{m/q} \leq e$, from (29) we get 
\[d(K_{m+q}(g), Z^+_q(g)) \leq C\sup_{\theta \in \mathbb{R}^n} \frac{\operatorname{Vol}(K_{m+1}(g) \cap \{(x, \theta) \geq 0\})^{1/q}}{\inf_{\theta \in \mathbb{R}^n} \operatorname{Vol}(K_{m+1}(g) \cap \{(x, \theta) \geq 0\})^{1/q}},
\]
for a universal constant $C$. Since $g$ has its barycenter at the origin, so does $K_{m+1}(g)$, and we deduce from a classical result of Grünbaum \[13\] that 
\[\frac{\operatorname{Vol}(K_{m+1}(g) \cap \{(x, \theta) \geq 0\})^{1/q}}{\inf_{\theta \in \mathbb{R}^n} \operatorname{Vol}(K_{m+1}(g) \cap \{(x, \theta) \geq 0\})^{1/q}} \leq (e-1)^{1/q} \leq e-1.
\]

Thus (28) is proved.

It is now enough to establish that $d(K_{m+q}, K_{m+p}) \leq c$, where $q = \max(m, p)$. For $q = p$, this is obvious, so we may assume that $q = m \geq p$. Then $m/2 \leq m+p \leq m+q = 2m$ and using (25) for $a = m+p \leq b = 2m$, we deduce that 
\[d(K_{m+p}(g), K_{2m}(g)) \leq ce^4\frac{2m}{m+p} \leq 4ce^4.
\]

**Proof of inequality (22).** Our goal is to estimate 
\[\frac{d}{dp} \log((\mathbb{E}|X|^p)^{1/p})
\]
\[= \frac{d}{dp} \log((\mathbb{E}h_{k,p}(U))^{1/p}) + \frac{d}{dp} \left( \frac{1}{p} \log \frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)} \right),
\]
As already mentioned in (15), by concavity of $p \mapsto \frac{d}{dp} \log \Gamma(p)$, we have 
\[\frac{d}{dp} \left( \frac{1}{p} \log \frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)} \right) \leq 0.
\]
We use the following notation. Let \((\Omega, \mu)\) be a measurable space. For any measurable function \(f : \Omega \to \mathbb{R}^+\), we set

\[
\mathbb{E}_\mu(f) = \int f \, d\mu \quad \text{and} \quad \text{Ent}_\mu(f) = \mathbb{E}_\mu(f \log f) - \mathbb{E}_\mu(f) \log \mathbb{E}_\mu(f).
\]

Let \(w\) be the density of the distribution of \(X\) on \(\mathbb{R}^n\). Since \(X\) is \((-1/r)\)-concave, \(w\) is \((-1/(r + n))\)-concave on \(\mathbb{R}^n\). To any fixed \(u \in \text{SO}(n)\), we associate the measure \(\mu_u\) on \(\mathbb{R}^+\) with density

\[
t \mapsto |S^{k-1}| t^{k-1} \pi_{u(E_0)} w(tu(\theta_0))
\]

so that

\[
h_{k,p}(u) = |S^{k-1}| \int_0^\infty t^{p+k-1} \pi_{u(E_0)} w(tu(\theta_0)) \, dt = \mathbb{E}_{\mu_u}(t^p).
\]

Define also \(\mu_{k,p}\) to be the measure on \(\mathbb{R}^+\) with density

\[
t \mapsto |S^{k-1}| t^{k-1} \mathbb{E}_{\pi_{U(E_0)}} w(tU(\theta_0)).
\]

Then \(\mathbb{E} h_{k,p}(U) = \mathbb{E}_U \mathbb{E}_{\mu_u}(t^p) = \mathbb{E}_{\mu_{k,p}}(t^p)\). Since \(w\) is a density of probability, \(\mu_{k,p}\) is a probability measure on \(\mathbb{R}^+\). A classical fact, verified by direct computation, is that

\[
\frac{d}{dp} \log((\mathbb{E}_\mu(f^p))^{1/p}) = \frac{1}{p^2} \text{Ent}_\mu(f^p) - \mathbb{E}_\mu(f^p).
\]

Therefore

\[
\frac{d}{dp} \log((\mathbb{E} h_{k,p}(U))^{1/p}) = \frac{d}{dp} \log((\mathbb{E}_{\mu_{k,p}}(t^p))^{1/p}) = \frac{1}{p^2} \text{Ent}_{\mu_{k,p}}(t^p) = \frac{1}{p^2} \mathbb{E} h_{k,p}(U).
\]

The numerator can be decomposed into two terms:

\[
\text{Ent}_{\mu_{k,p}}(t^p) = \mathbb{E}_U \text{Ent}_{\mu_U}(t^p) + \mathbb{E}_U \mathbb{E}_{\mu_U}(t^p) = \mathbb{E}_U \text{Ent}_{\mu_U}(t^p) + \mathbb{E}_U h_{k,p}(U).
\]

To control the second term, we use the log-Sobolev inequality \([2]\):

\[
\frac{1}{p^2} \mathbb{E} h_{k,p}(U) \leq \frac{c}{p^{2n}} \frac{\mathbb{E}((\nabla \log h_{k,p})^2(U)h_{k,p}(U))}{\mathbb{E} h_{k,p}(U)} \leq \frac{cL_{k,p}^2}{p^{2n}}.
\]

To control the first term, we start by observing that for a fixed \(u \in \text{SO}(n),\)

\[
\frac{1}{p^2} \mathbb{E}_{\mu_u}(t^p) = \frac{d}{dp} \log((\mathbb{E}_{\mu_u}(f^p))^{1/p}) = \frac{d}{dp} \left(\frac{1}{p} \log h_{k,p}(u)\right)
\]

\[
= \frac{d}{dp} \left(\log \frac{h_{k,p}(u)}{B(p + k, r - p)} - \log \frac{h_{k,0}(u)}{B(k, r)} + \log \frac{B(p + k, r - p)}{B(k, r)} + \log h_{k,0}(u)\right).
\]
By Corollary 9, the map \( p \mapsto h_{k,p}(u) \) \( B(p+k,r-p) \) is log-concave on \((-k+1, r)\). This implies that

\[
\frac{d}{dp} \left( \log \frac{h_{k,p}(u)}{B(p+k,r-p)} - \log \frac{h_{k,0}(u)}{B(k,r)} \right) \leq 0.
\]

We know from Lemma 11 that, for all \( p \in [-\frac{k-1}{2}, \frac{r-1}{2}] \),

\[
\frac{d}{dp} \left( \frac{1}{p} \log \frac{B(k+p,r-p)}{B(k,r)} \right) \leq C \left( \frac{1}{k-1} + \frac{1}{r-1} \right).
\]

Therefore, for any fixed \( u \in SO(n) \),

\[
\frac{1}{p^2} \text{Ent}_{\mu_u}(t^p) \leq Ch_{k,p}(u) \left( \frac{1}{k-1} + \frac{1}{r-1} \right) - \frac{1}{p^2} h_{k,p}(u) \log h_{k,0}(u).
\]

Integrating over \( u \in SO(n) \), we deduce that

\[
\frac{1}{p^2} \mathbb{E} \text{Ent}_{\mu_U}(t^p) \leq C \left( \frac{1}{k-1} + \frac{1}{r-1} \right) + \frac{1}{p^2} \mathbb{E} h_{k,p}(U) \log(h_{k,0}(U)^{-1}) \mathbb{E} h_{k,p}(U).
\]

From the Jensen and Hölder inequalities,

\[
\frac{\mathbb{E}(h_{k,p}(U) \log h_{k,0}(U)^{-1})}{\mathbb{E} h_{k,p}(U)} \leq \log \left( \frac{\mathbb{E}(h_{k,p}(U)h_{k,0}(U)^{-1})}{\mathbb{E} h_{k,p}(U)} \right)
\]

\[
\leq \log \left( \frac{(\mathbb{E} h_{k,p}(U)^2)^{1/2}}{\mathbb{E} h_{k,p}(U)} \right) + \log \left( ((\mathbb{E}(h_{k,0}(U)^{-2}))^{1/2}) \right).
\]

From (3), the first term is upper bounded by \((c/n) L_{k,p}^2\). For the second term, we first use (3) with \( f = h_{k,0}^{-1}, q = 2 \) and \( r = 0 \), then we use (3) again with \( f = h_{k,0}, q = 1 \) and \( r = 0 \). Since \( \mathbb{E} h_{k,0}(U) = \mathbb{E} \mu_{k,0}(1) = 1 \), we deduce that this term is bounded by \((3c/n) L_{k,0}^2\). Combining this last inequality with (32), (31) and (30), we conclude that

\[
\frac{d}{dp} \log((\mathbb{E} |X|_2)^{1/p}) \leq \frac{c}{p^2 n} \left( 2L_{k,p}^2 + 3L_{k,0}^2 \right) + \frac{C}{k-1} + \frac{C}{r-1}.
\]

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Matthieu Fradelizi, Olivier Guédon, Alain Pajor
Université Paris-Est
Laboratoire d’Analyse et Mathématiques Appliquées (UMR 8050)
UPEMLV
F-77454 Marne-la-Vallée Cedex 2, France
E-mail: matthieu.fradelizi@u-pem.fr
olivier.guedon@u-pem.fr
alain.pajor@u-pem.fr

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