

Regularity of $\int_{\Omega} |\nabla u|^2 + \lambda \int_{\Omega} |u - f|^2$ and Some Gap Phenomenon

R. HADIJI¹ and F. ZHOU^{1,2}

¹Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie, 4, place Jussieu, 75252 Paris cedex 05, France.

²University of Wuhan, Department of Mathematics, 430072 Wuhan, China.

(Received: 11 June 1992; accepted: 1 July 1992)

Abstract. We study the regularity of the minimizer u_{λ} for the functional $F_{\lambda}(u, f) = \int_{\Omega} |\nabla u|^2 + \lambda \int_{\Omega} |u - f|^2$ over all maps $u \in H^1(\Omega, S^2)$. We prove that for some suitable functions f every minimizer u_{λ} is smooth in Ω if $\lambda \leq \lambda_0$ and for the same functions f , u_{λ} has singularities when λ is large enough.

Résumé. On étudie la régularité des minimiseurs u_{λ} du problème de minimisation $\min_{u \in H^1(\Omega, S^2)} (\int_{\Omega} |\nabla u|^2 + \lambda \int_{\Omega} |u - f|^2)$. On montre que pour certaines fonctions f , u_{λ} est régulière lorsque $\lambda \leq \lambda_0$ et pour les mêmes f , si λ est assez grand, alors u_{λ} possède des singularités.

Mathematics Subject Classifications (1991). 35K45, 58GXX, 58E35.

Key words. Gap phenomenon, regularity, singularities, sphere, minimizers, infimum.

0. Introduction

Let Ω be the unit ball of \mathbb{R}^3 . Set

$$H^1(\Omega, S^2) = \{u \in H^1(\Omega, \mathbb{R}^3); u(x) \in S^2 \text{ a.e.}\}$$

where S^2 is the unit sphere of \mathbb{R}^3 . For $\lambda > 0$ and $f \in L^2(\Omega, S^2)$, let

$$F_{\lambda}(u, f) = \int_{\Omega} |\nabla u|^2 + \lambda \int_{\Omega} |u - f|^2. \quad (1)$$

Consider now the following minimization problems

$$E_{\lambda}(f) = \inf_{u \in H^1(\Omega, S^2)} F_{\lambda}(u, f) \quad (2)$$

and

$$E_{\lambda, \text{reg}}(f) = \inf_{u \in C^0 \cap H^1(\Omega, S^2)} F_{\lambda}(u, f). \quad (3)$$

Note that the functional in (1) is lower semi-continuous on $H^1(\Omega, S^2)$, so that the infimum in (2) is achieved. We recall that if u_λ is a minimizer for (2) then u_λ is a weak solution of the Euler equation associated to (2):

$$-\Delta u = u|\nabla u|^2 + \lambda[f - \langle u, f \rangle u]. \tag{4}$$

It is well known (see [10, 11]) that any minimizer of (2) is smooth except at a finite number of points.

It is very natural to ask whether the minimizer u_λ is regular. This question is closely related to an existence of a gap phenomenon (see [7] or [1]). We recall that Bethuel and Brezis have shown that there exists some regular function f with values in \mathbb{R}^3 such that we have the following gap phenomenon

$$E_1(f) < E_{1,reg}(f).$$

In this paper, we want to study the regularity of minimizer of (2). Note that the two terms of $F_\lambda(u, f)$ have not the same homogeneity. The other difficulty comes from the fact that the possible singularities of minimizer u_λ may get more and more closer to the boundary as λ goes to 0. We overcome this difficulty by choosing some suitable function f . More precisely, we will assume that f satisfies the following conditions

$$\int_\Omega f \neq 0; \tag{5}$$

There is some $0 \leq r_0 < 1$ such that the point $\frac{\int_\Omega f}{|\int_\Omega f|}$ and the set $f(\Omega \setminus B(0, r_0))$ are included in the same hemisphere of S^2 . (6)

Our main results are the following:

THEOREM 1. *Suppose that f is a measurable function with values into S^2 satisfying (5), (6). Then there is some $\lambda_0 > 0$ such that for every $\lambda \leq \lambda_0$, u_λ is regular in Ω . In particular, we have*

$$E_\lambda(f) = E_{\lambda,reg}(f).$$

REMARK 1. We do not know whether conditions (5) and (6) are really necessary.

THEOREM 2. *Let f be any function in $H^1(\Omega, S^2)$ which is not a strong limit of smooth maps. Then there exists $\lambda_1 > 0$ such that for every $\lambda \geq \lambda_1$, every u_λ is not regular in Ω . In particular, we have*

$$E_\lambda(f) < E_{\lambda,reg}(f).$$

REMARK 2. 1. Let f be any function in $H^1(\Omega, S^2)$ which is not a strong limit of smooth maps. Then there exists a sequence of smooth functions f_n which converges to f in $L^2(B^3, S^2)$. For n large enough we have

$$E_\lambda(f_n) < E_{\lambda,reg}(f_n).$$

2. The case of f with axial symmetry: Consider the case where f is radial symmetric with image contained in an hemisphere (without loss of generality $S^{2+} = \{x \in S^2; x_3 \geq 0\}$). By radial symmetric we mean $f = \sin \varphi(|x|)(x/|x|, 0) + \cos \varphi(|x|)(0, 0, 1)$ for some $\varphi: [0, 1] \rightarrow [0, \pi]$. I. Shafrir has informed us that modifying an argument in [3] one may infer that for each $\lambda > 0$, u_λ is unique and symmetric.

We also consider the following analogous problem with not prescribed constant boundary data. Namely, let us denote by $H_c^1(B^3, S^2)$ the set of maps $u \in H^1(\Omega, S^2)$ such that u is a constant on ∂B^3 , then we study the problem:

$$\min_{u \in H_c^1(B^3, S^2)} F_\lambda(u, f) \tag{7}$$

and we obtain the following

THEOREM 3. *Let f be any measurable function with values in S^2 . Then for all $\lambda < 2^{-2/3}$, every minimizer u_λ of (7) is regular on Ω .*

It is easy to prove that for any f , $E_\lambda(f)$ (resp. $E_{\lambda,reg}(f)$) is a continuous, nondecreasing function of λ . If we denote by $I(f) = \{\lambda \geq 0; E_\lambda(f) = E_{\lambda,reg}(f)\}$ then $I(f)$ is closed set of \mathbb{R}^+ . Using Theorem 1 and Theorem 2, we can prove that for some choice of f (for example the dipole function see [1] or [4]) $I(f)$ is a compact subset of \mathbb{R}^+ with nonempty interior. We conjecture that in this case, $I(f)$ is an interval of \mathbb{R}^+ .

In order to establish Theorem 1, one uses the well known ε -regularity Theorem (see [2, 3, 10, 11]) and other techniques introduced by Bethuel–Brezis, and Hardt–Kinderlehrer–Lin in variational problem involving S^2 -valued maps. Our proof is also based on a Theorem of classification of singularities. More precisely, we describe the nature of the singularities when they really occur. We prove that all singularity has a simple form: if y is a singularity then for a rotation R , $u_\lambda(x) \simeq \pm R(x - y/|x - y|)$ as x tends to y . This description was first given by [4] for minimizers harmonic maps.

In the proof of Theorem 3 we use the same argument as above combined with a modification of the monotonicity formula (see Lemma 2).

In the case where f is a constant, it can be proved that (4) is just the elliptic type of the Landau–Lifshitz equation independent of time without boundary data (see [6]). We know that (see [7]) in this case, for any $C \in S^2$ there is a weak smooth solution of (4) with boundary value C .

This paper is organized as follows:

I. Classification of energy minimizing maps

Step 1. Any minimizer u_λ satisfies a reverse Hölder inequality.

Step 2. A monotonicity formula.

II. Proof of Theorem 1

Step 1. An ε -regularity Theorem.

Step 2. A regularity criterion on compact set.

Step 3. Proof of Theorem 1 completed.

III. Proof of Theorem 2

IV. Proof of Theorem 3

I. Classification of Energy Minimizing Maps

We start with the following definition. Let Ω_1 be a bounded domain in \mathbb{R}^3 , we say that u is a w -minimizer associated to λ, f and Ω_1 if for every $v \in H^1(\Omega, S^2)$ such that $u = v$ on $\partial\Omega_1$, we have

$$\int_{\Omega_1} |\nabla u|^2 + \lambda \int_{\Omega_1} |u - f|^2 \leq \int_{\Omega_1} |\nabla v|^2 + \lambda \int_{\Omega_1} |v - f|^2.$$

In order to establish the classification of the singularities Theorem, we need the following

PROPOSITION 1. *Let (f_j) be a sequence of maps in $L^2(\Omega, S^2)$, (λ_j) in a sequence of positive numbers, and (u_j) a sequence of w -minimizer associated to λ_j and f_j . Suppose that $f_j \rightarrow f$ in $L^2(\Omega, S^2)$, $\lambda_j \rightarrow \lambda$ and $u_j \rightarrow u$ weakly in $H^1(\Omega, S^2)$, then u is a w -minimizer associated to λ and f .*

The proof of this Proposition is given in the appendix.

For $\lambda = 0$, it is known (see [4]) that the singularities of w -minimizers have the form $\pm R(x - y/|x - y|)$ as x tends to y for some rotation R of \mathbb{R}^3 . In the following we establish the same property for $\lambda > 0$. We shall show that

THEOREM 4. *Suppose that $y \in \Omega$ is a singularity of u_λ , then we have $u_\lambda(x) \simeq \pm R(x - y/|x - y|)$ as x tends to y where R is a rotation of \mathbb{R}^3 . In particular all the singularities of u_λ are of degree ± 1 .*

The proof of Theorem 4 relies on several steps.

1.1. A REVERSE HÖLDER INEQUALITY

We shall follow a technique used in [8] and [3] to prove a Hölder reverse inequality.

LEMMA 1. *There exist constants $q > 2$ and $C > 0$ such that*

$$\left(\int_{B_r} |\nabla u_\lambda|^q \right)^{\frac{1}{q}} \leq C \left(\int_{B_{2r}} |\nabla u_\lambda|^2 \right)^{\frac{1}{2}} + C\lambda^{\frac{3}{2}} \tag{8}$$

for every ball B_r such that $B_{2r} \subset \Omega$.

Proof of Lemma 1. First we claim that for any ball $B_r \subset \Omega$ we have

$$\int_{B_r} |\nabla u_\lambda|^2 \leq \int_{B_r} |\nabla v|^2 + \frac{16\pi}{3} \lambda r^3 \tag{9}$$

for every $v \in H^1(B_r, S^2)$ such that $v = u$ on ∂B_r . Indeed, let v be as above, we define $w: \Omega \rightarrow S^2$ by $w(x) = u_\lambda(x)$ if $x \in \Omega \setminus B_r$ and $w(x) = v(x)$ if $x \in B_r$, so $w \in H^1(\Omega, S^2)$. Using the fact that u_λ is a minimizer we obtain

$$\int_{B_r} |\nabla u_\lambda|^2 + \lambda \int_{B_r} |u_\lambda - f|^2 \leq \int_{B_r} |\nabla v|^2 + \lambda \int_{B_r} |v - f|^2,$$

hence

$$\begin{aligned} \int_{B_r} |\nabla u_\lambda|^2 &\leq \int_{B_r} |\nabla v|^2 + 2\lambda \int_{B_r} \langle u_\lambda - v, f \rangle \\ &\leq \int_{B_r} |\nabla v|^2 + \frac{16\pi}{3} \lambda r^3, \end{aligned}$$

which is (9).

Now, using a result of ([8], Appendix), we have for all $\zeta \in \mathbb{R}^3$

$$\min_{\substack{v \in H^1(B_r, S^2) \\ v = u_\lambda \text{ on } \partial B_r}} \int_{B_r} |\nabla v|^2 \leq C \|\nabla_T u_\lambda\|_{L^2(\partial B_r)} \|u_\lambda - \xi\|_{L^2(\partial B_r)} \tag{10}$$

where C is a universal constant.

Combining (9) and (10) we find

$$\int_{B_r} |\nabla u_\lambda|^2 \leq C \|\nabla_T u_\lambda\|_{L^2(\partial B_r)} \|u_\lambda - \xi\|_{L^2(\partial B_r)} + C\lambda r^3.$$

We now follow the argument of [8] or [3] to deduce that for every $\varepsilon > 0$ and $r < \rho < 2r$

$$\int_{B_r} |\nabla u_\lambda|^2 \leq \varepsilon r \int_{\partial B_r} |\nabla_T u_\lambda|^2 + C(\varepsilon r)^{-1} \int_{\partial B_r} |u_\lambda - \xi|^2 + C\lambda r^3. \tag{11}$$

Integrating (11) between r and $2r$ we obtain

$$\int_{B_r} |\nabla u_\lambda|^2 \leq \varepsilon r^{-1} \int_{B_{2r}} |\nabla u_\lambda|^2 + C(\varepsilon r)^{-1} \int_{B_{2r}} |u_\lambda - \xi|^2 + C\lambda r^3.$$

Choosing $\xi = \bar{u}_\lambda = 1/|B_{2r}| \int_{B_{2r}} u_\lambda$ and applying the Hölder inequality, we find

$$\int_{B_r} |\nabla u_\lambda|^2 \leq 2\delta \int_{B_{2r}} |\nabla_T u_\lambda|^2 + \frac{C}{\delta r^3} \left(\int_{B_{2r}} |\nabla u_\lambda| \right)^2 + C\lambda r^3$$

where $\delta = \varepsilon/r$. Thus

$$\frac{1}{|B_r|} \int_{B_{2r}} |\nabla u_\lambda|^2 \leq \frac{16\delta}{|B_{2r}|} \int_{B_{2r}} |\nabla u_\lambda|^2 + \frac{C}{\delta |B_{2r}|} \left(\int_{B_{2r}} |\nabla u| \right)^2 + C\lambda$$

Fixing $\delta < \frac{1}{16}$, we may now apply a result of Giaquinta ([5], chap. V, Prop. 1.1] to conclude the existence of some $q > 2$ such that (8) is satisfied. This proves Lemma 1.

1.2. A MONOTONICITY FORMULA

In this step, we establish a monotonicity formula as a variant of the well known monotonicity formula for a standard minimizing harmonic map u

$$\frac{d}{dr} \left(\frac{1}{r} \int_{B_r} |\nabla u|^2 \right) \geq 0.$$

Here we have the following

LEMMA 2. *For any $a \in \Omega$ and $r < \text{dist}(a, \partial\Omega)$ we have*

$$\frac{d}{dr} \left(\frac{1}{r} \int_{B_r(a)} |\nabla u_\lambda|^2 + \frac{\lambda}{r} \int_{B_r(a)} |u_\lambda - f|^2 + \frac{8\pi}{3} \lambda r^2 \right) \geq 0 \tag{12}$$

In particular $1/r \int_{B_r(a)} |\nabla u_\lambda|^2 + \frac{32}{3} \pi \lambda r^2$ is nondecreasing in r .

Proof of Lemma 2. We can assume without loss of generality that $a = 0$ and we write $B_r = B_r(a)$, and $S_r = \partial B_r$. Set $u'_\lambda(x) = u_\lambda(x)$ for $x \in \Omega \setminus B_r$ and $u_\lambda(r(x/|x|))$ for $x \in B_r$. Since $u'_\lambda \in H^1(\Omega, S^2)$, we have

$$\int_{B_r} |\nabla u_\lambda|^2 + \lambda \int_{B_r} |u_\lambda - f|^2 \leq \int_{S_r} |\nabla_T u'_\lambda|^2 + \lambda \int_{B_r} |u'_\lambda - f|^2. \tag{13}$$

On the other hand, let

$$E(r) = \frac{1}{r} \int_{B_r} |\nabla u_\lambda|^2 + \lambda \int_{B_r} |u_\lambda - f|^2 + \frac{8\pi}{3} \lambda r^2.$$

We have

$$\frac{dE(r)}{dr} = \frac{1}{r^2} \left(\int_{S_r} |\nabla_T u_\lambda^r|^2 + r \int_{S_r} \left| \frac{\partial u_\lambda}{\partial r} \right|^2 + \lambda r \int_{S_r} |u_\lambda - f|^2 - \int_{B_r} |\nabla u_\lambda|^2 - \lambda \int_{B_r} |u_\lambda - f|^2 + \frac{16\pi}{3} \lambda r^3 \right).$$

Combining the last equality with (13) we obtain

$$\begin{aligned} \frac{dE(r)}{dr} &= \frac{1}{r^2} \left(r \int_{S_r} \left| \frac{\partial u_\lambda}{\partial r} \right|^2 + \lambda r \int_{S_r} |u_\lambda - f|^2 - \lambda \int_{B_r} |u_\lambda^r - f|^2 + \frac{16\pi}{3} \lambda r^3 \right) \\ &\geq \frac{1}{r} \int_{S_r} \left| \frac{\partial u_\lambda}{\partial r} \right|^2 \geq 0. \end{aligned}$$

This proves (12). Moreover, we have

$$\begin{aligned} \frac{d}{dr} \left(\lambda \int_{B_r} |u_\lambda - f|^2 \right) &= \frac{\lambda}{r} \int_{S_r} |u_\lambda - f|^2 - \frac{\lambda}{r^2} \int_{B_r} |u_\lambda - f|^2 \\ &\geq 16\pi\lambda r. \end{aligned}$$

It follows that

$$\frac{d}{dr} \left(\frac{1}{r} \int_{B_r(a)} |\nabla u_\lambda|^2 \right) + 16\pi\lambda r + \frac{16}{3}\pi\lambda r \geq \frac{d}{dr}(E(r)) \geq 0.$$

Therefore $1/r \int_{B_r(a)} |\nabla u_\lambda|^2 + \frac{32}{3}\pi\lambda r^2$ is monotonically nondecreasing in r .

COROLLARY 1. *Let a be any point of Ω , then there exists a constant C depending only on a such that for every $r < \text{dist}(a, \partial\Omega)$ we have*

$$\frac{1}{r} \int_{B_r} |\nabla u_\lambda|^2 \leq C \tag{14}$$

and so the limit in (14) exists as r tends to zero.

Proof of Theorem 4 completed. We write again $B_\sigma = B_\sigma(y)$ and $S_\sigma = \partial B_\sigma$ for $0 < \sigma < \text{dist}(y, \partial\Omega)$. Let us define for $x \in \Omega$

$$u_\lambda^\sigma = u_\lambda(\sigma x + y),$$

and

$$f^\sigma = f(\sigma x + y).$$

Then $u_\lambda^\sigma \in H^1(\Omega, S^2)$ and $u_\lambda^\sigma|_{\partial\Omega} = u_\lambda|_{S_\sigma}$. Using Corollary 1, we obtain

$$\int_\Omega |\nabla u_\lambda^\sigma|^2 = \frac{1}{\sigma} \int_{B_r} |u_\lambda|^2 \leq C \tag{15}$$

for some constant $C > 0$ and for all $0 < \sigma < \text{dist}(y, \partial\Omega)$.

On the other hand, since u_λ is a minimizer, we claim that u_λ^σ is a w -minimizer associated to $\sigma^2\lambda$ and f^σ . In fact, let w be in $H^1(\Omega, S^2)$ such that $w = u_\lambda^\sigma$ on $\partial\Omega$. We set

$$w_\sigma(x) = w\left(\frac{x - y}{\sigma}\right) \quad x \in B_\sigma.$$

Thus $w_\sigma \in H^1(B_\sigma, S^2)$ satisfying

$$w_\sigma|_{S_\sigma} = w|_{\partial\Omega} = u_\lambda^\sigma|_{\partial\Omega} = u_\lambda|_{S_\sigma}$$

then we obtain

$$\begin{aligned} \int_{B_\sigma} |\nabla u_\lambda|^2 + \lambda \int_{B_\sigma} |u_\lambda - f|^2 &\leq \int_{B_\sigma} |\nabla w_\sigma|^2 + \lambda \int_{B_\sigma} |u_\sigma - f|^2 \\ &= \sigma \left(\int_\Omega |\nabla w|^2 + \lambda \sigma^2 \int_\Omega |w - f^\sigma|^2 \right). \end{aligned}$$

It is easy to see that

$$\frac{1}{\sigma} \left(\int_{B_\sigma} |\nabla u_\lambda|^2 + \lambda \int_{B_\sigma} |u_\lambda - f|^2 \right) = \int_\Omega |\nabla u_\lambda^\sigma|^2 + \lambda \sigma^2 \int_\Omega |u_\lambda^\sigma - f^\sigma|^2,$$

hence

$$\int_\Omega |\nabla u_\lambda^\sigma|^2 + \lambda \sigma^2 \int_\Omega |u_\lambda^\sigma - f^\sigma|^2 \leq \int_\Omega |\nabla w|^2 + \lambda \sigma^2 \int_\Omega |w - f^\sigma|^2$$

and our assertion is proved.

By (15), there exists some $v_\lambda \in H^1(\Omega, S^2)$ such that (for a subsequence if necessary)

$$u_\lambda^\sigma \rightarrow v_\lambda \quad \text{weakly in } H^1(\Omega, S^2) \text{ as } \sigma \rightarrow 0.$$

We claim that

$$v_\lambda(x) = w_\lambda \left(\frac{x}{|x|} \right) \quad \text{for some } w_\lambda \in H^1(S^2, S^2).$$

Indeed, by Lemma 2, we have

$$D(r) = \frac{d}{dr} E(r) \geq \frac{1}{r} \int_{S_r} \left| \frac{\partial u_\lambda}{\partial r} \right|^2 \geq 0.$$

Since $\lim_{r \rightarrow 0^+} E(r)$ exists, we have $\int_0^\sigma D(r) dr = E(\sigma) - \lim_{r \rightarrow 0^+} E(r)$ exists and tends to zero as $\sigma \rightarrow 0$, but

$$\int_{B_\sigma} \frac{1}{r} \left| \frac{\partial u_\lambda}{\partial r} \right|^2 = \int_0^\sigma \frac{1}{r} \int_{S_r} \left| \frac{\partial u_\lambda}{\partial r} \right|^2 \leq \int_0^\sigma D(r) dr.$$

It follows that $\int_{B_r} 1/r |\partial u_\lambda / \partial r|^2 \rightarrow 0$ as $\sigma \rightarrow 0$. Since $\int_\Omega 1/r |\partial u_\lambda / \partial r|^2 = \int_{B_r} 1/r |\partial u_\lambda / \partial r|^2$ thus our claim follows.

Following the strategy of [3] or [10], we can prove that $u_\lambda^\sigma \rightarrow v_\lambda$ strongly in $H^1(\Omega, S^2)$. We use the fact that

$$-\Delta u_\lambda^\sigma = u_\lambda^\sigma |\nabla u_\lambda^\sigma|^2 + \sigma^2 \lambda [f^\sigma - \langle u_\lambda^\sigma, f^\sigma \rangle u_\lambda^\sigma]$$

and

$$\int_\Omega |\nabla u_\lambda^\sigma|^2 \leq C.$$

We deduce from standard elliptic estimates that ∇u_λ^σ is relatively compact in $L^1_{loc}(\Omega)$ and therefore we may assume that $\nabla u_\lambda^{\sigma_n} \rightarrow \nabla v_\lambda$ a.e. on Ω (for an appropriate subsequence). On the other hand, using Lemma 1, we know that

$$\int_\Omega |\nabla u_\lambda^\sigma|^q \leq C \quad \text{for some } q > 2.$$

Thus

$$\nabla u_\lambda^{\sigma_n} \rightarrow \nabla v_\lambda \quad \text{in } L^2(\Omega, \mathbb{R}^3).$$

Finally, applying Proposition 1 we deduce that v_λ is a w -minimizer associated to $\lim(\sigma_n^2 \lambda) = 0$. By Theorem 7.3 and Theorem 7.4 of [4], we conclude that $w_\lambda = \pm R$ where R is a rotation of \mathbb{R}^3 and in particular $\text{deg}(u_\lambda, y) = \pm 1$. This completes the proof of Theorem 4.

II. Proof of Theorem 1

II.1 ε -REGULARITY

For the proof of Theorem 1 we need some preliminary regularity results. The first is the following ε -regularity Theorem of [10].

THEOREM 5 [10]. *There exists some ε_0 independent of λ such that if $\lambda < \varepsilon_0$ and*

$$\frac{1}{r} \int_{B_r} |\nabla u_\lambda|^2 < \varepsilon_0$$

for some ball B_r , then u_λ is Hölder continuous on $B_{r/2}$ and $|u_\lambda(x) - u_\lambda(y)| \leq C|x - y|^\alpha$ for all x, y in $B_{r/2}$ where α and C are absolute constants independent of λ .

For the proof of this Theorem we refer to ([10], Th. 3.1.) or ([11], regularity estimate 1.5.).

II.2. A REGULARITY CRITERION ON COMPACT SUBSET

The next result asserts that the possible singularities of u_λ are close to the boundary as λ goes to zero. This phenomenon is due to the fact that the boundary value of u_λ is not prescribed. More precisely we have

PROPOSITION 2. *Given an arbitrary function f in $L^2(\Omega, S^2)$. For every compact $K \subset \Omega$, there is some $\lambda_0(K) = \text{dist}(K, \partial\Omega)$ such that for every $\lambda < \lambda_0(K)$, u_λ is smooth on K .*

Proof of Proposition 2. First, we see that

$$\int_{\Omega} |\nabla u_\lambda|^2 + \lambda \int_{\Omega} |u_\lambda - f|^2 \leq \lambda \int_{\Omega} |P - f|^2$$

for all $P \in S^2$, hence

$$\int_{\Omega} |\nabla u_\lambda|^2 \leq C\lambda \|u_\lambda - P\|_{L^2(\Omega, S^2)}. \tag{16}$$

In particular, $u_\lambda \rightarrow P_0$ strongly on $H^1(\Omega, S^2)$ where $P_0 \in S^2$. For the proof of Proposition 2, we argue by contradiction. Let $K \subset \Omega$, be a compact set. We can assume that there exist a sequence (a_n) of singularities such that $a_n \rightarrow a \in K$ as $n \rightarrow \infty$ and $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. Applying the monotonicity formula (11) we obtain (for $\lambda_n < \text{dist}(K, \partial\Omega)$)

$$\frac{1}{r} \int_{B_r(a_n)} |\nabla u_{\lambda_n}|^2 \leq \frac{1}{\lambda_n} \int_{B_{\lambda_n}(a_n)} |\nabla u_{\lambda_n}|^2 + \frac{32}{3} \lambda_n \pi (\lambda_n^2 - r^2).$$

By (16) we have

$$\frac{1}{r} \int_{B_r(a_n)} |\nabla u_{\lambda_n}|^2 \leq 2 \|u_{\lambda_n} - P_0\|_{L^2(\Omega, S^2)} + \frac{32}{3} \pi \lambda_n^3.$$

Using Theorem 3 we obtain

$$7\pi \leq \frac{1}{r} \int_{B_r(a_n)} |\nabla u_{\lambda_n}|^2 \leq 2 \|u_{\lambda_n} - P_0\|_{L^2(\Omega, S^2)} + \frac{32}{2} \pi \lambda_n^3,$$

which is a contradiction for n large enough.

II.3. PROOF OF THEOREM 1 COMPLETED

First we prove the following

LEMMA 3. *Let (λ_n) be any sequence of positive numbers going to zero, then $u_{\lambda_n} \rightarrow \int f/|\int f|$ strongly in $H^1(\Omega, S^2)$ as $n \rightarrow \infty$.*

Proof of Lemma 3. Since $\int_{\Omega} |\nabla u_{\lambda_n}|^2 \leq 2\lambda \int_{\Omega} \langle u_{\lambda_n} - P, f \rangle$ for all $P \in S^2$ we have for all n

$$\int_{\Omega} |\nabla u_{\lambda_n}|^2 \leq 4|\Omega|\lambda_n \leq C, \tag{17}$$

so we may as well assume that u_{λ_n} tends to a constant map $P \in S^2$ strongly in $H^1(\Omega, S^2)$. Assuming that $P \neq \int f / |\int f|$, then there is a rotation R of \mathbb{R}^3 such that $R \neq Id$, $R(P) = \int f / |\int f|$, and $R(u_{\lambda_n}) \rightarrow R(P)$ strongly in $H^1(\Omega, S^2)$. On the other hand, since u_{λ_n} is a minimizer, we have

$$\int_{\Omega} |\nabla u_{\lambda_n}|^2 + \lambda_n \int_{\Omega} |u_{\lambda_n} - f|^2 \leq \int_{\Omega} |\nabla R(u_{\lambda_n})|^2 + \lambda_n \int_{\Omega} |R(u_{\lambda_n}) - f|^2,$$

hence

$$\int_{\Omega} |u_{\lambda_n} - f|^2 \leq \int_{\Omega} |R(u_{\lambda_n}) - f|^2.$$

Passing to the limit, we obtain

$$\int_{\Omega} |P - f|^2 \leq \int_{\Omega} |R(P) - f|^2.$$

Thus

$$P \int_{\Omega} f \geq \left| \int_{\Omega} f \right|.$$

This is impossible unless $P = \int f / |\int f|$ which proves the Lemma.

Secondly, we claim that there is some λ_1 such that

$$u_{\lambda}(x) \in S^2 \cap B(P, \frac{1}{2}) \quad \forall x \in \partial B_{r_0} \quad \text{and} \quad \forall \lambda \leq \lambda_1. \tag{18}$$

We postpone the proof of the claim and complete the proof of Theorem 1. We can assume without loss of generality, that P is the north pole and $f(\Omega \setminus B(0, r_0)) \subset S^2_+$. Set for $\lambda \leq \lambda_1$

$$\begin{cases} w_{\lambda} = (u_{\lambda}^1, u_{\lambda}^2, u_{\lambda}^3) & \text{on } B(0, r_0), \\ = (u_{\lambda}^1, u_{\lambda}^2, |u_{\lambda}^3|) & \text{on } \Omega \setminus B(0, r_0). \end{cases}$$

By (18), $w_{\lambda} \in H^1(\Omega, S^2)$ and $\int_{\Omega} \langle u_{\lambda} - w_{\lambda}, f \rangle \geq 0$. Since f satisfies (6), we deduce that $u_{\lambda}^3 = |u_{\lambda}^3|$ on $\Omega \setminus B(0, r_0)$, so $u_{\lambda} = w_{\lambda}$. By Theorem 4, we cannot have singularities on $\Omega \setminus B(0, r_0)$. On the other hand, by Proposition 2, u_{λ} has no singularities on $B(0, r_0)$ if $\lambda \leq 1 - r_0$. Thus we have shown Theorem 1.

Proof of claim. Let $0 < r_1 < 1 - r_0$ there exists a finite number of points $x_i, i = 1, \dots, n$ of ∂B_{r_0} such that $\partial B_{r_0} \subset \cup B(x_i, r_1/2)$. Thus if we choose

$\lambda_0 \leq \min(1 - r_0 - r_1, r_1 \varepsilon_0 / 4|\Omega|)$, we have for all $\lambda \leq \lambda_0$

$$\frac{1}{r_1} \int_{B(x_i, r_1)} |\nabla u_\lambda|^2 \leq \frac{4|\Omega|}{r_1} \lambda \leq \varepsilon_0.$$

This implies, by the ε -regularity (see Theorem 5), that u_λ is Hölder continuous on $B(x_i, r_1/2)$ for all $\lambda \leq \lambda_0$ and for all $i = 1, \dots, n$. Moreover, we have $\forall x, y \in B(x_i, r_1/2)$

$$|u_\lambda(x) - u_\lambda(y)| \leq C|x - y|^\alpha.$$

Suppose, by contradiction, that (18) is false. We could find a sequence of positive numbers (λ_n) going to zero and a sequence of points (x_n) of ∂B_{r_0} such that

$$|u_{\lambda_n}(x_n) - P| \geq \frac{1}{2}. \tag{19}$$

By the compactness of ∂B_{r_0} , we can assume that $x_n \rightarrow x$ as $n \rightarrow \infty$ and $x \in \partial B_{r_0} \cap B(x_i, r_1/2)$. By (17), we have also (for a subsequence if necessary) $u_{\lambda_n}(x) \rightarrow P$ as $n \rightarrow \infty$ for almost everywhere x . Then for some $y \in B(x_i, r_1/2)$, we have $u_{\lambda_n}(y) \rightarrow P$. But

$$\begin{aligned} |u_{\lambda_n}(x_n) - P| &\leq |u_{\lambda_n}(x_n) - u_{\lambda_n}(y)| + |u_{\lambda_n}(y) - P| \\ &\leq C|x_n - y|^\alpha + |u_{\lambda_n}(y) - P| \\ &< \frac{1}{4}. \end{aligned}$$

for n large enough. This contradicts (19).

In what follows u_λ denote a minimizer for (2) and C is a generic constant.

III. Proof of Theorem 2

First we see that

$$E_\lambda(f) \leq \int_\Omega |\nabla f|^2 < +\infty \quad \forall \lambda \leq 0. \tag{20}$$

Let $(\lambda_n), (\varepsilon_n)$ be two sequences of positive numbers such that $\lambda_n \rightarrow \infty$ and $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. For every $n \in \mathbb{N}$ there is a map (u_n) in $C^1(\bar{\Omega}, S^2)$ such that $F_{\lambda_n}(u_n, f) \leq E_{\lambda_n, \text{reg}}(f) + \varepsilon_n$. Note that we can assume that $E_{\lambda_n, \text{reg}}(f)$ is bounded (otherwise by (20), our conclusion is satisfied). Thus we have (for a subsequence still denoted by (u_n))

$$u_n \rightarrow u_0 \text{ weakly in } H^1(\Omega, S^2) \text{ as } n \rightarrow \infty$$

and

$$\lambda_n \int_\Omega |u_n - f|^2 \rightarrow 0.$$

We deduce that $u_0 = f$. Since f is not a strong limit of smooth maps, we obtain

$$\int_{\Omega} |\nabla f|^2 < \alpha = \liminf \int_{\Omega} |\nabla u_n|^2.$$

Then, for every $\varepsilon > 0$, there is some N such that for all $n \geq N$ we have

$$\begin{aligned} E_{\lambda_n}(f) &\leq \int_{\Omega} |\nabla f|^2 < \alpha - \varepsilon \leq \int_{\Omega} |\nabla u_n|^2 \\ &\leq \int_{\Omega} |\nabla u_n|^2 + \lambda_n \int_{\Omega} |u_n - f|^2 \\ &\leq E_{\lambda_n, \text{reg}}(f) + \varepsilon_n. \end{aligned}$$

Thus the conclusion holds for n large enough. This completes the proof of Theorem 2.

We deduce the following:

COROLLARY 2. *There exists some $R_0 > 0$ such that for every $R \geq R_0$, we have*

$$\begin{aligned} \min_{u \in H^1(B(0,R), S^2)} \left(\int_{B(0,R)} |\nabla u|^2 + \int_{B(0,R)} |u - f_R|^2 \right) \\ < \inf_{C^1(B(0,R), S^2)} \left(\int_{B(0,R)} |\nabla u|^2 + \int_{B(0,R)} |u - f_R|^2 \right). \end{aligned}$$

where $f_R(x) = f(x/R) \forall x \in B(0, R)$.

IV. Proof of Theorem 3

For the proof, we use essentially the same method as above with the following modification:

LEMMA 4. *Let u_λ be any minimizer of (7) and let a be in Ω , then we have*

$$r \rightarrow \frac{1}{r} \int_{B(a) \cap \Omega} |\nabla u_\lambda|^2 + \frac{32}{3} \pi \lambda r^2$$

is nondecreasing in r .

The proof of this Lemma uses the same ingredients as in Lemma 2, we only have to replace the comparison function in (13) by the following:

If $r < \text{dist}(a, \partial\Omega)$ we set

$$\begin{cases} w_\lambda = u_\lambda \left(r \frac{x - a}{|x - a|} + a \right) & \text{on } B(a, r_0) \\ = u_\lambda(x) & \text{on } \Omega \setminus B(a, r_0). \end{cases}$$

If $r \geq \text{dist}(a, \partial\Omega)$ we take

$$\begin{cases} w_\lambda = u_\lambda \left(r \frac{x-a}{|x-a|} + a \right) & \text{if } x \in B(a, r_0) \text{ and } r \frac{x-a}{|x-a|} + a \in \Omega \\ = u_\lambda(x) & \text{on } \Omega \setminus B(a, r_0) \\ = P & \text{otherwise} \end{cases}$$

where P is the trace of u_λ on $\partial\Omega$.

Returning to the proof of Theorem 3, let $a \in \Omega$, $0 < r < \lambda$; by Lemma 4 we have

$$\begin{aligned} \frac{1}{r} \int_{B(a) \cap \Omega} |\nabla u_\lambda|^2 &\leq \frac{1}{\lambda} \int_{B(a) \cap \Omega} |\nabla u_\lambda|^2 + \frac{32}{3} \pi \lambda (\lambda^2 - r^2) \\ &\leq \frac{1}{\lambda} \int_{B(a) \cap \Omega} |\nabla u_\lambda|^2 + \frac{32}{3} \pi \lambda^3. \end{aligned}$$

Using (17), we find that if $\lambda < 2^{-\frac{2}{3}}$

$$\lim_{r \rightarrow 0} \frac{1}{r} \int_{B(a) \cap \Omega} |\nabla u_\lambda|^2 \leq \frac{16\pi}{3} + \frac{32\pi}{3} \lambda^3 < 8\pi.$$

We deduce again from Theorem 4, that a is a regular point of u_λ . This completes the proof of Theorem 3.

Appendix: Proof of Proposition 1

For simplicity, we may assume that $\Omega = B_{p+1}$; $p > 0$. Let (u_j) be a sequence of minimizers on $H^1(\Omega, S^2)$, so that $u_j \in H^1(B^3, S^2)$ and $u_j|_{B^3}$ are w -minimizers. Let $w \in H^1(B^3, S^2)$ with $w = u$ on ∂B^3 . Given $\delta > 0$, choose $\eta \in H^{1,\infty}(B^3)$ such that

$$\begin{cases} \eta = 1 & \text{on } B_{1-\delta}^3 \\ \eta = 0 & \text{on } \partial B^3 \\ |\nabla \eta| \leq \frac{1}{\delta} & \text{on } B^3 \setminus B_{1-\delta}^3. \end{cases}$$

As in [8] let us set

$$v_j = (1 - \eta)u_j + \eta w \quad \text{on } B^3.$$

We have

$$\begin{aligned} |v_j| &= |w| = 1 && \text{on } \partial B_{1-\delta}^3 \\ |v_j| &= |u_j| = 1 && \text{on } \partial B^3. \end{aligned}$$

Applying the extension Lemma A.1. in [3] we may find a function $w_j \in H^1(B^3 \setminus B_{1-\delta}^3, S^2)$ such that

$$\int_{B^3 \setminus B_{1-\delta}^3} |\nabla w_j|^2 \leq C \int_{B^3 \setminus B_{1-\delta}^3} |\nabla v_j|^2$$

$$w_j = v_j = w \quad \text{on } \partial B_{1-\delta}^3$$

and

$$w_j = v_j = u_j \quad \text{on } B^3.$$

Extending w_j to all the ball B^3 by letting $w_j = w$ on $B_{1-\delta}$. We have then,

$$\int_{B^3} |\nabla u_j|^2 + \lambda_j \int_{B^3} |u_j - f_j|^2 \leq \int_{B^3} |\nabla w_j|^2 + \lambda_j \int_{B^3} |w_j - f_j|^2.$$

Using the assumption on (u_j) , we see that given $\varepsilon > 0$, for j large enough

$$\int_{B^3} |\nabla u|^2 + \lambda \int_{B^3} |u - f|^2 - \frac{\varepsilon}{2} \leq \int_{B^3} |\nabla w_j|^2 + \lambda_j \int_{B^3} |w_j - f_j|^2$$

$$\leq \int_{B^3} |\nabla w|^2 + \lambda_j |w - f_j|^2$$

$$+ 4\lambda_j |B^3 \setminus B_{1-\delta}^3| + C \int_{B^3 \setminus B_{1-\delta}^3} |\nabla v_j|^2 \tag{21}$$

It suffices now to estimate the last term in the right-hand side of (21). Using Poincaré's inequality and Theorem 2, we see that

$$\limsup_{j \rightarrow \infty} \int_{B^3 \setminus B_{1-\delta}^3} |\nabla v_j|^2 \leq C |B^3 \setminus B_{1-\delta}^3|^{1-2/q} + 2 \int_{B^3 \setminus B_{1-\delta}^3} |\nabla w|^2$$

$$+ C \int_{B^3 \setminus B_{1-\delta}^3} |\nabla(w - u)|^2.$$

Choosing δ small enough to make the last two terms of the right-hand side less than $\varepsilon/2$, we obtain the conclusion of Proposition 1.

Acknowledgement

The authors would like to thank Professor F. Bethuel and Professor H. Brezis for their very valuable advice, many helpful conversations and hearty encouragement.

References

1. Brezis, H.: S^k -valued maps with singularities, in *Topics in the Calculus of Variations* (M. Giaquinta, ed.), Lecture Notes, Vol. 1365, Springer (1989), 1–30.
2. Bethuel, F. and Brezis, H.: Minimisation de $\int |\nabla(u - x/|x|)|^2$ et divers phénomènes de gap, *C.R. Acad. Sci. Paris* **310** (1990), 859–864.
3. Bethuel, F. and Brezis, H.: Regularity of minimizers of relaxed problems for harmonic maps, *Journal of Functional Analysis* **101** (1991), 145–161.
4. Brezis, H., Coron, J. M. and Lieb, E.: Harmonic maps with defects, *Comm. Math. Physics* **107** (1986), 649–705.
5. Giaquinta, M.: Multiple integrals in the calculus of variations and nonlinear elliptic systems, *Annals Math. Study* **105**, Princeton (1983).
6. Guo, B. and Hong, M. C.: The Landau–Lifshitz equation of the ferromagnetic spin chain and harmonic maps, to appear.
7. Hong, M. C.: The Landau–Lifshitz equation with the external field – A new extension for harmonic maps with values in S^2 , to appear.
8. Hardt, R., Kinderlehrer, D. and Lin, F. H.: Stable defects of minimizers of constrained variational principles, *Ann. I.H.P. Analyse Nonlinéaire* **5** (1988), 297–322.
9. Kaniel, S. and Shafrir, I.: A new symmetrization method for vector valued maps (in preparation).
10. Schoen, R. and Uhlenbeck, K.: A regularity theory for harmonic maps, *J. Diff. Geom.* **17** (1982), 307–335.
11. Schoen, R. and Uhlenbeck, K.: Boundary regularity and the Dirichlet problem of harmonic maps, *J. Diff. Geom.* **18** (1983), 253–268.