ON A CLASS OF GINZBURG - LANDAU EQUATIONS WITH WEIGHT.

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1. Introduction.

Let G be the unit disc in R^2 and let p be a smooth map from $\overline{G}$ into R such that $\alpha \leq p(x) \leq \beta$ for all $x \in \overline{G}$, $\alpha > 0$, $\beta > 0$. Fix a boundary condition g from ∂G into S^1 which is smooth. Set $d = deg(g, \partial G)$ and suppose that $d \geq 0$. Consider the Ginzburg-Landau type functional

$$E_\varepsilon(u) = \frac{1}{2}\int_G p|\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_G p(1-|u|^2)^2$$

which is defined for maps u in the class

$$H^1_g(G, C) = \{u \in H^1(G, C) \quad ; \quad u = g \quad \text{on} \quad \partial G\}.$$

It is easy to see that the minimization problem

$$\min_{u \in H^1_g} E_\varepsilon(u_\varepsilon)$$

is achieved by some u_ε that is smooth and satisfies the Euler equation

$$(1.1) \qquad \begin{cases} -div(p\nabla u_\varepsilon) = \dfrac{p}{\varepsilon^2}u_\varepsilon(1-|u_\varepsilon|^2) & \text{on} \quad G \\ u_\varepsilon = g \quad \text{on} \quad \partial G. \end{cases}$$

In the case where $p = 1$, this problem was studied by F. Bethuel, H. Brezis and F. Helein in $[BBH]_1$ and $[BBH]_2$. Our work is motivated by problem 4 in $[BBH]_2$: study the asymptotic behavior for minimizing solutions of (1.1) in the case where $p(x_1, x_2) = x_1$ and $G = \{(x_1, x_2) \in R^2 \quad ; \quad (x_1-1)^2 + x_2^2 < R^2\}$, $R < 1$. This arises when dealing with the cross section of a 3-dimensional solid torus having axial symmetry.

In the case where $d = 0$ we have exactly the same result as in $[BBH]_1$. Suppose that $d > 0$. Set

$$p_0 = \min_{x \in \overline{G}} p(x).$$

Define

$$\Lambda_1 = \{x \in \overline{G} \quad ; \quad p(x) = p_0\}$$

and

$$\Lambda_2 = \{x \in G \quad ; \quad p(x) = p_0\}.$$

Our results concern two cases

$$\text{card}\Lambda_2 \geq d \tag{1.2}$$

$$\Lambda_1 = \{a\}. \tag{1.3}$$

In this paper we study the problem of the convergence of (u_ε) to some limit u_0 and the singularities of u_0. Our main results concerning the convergence of (u_ε) are the following

THEOREM 1.1. Assume that $\text{card}\Lambda_2 \geq d$. Then there exist a subsequence ε_n tending to 0 and exactly d points $a_1, ..., a_d \in \Lambda_2$ such that (u_{ε_n}) converges to a map $u_0 \in C^\infty(\overline{G}\backslash\{a_i\}, S^1)$ in $C^{1,\alpha}(\overline{G}\backslash\{a_i\})$ for all $\alpha < 1$. The map u_0 is defined by

$$u_0(z) = \frac{z - a_1}{|z - a_1|} ... \frac{z - a_d}{|z - a_d|} e^{i\phi}$$

and

$$\begin{cases} -div(p\nabla\phi) = \nabla p.(\nabla\theta_1 + ... + \nabla\theta_d) \quad \text{in} \quad G \\ \phi = \phi_0 \quad \text{on} \quad \partial G \end{cases} \tag{1.4}$$

where

$$e^{i\theta_j} = \frac{z - a_j}{|z - a_j|} \quad \text{for} \quad j = 1, ..., d,$$

$$\nabla\theta_k = (-\frac{y - \beta_k}{|z - a_k|}, \frac{x - \alpha_k}{|z - a_k|}) \quad \text{where} \quad a_k = \alpha_k + i\beta_k \quad k = 1, ..., d$$

and

$$e^{i\phi_0} = \frac{|z - a_1|}{z - a_1} ... \frac{|z - a_d|}{z - a_d} g(z) \qquad \text{on} \quad \partial G. \tag{1.5}$$

THEOREM 1.2. Suppose that $\Lambda_1 = \{a\}$. Then we have

(i) *If $a \in \partial G$, the whole sequence (u_ε) converges in $C^{1,\alpha}(\overline{G}\backslash\{a\})$ for all $\alpha < 1$ to the map*

$$u_0(z) = (\frac{z - a}{|z - a|})^{2d} e^{i\phi}$$

where ϕ is defined by

$$\begin{cases} -div(p\nabla\phi) = 2d\nabla p.\nabla\theta \quad in \quad G \\ \phi = \phi_0 \quad on \quad \partial G \end{cases} \tag{1.7}$$

and ϕ_0 is such that for all z in ∂G

$$e^{i\phi_0} = \frac{g(z)}{(-az)^d}. \tag{1.8}$$

(ii) *If $a \in G$ we have the same conclusion as in (i) except that to replace $2d$ by d in (1.6), (1.7) and (1.8).*

REMARK 1.1. The definition of ϕ_0 in (1.8) is justified by the fact that, if $a \in \partial G$ we have

$$(\frac{z-a}{|z-a|})^{2d} = (-az)^d \quad \text{for all } z \text{ in } \partial G.$$

By Theorem 1.1, we know that the singularities are in Λ_2. Our next result tells us where to find them in Λ_2. For this purpose, let introduce, as in $[BBH]_2$ the renormalized energy $W(\overline{a}, \overline{d}, g, p)$ associated to a given configuration $\overline{a} = (\overline{a}_1, ..., \overline{a}_k)$ of distinct points in G with associated degrees $\overline{d} = (d_1, ..., d_k) \in Z^k$ such that $\sum_{i=1}^k d_i = d$. We define

$$W(\overline{a}, \overline{d}, g, p) = -\pi \sum_{j \neq i} p(\overline{a}_i) d_i d_j \log|\overline{a}_i - \overline{a}_j| + \frac{1}{2} \int_{\partial G} \Phi_0 (g \times g_\tau) - \pi \sum_{j=1}^k d_j R_0(\overline{a}_j)$$

where Φ_0 is the unique solution of

$$(1.9) \qquad \begin{cases} div(\frac{1}{p} \nabla \Phi_0) = 2\pi \sum_{i=1}^k d_i \delta_{\overline{a}_i} & \text{on} \quad G \\ \frac{1}{p} \frac{\partial \Phi_0}{\partial \nu} = g \times g_\tau & \text{on} \quad \partial G \\ \int_{\partial G} \Phi_0 = 0 \end{cases}$$

and

$$(1.10) \qquad R_0 = \Phi_0 - \sum_{i=1}^k d_i p(\overline{a}_i) \log|x - \overline{a}_i|.$$

The localization of the points (a_i) in Theorem 1.1 is governed by

$$W = W((\overline{a}_1, ..., \overline{a}_d), (1, ..., 1), g, p)$$

through the following

THEOREM 1.3. Let $(a_1, ..., a_d)$ be as in Theorem 1.1. Then $(a_1, ..., a_d)$ minimizes W on Λ_2^d.

These results were announced in [BH]. The same problem, under more general assumptions, has been recently considered by N. André and I. Shafrir, see their announcement [AH].

2. The convergence of (u_{ε_n}).

2.1. Proof of Theorem 1.1.

Using the same proof as in $[ST]$ we obtain

$$\frac{1}{\varepsilon^2}\int_G (1-|u_\varepsilon|^2)^2 \le C = C(p,g). \tag{2.1}$$

As in $[BBH]_2$, we have the existence of $\lambda > 0$ and a collection of balls $B(x_j^\varepsilon, \lambda\varepsilon)$, $j = 1,...,J$ with $J \le J_0$ independently of ε such that

$$\{x \in \overline{G} \quad ; \quad |u_\varepsilon(x)| \le \frac{1}{2}\} \quad \subset \quad \cup_{j=1}^J B(x_j^\varepsilon, \lambda\varepsilon).$$

Given any subsequence ε_n tending to 0 we may assume that $x_j^{\varepsilon_n}$ tend to $b_j \in \overline{G}$ for $j = 1,...,N$, $b_i \ne b_j$ for $i \ne j$.

The convergence of a subsequence (u_{ε_n}) is an adaptation of results of $[BBH]_2$. The next two lemmas are still valid when $\text{card}\Lambda_2 < d$.

LEMMA 2.1. For every $\sigma > 0$ small enough there exists a constant $C = C(\sigma, p, g)$ such that

$$E_\varepsilon(u_\varepsilon) \le \pi d(p_0 + \sigma)\log\frac{1}{\varepsilon} + C. \tag{2.2}$$

Proof of Lemma 2.1. Let $\sigma > 0$ be small enough. There exist $\rho > 0$ and d distinct points $x_1,...,x_d$ in G such that $p(x) \le p_0 + \sigma = p_0'$ for all $x \in B(x_i,\rho) = B_i$ and $B_i \cap B_j = \emptyset$ for $i \ne j$. Consider

$$I(x_i,\varepsilon,\rho,p) = \min_{u\in E_i}\{\frac{1}{2}\int_{B_i} p|\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_{B_i} p(1-|u|^2)^2\} \tag{2.3}$$

where

$$E_i = \{u \in H^1(B_i) \quad ; \quad u = \frac{x-x_i}{|x-x_i|} \quad \text{on} \quad \partial B_i\}.$$

We have as in $[BBH]_2$

$$I(x_i,\varepsilon,\rho,p) \le \pi p(x_i)\log\frac{\rho}{\varepsilon} + C.$$

Set

$$\begin{aligned} u(x) &= \frac{x-x_i}{|x-x_i|} \quad \text{in} \quad \partial B_i \quad i = 1,...,d \\ u(x) &= v_{i\varepsilon} \quad \text{on} \quad B_i \\ |u| &= 1 \quad \text{on} \quad G\setminus \cup_{i=1}^d B_i \end{aligned}$$

where $v_{i\varepsilon}$ is a minimizer for the problem (2.3). We have

$$\begin{aligned} E_\varepsilon(u_\varepsilon) \le E_\varepsilon(u) &= \frac{1}{2}\int_{G\setminus\cup_{i=1}^d B_i} p|\nabla u|^2 + \sum_{i=1}^d I(x_i,\varepsilon,\rho,p) \\ &\le \pi d p_0'\log\frac{1}{\varepsilon} + C. \end{aligned}$$

REMARK 2.2. In the case where card$\Lambda_2 \geq d$ we can take $x_i \in \Lambda_2$ and we obtain

$$E_\varepsilon(u_\varepsilon) \leq \pi d p_0 \log\frac{1}{\varepsilon} + C. \tag{2.4}$$

LEMMA 2.2. Suppose that $deg(u_{\varepsilon_n}, \partial B(b_j,\rho)) = d_j \neq 0$. Then we have $b_j \in \Lambda_1$.

Proof of Lemma 2.2. Arguing by contradiction, suppose that there exist $b_j \notin \Lambda_1$ and $p_0'' > p_0$ such that

$$p(x) \geq p_0'' \quad \text{for all } x \in B(b_j,\rho);$$

we choose $\rho > 0$ such that $B(b_i,\rho) \cap B(b_j,\rho) = \emptyset$ for $i \neq j$. Since we have (2.1), using a result of [HS] we obtain

$$\frac{1}{2}\int_{B(b_j,\rho)} p|\nabla u_\varepsilon|^2 \geq \pi|d_j|p_0''\log\frac{\rho}{\varepsilon} - C \tag{2.5}$$

where C depends only on g. On the other hand, we have for $k \neq j$, $k = 1,...,N$

$$\frac{1}{2}\int_{B(b_k,\rho)} p|\nabla u_\varepsilon|^2 \geq \pi|d_k|p_0\log\frac{\rho}{\varepsilon} - C. \tag{2.6}$$

Combining (2.2), (2.5) and (2.6) we obtain a contradiction for σ small enough and this completes the proof of Lemma 2.2.

Using Lemma 2.1 and (2.6) for $k = 1,...,N$ we deduce that

$$\sum_{j=1}^{N} |d_j| \leq \frac{p_0+\sigma}{p_0} d.$$

Then we have $d_j \geq 0$ for all j. On the other hand we can prove as in $[BBH]_2$, ch.6 that $d_j = 1$ and that we have the convergence of (u_{ε_n}) to u_* in $C^{1,\infty}(\overline{G}\backslash\{a_1,...,a_d\}$ for all $\alpha < 1$ where $a_i \in \Lambda_2$. The limit u_* satisfies

$$\begin{cases} -div(p\nabla u_*) = pu_*|\nabla u_*|^2 & \text{in} \quad G\backslash\{a_1,...,a_d\} \\ \qquad u_* = g & \text{on} \quad \partial G. \end{cases} \tag{2.7}$$

We have

$$deg(u_*, a_j) = 1 \quad j = 1,...,d \tag{2.8}$$

and, as in $[BBH]_2$, Appendix 4

$$div(pu_* \times \nabla u_*) = 0 \quad \text{in} \quad D'(G) \tag{2.9}$$

and

$$u_* \in W^{1,p}(G) \quad \text{for all } p < 2. \tag{2.10}$$

Since u_* satisfies (2.6) and (2.7) we can write

$$u_*(z) = \frac{z-a_1}{|z-a_1|}...\frac{z-a_d}{|z-a_d|}e^{i\phi} \tag{2.11}$$

where ϕ verifies

$$\begin{cases} -div(p\nabla\phi) = \nabla p.(\nabla\theta_1 + ... + \nabla\theta_d) & \text{in} \quad G\backslash\{a_1, ..., a_d\} \\ \phi = \phi_0 \quad \text{on} \quad \partial G. \end{cases} \tag{2.12}$$

It follows from (2.10) and (2.11) that

$$div(pu_* \times \nabla u_*) = div(p\nabla\phi) + \nabla p.(\nabla\theta_1 + ... + \nabla\theta_d) \quad \text{in} \quad D'(G). \tag{2.13}$$

¿From (2.9) and (2.13) together with (2.12) we obtain (1.4) and (1.5). This concludes the proof of Theorem 1.1.

2.2. Proof of Theorem 1.2.

We give the proof of part (i) , we can use the same method as in (i) to obtain (ii).

The proof of Theorem 1.2. is based on the following upper bound for the energy of (u_ε) away from the absolute minimum of p. Let (u_ε) be any minimizer for (1.1) and let us denote by $a = (a, 0)$ the absolute minimum of p. Suppose that there is no point b_j having degree 0. We will see later that this is hold (see Remark 2.2). THEOREM 2.1. There exists a constant $C > 0$ depending only on R and g such that for any $\rho > 0$ we have

$$\frac{1}{2}\int_{G\backslash B(a,\rho)} p|\nabla u_\varepsilon|^2 \leq 2\pi d^2 p(a)\log\frac{1}{\rho} + C. \tag{2.14}$$

The proof of Theorem 2.1 relies on precise lower and upper bounds for $E_\varepsilon(u_\varepsilon)$. We start by the following proposition, where $x(\varepsilon) << y(\varepsilon)$ means that $\lim_{\varepsilon\to 0}\frac{x(\varepsilon)}{y(\varepsilon)} = 0$.

PROPOSITION 2.1. Let $\sigma_1(\varepsilon) << ... << \sigma_{k+1}(\varepsilon)$ be $k+1$ positive numbers such that $\sigma_i(\varepsilon)$ tend to 0 as ε tends to 0 for $i = 1, ..., k$ and such that $\sigma_{k+1} = \frac{1}{2}$. Let $d_1, ..., d_k$ be k integers satisfying $\sum_{i=1}^{i=k} d_i = d$. Then there exists a constant $C = C(p, g)$ such that

$$\frac{1}{2}\int_G p|\nabla u_\varepsilon|^2 \leq \pi\sum_{i=1}^{i=k}(p(a) + \sigma_i(\varepsilon))|d_i|\log\frac{\sigma_i(\varepsilon)}{\varepsilon}$$
$$+ 2p(a)\sum_{i=1}^{i=k}(\sum_{j=1}^{j=i} d_j)^2\log\frac{\sigma_{i+1}(\varepsilon)}{\sigma_i(\varepsilon)}) + C$$

Proof of Proposition 2.1. We need some preliminary results. First we have LEMMA 2.3. Let $a = (a_1, a_2) \in G$ and $\rho > 0$ such that $\overline{B}(a, \rho) \subset G$. Let v_ε be any minimizer for

$$I(a, \varepsilon, \rho, p) = \min_{u\in H^1_{u_0}(B(a,\rho))}\{\frac{1}{2}\int_{B(a,\rho)} p|\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_{B(a,\rho)} p(1-|u|^2)^2\}$$

where $u_0 = C_0 \frac{x-a}{|x-a|}$ and $|C_0| = 1$. There exists a constant $C > 0$ independing on ρ such that

$$\frac{1}{2}\int_{B(a,\rho)} p|\nabla v_\varepsilon|^2 \le \pi p(a)\log\frac{\rho}{\varepsilon} + C. \tag{2.15}$$

Proof of Lemma 2.3. We use the same idea as in $[BBH]_2$, Theorem 3.1. Let $0 < \varepsilon < \rho$ be given. Let u_0 be a minimizer for $I(a, 1, \varepsilon, p)$. Set

$$u(x) = u_0(x) \quad \text{if} \quad x \in B(a,\varepsilon)$$
$$u(x) = C_0\frac{x-a}{|x-a|} \quad \text{if} \quad x \in B(a,\rho)\backslash B(a,\varepsilon).$$

We have

$$\begin{aligned} I(a,\varepsilon,\rho,p) &\le \frac{1}{2}\int_{B(a,\rho)} p|\nabla u_0|^2 + \frac{1}{4\varepsilon^2}\int_{B(a,\varepsilon)} p(1-|u_0|^2)^2 \\ &+ \frac{1}{2}\int_{B(a,\rho)\backslash B(a,\varepsilon)} \frac{p(x)}{|x-a|^2} \\ &\le I(a,1,\varepsilon,p) + \pi p(a)\log\frac{\rho}{\varepsilon} + (\frac{1}{4\varepsilon^2} - \frac{1}{4})\int_{B(a,\varepsilon)} p(1-|u_0|^2)^2 \\ &\le \pi p(a)\log\frac{\rho}{\varepsilon} + C. \end{aligned}$$

LEMMA 2.4. Let $\sigma > 0$. We have

$$\frac{1}{2}\int_G p|\nabla u_\varepsilon|^2 \le \pi d(p(a)+\sigma)\log\frac{\sigma}{\varepsilon} + 2\pi d^2 p(a)\log\frac{1}{\sigma} + C \tag{2.16}$$

where C depends only on g.

Proof of Lemma 2.4. We can suppose that a is on the horizontal axis. Let $a = (a, 0)$ and let $a + \sigma = (a + \sigma, 0)$. Let $x_1, ..., x_d$ be d distinct points in

$$B(a+\sigma,\sigma) \cap \{(x_1,x_2) \in R^2 \quad ; \quad x_2 = 0\}.$$

Fix $\rho > 0$ such that $B(x_i, \rho) \cap B(x_j, \rho) = \emptyset$, for all $i \neq j$. Consider the function u from $B(a + \sigma, \sigma)$ into R defined by

$$u(x) = C_1(\frac{x-(a+\sigma)}{|x-(a+\sigma)|})^d$$

on $\partial B(a+\sigma, \sigma)$ where C_1 is a constant such that $C_1(\frac{x-(a+\sigma)}{|x-(a+\sigma)|})^d = g(a)$, and u is equal to a minimizer for $I(x_i, \varepsilon, \rho, p)$ on $\partial B(x_i, \rho)$, $i = 1, ..., d$. Then there exist a number $0 \le \sigma' \le \sigma$, and a constant C depending only on R and g such that

$$\int_{B(a+\sigma,\sigma)} p|\nabla u|^2 \le 2\pi d p(a+\sigma')\log\frac{\sigma}{\varepsilon} + C. \tag{2.17}$$

Indeed, letting $\rho = \frac{\sigma}{m}$, where m is a given integer and using the following change of variables $y = \frac{x-(a+\sigma)}{\sigma}$ we see that

$$\int_{B(a+\sigma,\sigma)\setminus\cup_{i=1}^{d}B(x_i,\rho)} p|\nabla u|^2 \quad \text{is independent of } \sigma. \tag{2.18}$$

By Lemma 2.3, we obtain

$$\frac{1}{2}\int_{\cup_{i=1}^{d}B(x_i,\rho)} p|\nabla u|^2 \leq \pi\sum_{i=1}^{d} p(x_i)\log\frac{\rho}{\varepsilon} + C. \tag{2.19}$$

Combining (2.18) and (2.19) we obtain (2.17). Then there exists $C_1 > 0$ such that

$$p(a+\sigma') \leq p(a) + C_1\sigma. \tag{2.20}$$

¿From (2.17) and (2.20) we obtain

$$\frac{1}{2}\int_{B(a+\sigma,\sigma)} \leq \pi d(p(a)+\sigma)\log\frac{\sigma}{\varepsilon} + C. \tag{2.21}$$

Next we extend u on $\overline{G}$ as in $[BBH]_2$, namely we define explicitly u on $\partial B(z, z-a)$, letting $z = (z, 0)$, where $a+\sigma \leq z \leq a+r$ and $r \in]2\sigma, R[$ by

$$u(x) = C_2\left(\frac{x-z}{|x-z|}\right)^2 \quad \text{for} \quad x \in \partial B(z, z-a)$$

where C_2 is a constant such that $u(a) = g(a)$.

Using the polar coordinates centered on a, we can write

$$u(x) = C_2 e^{2id\theta}.$$

Since $x_1 = a_1 + |x-a|cos\theta$ and $|\nabla u|^2 = \frac{4d^2}{r^2}$ we have

$$\int_{B((a+r,0),r)\setminus B(a+\sigma,\sigma)} p|\nabla u|^2 \leq 2\pi d^2 p(a)\log\frac{r}{\sigma} + C. \tag{2.22}$$

Combining (2.21) and (2.22) we get the conclusion of Lemma 2.4.

Proof of Proposition 2.1. Set $A = (0,0)$ and $a = (a,0)$. Fix $2k$ points $x_i : (a+\sigma_i, 0)$ and $z_i = (a + \frac{\sigma_{i+1}}{2}, 0)$ in $[b, A]$ for $i = 1, ..., k$. Iterating the construction of Lemma 2.4, we can define a map u as the following : in each disc $B(x_i, \sigma_i)$ we construct d_i disjoint discs centered on the set $\{(x_1, x_2) \quad \text{such that} \quad x_2 = 0\}$. Let denote by $\overline{B}_1$ the disc $B(a + \frac{\sigma_2}{2}, \frac{\sigma_2}{2})$. By Lemma 2.5 we can deduce that

$$\frac{1}{2}\int_{\overline{B}_1} p|\nabla u|^2 \leq (p(a)+\sigma_1(\varepsilon))|d_1|Log\frac{\sigma_1}{\varepsilon} + 2\pi d_1^2 p(a)\log\frac{\sigma_2(\varepsilon)}{\sigma_1(\varepsilon)} + C$$

where u is given as in the proof of Lemma 2.4 . We continue the construction by setting

$$u(x)=C(\frac{x-(a+\sigma_2\frac{A-a}{R})}{|x-(a+\sigma_2\frac{A-a}{R})|})^{d_1+d_2}$$

on $\partial B(a+\frac{\sigma_2}{2},\frac{\sigma_2}{2})$. The constant C is such that u coincides with the function defined on $\overline{B}_1$ and such that u coincides with each minimizer on each disc centered on $\{x_1=a+\sigma_2\}$. Let denote by $\overline{B}_2$ the disc $\overline{B}_2(a+\frac{\sigma_3}{2},\frac{\sigma_3}{2})$. By Lemma 2.4 we have

$$\begin{aligned}\frac{1}{2}\int_{\overline{B}_2}p|\nabla u|^2&\leq\frac{1}{2}\int_{\overline{B}_1}p|\nabla u|^2+\frac{1}{2}\int_{\overline{B}_2\backslash\overline{B}_1}p|\nabla u|^2\\&\leq(p(a)+\sigma_1(\varepsilon))|d_1|\log\frac{\sigma_1(\varepsilon)}{\varepsilon}+(p(a)+\sigma_2(\varepsilon))|d_2|\log\frac{\sigma_2(\varepsilon)}{\varepsilon}\\&+\pi(d_1+d_2)^2p(a)\log\frac{\sigma_3(\varepsilon)}{\sigma_2(\varepsilon)}+C.\end{aligned}$$

We shall use the same construction k times. In the last step u is defined on $\overline{B}_{k+1}(a+\frac{\sigma_{k+1}}{2},\frac{\sigma_{k+1}}{2})$ with $\sigma_{k+1}=\frac{1}{2}$. This concludes the proof of Proposition 2.1.

We have the following estimate PROPOSITION 2.2. Let $\rho>0$. There exist a subsequence ε_n tending to 0, k numbers $\sigma_i'(\varepsilon_n)$ tending to 0, $0<\sigma_1'(\varepsilon_n)<<...<<\sigma_k'(\varepsilon_n)$ such that $\sigma_{k+1}=\frac{1}{2}$ and k integers $d_1,...,d_k$ such that $\sum_{i=1}^{i=k}d_i=d$ such that

$$\begin{aligned}\frac{1}{2}\int_{B(a,\rho)}p|\nabla u_\varepsilon|^2&\geq\pi\sum_{i=1}^{i=k}|d_i|(p(a)+\sigma_i'(\varepsilon))\log\frac{\sigma_i'(\varepsilon)}{\varepsilon}\\&-2\pi p(a)\sum_{i=1}^{i=k}(\sum_{j=1}^{j=i}d_j)^2\log\frac{\sigma_{i+1}'(\varepsilon)}{\sigma_i'(\varepsilon)}-C\end{aligned}$$

where C depends only on g and G.

The proof of Proposition 2.2 requies the following variant of two results of $[BMR]$ and $[BBH]_2$.

LEMMA 2.5. Let $\varepsilon<R_0<R_1$ and $x_0\in G$. Suppose $u\in H_g^1$ satisfies $|u|\leq 1$ in G, $|u|\geq\frac{1}{2}$ in $A_{R_1,R_0}=(G\cap B(x_0,R_1))\backslash B(x_0,R_0)$, and

$$\frac{1}{\varepsilon^2}\int_{A_{R_1,R_0}}(1-|u|^2)^2\leq K$$

then

$$\frac{1}{2}\int_{A_{R_1,R_0}}p|\nabla u|^2\geq\pi d^2p(a)\log\frac{R_1}{R_0}-C$$

where

$$d=deg(u,G\cap\partial B(x_0,R_1))\quad\text{and}\quad C=C(G,g).$$

Proof of Lemma 2.5. We extend u to the full annulus

$$\overline{A}_{R_1,R_0} = B(x_0,R_1)\backslash B(x_0,R_0)$$

by letting

$$\begin{aligned} \overline{u}(x) &= u(x) \quad x \in A_{R_1,R_0} \\ \overline{u}(x) &= \overline{g}(x) \quad x \in \overline{A}_{R_1,R_0}\backslash A_{R_1,R_0} \subset U_0 \end{aligned}$$

where U_0 is a tubular neighborhood of ∂G and $\overline{g}$ extend g to a smooth function on U_0. We have

$$\int_{A_{R_1,R_0}} p|\nabla u|^2 \geq \int_{\overline{A}_{R_1,R_0}} p|\nabla \overline{u}|^2 - C(G,g)R_1.$$

We can write $x = re^{i\theta}$ and $\overline{u}(x) = \rho(x)e^{i\phi(x)}$ where $\rho = |\overline{u}|$ and $\phi = d\theta + \Psi$. Note that

$$\nabla\phi = \frac{d}{r}(ie^{i\theta}) + \nabla\Psi$$

thus

$$|\nabla\phi|^2 = \frac{d^2}{r^2} + \frac{2d}{r^2}\frac{\partial\Psi}{\partial\theta} + (\nabla\Psi)^2.$$

Hence

$$\begin{aligned} \int_{\overline{A}_{R_1,R_0}} p|\nabla u|^2 &= \int_{\overline{A}_{R_1,R_0}} p\rho^2|\nabla\phi|^2 + \int_{\overline{A}_{R_1,R_0}} p|\frac{\partial\overline{u}}{\partial\nu}|^2 \\ &\geq \int_{\overline{A}_{R_1,R_0}} p\rho^2|\nabla\phi|^2 \\ &= \int_{\overline{A}_{R_1,R_0}} p\frac{d^2}{r^2} - \int_{\overline{A}_{R_1,R_0}} p(1-\rho^2)\frac{d^2}{r^2} \\ &+ 2d\int_{\overline{A}_{R_1,R_0}} p\frac{\rho^2}{r^2}\frac{\partial\Psi_\varepsilon}{\partial\theta} + \int_{\overline{A}_{R_1,R_0}} p\rho^2|\nabla\Psi|^2 \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Treating each term separately we obtain

$$I_1 = \int_0^{2\pi}\int_0^{R_1} p(a + rcos\theta)\frac{d^2}{r^2}drd\theta = 2\pi p(a)d^2\log\frac{R_1}{R_0}$$

and

$$I_2 = d^2\int_{\overline{A}} p(a + rcos\theta)\frac{1-\rho^2}{r^2}.$$

By Cauchy-Schwarz we have

$$|I_2| \leq d^2(\pi K)^{\frac{1}{2}}a + R_2 d^2(\pi K)^{\frac{1}{2}}.$$

Since

$$\frac{1}{r}\frac{\partial\Psi}{\partial\theta}.(ie^{i\theta}) = \nabla\Psi.(iei\theta)$$

we have

$$I_3 = 2d\int_A cos\theta\nabla\Psi.(ie^{i\theta}) + 2d\int_A p\frac{\rho^2-1}{r^2}\nabla\Psi.(ie^{i\theta}).$$

Thus

$$|I_3| \leq 2d(\pi(R_1^2 - R_0^2)||\nabla\Psi||_{L^2(\overline{A})} + \frac{p(a)}{\varepsilon}(\int_{\overline{A}}(1-\rho^2)^2)^{\frac{1}{2}}||\nabla\Psi||_{L^2(\overline{A})})$$

and this implies

$$|I_3| \leq \frac{h^2}{2}(2d\pi(R_1^2 - R_0^2)) - 2dp(a)K^{\frac{1}{2}}) + \frac{1}{2h^2}\int_{\overline{A}}|\nabla\Psi|$$

for $h \neq 0$. Finally

$$I_4 = \int_{\overline{A}} p\rho^2|\nabla\Psi|^2 \geq \frac{p(a)}{4}\int_{\overline{A}}|\nabla\Psi|^2.$$

We obtain the conclusion of Lemma by choosing h such that

$$-\frac{p(a)}{4} + \frac{1}{2h^2} \leq 0.$$

We shall use the following which is due to $[BBH]_2$.

LEMMA 2.6. Let $\varepsilon > 0$, $R_1 > 0$, $\eta \in]0, R_1[$. Set $a = (a, 0)$ and let $u \in C^1_{loc}(B(a,R)\backslash\{a\}, R^2)$ be any function verifying

$$|u| \geq \frac{1}{2} \quad \text{in} \quad A_{\eta,R_1} = B(a,R_1)\backslash B(a,\eta)$$

$$u = \overline{g} \quad \text{in} \quad G'\backslash G \cap B(a,R_1)$$

$$deg(u, \partial B(a,R_1)) = d$$

$$\frac{1}{\varepsilon^2}\int_{\{x\in G;\eta\leq|x-a|\leq R_1\}}(1-|u|^2)^2 \leq K$$

for some constant K. Then we have

$$\frac{1}{2}\int_{A_{\eta,R_1}} p|\nabla u|^2 \geq 2\pi p(a)d^2\log\frac{R_1}{\eta} - C.$$

Proof of Proposition 2.2 completed. Using the same notation as in $[ST]$, there exist a subsequence ε_n and sequences $\sigma_i(\varepsilon_n)$ tending to 0, $i = 1, ..., k$ such that $\sigma_1(\varepsilon_n) << ... << \sigma_k(\varepsilon_n)$ and $\lambda > 0$ such that for all $j \in J$ there is some $i \in \{1, ..., k\}$

$$\sigma_i(\varepsilon_n) \leq p(x_j^{\varepsilon_n}) - p(a) \leq \lambda\sigma_i(\varepsilon_n).$$

Let denote by X_i the set $\{x \in G; \quad \sigma_i \leq p(x)-p(a) \leq \lambda\sigma_i\}$ and X_i' the set $\{x \in G; \quad \lambda\sigma_i \leq p(x) - p(a) \leq \sigma_{i+1}\}$

We can arrange the centers of the bad discs as in $[ST]$. We do this separatly in each X_i for $i = 1, ..., k$. We obtain

$$\frac{1}{2}\int_{X_i} p|\nabla u_\varepsilon|^2 \geq \frac{1}{2}\sum_{j\in J^l}\int_{A_{R^l,R_0^l}} p|\nabla u_\varepsilon|^2.$$

Moreover there is a constant M such that

$$R^L \geq \frac{\sigma}{M} \quad \text{and} \quad R_0^1 \leq M\varepsilon.$$

Hence we have

$$\frac{1}{2}\int_{X_i} p(x)|\nabla u_\varepsilon|^2 \geq \pi\sum_{l=1}^{L}\sum_{j\in J^l} p(x_j^\varepsilon)|d_{jR^l}|^2 \log\frac{R^{l+1}}{R_0^l}.$$

Then we have

$$\sum_{i=1}^{k}\frac{1}{2}\int_{X_i} p|\nabla u_\varepsilon|^2 \geq \sum_{i=1}^{k}\pi(p(a)+\sigma_i)|d_i|\log\frac{\sigma_i(\varepsilon)}{\varepsilon} - C \tag{2.23}$$

where

$$d_{jR} = deg(u_\varepsilon, \partial B(x_j, R)),$$

since

$$|d_i| = |\sum_{j\in J^l} d_{jR}| \leq \sum_{j\in J^l}|d_{jR}|^2.$$

Since p is smooth there exist σ_i' satisfying $\sigma_i' << \sigma_{i+1}'$, $\sigma_i' > \sigma_i$ and such that X_i' contains $B(a,\sigma_{i+1}')\backslash B(a,\sigma_i')$. By Lemma 2.6 we have

$$\frac{1}{2}\int_{X_i'} p|\nabla u_\varepsilon|^2 \geq 2\pi p(a)(\sum_{j=1}^{i} d_j)^2\log\frac{\sigma_{i+1}'(\varepsilon)}{\sigma_i'(\varepsilon)} - C. \tag{2.24}$$

Using the fact that the map $x \to (1+x)\log(1+x)$is nondecreasing we obtain

$$(p(a)+\sigma_i)\log\frac{\sigma_i}{\varepsilon} \geq (p(a)+\sigma_i')\log\frac{\sigma_i'}{\varepsilon} \tag{2.25}$$

Combining (2.23) ,(2.24)and (2.25) we obtain Proposition 2.2.

Remark 2.2. Suppose that there exist $j \in \{1, ..., N\}$ such that $deg(u_\varepsilon, b_j) = 0$. Using the same technique as in Proposition 2.1 and Proposition 2.2 we can show that $u_\varepsilon \in H^1_{loc}(G'\backslash\{a, b_j\})$. Arguing as in $[BBH]_2$ we see that this contradicts the fact that b_j is the limit of centers of bad discs.

Proof of Theorem 2.1. Combining Proposition 2.1 and Proposition 2.2 and choosing $\sigma_i = \sigma_i'$, we obtain Theorem 2.1.

Proof of Theorem 1.2 completed. We deduce from (2.14) that there exists a subsequence ε_n tending to 0 such that u_{ε_n} converges weakly in $H^1_{loc}(G'\backslash\{a\}, C)$. Clearly we have $u_* = g$ on ∂G and $|u_*| = 1$. Since $deg(u_*, a) = d$, we deduce that for $n \geq N$, $deg(u_{\varepsilon_n}, G\backslash B(a,\rho)) = 0$ and we can obtain the convergence of u_{ε_n} to u_* in $C^{1,\alpha}$ as in $[BBH]_2$. If $a \in G$, we conclude as in the proof of Theorem 1.1. If $a \in \partial G$, let ϕ_0 be such that $g(z) = (\frac{z-a}{|z-a|})^{2d} e^{i\phi_0}$. The limit u_* satisfies

$$\begin{cases} -div(p\nabla u_*) = pu_*|\nabla u_*|^2 \quad \text{in} \quad G \\ u_* = g \quad \text{on} \quad \partial G \end{cases} \tag{2.26}$$

and

$$deg(u_*, a) = d.$$

Hence there exists ϕ from G into R such that

$$u_* = (\frac{z-a}{|z-a|})^{2d} e^{i\phi}.$$

Since u_* verifies (2.26) we have

$$\begin{cases} -div(p\nabla\phi) = 2d\nabla p.\nabla\theta \quad \text{in} \quad G \\ \phi = \phi_0 \quad \text{on} \quad \partial G \end{cases}$$

which is the desired conclusion. The convergence of the full sequence u_ε follows from the uniqueness of the possible limit.

3. Renormalized energy.

Let $\overline{a} = (\overline{a}_1, ..., \overline{a}_k)$ be any configuration of distinct points in G with associated degrees $(d_1, ..., d_k)$ with $\sum_i d_i = d$.

We set

$$W(\overline{a}, \overline{d}, g, p) = -\pi \sum_{i\neq j} p(\overline{a}_i) d_i d_j \log|\overline{a}_i - \overline{a}_j| + \frac{1}{2}\int_{\partial G} \Phi_0 (g \times g_\tau) - \pi \sum_i d_i R_0(\overline{a}_i) \tag{3.1}$$

where Φ_0 is the solution of

$$\begin{cases} -div(\frac{1}{p}\nabla\Phi_0) = 2\pi \sum_j d_j \delta_{\overline{a}_j} \quad \text{in} \quad G \\ \frac{1}{p}\frac{\partial\Phi_0}{\partial\nu} = g \times g_\tau \quad \text{on} \quad \partial G \\ \int_{\partial G} \Phi_0 = 0 \end{cases} \tag{3.2}$$

and

$$R_0 = \Phi_0 - \sum_j d_j p(\overline{a}_j)\log|x - \overline{a}_j|. \tag{3.3}$$

We prove that the configuration $a = (a_1, ..., a_d)$ in Theorem1.1, with associated degrees $d_j = 1$, $j = 1, ..., d$, minimizes, on Λ_2 the renormalized energy

$$W(\overline{a}) = -\pi \sum_{i \neq j} p(\overline{a}_i)\log|\overline{a}_i - \overline{a}_j| + \frac{1}{2}\int_{\partial G} \Phi_0(g \times g_\tau) - \pi \sum_i R_0(\overline{a}_i). \tag{3.4}$$

We have the two following lemmas. First we have

LEMMA 3.1. Let $\overline{a} = (\overline{a}_j)$ be any configuration of d distinct points in G. There is some $\rho_0 > 0$, depending only on $\overline{a}$, such that for every $\rho < \rho_0$ and every $\varepsilon > 0$ we have

$$E_\varepsilon(u_\varepsilon) \leq \sum_j I_(\overline{a}_j, \varepsilon, \rho, p) + W(\overline{a}) + \pi \sum_j p(\overline{a}_j)\log\frac{1}{\rho} + o(1) \tag{3.5}$$

where $o(1)$ tends to 0 as ρ tends to 0 and depends only on $\overline{a}$ and g. If $\overline{a}_j$ are in Λ_2, then we have $O(\rho log\rho)$ instead of $o(1)$.

We also have

LEMMA 3.2. Let a be the configuration as in Theorem 1.1. Then, given any ρ sufficiently small, there is an integer $N(\rho)$ such that for every $n \geq N(\rho)$,

$$E_{\varepsilon_n}(u_{\varepsilon_n}) \geq \sum_j I_(a_j, \varepsilon_n, \rho, p) + W(a) + \pi d p_0 \log\frac{1}{\rho} + o(1) \tag{3.6}$$

where $o(1)$ depends only on a and g and tends to 0 as ρ tends to 0..

To prove Lemma 3.1, we use the two following lemmas.

LEMMA 3.3. Let u_0 be the canonical map associated to $(g, a, \overline{d})$. Suppose that a_i are in G, $i = 1, ..., k$. Then, as ρ tends to 0 we have

$$\frac{1}{2}\int_{\Omega_\rho} p|\nabla u_0|^2 = \pi \sum_i d_i^2 \log\frac{1}{\rho} + W + o(1) \tag{3.7}$$

where $\Omega_\rho = G \backslash \overline{B}(a_j, \rho)$.

Proof of Lemma 3.3. Recall that, as in $[BBH]_2$, u_0 is the solution for

$$\begin{cases} u_0 \times \dfrac{\partial u_0}{\partial x_1} = -\dfrac{1}{p}\dfrac{\partial \Phi_0}{\partial x_2} & \text{in} \quad G\backslash \cup_i \{a_i\} \\ u_0 \times \dfrac{\partial u_0}{\partial x_2} = \dfrac{1}{p}\dfrac{\partial \Phi_0}{\partial x_1} & \text{in} \quad G\backslash \cup_i \{a_i\} \end{cases} \tag{3.8}$$

and that we have

$$\int_{\Omega_\rho} p|\nabla u_0|^2 = \int_{\Omega_\rho} \frac{1}{p}|\nabla \Phi_0|^2. \tag{3.9}$$

We have

$$\int_{\Omega_\rho} \frac{1}{p}|\nabla\Phi_0|^2 = \int_{\partial G} \frac{1}{p}\frac{\partial\Phi_0}{\partial\nu}\Phi_0 - \sum_j \int_{\partial B(a_j,\rho)} \frac{1}{p}\frac{\partial\Phi_0}{\partial\nu}\Phi_0. \tag{3.10}$$

Let R_0 be defined as in (3.3). Then R_0 verifies

$$div(\frac{1}{p}\nabla R_0) = -\sum_j p(a_j)d_j\nabla\frac{1}{p}.\nabla \log|x-a_j| \quad \text{in} \quad G. \tag{3.11}$$

By standard results, we deduce from (3.11) that R_0 belongs to $C^0(G)$. We remark that if a_j are in Λ_2, $j = 1, ..., k$, then R_0 belongs to $C^\infty(G)$. Set

$$S_j(x) = \Phi_0 - d_j p(a_j)\log|x-a_j|. \tag{3.12}$$

Note that S_j satisfies, in some neighborhood of a_j, the equation

$$div(\frac{1}{p}\nabla S_j) = -p(a_j)d_j\nabla\frac{1}{p}.\nabla\log|x-a_j|. \tag{3.13}$$

Note that

$$S_j(x) = \Phi_0(x) - d_j p(a_j)\log\rho \quad \text{on} \quad \partial B(a_j,\rho)$$

$$\frac{\partial S_j}{\partial\nu} = \frac{\partial\Phi_0}{\partial\nu} - \frac{d_j p(a_j)}{\rho} \qquad \text{on} \quad \partial B(a_j,\rho)$$

and

$$S_j(a_j) = R_0(a_j) + \sum_{i\neq j} d_i p(a_i)\log|a_i - a_j|.$$

Then we have

$$\begin{aligned} \int_{\partial B(a_j,\rho)} \frac{1}{p}\frac{\partial\Phi_0}{\partial\nu}\Phi_0 &= \int_{B(a_j,\rho)} \frac{1}{p}|\nabla S_j|^2 + \int_{B(a_j,\rho)} div(\frac{1}{p}\nabla S_j)S_j \\ &+ d_j p(a_j)\log\rho\int_{B(a_j,\rho)} div(\frac{1}{p}\nabla S_j) \\ &+ \frac{d_j p(a_j)}{\rho}\int_{\partial B(a_j,\rho)}\frac{1}{p}S_j + \int_{\partial B(a_j,\rho)}\frac{1}{p}\frac{d_j^2 p(a_j)^2}{\rho}\log\rho. \end{aligned} \tag{3.14}$$

We have

$$\int_{\partial B(a_j,\rho)} \frac{1}{p}\frac{d_j^2 p(a_j)^2}{\rho} Log\rho = 2\pi d_j^2 p(a_j)\log\rho + O(\rho^2\log\rho).$$

We easily deduce from (3.13)

$$\int_{B(a_j,\rho)} div(\frac{1}{p}\nabla S_j)S_j = O(\rho)$$

and

$$\log\rho \int_{B(a_j,\rho)} div(\frac{1}{p}\nabla S_j) = O(\rho\log\rho).$$

We deduce from (3.13) that S_j is in $W^{1,p}(V(a_j))$, where $V(a_j)$ is a neighborhood of a_j, for all $p < 2$. In particular, S_j is continous in $V(a_j)$ and ∇S_j is in $L^2(V(a_j))$. Then we have

$$\int_{B(a_j,\rho)} \frac{1}{p}|\nabla S_j|^2 = o(1)$$

and

$$\frac{d_j p(a_j)}{\rho} \int_{\partial B(a_j,\rho)} \frac{1}{p} S_j = 2\pi d_j S_j(a_j) + o(1).$$

We deduce from these estimates combining with (3.10) and (3.14) that

$$\int_{\Omega_\rho} p|\nabla u_0|^2 = W(a, \overline{d}, g) + \pi \sum_i d_i^2 p(a_i) \log\frac{1}{\rho} + o(1)$$

and we have proved Lemma 3.3.

Let $\overline{u}_\rho$ be the unique minimizer for the problem

$$\min_{u \in \overline{E}_\rho} \int_{\Omega_\rho} p|\nabla u|^2$$

where

$$\overline{E}_\rho = \{v \in H^1(\Omega_\rho, S^1), \quad v = g \text{ on } \partial G \text{ and for all i there is } \alpha_i \text{ with } |\alpha_i| = 1 \text{ such that}$$

$$v(z) = \frac{\alpha_i}{\rho^{d_i}}(z - a_i)^{d_i} \quad \text{on} \quad \partial B(a_i, \rho)\}.$$

We have the following lemma LEMMA 3.4.

$$\frac{1}{2}\int_{\Omega_\rho} p|\nabla \overline{u}_\rho|^2 = \pi \sum_i d_i^2 p(a_i) \log\frac{1}{\rho} + W + o(1).$$

Proof of Lemma 3.4. We know that (see $[BBH]_2$)

$$\int_{\Omega_\rho} p|\nabla \overline{u}_\rho|^2 = \int_{\Omega_\rho} \frac{1}{p}|\nabla \overline{\Phi}_\rho|^2$$

where Φ_ρ is defined by

$$(3.15) \qquad \begin{cases} \overline{u}_\rho \times \dfrac{\partial \overline{u}_\rho}{\partial x_1} = -\dfrac{1}{p}\dfrac{\partial \overline{\Phi}_\rho}{\partial x_2} & \text{in} \quad G \setminus \cup_i \{a_i\} \\ \overline{u}_\rho \times \dfrac{\partial \overline{u}_\rho}{\partial x_2} = \dfrac{1}{p}\dfrac{\partial \overline{\Phi}_\rho}{\partial x_1} & \text{in} \quad G \setminus \cup_i \{a_i\} \end{cases}$$

and $\overline{\Phi}_\rho$ is the solution of the linear problem

$$(3.16)\qquad \begin{cases} div(\frac{1}{p}\nabla\overline{\Phi}_\rho) = 0 \quad \text{in} \quad \Omega_\rho \\ \quad \frac{1}{p}\frac{\partial\overline{\Phi}_\rho}{\partial\nu} = \frac{d_i}{\rho} \quad \text{on} \quad \partial B(a_i,\rho) \quad i = 1,...,k. \\ \quad \frac{1}{p}\frac{\partial\overline{\Phi}_\rho}{\partial\nu} = g\times g_\tau \quad \text{on} \quad \partial G. \end{cases}$$

We normalize $\overline{\Phi}_\rho$ by $\int_{\partial G}\overline{\Phi}_\rho = 0$. We define T_j^ρ , $j = 1,...,k$, by

$$(3.17)\qquad \begin{cases} div(\frac{1}{p}\nabla T_j^\rho) = -\nabla\frac{1}{p}.\nabla\log|x-a_j| \quad \text{in} \quad G\backslash B(a_j,\rho) \\ p(a_j)\frac{\partial T_j^\rho}{\partial\nu} = \frac{p(x)-p(a_j)}{\rho} \quad \text{on} \quad \partial B(a_j,\rho) \\ \quad \frac{1}{p}\frac{\partial T_j^\rho}{\partial\nu} = m_j^\rho \quad \text{on} \quad \partial G. \end{cases}$$

The constant m_j^ρ is defined by

$$(3.18)\qquad m_j^\rho.\ \text{mes}(\partial G) = \int_{\partial B(a_j,\rho)}\frac{p(x)-p(a_j)}{\rho p(x)p(a_j)} - \int_{G\backslash B(a_j,\rho)}\nabla\frac{1}{p}.\nabla\log|x-a_j|.$$

Set

$$(3.19)\qquad \Psi_\rho = \overline{\Phi}_\rho - \sum_j p(a_j)d_j(\log|x-a_j| + T_j^\rho).$$

The function Ψ_ρ satisfies

$$(3.20)\qquad \begin{cases} div(\frac{1}{p}\nabla\Psi_\rho) = 0 \quad \text{in} \quad \Omega_\rho \\ \quad \frac{1}{p}\frac{\partial\Psi_\rho}{\partial\nu} = g\times g_\tau - \frac{1}{p}\sum_j d_j p(a_j)\frac{\partial}{\partial\nu}(\log|x-a_j| + T_j^\rho) \quad \text{on} \quad \partial G \\ \quad \frac{1}{p}\frac{\partial\Psi_\rho}{\partial\nu} = -\frac{1}{p}\sum_{j\neq i} d_j p(a_j)\frac{\partial}{\partial\nu}(\log|x-a_j| + T_j^\rho) \quad \text{on} \quad \partial B(a_i,\rho) \quad i = 1...k. \end{cases}$$

Let

$$(3.21)\qquad g_i = -\frac{1}{p}\sum_{j\neq i} d_j p(a_j)\frac{\partial}{\partial\nu}(\log|x-a_j| + T_j^\rho) \quad \text{on} \quad \partial B(a_i,\rho) \quad i = 1,...,k$$

and

$$f_\rho = g \times g_\tau - \frac{1}{p}\sum_j d_j p(a_j)\frac{\partial}{\partial \nu}(\log|x-a_j| + T_j^\rho) \quad \text{on} \quad \partial G. \tag{3.22}$$

We have

$$\int_{\partial B(a_i,\rho)} g_i = -\sum_{j\neq i} p(a_j)d_j \int_{\partial B(a_i,\rho)} div(\frac{1}{p}\nabla(\log|x-a_j| + T_j^\rho)) = 0 \quad i = 1,...,k \tag{3.23}$$

by (3.17), and then

$$\int_{\partial G} f_\rho = 0. \tag{3.24}$$

Let Ψ_ρ^* be defined by

$$\begin{cases} \dfrac{\partial \Psi_\rho^*}{\partial x_1} = -\dfrac{1}{p}\dfrac{\partial \Psi_\rho}{\partial x_2} \quad \text{in} \quad \Omega_\rho \\ \dfrac{\partial \Psi_\rho^*}{\partial x_2} = \dfrac{1}{p}\dfrac{\partial \Psi_\rho}{\partial x_1} \quad \text{in} \quad \Omega_\rho. \end{cases} \tag{3.25}$$

Note that Ψ_ρ^* is well defined globally on Ω_ρ since $\int_\Gamma \frac{1}{p}\frac{\partial \Psi_\rho}{\partial \nu} = 0$ for each connected component Γ of ∂G, by (3.23) and (3.24). The function Ψ_ρ^* satisfies

$$\begin{cases} div(p\nabla \Psi_\rho^*) = 0 \quad \text{in} \quad \Omega_\rho \\ \dfrac{\partial \Psi_\rho^*}{\partial \tau} = f_\rho \quad \text{on} \quad \partial G \\ \dfrac{\partial \Psi_\rho^*}{\partial \tau} = g_i \quad \text{on} \quad \partial B(a_i,\rho) \quad i = 1,...,k. \end{cases} \tag{3.26}$$

Then we have

$$\begin{cases} div(p\nabla \Psi_\rho^*) = 0 \quad \text{in} \quad \Omega_\rho \\ \Psi_\rho^* = F_\rho \quad \text{on} \quad \partial G \\ \Psi_\rho^* = G_i \quad \text{on} \quad \partial B(a_i,\rho) \quad i = 1,...,k \\ \displaystyle\int_{\partial B(a_i,\rho)} p\frac{\partial \Psi_\rho^*}{\partial \nu} = 0 \quad i = 1,...,k \end{cases} \tag{3.27}$$

where F_ρ and G_i are primitives of f_ρ and g_i on ∂G and $\partial B(a_i,\rho)$ respectively. We deduce from (3.22) and (3.18) that we can extract a subsequence still denoted f_ρ such that f_ρ converges uniformly on ∂G to a function f as ρ tends to 0. Let F be the limit of F_ρ as ρ tends to 0. Then we have $\frac{\partial F}{\partial \tau} = f$. Let Ψ^* be the solution

$$\begin{cases} div(p\nabla \Psi^*) = 0 \quad \text{in} \quad G \\ \Psi^* = F \quad \text{on} \quad \partial G. \end{cases} \tag{3.28}$$

We have to apply the following lemma to $\Psi_\rho^* - \Psi^*$ (the proof of this lemma is the same as the proof of Lemma 1.4 in $[BBH]_2$). LEMMA 3.5. Let v be a function satifying

$$\begin{cases} div(p\nabla v) = 0 \quad \text{in} \quad \Omega_\rho \\ \displaystyle\int_{\partial B(a_j,\rho)} p\frac{\partial v}{\partial \nu} = 0 \quad j = 1, ..., k \end{cases}$$

then

$$\sup_{\Omega_\rho} v - \inf_{\Omega_\rho} v \leq \sum_j (\sup_{\partial B(a_j,\rho)} v - \inf_{\partial B(a_j,\rho)} v) + \sup_{\partial G} v - \inf_{\partial G} v.$$

Applying Lemma 3.5 to $v = \Psi_\rho^* - \Psi^*$ we have

$$\begin{aligned} ||\Psi_\rho^* - \Psi^*||_{L^\infty(\Omega_\rho)} \leq \sum_j \sup_{\partial B(a_j,\rho)} (G_j - \Psi^*) - \inf_{\partial B(a_j,\rho)} (G_j - \Psi^*) \\ + \sup_{\partial G}(F - F_\rho) - \inf_{\partial G}(F - F_\rho). \end{aligned}$$

Since $||g_j||_{L^\infty(\partial B(a_j,\rho))} \leq C$, we have

$$||\Psi_\rho^* - \Psi^*|| = O(\rho). \tag{3.29}$$

¿From (3.27), (3.28), (3.29) and standard elliptic estimates we deduce that for every compact subset $K \subset \overline{G} \setminus \cup_j \{a_j\}$ we have

$$||\nabla(\Psi_\rho^* - \Psi^*)||_{L^\infty(K)} \leq C_K \rho. \tag{3.30}$$

Let Ψ be defined as a solution of

$$\begin{cases} \dfrac{\partial \Psi}{\partial x_1} = p\dfrac{\partial \Psi^*}{\partial x_2} \quad \text{in} \quad G \\ \dfrac{\partial \Psi}{\partial x_2} = -p\dfrac{\partial \Psi^*}{\partial x_1} \quad \text{in} \quad G \end{cases} \tag{3.31}$$

so that Ψ satisfies

$$\begin{cases} div(\dfrac{1}{p}\nabla \Psi) = 0 \quad \text{in} \quad G \\ \dfrac{1}{p}\dfrac{\partial \Psi}{\partial \nu} = f \quad \text{on} \quad \partial G. \end{cases} \tag{3.32}$$

Define $\overline{R}_0 = \Phi_0 - \sum_j p(a_j) d_j(\log|x - a_j| + \overline{T}_j)$ where $\overline{T}_j$ is a solution of

$$\begin{cases} div(\dfrac{1}{p}\nabla \overline{T}_j) = -\nabla \dfrac{1}{p}.\nabla \log|x - a_j| \quad \text{in} \quad G \\ \dfrac{1}{p}\dfrac{\partial \overline{T}_j}{\partial \nu} = m_j \quad \text{on} \quad \partial G. \end{cases} \tag{3.33}$$

The constant m_j is the limit of m_j^ρ as ρ tends to 0. The function $\overline{R}_0$ verifies (3.32). Since the solution of (3.32) is unique up to an additive constant we may choose $\Psi = \overline{R}_0$. From (3.25) and (3.31) we conclude that

$$||\nabla(\Psi_\rho - \overline{R}_0)||_{L^\infty(K)} \leq C_K\rho. \tag{3.34}$$

By our normalization choice we have

$$\int_{\partial G} (\Psi_\rho - \overline{R}_0) = 0. \tag{3.35}$$

It follows from (3.34) and (3.35) that

$$||\Psi_\rho - \overline{R}_0||_{L^\infty(K)} \leq C_K\rho. \tag{3.36}$$

In particular we have

$$||\Psi_\rho - \overline{R}_0||_{L^\infty(\partial G)} \leq C_K\rho. \tag{3.37}$$

We have now to prove

$$||\Psi_\rho||_{L^\infty(\partial B(a_i,\rho))} \leq C. \tag{3.38}$$

Set

$$\omega_i(x) = \sum_{i\neq j} p(a_j)d_j(\log|x-a_j| + T_j^\rho).$$

Fix $\alpha > 0$ so that $\overline{B}(a_i, \alpha) \subset G$ and $\overline{B}(a_i, \alpha)$ does not contain any other point a_j, $j \neq i$. From (3.20) and (3.17) we deduce that

$$div(\frac{1}{p}\nabla(\Psi_\rho + \omega_i)) = 0 \quad \text{in} \quad B(a_i,\alpha)\backslash B(a_i,\rho)$$

and

$$\frac{\partial}{\partial\nu}(\Psi_\rho + \omega_i) = 0 \quad \text{on} \quad \partial B(a_i,\rho).$$

It follows from the maximum principle that

$$||\Psi_\rho + \omega_i||_{L^\infty(B(a_i,\alpha)\backslash B(a_i,\rho))} \leq ||\Psi_\rho + \omega_i||_{L^\infty(\partial B(a_i,\alpha))} \leq C.$$

In particular

$$||\Psi_\rho + \omega_i||_{L^\infty(\partial B(a_i,\rho))} \leq C$$

and this implies (3.38).

Proof of Lemma 3.4 completed. We have

$$\int_{\Omega_\rho} p|\nabla\overline{u}_\rho|^2 = \int_{\Omega_\rho} \frac{1}{p}|\nabla\overline{\Phi}_\rho|^2$$

and

$$\int_{\Omega_\rho} \frac{1}{p}|\nabla\overline{\Phi}_\rho|^2 = \int_{\Omega_\rho}\frac{1}{p}|\nabla\Psi_\rho|^2 + 2\int_{\Omega_\rho}\frac{1}{p}\nabla\Psi_\rho.\nabla(\sum_j p(a_j)d_j(\log|x-a_j| + T_j^\rho))$$
$$+\int_{\Omega_\rho}\frac{1}{p}|\nabla\sum_j p(a_j)d_j(\log|x-a_j|+T_j^\rho)|^2. \tag{3.39}$$

We have

$$\int_{\Omega_\rho}\frac{1}{p}|\nabla\Psi_\rho|^2 = \int_{\partial G} f_\rho\Psi_\rho - \sum_j\int_{\partial B(a_j,\rho)} g_j\Psi_\rho.$$

Using (3.37) and (3.38) , we obtain

$$\int_{\Omega_\rho}\frac{1}{p}|\nabla\Psi_\rho|^2 = \int_{\Omega_\rho}\frac{1}{p}\frac{\partial\Psi_\rho}{\partial\nu}\overline{R}_0 + O(\rho). \tag{3.40}$$

We conclude, using (3.40) and (3.34)

$$\int_{\Omega_\rho}\frac{1}{p}|\nabla\Psi_\rho|^2 = \int_{\Omega_\rho}\frac{1}{p}|\nabla\overline{R}_0|^2 + O(\rho). \tag{3.41}$$

Integrating by parts we have

$$\int_{\Omega_\rho}\frac{1}{p}\nabla(\Psi_\rho - \overline{R}_0).\nabla\sum_j p(a_j)d_j\log|x-a_j|$$
$$= -\sum_i\int_{\partial B(a_i,\rho)}(g_i - \frac{1}{p}\frac{\partial\overline{R}_0}{\partial\nu})\sum_j p(a_j)d_j\log|x-a_j|\,. \tag{3.42}$$
$$+\int_{\partial G}(f_\rho - f)\sum_j p(a_j)d_j\log|x-a_j|.$$

We deduce from (3.42) that

$$\int_{\Omega_\rho}\frac{1}{p}\nabla(\Psi_\rho - \overline{R}_0).\nabla\sum_j p(a_j)d_j\log|x-a_j| = O(\rho\log\rho). \tag{3.43}$$

We now estimate

$$\int_{\Omega_\rho}\frac{1}{p}\nabla(\Psi_\rho - \overline{R}_0).\nabla T_i^\rho$$
$$= -\int_{\Omega_\rho}(\Psi_\rho - \overline{R}_0)div(\frac{1}{p}\nabla T_i^\rho) + \int_{\partial\Omega_\rho}(\Psi_\rho - \overline{R}_0)\frac{1}{p}\frac{\partial T_i^\rho}{\partial\nu}$$
$$= -\int_{\Omega_\rho}(\Psi_\rho - \overline{R}_0)div(\frac{1}{p}\nabla T_i) + \int_{\partial\Omega_\rho}(\Psi_\rho - \overline{R}_0)\frac{1}{p}\frac{\partial T_i}{\partial\nu} \tag{3.44}$$
$$+\int_{\partial G}(\Psi_\rho - \overline{R}_0)(\frac{1}{p}\frac{\partial(T_i^\rho - \overline{T}_i)}{\partial\nu}) - \sum_i\int_{\partial B(a_i,\rho)}(\Psi_\rho - \overline{R}_0)(\frac{1}{p}\frac{\partial(T_i^\rho - \overline{T}_i)}{\partial\nu}).$$

Using (3.17), (3.18) and (3.33) we have

$$\frac{\partial (T_i^\rho - \overline{T}_i)}{\partial \nu} = O(\rho). \tag{3.45}$$

On the other hand, we have by (3.17) and (3.33)

$$|\frac{\partial T_i^\rho}{\partial \nu}| \leq C \quad \text{and} \quad \overline{T}_i \in W^{2,p}(G) \quad \text{for all } p < 2. \tag{3.46}$$

We deduce from (3.44), (3.45) and (3.46), using

$$||\Psi_\rho - \overline{R}_0||_{L^\infty(\partial B(a_i,\rho))} \leq C$$

and (3.37) that

$$\int_{\Omega_\rho} \frac{1}{p} \nabla(\Psi_\rho - \overline{R}_0).\nabla T_i^\rho = \int_{\Omega_\rho} \frac{1}{p} \nabla(\Psi_\rho - \overline{R}_0)\nabla \overline{T}_i + o(1). \tag{3.47}$$

¿From (3.47) and the fact that $\overline{T}_i \in W^{2,p}$ for all $p < 2$ we deduce that

$$\int_{\Omega_\rho} \frac{1}{p} \nabla(\Psi_\rho - \overline{R}_0).\nabla T_i^\rho = o(1). \tag{3.48}$$

¿From (3.43) and (3.48) we deduce

$$\int_{\Omega_\rho} \frac{1}{p} \nabla(\Psi_\rho - \overline{R}_0).\nabla \sum_j p(a_j) d_j (\log|x - a_j| + T_j^\rho) = o(1). \tag{3.49}$$

We deduce from (3.39), (3.41) and (3.49) that

$$\begin{aligned}\int_{\Omega_\rho} \frac{1}{p} |\nabla \overline{\Phi}_\rho|^2 = \int_{\Omega_\rho} \frac{1}{p} |\nabla \overline{R}_0|^2 + 2\int_{\Omega_\rho} \frac{1}{p} \nabla \overline{R}_0.\nabla (\sum_j p(a_j) d_j (\log|x - a_j| + T_j^\rho)) \\ + \int_{\Omega_\rho} \frac{1}{p} |\nabla \sum_j p(a_j) d_j (\log|x - a_j| + T_j^\rho)|^2 + o(1).\end{aligned} \tag{3.50}$$

On the other hand we have from the definition of $\overline{R}_0$

$$\int_{\Omega_\rho} \frac{1}{p} |\nabla \Phi_0|^2 = \int_{\Omega_\rho} |\nabla (\overline{R}_0 + \sum_j d_j p(a_j)(\log|x - a_j| + \overline{T}_j))|^2. \tag{3.51}$$

¿From (3.50) and (3.51) we deduce

$$\begin{aligned}&\int_{\Omega_\rho} \frac{1}{p} |\nabla \overline{\Phi}_\rho|^2 = \int_{\Omega_\rho} \frac{1}{p} |\nabla \Phi_0|^2 \\ &- 2\int_{\Omega_\rho} \frac{1}{p} \nabla \overline{R}_0.\nabla \sum_j d_j p(a_j)(T_j^\rho - \overline{T}_j) - \int_{\Omega_\rho} \frac{1}{p} |\nabla \sum_j d_j p(a_j)(T_j^\rho - \overline{T}_j)|^2 \\ &- 2\int_{\Omega_\rho} \frac{1}{p} \nabla \sum_j d_j p(a_j) \log|x - a_j|.\nabla (T_j^\rho - \overline{T}_j) + o(1).\end{aligned} \tag{3.52}$$

Integrating by parts we deduce from (3.52)

$$
\begin{aligned}
&\int_{\Omega_\rho} \frac{1}{p}|\nabla \overline{u}_\rho|^2 = \int_{\Omega_\rho} \frac{1}{p}|\nabla \Phi_0|^2 + 2\sum_j d_j p(a_j) \int_{\partial\Omega_\rho} \frac{1}{p}\overline{R}_0 \frac{\partial}{\partial \nu}(T_j^\rho - \overline{T}_j) \\
(3.53)\qquad &\sum_j d_j p(a_j) \int_{\partial\Omega_\rho} (T_j^\rho - \overline{T}_j)\frac{1}{p}\frac{\partial}{\partial \nu}(T_j^\rho - \overline{T}_j) \\
&+ \sum_j d_j p(a_j) \int_{\partial\Omega_\rho} \log|x-a_j| \frac{1}{p}\frac{\partial}{\partial \nu}(T_j^\rho - \overline{T}_j) + o(1).
\end{aligned}
$$

We deduce from (3.53), combining with (3.46) that

$$\int_{\Omega_\rho} \frac{1}{p}|\nabla \overline{\Phi}_\rho|^2 = \int_{\Omega_\rho} \frac{1}{p}|\nabla \Phi_0|^2 + o(1).$$

Applying Lemma 3.3, we obtain the desired conclusion.

Proof of Lemma 3.1. We apply Lemma 3.4 to the configuration $\overline{a}$. This yields, for every $\rho < \rho_0$, some map $\overline{u}_\rho$ on ∂G , $\overline{u}_\rho(z) = \alpha_i \frac{z-\overline{a}_j}{|z-\overline{a}_j|}$ on $\partial B(\overline{a}_j, \rho)$ with $|\alpha_j| = 1$ and

$$\frac{1}{2}\int_{\Omega_\rho} p|\nabla \overline{u}_\rho|^2 = \pi \sum_j p(\overline{a}_j)\log\frac{1}{\rho} + W(\overline{a}) + o(1).$$

On the other hand, for each j we may find some v_j from $B(a_j,\rho)$ in C such that $v_j(z) = \alpha_i \frac{z-\overline{a}_j}{|z-\overline{a}_j|}$ on $\partial B(\overline{a}_j, \rho)$ with $|\alpha_j| = 1$ and

$$\frac{1}{2}\int_{B(\overline{a}_j,\rho)} p|\nabla v_j|^2 + \frac{1}{4\varepsilon^2}\int_{B(\overline{a}_j,\rho)} p(|v_j|^2-1)^2 = I(\overline{a}_j, \varepsilon, \rho, p).$$

Set $\omega = \overline{u}_\rho$ in Ω_ρ and $\omega = v_j$ in $B(\overline{a}_j, \rho)$, $j = 1, ..., d$.

We see that

$$E_\varepsilon(\omega) = \sum_j I(\overline{a}_j, \varepsilon, \rho, p) + W(\overline{a}) + \pi \sum_i p(\overline{a}_i)\log\frac{1}{\rho} + o(1).$$

We have the desired conclusion.

Proof of Lemma 3.2. Recall that u_{ε_n} converges in $H^1_{loc}(G\backslash \cup_j \{a_j\})$ to u_* and therefore, for every fixed $\rho < \rho_1$, $\frac{1}{2}\int_{\Omega_\rho} p|\nabla u_{\varepsilon_n}|^2$ tends to $\frac{1}{2}\int_{\Omega_\rho} p|\nabla u_*|^2$. In particular, there is an integer $N_1 = N_1(\rho)$ such that, for every $n > N_1$,

$$\frac{1}{2}\int_{\Omega_\rho} p|\nabla u_{\varepsilon_n}|^2 \geq \frac{1}{2}\int_{\Omega_\rho} |\nabla u_*|^2 - \rho^2. \tag{3.54}$$

On the other hand we have by Lemma 3.3 that

$$\frac{1}{2}\int_{\Omega_\rho} p|\nabla u_*|^2 = \pi \sum_j p(a_j)\log\frac{1}{\rho} + W(a) + o(1). \tag{3.55}$$

Combining (3.54) and (3.55) we see that, for $n \geq N_1(\rho)$ we have

$$\begin{aligned} &\frac{1}{2}\int_{\Omega_\rho} p|\nabla u_{\varepsilon_n}|^2 + \frac{1}{4\varepsilon_n^2}\int_{\Omega_\rho} p(|u_{\varepsilon_n}|^2-1)^2 \\ &\geq \pi \sum_j p(a_j)\log\frac{1}{\rho} + W(a) + o(1). \end{aligned} \tag{3.56}$$

We now turn to energy estimates on the balls $B(a_j,\rho)$. We claim that given any ρ, $\rho < \rho_1$ there is an integer $N_2(\rho)$ such that, for $n \geq N_2$

$$\begin{aligned} &\frac{1}{2}\int_{B(a_j,\rho)} p|\nabla u_{\varepsilon_n}|^2 + \frac{1}{4\varepsilon_n^2}\int_{B(a_j,\rho)} p(|u_{\varepsilon_n}|^2-1)^2 \\ &\geq I(a_j,\varepsilon_n,\rho,p) + O(\rho). \end{aligned} \tag{3.57}$$

Combining (3.56) and (3.57) we are led to the conclusion of Lemma 3.2.

Proof of Claim (3.57). The proof is directly adapted from the proof of Claim (10) in $[BBH]_2$ chap.8. We know that, given any $\rho < \rho_1$, we may find some integer $N_3(\rho)$ such that, for every $N \geq N_3$ we have

$$||u_{\varepsilon_n} - u_*||_{L^\infty(B(a_j,\rho)\setminus B(a_j,\frac{\rho}{2}))} \leq \rho^2 \tag{3.58}$$

and

$$||\nabla u_{\varepsilon_n} - \nabla u_*||_{L^\infty(B(a_j,\rho)\setminus B(a_j,\frac{\rho}{2}))} \leq \rho. \tag{3.59}$$

We may also assume that, for $n \geq N_3$, we have

$$\frac{1-|u_{\varepsilon_n}|^2}{\varepsilon_n^2} \leq |\nabla u_*|^2 + 1 \quad \text{in} \quad B(a_j,\rho)\setminus B(a_j,\frac{\rho}{2}). \tag{3.60}$$

We know that near each a_j we have

$$u_*(z) = e^{i(\theta + H_j(z))}$$

where H_j is some smooth function near a_j. Thus we have

$$|\nabla u_*| \leq \frac{2}{\rho} + O(1) \quad \text{in} \quad B(a_j,\rho)\setminus B(a_j,\frac{\rho}{2}). \tag{3.61}$$

Combining (3.60) and (3.61) we find, for $n \geq N_3$

$$\frac{1-|u_{\varepsilon_n}|^2}{\varepsilon_n^2} \leq C\rho^2 + C_1 = K(\rho) \quad \text{in} \quad B(a_j,\rho)\setminus B(a_j,\frac{\rho}{2}) \tag{3.62}$$

where C and C_1 denote constants independant of n and ρ. Consider the map

$$\omega_n(z) = (\frac{2|z-a_j|}{\rho} - 1)(u_*(z) - u_{\varepsilon_n}(z)) + u_{\varepsilon_n}(z) \tag{3.63}$$

defined for $z \in B(a_j,\rho)\backslash B(a_j,\frac{\rho}{2})$. We have

LEMMA 3.6. There is an integer $N_4(\rho)$ such that for every $n \geq N_4$ we have

$$||\omega_n - u_{\varepsilon_n}||_{L^\infty(B(a_j,\rho)\backslash B(a_j,\frac{\rho}{2}))} \leq C\rho^2, \tag{3.64}$$

$$||\nabla\omega_n - \nabla u_{\varepsilon_n}||_{L^\infty(B(a_j,\rho)\backslash B(a_j,\frac{\rho}{2}))} \leq C\rho \tag{3.65}$$

and

$$|\omega_n(z)|^2 \geq 1 - C\rho^4 \quad \text{in} \quad B(a_j,\rho)\backslash B(a_j,\frac{\rho}{2}). \tag{3.66}$$

Proof of Lemma 3.6. We have by (3.63)

$$|\omega_n(z) - u_{\varepsilon_n}(z)| \leq |u_*(z) - u_{\varepsilon_n}(z)|$$

and (3.64) follows from (3.58). Differentiating (3.63) we see that

$$|\nabla\omega_n(z) - \nabla u_{\varepsilon_n}(z)| \leq \frac{C}{\rho}|u_* - u_{\varepsilon_n}| + |\nabla u_* - \nabla u_{\varepsilon_n}|$$

and (3.65) follows from (3.58) and (3.59). The proof of (3.66) relies on the following variant of the parallelogram identity

$$\begin{aligned} &|ta + (1-t)b|^2 = t|a|^2 + (1-t)|b|^2 - t(1-t)(a-b)^2 \\ &\geq t|a|^2 + (1-t)|b|^2 - \frac{1}{4}|a-b|^2 \quad \text{for all } t \text{ in } [0,1]. \end{aligned} \tag{3.67}$$

We apply (3.67) with $a = u_*(z)$, $b = u_{\varepsilon_n}(z)$ and $t = \frac{2|z-a_j|}{\rho} - 1$. This yields, using (3.62)

$$|\omega_n(z)|^2 \geq 1 - 2K(\rho)\varepsilon_n^2 - \frac{1}{4}|u_*(z) - u_{\varepsilon_n}(z)|. \tag{3.68}$$

We finally choose $N_4(\rho) \geq N_3(\rho)$ such that

$$K(\rho)\varepsilon_n^2 \leq \rho^4 \quad \text{for all } n \geq N_4(\rho) \tag{3.69}$$

and then (3.66) follows from (3.68), (3.69) and (3.58). We may return to the proof of Claim (3.57).

Proof of Claim (3.57) completed. We set

$$R = R(n,\rho) = (1 - K(\rho)\varepsilon - n^2)^{\frac{1}{2}}. \tag{3.70}$$

We may assume that $n \geq N_4(\rho)$ and $\rho < 1$ so that R is well defined. Consider the map $P = P(n, \rho)$ from $C\backslash\{0\}$ into itself defined by

$$P\zeta = \zeta \quad \text{if} \quad |\zeta| \geq R \quad \text{and} \quad ||\nabla P(\zeta)|| = \frac{R}{|\zeta|} \quad \text{if} \quad |\zeta| < R.$$

We have

$$||\nabla P(\zeta)|| = 1 \quad \text{if} \quad |\zeta| \geq R \quad \text{and} \quad ||\nabla P(\zeta)|| = \frac{R}{|\zeta|} \quad \text{if} \quad |\zeta| < R. \tag{3.71}$$

Consider the map v_n from $B(a_j, \rho)$ into C defined by

$$\begin{cases} v_n(z) = u_{\varepsilon_n}(z) & \text{if} \quad z \in B(a_j, \frac{\rho}{2}) \\ v_n(z) = P\omega_n(z) & \text{if} \quad z \in B(a_j, \rho)\backslash B(a_j, \frac{\rho}{2}). \end{cases} \tag{3.72}$$

On $\partial B(a_j, \rho)$ we have

$$v_n(z) = u_*(z).$$

It follows from the definition of $I(a_j, \varepsilon_n, \rho, p)$ that

$$\frac{1}{2}\int_{B(a_j,\rho)} p|\nabla v_n|^2 + \frac{1}{4\varepsilon_n^2}\int_{B(a_j,\rho)} p(|v_n|^2 - 1)^2 \geq I(a_j, \varepsilon_n, \rho, p). \tag{3.73}$$

(Note that on $\partial B(a_j, \rho)$, $\omega_n = u_{\varepsilon_n}$ and thus $v_n = u_{\varepsilon_n}$, since $|u_{\varepsilon_n}| \geq R$ by (3.62), hence $v_n \in H^1(B(a_j, \rho))$. From (3.72) and (3.73) we deduce that

$$\frac{1}{2}\int_{B(a_j,\rho)} p|\nabla u_{\varepsilon_n}|^2 + \frac{1}{4\varepsilon_n^2}\int_{B(a_j,\rho)} p(|u_{\varepsilon_n}|^2 - 1)^2 \geq I_{a_j}(\varepsilon_n, \rho) - U - V \tag{3.74}$$

where

$$U = \frac{1}{2}\int_{B(a_j,\rho)\backslash B(a_j,\frac{\rho}{2})} p(|\nabla v_n|^2 - |\nabla u_{\varepsilon_n}|^2) \tag{3.75}$$

and

$$V = \frac{1}{4\varepsilon_n^2}\int_{B(a_j,\rho)\backslash B(a_j,\frac{\rho}{2})} p((|v_n|^2 - 1)^2 - (|u_{\varepsilon_n}|^2 - 1)^2). \tag{3.76}$$

We first estimate V. Since $|\omega_n| \leq 1$ we have $R \leq |v_n| \leq 1$ and therefore

$$(|v_n|^2 - 1)^2 \leq (1 - R^2)^2 = K^2(\rho)\varepsilon_n^4.$$

It follows from (3.62) and (3.69) that

$$V \leq \pi\rho^2 K^2(\rho)\varepsilon_n^2 \leq C\rho^4 \quad \text{for every } n \geq N_4(\rho). \tag{3.77}$$

We now estimate U. We have in $B(a_j,\rho)\backslash B(a_j,\frac{\rho}{2})$

$$|\nabla v_n|^2 \leq ||\nabla P(\omega_n)||^2|\nabla\omega_n|^2 \leq \frac{1}{1-C\rho^4}|\nabla\omega_n|^2 \tag{3.78}$$

by (3.71) and (3.66). From (3.77) and (3.65) we deduce that

$$|\nabla v_n|^2 \leq \frac{1}{1-C\rho^4}(|\nabla u_{\varepsilon_n}|^2 + 2C\rho|\nabla\omega_n|^2 + C^2\rho^2). \tag{3.79}$$

Hence

$$|\nabla v_n|^2 - |\nabla u_{\varepsilon_n}|^2 \leq C(\rho^4|\nabla u_{\varepsilon_n}|^2 + \rho|\nabla u_{\varepsilon_n}| + \rho^2). \tag{3.80}$$

On the other hand it follows from (3.61) that

$$||\nabla u_*||_{L^\infty(B(a_j,\rho)\backslash B(a_j,\frac{\rho}{2}))} \leq \frac{C}{\rho}. \tag{3.81}$$

Combining (3.80) and (3.59) we are led to

$$||\nabla u_{\varepsilon_n}||_{L^\infty(B(a_j,\rho)\backslash B(a_j,\frac{\rho}{2}))} \leq \frac{C}{\rho}. \tag{3.82}$$

Going back to (3.80) we obtain

$$U \leq C\rho^2. \tag{3.83}$$

The desired estimate (3.57) with $N_2(\rho) = N_4(\rho)$ follows from (3.74), (3.77) and (3.83).

Proof of Theorem 1.3 completed. Using Lemma 3.1 and Lemma 3.2 we obtain for $\rho < min(\rho_0,\rho_1)$ and $n \geq N(\rho)$

$$\sum_i I(a_i,\varepsilon_n,\rho,p) + W(a) \leq \sum_i I(\overline{a}_i,\varepsilon_n,\rho,p) + W(\overline{a}) + o(1) \tag{3.84}$$

where $o(1)$ stands for a quantity that tends to 0 as ρ tends to 0 and that depends only on $\overline{a}$ and g. Recall that

$$I(\overline{a}_i,\varepsilon,\rho,p) = \min_{u\in E_i}\{\frac{1}{2}\int_{B_i} p|\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_{B_i} p(1-|u|^2)^2\}$$

where

$$E_i = \{u \in H^1(B_i) \quad u = \alpha\frac{z-a_i}{|z-a_i|} \quad \text{on} \quad \partial B_i\} \quad ; \quad B_i = B(a_i,\rho).$$

Set

$$I^*(\varepsilon,\rho) = I(a_i,\varepsilon,\rho,1).$$

Using $[BBH]_2$ we have

$$I^*(\varepsilon,\rho)=\pi\log\frac{\rho}{\varepsilon}+\gamma+o(1) \tag{3.85}$$

where $o(1)$ tends to 0 as ε tends to 0 and ρ is fixed and γ is an universal constant.

We have

$$I(\overline{a}_i,\varepsilon,\rho,p)\geq p_0I^*(\varepsilon,\rho)=\pi p_0\log\frac{\rho}{\varepsilon}+\gamma p_0+o(1). \tag{3.86}$$

On the other hand, let $u_i(x)$ be the map that realizes $I(\overline{a}_i,\varepsilon,(\varepsilon\rho)^{\frac12},1)$ and set

$$u(x)=u_i(x)\quad\text{on } B(\overline{a}_i,(\varepsilon\rho)^{\frac12})$$

and

$$u(x)=\frac{x-\overline{a}_i}{|x-\overline{a}_i|}\quad\text{on}\quad B(\overline{a}_i,\rho)\backslash B(\overline{a}_i,(\varepsilon\rho)^{\frac12}).$$

We have

$$\begin{aligned}I(\overline{a}_i,\varepsilon,\rho,p)\leq&\frac12\int_{B(a_i,(\varepsilon\rho)^{\frac12})}p|\nabla u|^2+\frac{1}{4\varepsilon^2}\int_{B(\overline{a}_i,(\varepsilon\rho)^{\frac12})}p(1-|u|^2)^2\\&+\frac12\int_{B(\overline{a}_i,\rho)\backslash B(\overline{a}_i,(\varepsilon\rho)^{\frac12})}\frac{p(x)}{|x-\overline{a}_i|^2}.\end{aligned}$$

Since

$$|p(x)-p(\overline{a}_i)|\leq C(\varepsilon\rho)^{\frac12}\quad\text{for}\quad x\in B(\overline{a}_i,(\varepsilon\rho)^{\frac12})$$

we obtain

$$\begin{aligned}I(\overline{a}_i,\varepsilon,\rho,p)\leq&(p_0+C(\varepsilon\rho)^{\frac12})I^*(\varepsilon,(\varepsilon\rho)^{\frac12})+\pi p(\overline{a}_i)Log\frac{\rho}{(\rho\varepsilon)^{\frac12}}\\&+\frac12\int_{B(\overline{a}_i,\rho)\backslash B(\overline{a}_i,(\varepsilon\rho)^{\frac12})}\frac{|p(x)-p(\overline{a}_i)|}{|x-\overline{a}_i|^2}.\end{aligned}$$

Thus

$$I(\overline{a}_i,\varepsilon,\rho,p)\leq(p_0+C(\varepsilon\rho)^{\frac12})I^*(\varepsilon,(\varepsilon\rho)^{\frac12})+\pi p(\overline{a}_i)\log\frac{\rho}{(\varepsilon\rho)^{\frac12}}+o(1)$$

where $o(1)$ stands for a quantity that tends to 0 as ε or ρ tends to 0. Using (3.85) we obtain

$$I(\overline{a}_i,\varepsilon,\rho,p)\leq\pi p_0\log\frac{\rho}{\varepsilon}+p_0\gamma+o(1) \tag{3.87}$$

where $o(1)$ tends to 0 as ε tends to 0 and ρ is fixed. Combining (3.86) and (3.87) we have

$$I(\overline{a}_i,\varepsilon,\rho,p)=\pi p_0\log\frac{\rho}{\varepsilon}+p_0\gamma+o(1) \tag{3.88}$$

where $o(1)$ tends to 0 as ε tends to 0 and ρ is fixed. Inserting (3.88) in (3.84) and passing to the limit first as ε tends to 0 and then as ρ tends to 0, we obtain the desired conclusion.

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