ASYMPTOTIC BEHAVIOR OF MINIMIZERS OF A GINZBURG - LANDAU EQUATION WITH WEIGHT NEAR THEIR ZEROES.

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1. Introduction.

Let G be a smooth bounded domain in $\mathbf{R}^2$ and p a smooth positive bounded map from $\overline{G}$ into $\mathbf{R}$. For a given map $g \in \mathbf{C}^\infty(\partial G, S^1)$ such that $deg(g, \partial G) = d > 0$, we consider the problem

$$\min_{u \in H^1_g(G, \mathbf{C})} E_\varepsilon(u) \tag{1.1}$$

where E_ε is the Ginzburg-Landau type functional

$$E_\varepsilon(u) = \frac{1}{2}\int_G p \mid \nabla u \mid^2 + \frac{1}{4\varepsilon^2}\int_G p(1- \mid u \mid^2)^2 \tag{1.2}$$

and

$$H^1_g(G, \mathbf{C}) = \{u \in H^1(G, \mathbf{C}); u = g \text{ on } \partial G\}.$$

The problem (1.1) is a generalized form of a problem introduced and completely studied by Bethuel, Brezis and Hélein in [BBH1,2]. These authors proved in particular that in a starshaped domain G and with $p = 1$ there exists a subsequence ε_n, exactly d singularities $a_1, ..., a_d$ in G and a smooth harmonic map $u_\star$ in $G \backslash \{a_1, ..., a_d\}$ such that $u_{\varepsilon_n} \to u_\star$ as ε_n tends to 0 in $\mathcal{C}^{1,\alpha}_{loc}(\overline{G} \backslash \{a_1, ..., a_d\})$ for all $0 < \alpha < 1$. Moreover, $(a_1, ...a_d)$ minimizes a renormalized energy W. For $\varepsilon_n < \varepsilon_0$ depending only on g and G, u_{ε_n} has exactly d zeroes $x_1^{\varepsilon_n}, ..., x_d^{\varepsilon_n}$. These results have been generalized to arbitrary smooth domains (see [St], [dPF]).

In [CM], Comte and Mironescu estimate the rate of convergence of the zero $x_i^{\varepsilon_n}$ to a_i if $(a_1, ...a_d)$ is a nondegenerate critical point of W. For a general bounded smooth and positive p the problem (1.1) is studied in [AS1,2] by André and Shafrir and in [BH1,2].

Recall that (1.1) is an approximation of the Ginzburg-Landau model for the energy of superconductors. The weight p is modeling the problem of pinning of vortices, the physical motivations are to study a thin film with variable thickness (see [DG]), or to introduce impurities in the superconductor (see [R]).

Let us summarize some results obtained in [AS1,2] and [BH1,2]. We denote

$$p_0 = \min_{x \in \overline{G}} p(x)$$

and

$$\Lambda_1 = \{x \in \overline{G}; p(x) = p_0\} \ ; \ \Lambda_2 = \{x \in G; p(x) = p_0\}.$$

We know that there exists m points $a_1, ..., a_m$ in Λ_1, associated to the positive degrees $(d_1, ..., d_m)$, $\sum_{i=1}^m d_i = d$, a subsequence u_{ε_n} and a map u^* such that u_{ε_n} converges to u^*. For n large enough there exists exactly d_i zeroes for u_{ε_n} in a neighborhood of a_i denoted $x_{ij}^{\varepsilon_n}$, $j = 1, ..., d_i$. There exists $\alpha > 0$ such that $\mid x_{ij}^{\varepsilon_n} - a_i \mid \geq \alpha \varepsilon_n$ and $deg(u_{\varepsilon_n}, \partial B(x_{ij}^{\varepsilon_n}, \alpha \varepsilon_n)) = 1$.

Now we suppose that for all $y \in \Lambda_1$ there exists $C_y > 0$ such that

$$p(x) = p_0 + C_y \mid x - y \mid^2 + o(\mid x - y \mid^2).$$

In what follows we denote $a_i \in \Lambda_2$ for $i = 1, ..., l$ and $a_i \in \Lambda_1 \backslash \Lambda_2$ for $i = l+1, ..., m$. Let $l_2 = Card \Lambda_2$. If $l_2 \geq d$ we have $m = d$, $\{a_1, ..., a_d\} \subset \Lambda_2$ and minimizes $W_G(a, 1, ..., 1, g, p)$, that is defined in (2.1) of the present paper with $G = \Omega$ and $g = h_0$. If $l_2 < d$, the integers $m, l, d_1, ..., d_m$ realizes

$$F(d) = \min\{\sum_{i=1}^{l} \frac{d_i^2 - d_i}{2} + \sum_{i=l+1}^{m} \frac{2d_i^2 - d_i}{2}; \sum_{i=1}^{m} d_i = d\},$$

(the cases $l = m$ and $l = 0$ are included). In particular we have

$$\Lambda_2 \subset \{a_1, ..., a_m\}.$$

The results related to the zeroes of u_{ε_n} are the following. The sequence

$$\mid x_{ij}^{\varepsilon_n} - x_{ik}^{\varepsilon_n} \mid^2 \log \frac{1}{\varepsilon_n}$$

tends to a positive constant as ε_n tends to 0, for all i and all $j \neq k$. Moreover

$$\mid x_{ij}^{\varepsilon_n} - a_i \mid^2 \log \frac{1}{\varepsilon_n}$$

is bounded for all i and all $j = 1, ..., d_i$. More precisely, if $i = 1, ..., l$ and $d_i = 1$, $\mid x_{i1}^{\varepsilon_n} - a_i \mid^2 \log \frac{1}{\varepsilon_n}$ tends to 0. If $i = 1, ..., l$ and $d_i > 1$, $\mid x_{ij}^{\varepsilon_n} - a_i \mid^2 \log \frac{1}{\varepsilon_n}$ tends to a positive constant for at least $d_i - 1$ zeroes $x_{ij}^{\varepsilon_n}$. If $i = l+1, ..., m$, $\mid x_{ij}^{\varepsilon_n} - a_i \mid^2 \log \frac{1}{\varepsilon_n}$ and $dist(x_{ij}^{\varepsilon_n}, \partial G)$ tend to positive constants. In the particular case $d_i = 1$ these constants are both equal to $\frac{p_0}{2C_{a_i}}$. In the present paper we complete the results given in [AS1,2] and [BH,12] concerning the choice of the location of the singularities $(a_1, ..., a_m)$ and the estimate of the energy $E_\varepsilon(u_\varepsilon)$, that we do in a general setting. We denote

$$\sigma_n = \frac{1}{\log^{\frac{1}{2}} \frac{1}{\varepsilon_n}},$$

$$\omega_{ij}^{\varepsilon_n} = \frac{x_{ij}^{\varepsilon_n} - a_i}{\sigma_n}, \qquad \omega_{ij} = \lim_{\varepsilon_n \to 0} \omega_{ij}^{\varepsilon_n} \qquad \text{and} \qquad y_{ij}^{\varepsilon_n} = a_i + \sigma_n \omega_{ij}.$$

This work concerns also the characterization for all i of the configuration $(\omega_{i1}, ..., \omega_{id_i})$ and the estimate of the rate of convergence of the sequence $x_{ij}^{\varepsilon_n} - y_{ij}^{\varepsilon_n}$ to 0. We set for $i = 1, ..., l$ and for $(\eta_1, ..., \eta_{d_i}) \in (\mathbf{R}^2)^{d_i}$

$$H_i(\eta_1, ..., \eta_{d_i}) = \pi \sum_{j \neq k} \log \frac{1}{\mid \eta_j - \eta_k \mid}$$

and for $i = l+1, ..., m$ and $(\eta_1, ..., \eta_{d_i}) \in (\mathbf{R}_+^2)^{d_i}$

$$H_i(\eta_1, ..., \eta_{d_i}) = \pi \sum_{j \neq k} \log \frac{1}{\mid \eta_j - \eta_k \mid} + \pi \sum_{j,k} \log \frac{1}{\mid \eta_j - \overline{\eta}_k \mid}$$

where

$$\mathbf{R}_+^2 = \{(x_1, x_2) \in \mathbf{R}^2, x_2 > 0\}$$

and $\overline{\eta}_k = r(\eta_k)$, r being the reflection with respect to $\{(x_1, x_2) \in \mathbf{R}^2, x_2 = 0\}$. Let us recall (see [AS2]) that for $i = 1, ..., l$, $(\omega_{i1}, ..., \omega_{id_i})$ realizes the minimum in $\mathbf{R}^2$ of

$$G_i(\eta_1, ..., \eta_{d_i}) = H_i(\eta_1, ..., \eta_{d_i}) + \pi C_{a_i} \sum_{i=1}^{d_i} \mid \eta_j \mid^2 .$$

We have the following theorems

THEOREM 1. Let $i = 1, ..., l$. If $d_i = 1$,

$$\mid x_i^{\varepsilon_n} - a_i \mid = O(\frac{\varepsilon_n^{\frac{1}{3}}}{\log^{\frac{1}{2}} \frac{1}{\varepsilon_n}}).$$

If $d_i > 1$ and if the configuration $(\omega_{i1}, ..., \omega_{id_i})$ realizes a nondegenerate minimum of G_i, then

$$\mid x_{ij}^{\varepsilon_n} - y_{ij}^{\varepsilon_n} \mid = O(\frac{1}{\log^{\frac{3}{2}} \frac{1}{\varepsilon_n}} \log\log \frac{1}{\varepsilon_n}).$$

THEOREM 2. For $i = l+1, ..., m$ we assume that, locally near a_i, G is the half plane $\mathbf{R}^2_+$. The configuration $(\omega_{i1}, ..., \omega_{id_i})$ realizes the minimum of G_i in $(\mathbf{R}^2_+)^{d_i}$. If it is a nondegenerate point, then

$$\mid x_{ij}^{\varepsilon_n} - y_{ij}^{\varepsilon_n} \mid = O(\frac{1}{\log^{\frac{5}{8}} \frac{1}{\varepsilon_n}} \log^{\frac{1}{2}}(\log \frac{1}{\varepsilon_n})).$$

Next we set

$$A_i = \frac{d_i^2 - d_i}{2} \quad i = 1, ..., l$$

and

$$A_i = \frac{2d_i^2 - d_i}{2} \quad i = l+1, ..., m.$$

We consider the universal constant γ defined in [BBH2] by

$$\gamma = \lim_{\varepsilon \to 0} (I(\varepsilon, 1) - \pi \log \frac{1}{\varepsilon}),$$

$$I(\varepsilon, 1) = \min_{u \in H^1_{\frac{x}{|x|}}} (\frac{1}{2} \int_{B(0,1)} \mid \nabla u \mid^2 + \frac{1}{4\varepsilon^2} \int_{B(0,1)} (1 - \mid u \mid^2)^2).$$

For $b = (b_1, ..., b_m)$, $W_G(b, d_1, ..., d_m, g, p)$ is defined by (2.1) with $\Omega = G$, and $h_0 = g$.

THEOREM 3. The configuration $(a_1, ..., a_m)$ minimizes in $\Lambda_2^l \times (\Lambda_1 \backslash \Lambda_2)^{m-l}$ the map

$$\overline{W}(b) = W_G(b, d_1, ..., d_m, g, p) + \min_{\{\eta; C_{b_i} \sum_{j=1}^{d_i} |\eta_j|^2 = p_0 A_i\}} \sum_{i=1}^{m} H_i$$

and there exists a map $X(\varepsilon)$ that tends to 0 as ε tends to 0 such that

$$E_\varepsilon(u_\varepsilon) = \pi d p_0 \log \frac{1}{\varepsilon} + \pi p_0 F(d)(\log\log \frac{1}{\varepsilon} + 1) + \min_{\Lambda_2^l \times (\Lambda_1 \backslash \Lambda_2)^{m-l}} \overline{W} + d p_0 \gamma + X(\varepsilon).$$

2. Proof of Theorems 1, 2 and 3.

The domain Ω will be G or $B(0,1)$. Let $h_0 \in C^\infty(\partial\Omega, S^1)$ be such that $deg(h_0, \partial\Omega) = d$. Let $m \in \mathbf{N}^*$. We consider the positive integers $d_1, ..., d_m$ that verify $\sum_{j=1}^m d_j = d$ and $b = (b_1, ..., b_m) \in \Omega^l \times (\partial\Omega)^{m-l}$. We denote $\overline{d}_j = d_j$ for $j = 1, ..., l$, $\overline{d}_j = 2d_j$, $j = l+1, ..., m$ and $\overline{d} = \sum_{j=1}^m \overline{d}_j$. In the sequel we distinguish two cases. In the case I, the points b_j are fixed in $\overline{\Omega}$, $b_j \in \Omega$ for $j = 1, ..., l$, $b_j \in \partial\Omega$ for $j = l+1, ...m$ and in a neighborhood of b_j we have $p(x) = p_0 + C \mid x - b_j \mid^2 + o(\mid x - b_j \mid^2)$. In the case II we have $b_j \in \Omega$ for all $j = 1, ..., m$, that is $l = m$, and there exists $\delta > 0$ such that

$b=(b_1,...,b_m)\in\mathcal{A}_\delta$ where we denote $\mathcal{A}_\delta$ the set of the $b=(b_1,...,b_m)\in\Omega^m$ verifying $\mid b_j-b_k\mid\geq 2\delta$, $j\neq k$, the distance from b_j to $\partial\Omega$ is greater than a positive constant and is not taken into account. Moreover there exists $a_j\in\Omega$ with $\mid a_j-b_j\mid\leq\delta$ and with $p(x)=p_0+C\mid x-a_j\mid^2+o(\mid x-a_j\mid^2)$ in a neighborhood of a_j. Let $h=(h_1,...,h_m)\in\mathcal{C}^\infty(S^1,S^1)^m$ be such that

$$deg(h_j,S^1)=d_j,\quad j=1,...,l$$

and

$$deg(h_j,S^1)=2d_j,\quad j=l+1,...,m.$$

In the case I, since $\partial\Omega$ is regular, there exists, for $j=l+1,...,m$, a neighborhood $\mathcal{U}_j$ of b_j and a conformal change of variables $H_j:Q\to\mathcal{U}_j$, such that $H_j(Q+)=\mathcal{U}_j\cap\Omega$ and $H_j(Q_0)=\mathcal{U}_j\cap\partial\Omega$, where $Q=B(0,1)$, $Q_+=\{(x_1,x_2)\in Q,x_2>0\}$, $Q_0=\{(x_1,x_2)\in Q,x_2=0\}$. Let r be the reflection with respect to Q_0. We suppose that

$$h_j(\frac{x-b_j}{\mid x-b_j\mid})=h_j(\frac{H_joroH_j^{-1}(x)-b_j}{\mid H_joroH_j^{-1}(x)-b_j\mid})\text{ for all }x\in\mathcal{U}_j\ .$$

In the sequel we may assume for simplicity, that there exists $\beta>0$ such that

$$\Omega\cap B(b_j,\beta)\text{ is the half disc }\{(x_1,x_2);x_1^2+x_2^2\leq\beta^2,x_2>0\},$$

and

$$\partial\Omega\cap B(b_j,\beta)=\{(x_1,x_2),x_1^2+x_2^2\leq\beta^2,x_2=0\}.$$

We suppose that the balls $B(b_j,\beta)$, $j=1,...,m$, are disjoint. Now h_j verifies, for all $x\in B(b_j,\beta)$,

$$h_j(\frac{x-b_j}{\mid x-b_j\mid})=h_j(\frac{r_j(x)-b_j}{\mid r_j(x)-b_j\mid})$$

where r_j is the reflection associated to the flat boundary $\partial\Omega\cap B(b_j,\beta)$, $j=l+1,...m$. We suppose that

$$h_j(\frac{x-b_j}{\rho})=h_0(x)\text{ for }j=l+1,...,m\text{ and for all }x\in\partial\Omega\cap\partial B(b_j,\rho)\ .$$

Let $\mathcal{M}>0$ be fixed. We denote $\mathcal{H}_\mathcal{M}$ the set of the regular maps $h:S^1\to S^1$ such that $\mid\frac{\partial^k h}{\partial\tau^k}\mid\leq\mathcal{M}$, $k\leq 3$. In the sequel we suppose that h_0 is in $\mathcal{H}_\mathcal{M}$ and that $h_1,....,h_m$ are as above and are in $\mathcal{H}_\mathcal{M}$. We define for $\rho<\beta$ in the case I and for $\rho<\delta$ in the case II

$$\gamma_0=(\partial\Omega\backslash\cup_{j=l+1}^m B(b_j,\rho))\cup_{j=l+1}^m(\partial B(b_j,\rho)\cap\Omega),$$

$$\gamma_j=\partial B(b_j,\rho),\quad j=1,...,l,$$

and

$$g_0(x)=\begin{cases}h_0(x)\text{ on }\partial\Omega\backslash\cup_{j=l+1}^m B(b_j,\rho)\\ h_j(\frac{x-b_j}{\mid x-b_j\mid})\text{ on }\partial B(b_j,\rho)\cap\Omega,\quad j=l+1,...,m,\end{cases}$$

$$g_j(x) = h_j(\frac{x-b_j}{\mid x-b_j \mid}) \text{ on } \partial B(b_j,\rho), \quad j=1,...,l.$$

We set

$$\Omega_\rho^b = \Omega \backslash \cup_{j=1}^m B(b_j,\rho).$$

We consider the following minimization problem

$$(E_\rho^b) \qquad \min_{u\in\mathcal{E}_\rho} \int_{\Omega_\rho^b} p \mid \nabla u \mid^2$$

where

$$\mathcal{E}_\rho = \{v \in H^1(\Omega_\rho^b, S^1); \forall j=0,...,m, \exists \alpha_j \in \mathbf{C}, \mid \alpha_j \mid = 1 \text{ such that } v(z) = \alpha_j g_j \text{ on } \gamma_j\}.$$

We turn to the definition of a renormalized energy in the sense of [BBH2], with an observation of Ragazzo (see [B]). This definition is the same as in [BH2] with a minor modification. We define

$$(2.1) \qquad W_\Omega(b,d_1,...,d_m,h_0,p) = -\pi \sum_{i\neq j}^m d_i \bar{d}_j p(b_i) \log|b_i-b_j| - \pi \sum_{i,j} d_i \bar{d}_j R(b_i,b_j),$$

where the function $R_j = R(.,b_j)$, $j=1,...,m$ is defined by

$$(2.2) \qquad \begin{cases} div(\frac{1}{p}\nabla R_j) = -p(b_j)\nabla\frac{1}{p}\nabla \log|x-b_j| & \text{in} \quad \Omega \\ \frac{1}{p}\frac{\partial R_j}{\partial \nu} = (1-\frac{p(b_j)}{p})\frac{\partial \log|x-b_j|}{\partial \nu} + \frac{1}{\bar{d}}\frac{\partial \varphi_0}{\partial \tau} & \text{on} \quad \partial\Omega, \end{cases}$$

with the normalization condition

$$(2.3) \qquad \int_{\partial\Omega} (h_0 \times (h_0)_\tau) R_j = -p(b_j) \int_{\partial\Omega} (g \times g_\tau) \log|x-b_j|.$$

The function φ_0 is defined on $\partial\Omega$ by

$$h_0(z) = \Pi_{j=1}^m (\frac{z-b_i}{|z-b_i|})^{\bar{d}_j} e^{i\varphi_0(z)}.$$

The function R_j exists in $\overline{\Omega}$. Indeed, we have

$$-p(b_j)\int_\Omega \nabla\frac{1}{p}.\nabla \log|x-b_j| = \int_{\partial\Omega} ((1-\frac{p(b_j)}{p})\frac{\partial \log|x-b_j|}{\partial\nu} + \frac{1}{\bar{d}}\frac{\partial\varphi_0}{\partial\tau}).$$

We note that R is the regular part of the function $H_j = H(.,b_j)$ defined by

$$H(x,b_j) = R(x,b_j) + p(b_j)\log|x-b_j|$$

that is the solution of

$$
\text{(2.4)} \qquad \begin{cases} div(\frac{1}{p}\nabla H_j) = 2\pi\delta_{b_j} \quad \text{in} \quad \Omega \\ \dfrac{1}{p}\dfrac{\partial H_j}{\partial\nu} = \dfrac{\partial \log|x-b_j|}{\partial\nu} + \dfrac{1}{\overline{d}}\dfrac{\partial\varphi_0}{\partial\tau} \quad \text{on} \quad \partial\Omega, \end{cases}
$$

with the normalization condition

$$
\text{(2.5)} \qquad \int_{\partial\Omega} (h_0 \times (h_0)_\tau) H_j = 0.
$$

Note that, for $j = l+1, ..., m$, $\frac{\partial \log|x-b_j|}{\partial\nu}$ is in the dual of $W^{1-\frac{1}{p},p}(\partial\Omega)$ for all $p > 2$ and that if $f \in W^{1-\frac{1}{p},p}(\partial\Omega)$ we have

$$
< \frac{\partial \log|x-b_j|}{\partial\nu}, f >= \int_\Omega \nabla \log|x-b_j|.\nabla Pf
$$

where $P : W^{1-\frac{1}{p},p}(\partial\Omega) \to W^{1,p}(\Omega)$ is a continuous linear operator. Let us define Φ_0 by

$$
\Phi_0(x) = \sum_{j=1}^{m} \overline{d}_j H(x, b_j).
$$

We define $W^1(h_j) = \frac{\overline{d}_j^2}{2} \int_{B(0,1)} \mid \nabla\Psi \mid^2$, Ψ being the solution of

$$
\text{(2.6)} \qquad \begin{cases} \Delta\Psi = 0 \text{ in } B(0,1) \\ \dfrac{\partial\Psi}{\partial\nu} = \dfrac{h_j \times (h_j)_\tau}{\overline{d}_j} - 1 \text{ on } S^1 \\ \displaystyle\int_{S^1} (h_j \times (h_j)_\tau)\Psi = 0. \end{cases}
$$

Let us remark that $W^1(h_j)$ is the renormalized energy in $B(0,1)$, for the boundary value h_j, a singularity of degree $\overline{d}_j$ at the point 0 and for p constant and equal to 1.

We note that we have in the case I

$$
\text{(2.7)} \qquad || R_j ||_{W^{2,2}(\Omega)} \leq M
$$

In the case II, we have for all $1 \leq q < 2$

$$
|| R_j ||_{W^{2,q}(\Omega)} \leq M,
$$

and these estimates are valid uniformly for h_0 in $\mathcal{H}_{\mathcal{M}}$. More precisely, we claim that there exists a map θ_{b_j} which is bounded in $W^{2,2}(\Omega)$, uniformly for $h_0 \in \mathcal{H}_M$ and for $b \in \mathcal{A}_\delta$ such that, for $j = 1, ..., m$,

$$
\text{(2.8)} \qquad R(b_j, x) = -\frac{p^2(b_j)}{2}\nabla\frac{1}{p}(b_j).(x-b_j)\log \mid x - b_j \mid + \theta_{b_j}(x).
$$

Indeed, one can applies standard elliptic estimates (see [GT]) to the equation satisfied by the map θ_{b_i}, which is

$$\begin{cases} \operatorname{div}(\frac{1}{p}\nabla\theta_{b_i}) = \frac{p^2(b_i)}{2}\nabla\frac{1}{p}.((x-b_i).\nabla\frac{1}{p}(b_i)\nabla\log\mid x-b_i\mid +\nabla\frac{1}{p}(b_i)\log\mid x-b_i\mid) \\ +\frac{p(b_i)}{p}(p(b_i)\nabla\frac{1}{p}(b_i).\nabla\log\mid x-b_i\mid -p\nabla\frac{1}{p}.\nabla\log\mid x-b_i\mid \text{ in } \Omega \\ \frac{1}{p}\frac{\partial\theta_{b_i}}{\partial\nu} = (1-\frac{p(b_i)}{p})\frac{\partial}{\partial\nu}\log\mid x-b_i\mid +\frac{1}{\bar{d}}\frac{\partial\varphi_0}{\partial\tau} \\ -\frac{p^2(b_i)}{2p}\nabla\frac{1}{p}(b_i).(x-b_i)\frac{\partial}{\partial\nu}\log\mid x-b_i\mid -\frac{p^2(b_i)}{2p}\log\mid x-b_i\mid\nabla\frac{1}{p}(b_i).\nu \text{ on } \partial\Omega. \end{cases}$$

Thus we have the following estimates, uniformly for $h_0 \in \mathcal{H}_{\mathcal{M}}$,

$$\mid R(x,b_j)\mid = O(C\mid b_j-a_j\mid\mid x-b_j\mid\log\frac{1}{\mid x-b_j\mid}) + O(1) \tag{2.9}$$

and

$$\mid\nabla R(x,b_j)\mid = O(C\mid b_j-a_j\mid\log\frac{1}{\mid x-b_j\mid}) + O(1). \tag{2.10}$$

Let us derive an estimate, in the case II, which will be useful in the sequel. Let c_j be a point of Ω that realizes the same conditions that b_j. We define

$$\overline{R}_{b,c}(x) = \sum_{i=1}^{m}(R(b_i,x) - R(c_i,x))$$

and

$$\overline{\theta}_{b,c} = \sum_{i=1}^{m}(\theta_{b_i} - \theta_{c_i}).$$

Using standard elliptic estimates, we obtain

$$\mid\overline{\theta}_{b,c}\mid\mid_{W^{2,2}(\Omega)} = \sum_i O(C\mid b_i-c_i\mid),$$

thus (2.8) gives

$$\mid\overline{R}_{b,c}\mid\mid_{L^\infty(\Omega)} = \sum_i O(C\mid b_i-c_i\mid). \tag{2.11}$$

In the next two results, the parameters δ and C are taken into account only in the case II. We have the following theorem

THEOREM 4. Let $\hat{u}_\rho^b$ be a minimizer for (E_ρ^b). We have, uniformly for $(h_0, h_1, ..., h_m)$ in $\mathcal{H}_{\mathcal{M}}^{m+1}$, in the case I

$$\frac{1}{2}\int_{\Omega_\rho^b} p \mid \nabla \hat{u}_\rho^b \mid^2 = \pi \sum_{j=1}^{l} d_j^2 p_0 \log\frac{1}{\rho} + 2\pi \sum_{j=l+1}^{m} d_j^2 p_0 \log\frac{1}{\rho} + W_\Omega(b, d_1, ..., d_m, h_0, p)$$

$$+ \sum_{j=1}^{l} p_0 W^1(h_j) + \sum_{j=l+1}^{m} \frac{p_0}{2} W^1(h_j) + O(\rho \log^2 \frac{1}{\rho}),$$

and in the case II,

$$\frac{1}{2}\int_{\Omega_\rho^b} p \mid \nabla \hat{u}_\rho^b \mid^2 = \pi \sum_{j=1}^{m} d_j^2 p(b_j) \log\frac{1}{\rho} + W_\Omega(b, d_1, ..., d_m, h_0, p)$$

$$+ \sum_{j=1}^{k} \frac{p(b_j)^2}{p_0} W^1(h_j) + \sum_{j \neq k} O(\frac{\rho}{\mid b_j - b_k \mid}) + \sum_j O(C\rho \mid b_j - a_j \mid \log\frac{1}{\rho}).$$

We need the following proposition

PROPOSITION 1. We have, uniformly for $(h_0, h_1, ..., h_m)$ in $\mathcal{H}_{\mathcal{M}}^{m+1}$,

$$\frac{1}{2}\int_{\Omega_\rho} \frac{1}{p} \mid \nabla \Phi_0 \mid^2 = \pi \sum_{j=1}^{l} d_j^2 p(b_j) \log\frac{1}{\rho} + 2\pi \sum_{j=l+1}^{m} d_j^2 p(b_j) \log\frac{1}{\rho} + W_\Omega(b, d_1, ..., d_m, h_0, p)$$

$$+ X(\rho),$$

where $X(\rho) = O(\rho \log\frac{1}{\rho})$ in the case I and $X(\rho) = \sum_j O(C\rho \mid b_j - a_j \mid \log\frac{1}{\rho}) + \sum_{j \neq k} O(\frac{\rho}{|b_j - b_k|})$, in the case II.

We postpone he proof of Proposition 1 and Theorem 4 to the section 3 and we present the proofs of Theorems 1, 2 and 3. We start with some notations. Let $a_i \in \overline{G}$ and let x_{ij}^ε, $j = 1, ..., d_i$ be the zeroes of u_ε which tend to a_i. Let $\rho > 0$ be such that $\mid a_i - a_j \mid > 2\rho$ for $i \neq j$ and $0 < R < \rho$ such that $B(x_{ij}^\varepsilon, R) \subset B(a_i, \rho)$, $j = 1, ..., d_i$. We set $h_i(e^{i\theta}) = \frac{u_\varepsilon}{|u_\varepsilon|}(a_i + \rho e^{i\theta})$ and $k_{ij}(e^{i\theta}) = \frac{u_\varepsilon}{|u_\varepsilon|}(x_{ij}^\varepsilon + Re^{i\theta})$. For $i = l+1, ..., m$ we define h_i on S^1 by reflection with respect to the flat boundary $\partial G \cap B(a_i, \rho)$. We note that k_{ij} and h_i are in $\mathcal{H}_{\mathcal{M}}$, $j = 1, ..., d_i$ for some $\mathcal{M}$ independent of ε. Indeed, we follow the proof of Proposition 1.2, Part II in [AS2], we deduce that for any $k \geq 1$, there exists M independent of ε such that

$$|D^k u_\varepsilon(x)| \leq \frac{M}{|x_\varepsilon - x|^k} \text{ in } B(x_{ij}^\varepsilon, R_0) \backslash B(x_{ij}^\varepsilon, \alpha\varepsilon)$$

and

$$|D^k u_\varepsilon(x)| \leq \frac{M}{|x - a_i|^k} \text{ in } B(a_i, \rho_1) \backslash B(a_i, \rho_0)$$

where $B(x_{ij}^\varepsilon, R_0)$ does not contain any other zero of u_ε than x_{ij}^ε and ρ_0 is such that $x_{ij}^\varepsilon \in B(a_i, \rho_0)$, $j = 1, ..., d_i$. This implies that k_{ij} and h_i are in $\mathcal{H}_\mathcal{M}$, $j = 1, ..., d_i$ for some $\mathcal{M}$. In all what follows we denote $\varepsilon = \varepsilon_n$.

Proof of Theorem 1. We suppose now that $a_i \in G$ and $d_i = 1$. We denote $a = a_i$ and x^ε the zero of u_ε that tends to a. We define $h = h_i$ and $k = k_{ij}$ as below. We know that $\alpha\varepsilon \leq \mid x^\varepsilon - a \mid = o(\frac{1}{\log^{\frac{1}{2}} \frac{1}{\varepsilon}})$. Let us give a lower bound for $E_\varepsilon(u_\varepsilon)$. As in [CM], proof of Theorem 5, we have

$$E_\varepsilon(u_\varepsilon, G\backslash B(a,\rho)) \geq \frac{1}{2}\int_{G\backslash B(a,\rho)} p|\nabla(\frac{u_\varepsilon}{|u_\varepsilon|})|^2 + O(\frac{\varepsilon^2}{\rho^2})$$

thus

$$\begin{aligned} E_\varepsilon(u_\varepsilon) \geq & \frac{1}{2}\int_{G(a,\rho)} p|\nabla(\frac{u_\varepsilon}{|u_\varepsilon|})|^2 + \frac{1}{2}\int_{B(a,\rho)\backslash B(x^\varepsilon,R)} p|\nabla(\frac{u_\varepsilon}{|u_\varepsilon|})|^2 + p_0 I(\varepsilon, R, k) \\ & + O(\frac{\varepsilon^2}{R^2}). \end{aligned}$$

Applying Theorem 4, case II we infer

$$\begin{aligned} E_\varepsilon(u_\varepsilon) \geq & \frac{1}{2}\int_{G\backslash B(a,\rho)} p|\nabla(\frac{u_\varepsilon}{|u_\varepsilon|})|^2 + \pi p(x^\varepsilon)\log\frac{1}{R} + W_{B(a,\rho)}(x^\varepsilon, 1, h, p) \\ & + \frac{p^2(x^\varepsilon)}{p_0} W^1(k) + p_0 I(\varepsilon, R, k) + O(R) \\ & + O(\mid x^\varepsilon - a \mid R \log\frac{1}{R}) + O(\frac{\varepsilon^2}{R^2}). \end{aligned} \tag{2.12}$$

We turn now to an upper bound for $E_\varepsilon(u_\varepsilon)$. We construct a function w equal to $\frac{u_\varepsilon}{|u_\varepsilon|}$ in $G\backslash B(a,\rho)$, such that w realizes (E_R^a) in $B(a,\rho)\backslash B(a,R)$ with the boundary dates h and k and realizes $I(\varepsilon, R, k)$ in $B(a,R)$.

Theorem 4 case II gives

$$\begin{aligned} E_\varepsilon(u_\varepsilon) \leq & \frac{1}{2}\int_{G\backslash B(a,\rho)} p|\nabla(\frac{u_\varepsilon}{|u_\varepsilon|})|^2 + \pi p_0 \log\frac{1}{R} + W_{B(a,\rho)}(a, 1, h, p) + p_0 W^1(k) \\ & + (p_0 + CR^2) I(\varepsilon, R, k) + O(R) + o(R^2 I(\varepsilon, R, k)). \end{aligned} \tag{2.13}$$

Now, using the fact that $I(\varepsilon, R, k) = O(\log\frac{R}{\varepsilon})$, soustraying (2.13) and (2.12) we deduce

$$\begin{aligned} & \pi C|x^\varepsilon - a|^2 \log\frac{1}{R} + \frac{p^2(x^\varepsilon) - p_0^2}{p_0} W^1(k) + W_{B(a,\rho)}(x^\varepsilon, 1, h, p) - W_{B(a,\rho)}(a, 1, h, p) \\ & \leq O(\frac{\varepsilon^2}{R^2}) + O(R) + O(\mid x^\varepsilon - a \mid R \log\frac{1}{R}) + O(R^2 \log\frac{1}{\varepsilon}). \end{aligned} \tag{2.14}$$

We have

$$p^2(x^\varepsilon) - p_0^2 = O(\mid x^\varepsilon - a \mid^2),$$

and

$$W_{B(a,\rho)}(x^\varepsilon,1,h,p)-W_{B(a,\rho)}(a,1,h,p)=R_{B(a,\rho)}(x^\varepsilon,x^\varepsilon)-R_{B(a,\rho)}(a,a)$$

where $R_{B(a,\rho)}$ is defined in (2.2) with $\Omega=B(a,\rho)$ and $h_0=h$. By renormalization we get

$$R_{B(a,\rho)}(x^\varepsilon,x)=R(\omega^\varepsilon,y)$$

where $\omega^\varepsilon=\frac{x^\varepsilon-a}{\rho}$, $y=\frac{x-a}{\rho}$ and $R(\omega^\varepsilon,.)$ is defined by (2.2) with $\Omega=B(0,1)$, $h_0=h$ and p is replaced by $\tilde{p}$, $\tilde{p}(y)=p(a+\rho y)$. Thus

$$R_{B(a,\rho)}(x^\varepsilon,x^\varepsilon)-R_{B(a,\rho)}(a,a)=R(\omega^\varepsilon,\omega^\varepsilon)-R(0,0).$$

Using the notation of (2.11) we set

$$R(\omega^\varepsilon,\omega^\varepsilon)-R(0,0)=\overline{R}_{\omega^\varepsilon,0}(\omega^\varepsilon)+\overline{R}_{\omega^\varepsilon,0}(0).$$

In (2.11) we replace C by ρ^2, in view of the definition of $\tilde{p}$, and we get

$$R(\omega^\varepsilon,\omega^\varepsilon)-R(0,0)=O(\rho^2\mid\omega^\varepsilon\mid),$$

that gives

$$W_{B(a,\rho)}(x^\varepsilon,1,h,p)-W_{B(a,\rho)}(a,1,h,p)=O(\rho\mid x_\varepsilon-a\mid).$$

Now we turn to (2.14) and we choose $\rho=\lambda\mid x^\varepsilon-a\mid$ for some given $\lambda>0$ independent of ε. We obtain

$$\begin{aligned}&\pi C|x^\varepsilon-a|^2\log\frac{1}{R}+O(\mid x_\varepsilon-a\mid^2)\le O(\frac{\varepsilon^2}{R^2})+O(R)\\&+O(\mid x^\varepsilon-a\mid R\log\frac{1}{R})+O(R^2\log\frac{1}{\varepsilon}).\end{aligned}\tag{2.15}$$

Let $X=\mid x^\varepsilon-a\mid$. We deduce from (2.15) that there exists $K_1>0$ and $K_2>0$ such that

$$K_1X^2\log\frac{1}{R}-K_2XR\log\frac{1}{R}\le O(\frac{\varepsilon^2}{R^2}+R+R^2\log\frac{1}{\varepsilon}).\tag{2.16}$$

The optimal choice for R is $R=\varepsilon^{\frac{2}{3}}$. This choice of R is possible if we suppose that there exists $\mu>0$ such that $\mid x^\varepsilon-a\mid\ge\mu\varepsilon^{\frac{2}{3}}$. In this case (2.16) gives

$$K_1X^2\log\frac{1}{\varepsilon}-K_2X\varepsilon^{\frac{2}{3}}\log\frac{1}{\varepsilon}\le O(\varepsilon^{\frac{2}{3}})$$

and consequently

$$X=O(\frac{\varepsilon^{\frac{1}{3}}}{\log^{\frac{1}{2}}\frac{1}{\varepsilon}}).$$

We have proved the first part of Theorem 1.

We suppose now that $a_i \in G$ and $d_i > 1$. We denote $a_i = a$, $x^\varepsilon_{ij} = x^\varepsilon_j$ and $\sigma = \frac{1}{\log^{\frac{1}{2}} \frac{1}{\varepsilon}}$. We choose ρ of the order of σ. We denote $\omega^\varepsilon_j = \frac{x^\varepsilon_j - a}{\rho}$ and $\omega_j = \lim_{\varepsilon \to 0} \omega^\varepsilon_j$. We set $\tilde{p}(x) = p(a + \rho x)$ and $\frac{\tilde{u}_\varepsilon}{|\tilde{u}_\varepsilon|}(x) = \frac{u_\varepsilon}{|u_\varepsilon|}(a + \rho x)$. We have the following lower bound for $E_\varepsilon(u_\varepsilon)$

$$E_\varepsilon(u_\varepsilon) \geq \frac{1}{2} \int_{G \setminus B(a,\rho)} p \mid \nabla \frac{u_\varepsilon}{\mid u_\varepsilon \mid} \mid^2 + O(\frac{\varepsilon^2}{\rho^2}) + \int_{B(0,1) \setminus \cup_j B(\omega^\varepsilon_j, \frac{R}{\rho})} \tilde{p} \mid \nabla \frac{\tilde{u}_\varepsilon}{\mid \tilde{u}_\varepsilon \mid} \mid^2$$
$$+ \sum_j \int_{B(x^\varepsilon_j, R)} p \mid \nabla \frac{u_\varepsilon}{\mid u_\varepsilon \mid} \mid^2 + O(\frac{\varepsilon^2}{R^2}).$$

By Theorem 4, case II, we have

$$\int_{B(0,1) \setminus \cup_j B(\omega^\varepsilon_j, \frac{R}{\rho})} \tilde{p} \mid \nabla \frac{\tilde{u}_\varepsilon}{\mid \tilde{u}_\varepsilon \mid} \mid^2 \geq \pi \sum_j p(x^\varepsilon_j) \log \frac{\rho}{R} + W_{B(0,1)}(\omega^\varepsilon, 1, ..., 1, h, \tilde{p})$$
$$+ \sum_j \frac{p^2(x^\varepsilon_j)}{p_0} W^1(k_j) + O(\frac{R}{\rho}) + O(R\rho \log \frac{\rho}{R}).$$

We have used the fact that $\tilde{p}(x) = p_0 + O(\rho^2 \mid x \mid^2)$. Thus the constant C used in Theorem 4 case II is here ρ^2. Moreover

$$\int_{B(x^\varepsilon_j, R)} p \mid \nabla \frac{u_\varepsilon}{\mid u_\varepsilon \mid} \mid^2 \geq (p_0 + C \min_{x \in B(x^\varepsilon_j, R)} \mid x - a \mid^2) \int_{B(x^\varepsilon_j, R)} \mid \nabla \frac{u_\varepsilon}{\mid u_\varepsilon \mid} \mid^2$$
$$\geq (p_0 + C\rho^2 (\mid \omega^\varepsilon_j \mid - \frac{R}{\rho})^2) I(\varepsilon, R, k_j)$$

and

$$I(\varepsilon, R, k_j) = \pi \log \frac{R}{\varepsilon} + O(1).$$

We are led to the following lower bound (ρ being of the order of σ)

$$\begin{aligned} E_\varepsilon(u_\varepsilon) &\geq \frac{1}{2} \int_{G \setminus B(a,\rho)} p \mid \nabla \frac{u_\varepsilon}{\mid u_\varepsilon \mid} \mid^2 + \pi \sum_j p(x^\varepsilon_j) \log \frac{\rho}{R} \\ &+ W_{B(0,1)}(\omega^\varepsilon, 1, ..., 1, h, \tilde{p}) + \sum_j \frac{p^2(x^\varepsilon_j)}{p_0} W^1(k_j) + \pi \sum_j p_0 I(\varepsilon, R, k_j) \\ &+ \sum_j C\pi\rho^2 \mid \omega^\varepsilon_j \mid^2 \log \frac{R}{\varepsilon} + O(\frac{\varepsilon^2}{R^2}) + O(R\sigma \log \frac{\sigma}{R}) \\ &+ O(\frac{R}{\sigma}) + O(R\sigma \log \frac{R}{\varepsilon}). \end{aligned} \tag{2.17}$$

We turn now to an upper bound for $E_\varepsilon(u_\varepsilon)$. Let y_j, $j = 1, ..., d_i$ be any points in $\mathbf{R}^2$ and $y^\varepsilon_j = a + \rho y_j$. We construct a map w such that $w = \frac{u_\varepsilon}{|u_\varepsilon|}$ in $G \setminus B(a, \rho)$. In $B(a, \rho) \setminus \cup_j B(y^\varepsilon_j, R)$

we set $w(x)=\tilde{w}(\frac{x-a}{\rho})$ where $\tilde{w}$ realizes $(E^{y}_{\frac{R}{\rho}})$ in $B(0,1)\backslash\cup_j B(y_j,\frac{R}{\rho})$ with the weight $\tilde{p}$ and the boundary conditions h and k_j, $j=1,...,d_i$. In $B(y^{\varepsilon}_j,R)$, w realizes $I(\varepsilon,R,k_j)$. Using Theorem 4, case II we obtain

$$
\begin{aligned}
E_\varepsilon(u_\varepsilon) \leq {} & \frac{1}{2}\int_{G\backslash B(a,\rho)} p\mid \nabla\frac{u_\varepsilon}{\mid u_\varepsilon\mid}\mid^2 +\pi\sum_j p(y^{\varepsilon}_j)\log\frac{\rho}{R} \\
& + W_{B(0,1)}(y,1,...,1,h,\tilde{p}) + \sum_j \frac{p^2(y^{\varepsilon}_j)}{p_0}W^1(k_j) + \sum_j p_0 I(\varepsilon,R,k_j) \\
& + \sum_j C\pi\rho^2\mid y_j\mid^2\log\frac{R}{\varepsilon} + O(R\sigma\log\frac{R}{\varepsilon}) \\
& + O(\frac{R}{\sigma}) + O(R\sigma\log\frac{\sigma}{R}).
\end{aligned}
\tag{2.18}
$$

Soustraying (2.17) and (2.18) we are led to

$$
\begin{aligned}
& \pi\sum_j(p(x^{\varepsilon}_j)-p(y^{\varepsilon}_j))\log\frac{\rho}{R} + W_{B(0,1)}(\omega^{\varepsilon},1,...,1,h,\tilde{p}) - W_{B(0,1)}(y,1,...,1,h,\tilde{p}) \\
& + C\pi\sum_j(\mid\omega^{\varepsilon}_j\mid^2-\mid y_j\mid^2)\rho^2\log\frac{R}{\varepsilon} + \sum_j O(\sigma^2\mid\omega^{\varepsilon}_j-y_j\mid) \leq O(\frac{\varepsilon^2}{R^2}) \\
& + O(R\sigma\log\frac{\sigma}{R}) + O(\frac{R}{\sigma}) + O(R\sigma\log\frac{R}{\varepsilon}).
\end{aligned}
\tag{2.19}
$$

We have used that $p^2(x^{\varepsilon}_j)-p^2(y^{\varepsilon}_j)=O(\sigma^2\mid\omega^{\varepsilon}_j-y_j\mid)$. If we take the optimal choice $R=\varepsilon^{\frac{2}{3}}\sigma^{\frac{1}{3}}$, the right hand side of (2.19) is $O(\frac{\varepsilon^{\frac{2}{3}}}{\sigma^{\frac{2}{3}}})$ and we obtain

$$
\begin{aligned}
& \pi\sum_j C(\mid\omega^{\varepsilon}_j\mid^2-\mid y_j\mid^2)\rho^2\log\frac{\rho}{\varepsilon^{\frac{2}{3}}\sigma^{\frac{1}{3}}} + \pi\sum_j C(\mid\omega^{\varepsilon}_j\mid^2-\mid y_j\mid^2)\rho^2\log\frac{\sigma^{\frac{1}{3}}}{\varepsilon^{\frac{1}{3}}} \\
& + W_{B(0,1)}(\omega^{\varepsilon},1,...,1,h,\tilde{p}) - W_{B(0,1)}(y,1,...,1,h,\tilde{p}) \\
& \leq O(\frac{\varepsilon^{\frac{2}{3}}}{\sigma^{\frac{2}{3}}}) + \sum_j O(\sigma^2\mid\omega^{\varepsilon}_j-y_j\mid).
\end{aligned}
\tag{2.20}
$$

In view of (2.1) we have

$$W_{B(0,1)}(\omega^\varepsilon, 1, ..., 1, h, \tilde{p}) - W_{B(0,1)}(y, 1, ..., 1, h, \tilde{p}) = \pi \sum_{i\neq j} p_0 \log \frac{\mid y_i - y_j \mid}{\mid \omega_i^\varepsilon - \omega_j^\varepsilon \mid}$$

$$- \pi \sum_{i\neq j} (p(x_i^\varepsilon) - p(y_i^\varepsilon)) \log(\mid x_i^\varepsilon - x_j^\varepsilon \mid) - \pi \sum_{i\neq j} (p(y_i^\varepsilon) - p_0) \log \frac{\mid \omega_i^\varepsilon - \omega_j^\varepsilon \mid}{\mid y_i - y_j \mid}$$

$$- \pi \sum_{i,j} (R_{B(0,1)}(\omega_i^\varepsilon, \omega_j^\varepsilon) - R_{B(0,1)}(y_i, y_j))$$

$$= \pi \sum_{i\neq j} p_0 \log \frac{\mid y_i - y_j \mid}{\mid \omega_i^\varepsilon - \omega_j^\varepsilon \mid} + O(\sigma^2 \log \frac{1}{\sigma}) \sum_i \mid y_i - \omega_i^\varepsilon \mid$$

$$- \sum_{i,j} (R_{B(0,1)}(\omega_i^\varepsilon, \omega_j^\varepsilon) - R_{B(0,1)}(y_i, y_j)).$$

Using (2.11) (with $C = \rho^2$, $b_i = y_i$ and $c_i = \omega_i^\varepsilon$) we deduce

$$\sum_{i,j} (R_{B(0,1)}(\omega_i^\varepsilon, \omega_j^\varepsilon) - R_{B(0,1)}(y_i, y_j)) = \sum_i O(\sigma^2 \mid y_i - \omega_i^\varepsilon \mid)$$

and finally

$$\begin{aligned} &W_{B(0,1)}(\omega^\varepsilon, 1, ..., 1, h, \tilde{p}) - W_{B(0,1)}(y, 1, ..., 1, h, \tilde{p}) = \pi \sum_{i\neq j} p_0 \log \frac{\mid y_i - y_j \mid}{\mid \omega_i^\varepsilon - \omega_j^\varepsilon \mid} \\ &+ O(\sigma^2 \log \frac{1}{\sigma}) \sum_i \mid y_i - \omega_i^\varepsilon \mid . \end{aligned} \tag{2.21}$$

We define

$$H(y) = -\pi p_0 \sum_{i\neq j} \log \mid y_i - y_j \mid .$$

We are led by (2.20) to

$$\begin{aligned} &H(\omega_\varepsilon) - H(y) + C\pi \sum_i \mid \omega_i \mid^2 - C\pi \sum_i \mid y_i \mid^2 \leq O(\frac{\varepsilon^{\frac{2}{3}}}{\sigma^{\frac{2}{3}}}) \\ &+ \sum_i \mid y_i - \omega_i^\varepsilon \mid O(\sigma^2 \log \frac{1}{\sigma}). \end{aligned} \tag{2.22}$$

Letting $\varepsilon \to 0$ we find that ω minimizes $H(\eta) + C\pi \sum_i \mid \eta_i \mid^2$, that was proved in [AS]. If ω is a nondegenerate minimum of $H(\eta) + C\pi \sum_i \mid \eta_i \mid^2$, and if we set $y = \omega$ in (2.22) we obtain that for some $k > 0$

$$\mid \omega^\varepsilon - \omega \mid^2 - k \mid \omega^\varepsilon - \omega \mid \sigma^2 \log \frac{1}{\sigma} \leq O(\frac{\varepsilon^{\frac{2}{3}}}{\sigma^{\frac{2}{3}}}),$$

this gives

$$| \omega^\varepsilon - \omega | = O(\sigma^2 \log \frac{1}{\sigma}).$$

We have proved the second part of Theorem 1.

Proof of Theorem 2.

Let us suppose that the points $a_1, ..., a_l$ are in G and $a_{l+1}, ..., a_m$ in ∂G. As usual, the conclusion will follow from precise upper and lower bounds for the energy. We start with giving a lower bound for the energy. Using Theorem 4, case I, we deduce that

$$\begin{aligned} E_\varepsilon(u_\varepsilon, \Omega^a_{2\rho}) \geq \pi p_0 \sum_{i=1}^{l} d_i^2 \log \frac{1}{2\rho} + 2\pi p_0 \sum_{i=l+1}^{m} d_i^2 \log \frac{1}{2\rho} + W_G(a, d_1, ..., d_m, g, p) \\ + \sum_{i=1}^{l} p_0 W^1(h_i) + \sum_{i=l+1}^{m} \frac{p_0}{2} W^1(h_i) + O(\frac{\varepsilon^2}{\rho^2}) + O(\rho \log^2 \frac{1}{\rho}). \end{aligned} \tag{2.23}$$

On the other hand, for each i, we have

$$E_\varepsilon(u_\varepsilon, B(a_i, \rho) \cap G \backslash \cup_j B(x^\varepsilon_{ij}, R)) \geq \frac{1}{2} \int_{B(a_i,\rho)\cap G \backslash \cup_j B(x^\varepsilon_{ij},R)} p |\nabla \frac{u_\varepsilon}{|u_\varepsilon|}|^2 + O(\frac{\varepsilon^2}{R^2}). \tag{2.24}$$

Now, let $i = l+1, ..., m$, that is $a_i \in \partial G$. We can write

$$\begin{aligned} \frac{u_\varepsilon}{| u_\varepsilon |}(x) &= \Pi_j \frac{x - x^\varepsilon_{ij}}{| x - x^\varepsilon_{ij} |} \frac{x - \overline{x}^\varepsilon_{ij}}{| x - \overline{x}^\varepsilon_{ij} |} e^{i\Psi_\varepsilon(x)} \\ &= v_\varepsilon(x) e^{i\Psi_\varepsilon(x)}. \end{aligned} \tag{2.25}$$

We may suppose that locally, $\partial G \cap B(a_i, \rho)$ is flat and we may extend Ψ_ε in $B(a_i, \rho)$ by reflection. Following respectively the proof of the Theorem 4 and Theorem 5 in [BMR], we get respectively

$$\begin{aligned} \int_{B(a_i,\rho)\cap G\backslash\cup_j B(x^\varepsilon_{ij},R)} p|\nabla \frac{u_\varepsilon}{|u_\varepsilon|}|^2 \geq \frac{1}{2} \int_{B(a_i,\rho)\backslash\cup_j B(x^\varepsilon_{ij},R)\backslash\cup_j B(\overline{x}^\varepsilon_{ij},R)} p|\nabla v_\varepsilon|^2 \\ + O(R) + O(\sum_j \frac{| x^\varepsilon_{ij} - a_i |}{\rho}), \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} \frac{1}{2} \int_{B(a_i,\rho)\backslash\cup_j B(x^\varepsilon_{ij},R)\backslash\cup_j B(\overline{x}^\varepsilon_{ij},R)} p|\nabla v_\varepsilon|^2 \geq 2\pi p_0 d_i \log \frac{\rho}{R} - 2\pi p_0 \sum_{k\neq j} \log \frac{| x^\varepsilon_{ij} - x^\varepsilon_{ik} |}{\rho} \\ - 2\pi p_0 \sum_{j,k} \log \frac{| x^\varepsilon_{ij} - \overline{x}^\varepsilon_{ik} |}{\rho} + O(\sum_j \frac{| x^\varepsilon_{ij} - a_i |}{\rho}) + \sum_{j\neq k} O(\frac{R}{| x^\varepsilon_{ij} - x^\varepsilon_{ik} |}). \end{aligned} \tag{2.27}$$

Since $deg(\frac{u_\varepsilon}{|u_\varepsilon|}, a_i) = d_i$ we have

$$E_\varepsilon(u_\varepsilon, B(a_i, 2\rho) \cap G \backslash B(a_i, \rho)) \geq 2\pi p_0 d_i^2 \log 2.$$

We denote $C_i = C_{a_i}$. Using (2.24)-(2.27), we conclude that, for $a_i \in \partial G$,

$$\begin{aligned} & E_\varepsilon(u_\varepsilon, B(a_i, 2\rho) \cap G) \geq \pi p_0 d_i \log \frac{\rho}{R} + 2\pi p_0 d_i^2 \log 2 - \pi p_0 \sum_{k \neq j} \log \frac{\mid x_{ij}^\varepsilon - x_{ik}^\varepsilon \mid}{\rho} \\ & - p_0 \pi \sum_{j,k} \log \frac{\mid x_{ij}^\varepsilon - \overline{x}_{ik}^\varepsilon \mid}{\rho} + \sum_j (p_0 + C_i(\mid x_{ij}^\varepsilon - a_i \mid - R)^2) I(\varepsilon, R, k_{ij}) + O(\frac{\varepsilon^2}{R^2}) \\ & + O(\frac{\sigma}{\rho}) + O(\frac{R}{\sigma}) \end{aligned} \tag{2.28}$$

and for $a_i \in G$, we find

$$\begin{aligned} & E_\varepsilon(u_\varepsilon, B(a_i, 2\rho) \geq \pi p_0 d_i \log \frac{\rho}{R} + \pi p_0 d_i^2 \log 2 - \pi p_0 \sum_{k \neq j} \log \frac{\mid x_{ij}^\varepsilon - x_{ik}^\varepsilon \mid}{\rho} \\ & + \sum_j (p_0 + C_i(\mid x_{ij}^\varepsilon - a_i \mid - R)^2) I(\varepsilon, R, k_{ij}) + O(\frac{\varepsilon^2}{R^2}) \\ & + O(\frac{\sigma}{\rho}) + O(\frac{R}{\sigma}). \end{aligned} \tag{2.29}$$

In order to obtain an adequate upper bound for the energy, we construct the following map

$$\omega = u_0 \text{ in } G \backslash \cup_i B(a_i, 2\rho),$$

where u_0 is the canonical map associated to $(a_1, ..., a_m)$, $(d_1, ..., d_m)$, G and g (see [BH2]). Next, we extend ω to $B(a_i, \rho) \cap G$, for i in $\{l+1, ..., m\}$. Let y_{ij}^ε, $j = 1, ..., d_i$, be any points in $B(a_i, \rho) \cap G$ such that $dist(y_{ij}^\varepsilon, \partial G)$ and $\mid y_{ij}^\varepsilon - y_{ik}^\varepsilon \mid$ are of the order of σ for all $k \neq j$. Let $\varphi_0(x)$ be the smooth map defined on the flat boundary $\partial G \cap B(a_i, \rho)$ by

$$g(x) = e^{i\varphi_0(x)}.$$

We define for $0 \leq r \leq \rho$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$\overline{\varphi}_0(r, \theta) = (\frac{1}{2} + \frac{\theta}{\pi})\varphi_0(r, \frac{\pi}{2}) + (\frac{1}{2} - \frac{\theta}{\pi})\varphi_0(r, \frac{-\pi}{2}),$$

thus $\overline{\varphi}_0 = \varphi_0$ on $\partial G \cap B(a_i, \rho)$. Let $\theta_0 \in]0, \frac{\pi}{2}[$, we set for $\theta_0 \leq \theta \leq \frac{\pi}{2}$

$$l_\varepsilon(r, \theta) = \frac{\theta - \theta_0}{\frac{\pi}{2} - \theta_0}\overline{\varphi}_0(r, \theta) + \frac{\frac{\pi}{2} - \theta}{\frac{\pi}{2} - \theta_0}\varphi_0(a),$$

for $-\frac{\pi}{2} \leq \theta \leq -\theta_0$

$$l_\varepsilon(r,\theta) = \frac{\theta+\theta_0}{\frac{-\pi}{2}+\theta_0}\overline{\varphi}_0(r,\theta) + \frac{\frac{\pi}{2}+\theta}{\frac{\pi}{2}-\theta_0}\varphi_0(a)$$

and for $-\theta \leq \theta \leq \theta_0$

$$l_\varepsilon(r,\theta) = \varphi_0(a).$$

By reflection, we obtain l_ε in $B(a_i,\rho)$. Direct computation gives for $\theta_0 \leq \theta \leq \frac{\pi}{2}$

$$| \nabla l_\varepsilon | \leq \frac{M}{\frac{\pi}{2}-\theta_0}. \tag{2.30}$$

Finally, on each $B(a_i,\rho)\cap G\backslash \cup_j B(y_{ij}^\varepsilon, R)$ we set

$$\begin{aligned}\omega(x) &= \Pi_j \frac{x-y_{ij}^\varepsilon}{| x-y_{ij}^\varepsilon |}\frac{x-\overline{y}_{ij}^\varepsilon}{| x-\overline{y}_{ij}^\varepsilon |}e^{il_\varepsilon(x)}\\ &= v(x)e^{il_\varepsilon(x)}.\end{aligned}$$

Following the proof of the Theorem 4 in [BMR] we obtain

$$\begin{aligned}&\int_{B(a_i,\rho)\backslash\cup_j B(y_{ij}^\varepsilon,R)\backslash\cup_j B(\overline{y}_{ij}^\varepsilon,R)} p \mid \nabla\omega \mid^2 = \int_{B(a_i,\rho)\backslash\cup_j B(y_{ij}^\varepsilon,R)\backslash\cup_j B(\overline{y}_{ij}^\varepsilon,R)} p \mid \nabla v \mid^2\\ &+ O(\int_{B(a_i,\rho)\backslash\cup_j B(y_{ij}^\varepsilon,R)\backslash\cup_j B(\overline{y}_{ij}^\varepsilon,R)} p \mid \nabla l_\varepsilon \mid^2 + \sum_j \frac{| y_{ij}^\varepsilon - a_i |}{\rho} + R).\end{aligned}$$

Next, we use the proof of the Theorem 5 of [BMR] and we are led to

$$\begin{aligned}&\frac{1}{2}\int_{B(a_i,\rho)\backslash\cup_j B(y_{ij}^\varepsilon,R)\backslash\cup_j B(\overline{y}_{ij}^\varepsilon,R)} p \mid \nabla v \mid^2 = 2\pi p_0 d_i \log\frac{\rho}{R} - 2\pi p_0 \sum_{j\neq k}\log\frac{| y_{ij}^\varepsilon - y_{ik}^\varepsilon |}{\rho}\\ &- 2\pi p_0 \sum_{j,k}\log\frac{| y_{ij}^\varepsilon - \overline{y}_{ik}^\varepsilon |}{\rho} + O(\sum_j \frac{| y_{ij}^\varepsilon - a_i |}{\rho}) + \sum_{j\neq k} O(\frac{R}{| y_{ij}^\varepsilon - y_{ik}^\varepsilon |})\\ &+ \sum_j O(| y_{ij}^\varepsilon - a_i |^2 \log\frac{1}{R}) + O(\rho^2\log\frac{1}{\rho}).\end{aligned} \tag{2.31}$$

Combining (2.30) and (2.31) we obtain

$$\begin{aligned}&\frac{1}{2}\int_{B(a_i,\rho)\cap G\backslash\cup_j B(y_{ij}^\varepsilon,R)} p \mid \nabla\omega \mid^2 = \pi p_0 d_i \log\frac{\rho}{R} - \pi p_0 \sum_{j\neq k}\log\frac{| y_{ij}^\varepsilon - y_{ik}^\varepsilon |}{\rho}\\ &- \pi p_0 \sum_{j,k}\log\frac{| y_{ij}^\varepsilon - \overline{y}_{ik}^\varepsilon |}{\rho} + O(\frac{\rho}{\frac{\pi}{2}-\theta_0})^2 + O(\frac{\sigma}{\rho}) + O(\rho^2\log\frac{1}{\rho})\\ &+ O(\sigma^2\log\frac{1}{R}) + O(\frac{R}{\sigma}).\end{aligned} \tag{2.32}$$

Now, we extend ω to $B(a_i, 2\rho)\backslash B(a_i, \rho)$ as in [AS], Lemma 4.6. So, in a neighborhood of a_i we set

$$u_0 = e^{2id_i\theta}e^{i\psi_i}$$

and

$$\omega = e^{2id_i\theta}e^{i\phi_i}.$$

We have

$$\mid \nabla\phi_i \mid = O(\mid \nabla l_\varepsilon \mid) + O(\frac{\sigma}{\rho^2})$$

that gives by (2.30)

$$\mid \nabla\phi_i \mid = O(\frac{1}{\frac{\pi}{2} - \theta_0}) + O(\frac{\sigma}{\rho^2}).$$

For $\rho \le r \le 2\rho$ we set

$$\psi(r, \theta) = (2 - \frac{r}{\rho})\phi_i + (-1 + \frac{r}{\rho})\psi_i.$$

Thus,

$$\mid \frac{\partial\psi}{\partial r} \mid = O(\frac{1}{\frac{\pi}{2} - \theta_0}) + O(\frac{\sigma}{\rho^2})$$

and

$$\mid \frac{\partial\psi}{\partial\theta} \mid = O(\frac{1}{\frac{\pi}{2} - \theta_0}) + O(\frac{\sigma}{\rho^2}).$$

We set

$$\omega = e^{i\psi + 2id_i\theta} \text{ in } B(a_i, 2\rho)\backslash B(a_i, \rho).$$

We have

$$\mid \nabla\omega(x) \mid = \frac{1}{\mid x - a_i \mid} + O(\frac{\sigma}{\rho^2} + \frac{1}{\frac{\pi}{2} - \theta_0})$$

therefore

$$\int_{B(a_i,2\rho)\backslash B(a_i,\rho)} p \mid \nabla\omega \mid^2 = 4\pi p_0 d_i^2 \log 2 + O(\frac{\sigma^2}{\rho^2} + \frac{\rho^2}{(\frac{\pi}{2} - \theta_0)^2}). \tag{2.33}$$

On each $B(y_{ij}^\varepsilon, R)$ we take ω as a minimizer for $I(\varepsilon, R, k_{ij})$. Moreover, we may choose θ_0 such that $\frac{\pi}{2} - \theta_0$ is independent of ε. By (2.32) and (2.33), we obtain for $a_i \in \partial G$

$$\begin{aligned}\frac{1}{2}\int_{B(a_i,2\rho)\cap G} p \mid \nabla\omega \mid^2 &= 2\pi p_0 d_i \log\frac{\rho}{R} + 2\pi p_0 d_i^2 \log 2 - \pi p_0 \sum_{k\neq j} \log\frac{\mid y_{ij}^\varepsilon - y_{ik}^\varepsilon \mid}{\rho} \\ &- \pi p_0 \sum_{j,k} \log\frac{\mid y_{ij}^\varepsilon - \overline{y}_{ik}^\varepsilon \mid}{\rho} + \sum_j (p_0 + C_i(\mid y_{ij}^\varepsilon - a_i \mid + R)^2) I(\varepsilon, R, k_{ij}) \\ &+ O(\frac{\sigma}{\rho}) + O(\frac{R}{\sigma}) + O(\rho^2 \log\frac{1}{\rho}) + O(\sigma^2 \log\frac{1}{R}),\end{aligned} \tag{2.34}$$

For $a_i \in G$ we need not to use the function l_ε and it suffices to choose y^ε_{ij} such that $\mid y^\varepsilon_{ij} - y^\varepsilon_{ik} \mid$ is of the order of σ, for $j \neq k$ and the order of $\mid y^\varepsilon_{ij} - a_i \mid$ is not greater than σ for $j = 1, ..., d_i$. Hence, we obtain

$$\begin{aligned} \frac{1}{2}\int_{B(a_i,2\rho)} p \mid \nabla\omega \mid^2 &= \pi p_0 d_i \log\frac{\rho}{R} + \pi p_0 d_i^2 \log 2 - \pi p_0 \sum_{k\neq j} \log\frac{\mid y^\varepsilon_{ij} - y^\varepsilon_{ik} \mid}{\rho} \\ &+ \sum_j (p_0 + C_i(\mid y^\varepsilon_{ij} \mid + R)^2) I(\varepsilon, R, k_{ij}) + O(\frac{\sigma}{\rho}) + O(\frac{R}{\sigma}) + O(\rho^2 \log\frac{1}{\rho}) \\ &+ O(\sigma^2 \log\frac{1}{R}). \end{aligned} \tag{2.35}$$

Recall that

$$\frac{1}{2}\int_{G\setminus\cup_i B(a_i,2\rho)} p \mid \nabla u_0 \mid^2 = \frac{1}{2}\int_{G\setminus\cup_i B(a_i,2\rho)} \frac{1}{p} \mid \nabla\Phi_0 \mid^2 .$$

Let $a_i \in \partial G$. In order to study the zeroes of u_ε near a_i, we may choose $y^\varepsilon_{kj} = x^\varepsilon_{kj}$ for all $k \neq i$ and $j = 1, ..., d_k$. Let y_{ij} be any points in $\mathbf{R}^2_+$ and set $y^\varepsilon_{ij} = a_i + \sigma y_{ij}$. Using (2.23), (2.28), (2.29), (2.34), (2.35) and Proposition 1, we deduce that for all $i = l+1, ..., m$

$$\begin{aligned} &-\pi \sum_{k\neq j} p_0 \log\frac{\mid x^\varepsilon_{ij} - x^\varepsilon_{ik} \mid}{\rho} - \pi p_0 \sum_{j,k} \log\frac{\mid x^\varepsilon_{ij} - \overline{x}^\varepsilon_{ik} \mid}{\rho} + \sum_{i=1}^{l} p_0 W^1(h_i) + \sum_{i=l+1}^{m} \frac{p_0}{2} W^1(h_i) \\ &+ \sum_j (p_0 + C_i(\mid x^\varepsilon_{ij} - a_i \mid - R)^2) I(\varepsilon, R, k_{ij}) \\ &\leq -\pi \sum_{k\neq j} p_0 \log\frac{\mid y^\varepsilon_{ij} - y^\varepsilon_{ik} \mid}{\rho} - \pi p_0 \sum_{j,k} \log\frac{\mid y^\varepsilon_{ij} - \overline{y}^\varepsilon_{ik} \mid}{\rho} \\ &+ \sum_j (p_0 + C_i(\mid \sigma y_{ij} \mid + R)^2) I(\varepsilon, R, k_{ij}) + O(\frac{\varepsilon^2}{R^2}) + O(\frac{R}{\sigma}) + O(\frac{\sigma}{\rho}) \\ &+ O(\rho \log^2\frac{1}{\rho}) + O(\sigma^2 \log\frac{1}{R}). \end{aligned}$$

We are led to

$$\begin{aligned} &\sum_{i=1}^{l} p_0 W^1(h_i) + \sum_{i=l+1}^{m} \frac{p_0}{2} W^1(h_i) + H_i(\omega^\varepsilon_i) - H_i(y_i) + \pi C_i \sum_j \mid \omega^\varepsilon_{ij} \mid^2 - \pi C_i \sum_j \mid y_{ij} \mid^2 \\ &\leq O(\frac{\varepsilon^2}{R^2}) + O(\frac{R}{\sigma}) + O(\frac{\sigma}{\rho}) + O(\rho \log^2\frac{1}{\rho}) + O(\sigma^2 \log\frac{1}{R}). \end{aligned}$$

Now, we set $\rho = \frac{\sigma}{\lambda}$ and $R = \lambda\sigma$. The optimal choice for λ is $\lambda = \sigma^{\frac{1}{2}} \log\frac{1}{\sigma}$, thus

$$\begin{aligned} &\sum_{i=1}^{l} p_0 W^1(h_i) + \sum_{i=l+1}^{m} p_0 W^1(h_i) + H_i(\omega^\varepsilon_i) - H_i(y_i) + \pi C_i \sum_j \mid \omega^\varepsilon_{ij} \mid^2 - \pi C_i \sum_j \mid y_{ij} \mid^2 \\ &\leq O(\sigma^{\frac{1}{2}} \log\frac{1}{\sigma}). \end{aligned}$$

We know that for, $i = 1, ..., m$, $W^1(h_i) \geq 0$. Letting $\varepsilon \to 0$ we see that $(\omega_{i1}, ..., \omega_{id_i})$ minimizes $H_i(\eta_1, ...\eta_{d_i}) + C_i\pi \sum_{j=1}^{d_i} \mid \eta_j \mid^2$. Next we choose $y_{ik} = \omega_{ik}$ for $k = 1, ..., d_i$. If $(\omega_{i1}, ..., \omega_{id_i})$ realizes a nondegenerate minimum of $H_i(y_{i1}, ..., y_{id_i}) + \pi C_i \sum_j \mid y_{ij} \mid^2$ we get

$$\sum_k \mid \omega_{ik} - \omega_{ik}^\varepsilon \mid^2 = O(\sigma^{\frac{1}{2}} \log \frac{1}{\sigma}).$$

In any case, we obtain for all $i = 1, ..., m$

$$W^1(h_i) = O(\sigma^{\frac{1}{2}} \log \frac{1}{\sigma}).$$

We have proved Theorem 2.

We now turn to the proof of Theorem 3. Using (2.23), (2.28) and (2.29) we obtain

$$\begin{aligned}
E_\varepsilon(u_\varepsilon) \geq\ & \pi p_0 \sum_{i=1}^{l} d_i^2 \log \frac{1}{\rho} + 2\pi p_0 \sum_{i=l+1}^{m} d_i^2 \log \frac{1}{\rho} + \pi p_0 \sum_{i=1}^{m} d_i \log \frac{\rho}{R} \\
& + W_G(a, d_1, ..., d_m, g, p) - \pi p_0 \sum_{i=1}^{m} \sum_{k \neq j} \log \frac{\mid x_{ij}^\varepsilon - x_{ik}^\varepsilon \mid}{\rho} \\
& - \pi p_0 \sum_{i=l+1}^{m} \sum_{k,j} \frac{\log \mid x_{ij}^\varepsilon - \overline{x}_{ik}^\varepsilon \mid}{\rho} \\
& + \sum_{i,j} (p_0 + C_i(\mid x_{ij}^\varepsilon - a_i \mid - R)^2) I(\varepsilon, R, k_{ij}) \\
& + O(R\sigma \log \frac{R}{\varepsilon}) + O(\frac{\varepsilon^2}{R^2}) + O(\frac{\sigma}{\rho}) + O(\frac{R}{\sigma}) + O(R\sigma \log \frac{1}{\varepsilon}) + O(\rho \log^2 \frac{1}{\rho}).
\end{aligned} \tag{2.36}$$

Now, we take $\rho = \frac{\sigma}{\delta}$, and $R = \delta\sigma$ where δ is a constant independent of ε. As in [AS] we have $I(\varepsilon, R, k_{ij}) = \pi \log \frac{\delta\sigma}{\varepsilon} + \gamma + X(\varepsilon, \delta)$ where $X(\varepsilon, \delta)$ tends to 0 as ε tends to 0, δ being fixed. Setting $A_i = \frac{d_i^2 - d_i}{2}$ for $i = 1, ...l$ and $A_i = \frac{2d_i^2 - d_i}{2}$ for $i = l+1, ...m$, we obtain, using (2.36),

$$\begin{aligned}
E_\varepsilon(u_\varepsilon) \geq\ & W_G(a, d_1, ..., d_m, g, p) + \pi p_0 d \log \frac{1}{\varepsilon} + 2\pi p_0 \sum_{i=1}^{m} A_i \log \frac{1}{\sigma} + \sum_{i=1}^{m} H_i(\omega_i) \\
& + \sum_{i=1}^{m} \pi C_i \sum_j \mid \omega_{ij} \mid^2 + d p_0 \gamma + O(\delta) + X(\varepsilon, \delta),
\end{aligned} \tag{2.37}$$

where $X(\varepsilon, \delta)$ tends to o as ε tends to 0 and δ is fixed.

Using Theorem 2, we know that for all i, ω_i minimizes

$$H_i(\eta_1, ...\eta_{d_i}) + C_i\pi \sum_{j=1}^{d_i} \mid \eta_j \mid^2 .$$

This leads for all i to

$$C_i \sum_{j=1}^{d_i} \mid \eta_j \mid^2 = p_0 A_i.$$

Thus

$$H_i(\omega_{i1}, ...\omega_{id_i}) + C_i \pi \sum_{j=1}^{d_i} \mid \omega_{ij} \mid^2 = \min_{\{\eta; C_{a_i} \sum_{j=1}^{d_i} |\eta_j|^2 = p_0 A_i\}} H_i + \pi p_0 A_i. \tag{2.38}$$

Moreover, we know (see [BH]) that m and $(d_1, ..., d_m)$ realizes

$$F(d) = \min_{\{m, d_1, ... d_m; \sum_{i=1}^{m} d_i = d\}} \sum_{i=1}^{m} A_i.$$

Next, we may use the upper bound for $E_\varepsilon(u_\varepsilon)$ in the proof of Theorem 2 with m, $(d_1, ..., d_m)$, any m distinct points $(b_1, ..., b_m)$ in Λ^m, and with, for all i, $(\omega_{i1}, ..., \omega_{id_i})$ being chosen as in (2.38). We are led to the following:

the configuration $(a_1, ..., a_m)$ minimizes in $\Lambda_2^l \times (\Lambda_1 \backslash \Lambda_2)^{m-l}$ the map

$$\overline{W}(b) = W_G(b, d_1, ..., d_m, g, p) + \sum_i \min_{\{\eta; C_{b_i} \sum_{j=1}^{d_i} |\eta_j|^2 = p_0 A_i\}} H_i.$$

Finally, we obtain

$$E_\varepsilon(u_\varepsilon) = \pi d p_0 \log \frac{1}{\varepsilon} + \pi p_0 F(d)(\log \log \frac{1}{\varepsilon} + 1) + \min_{\Lambda_2^l \times (\Lambda_1 \backslash \Lambda_2)^{m-l}} \overline{W} + d p_0 \gamma + X(\varepsilon)$$

where $X(\varepsilon)$ tends to 0 as ε tends to 0, which is is the desired estimate.

3. Proof of Proposition 1 and Theorem 4.

Proof of Proposition 1. In the course of the proof, we shall take into account the two cases. We have

$$\int_{\Omega_\rho^b} \frac{1}{p} \mid \nabla \Phi_0 \mid^2 = \int_{\partial\Omega \backslash \cup_{i=l+1}^{m} B(b_i, \rho)} \frac{1}{p} \frac{\partial \Phi_0}{\partial \nu} \Phi_0 - \sum_{i=1}^{m} \int_{\partial B(b_i, \rho) \cap \Omega} \frac{1}{p} \frac{\partial \Phi_0}{\partial \nu} \Phi_0. \tag{3.1}$$

Using the fact that on $\partial\Omega \backslash \cup_{i=l+1}^{m} B(b_i, \rho)$, we have $\frac{1}{p} \frac{\partial \Phi_0}{\partial \nu} = h_0 \times (h_0)_\tau$ and by the normalization condition (2.5), we obtain that

$$\int_{\partial\Omega \backslash \cup_{i=l+1}^{m} B(b_i, \rho)} \frac{1}{p} \frac{\partial \Phi_0}{\partial \nu} \Phi_0 = - \sum_{i=1}^{m} \int_{B(b_i, \rho) \cap \partial\Omega} (g \times g_\tau) \Phi_0 = X(\rho) \tag{3.2}$$

where $X(\rho)=O(\rho\log\frac{1}{\rho})$ in the case I and $X(\rho)=0$ in the case II. We set

$$S_j(x)=\Phi_0(x)-\overline{d}_jp(b_j)\log|x-b_j|.$$

We have for $j=1,...,m$

$$\begin{aligned}\int_{\partial B(b_j,\rho)\cap\Omega}\frac{1}{p}\frac{\partial\Phi_0}{\partial\nu}\Phi_0=\int_{\partial B(b_j,\rho)\cap\Omega}\frac{1}{p}(\frac{\partial S_j}{\partial\nu}\\ +\overline{d}_jp(b_j)\frac{\partial\log|x-b_j|}{\partial\nu})(S_j+\overline{d}_jp(b_j)\log|x-b_j|).\end{aligned}\tag{3.3}$$

For $x\in\partial B(b_j,\rho)\cap\Omega$, we have $\frac{\partial\log|x-b_j|}{\partial\nu}=\frac{1}{\rho}$. Using (3.3) we have

$$\begin{aligned}\int_{\partial B(b_j,\rho)\cap\Omega}\frac{1}{p}\frac{\partial\Phi_0}{\partial\nu}\Phi_0=\int_{\partial B(b_j,\rho)\cap\Omega}\frac{1}{p}\frac{\partial S_j}{\partial\nu}S_j\\ +\frac{\overline{d}_jp(b_j)}{\rho}\int_{\partial B(b_j,\rho)\cap\Omega}\frac{S_j}{p}+\overline{d}_jp(b_j)\int_{\partial B(b_j,\rho)\cap\Omega}\frac{1}{p}\frac{\partial S_j}{\partial\nu}\log\rho\\ +\frac{\overline{d}_j^2p(b_j)^2}{\rho}\int_{\partial B(b_j,\rho)\cap\Omega}\frac{\log\rho}{p}.\end{aligned}\tag{3.4}$$

Let us estimate the first term of (3.4). By (2,7), (2.9) and (2.10) we obtain in the case I

$$\|S_j\|_{L^\infty(B(b_j,\rho))}=O(1)\text{ and }\|\nabla S_j\|_{L^\infty(B(b_j,\rho))}=O(1)$$

and in the case II

$$\|S_j\|_{L^\infty(B(b_j,\rho))}=\sum_{k\neq j}O(\log\frac{1}{|b_k-b_j|})$$

and for $x\in B(b_j,\rho)$

$$|\nabla S_j(x)|=O(C\,|b_j-a_j|\log\frac{1}{|x-b_j|}+\sum_{k\neq j}\frac{1}{|b_j-b_k|}).$$

Thus we have in the case I

$$\int_{\partial B(b_j,\rho)\cap\Omega}\frac{1}{p}\frac{\partial S_j}{\partial\nu}S_j=O(\rho)\tag{3.5}$$

and in the case II

$$\begin{aligned}\int_{\partial B(b_j,\rho)}\frac{1}{p}\frac{\partial S_j}{\partial\nu}S_j=\int_{B(b_j,\rho)}\operatorname{div}(\frac{1}{p}\nabla S_j)S_j+\int_{B(b_j,\rho)}\frac{1}{p}|\nabla S_j|^2\\ =\sum_{k\neq j}O(C\rho\,|b_j-a_j|\log\frac{1}{|b_j-b_k|}+\frac{\rho^2}{|b_j-b_k|^2}+C^2|b_j-a_j|^2\rho^2\log^2\frac{1}{\rho}).\end{aligned}\tag{3.6}$$

Next, we estimate the second term of (3.4). Let us prove that, respectively in the cases I and II, we have

$$\begin{aligned}&\frac{\overline{d}_j p(b_j)}{\rho}\int_{\partial B(b_j,\rho)\cap\Omega}\frac{S_j}{p} = 2\pi d_j S_j(b_j) + O(\rho)\\ &= \sum_{k\neq j} O(C\rho \mid b_j - a_j \mid \log\frac{1}{\rho} + \frac{\rho}{\mid b_j - b_k \mid} + C\rho^2 \log\frac{1}{\mid b_j - b_k \mid}).\end{aligned}\tag{3.7}$$

We first remark that we have respectively in the case I and in the case II

$$\begin{aligned}&\frac{1}{p(x)} - \frac{1}{p(b_j)} = O(C \mid x - b_j \mid^2), \text{ and}\\ &\frac{1}{p(x)} - \frac{1}{p(b_j)} = O(C \mid x - b_j \mid\mid b_j - a_j \mid + C \mid x - b_j \mid^2),\end{aligned}\tag{3.8}$$

this gives

$$\begin{aligned}&\frac{\overline{d}_j p(b_j)}{\rho}\int_{\partial B(b_j,\rho)\cap\Omega}\frac{S_j}{p} = \frac{\overline{d}_j}{\rho}\int_{\partial B(b_j,\rho)\cap\Omega} S_j + O(C\rho^2) \text{ in the case I}\\ &= \frac{\overline{d}_j}{\rho}\int_{\partial B(b_j,\rho)} S_j + \sum_{k\neq j} O((C\rho \mid b_j - a_j \mid + C\rho^2)\log\frac{1}{\mid b_j - b_k \mid}) \text{ in the case II.}\end{aligned}$$

Respectively in the cases I and II we have for $x \in \partial B(b_j,\rho)$

$$\begin{aligned}&\mid S_j(x) - S_j(b_j) \mid = O(\rho)\\ &\qquad = \sum_{k\neq j} O(C\rho \mid b_j - a_j \mid \log\frac{1}{\rho} + \frac{\rho}{\mid b_j - b_k \mid})).\end{aligned}\tag{3.9}$$

We have proved (3.7). Now we use

$$S_j(b_j) = \sum_i \overline{d}_i R(b_j,b_i) + \sum_{i\neq j}\overline{d}_i p(b_i)\log|b_i - b_j|$$

to infer, in view of (3.7)

$$\frac{\overline{d}_j p(b_j)}{\rho}\int_{\partial B(b_j,\rho)\cap\Omega}\frac{S_j}{p} = 2\pi d_j \sum_i \overline{d}_i R(b_j,b_i) + 2\pi d_j \sum_{i\neq j}\overline{d}_i p(b_j)\log|b_i - b_j| + X(\rho),\tag{3.10}$$

where $X(\rho)$ is equal respectively in the cases I and II to $O(\rho)$ and $\sum_{k\neq j} O(\frac{\rho}{|b_j - b_k|} + C\rho \mid b_j - a_j \mid \log\frac{1}{\rho} + C\rho^2\log\frac{1}{|b_j - b_k|})$. We estimate the third term of (3.4). We easily get respectively in the cases I and II

$$\begin{aligned}&\overline{d}_j p(b_j)\int_{\partial B(b_j,\rho)\cap\Omega}\frac{1}{p}\frac{\partial S_j}{\partial\nu}\log\rho = O(\rho\log\frac{1}{\rho})\\ &\qquad = O(C \mid a_j - b_j \mid \rho\log\frac{1}{\rho} + C\rho^2\log\frac{1}{\rho}).\end{aligned}\tag{3.11}$$

Finally, we evaluate the fourth term of (3.4). Using (3.8) we get in the case I

$$
\begin{aligned}
\frac{2\overline{d}_j^2 p(b_j)^2}{\rho}\int_{\partial B(b_j,\rho)\cap\Omega}\frac{\log\rho}{p} &= 2\pi\overline{d}_j^2 p(b_j)\log\rho + O(\rho^2\log\frac{1}{\rho}), \quad j=l+1,...,m \\
&= 4\pi d_j^2 p(b_j)\log\rho + O(\rho^2\log\frac{1}{\rho}), \quad j=1,...,l,
\end{aligned} \tag{3.12}
$$

and, in the case II

$$
\begin{aligned}
\frac{2d_j^2 p(b_j)^2}{\rho}\int_{\partial B(b_j,\rho)}\frac{\log\rho}{p} = 4\pi d_j^2 p(b_j)\log\rho + O(C\rho\delta\log\frac{1}{\rho} + C\rho^2\log\frac{1}{\rho} \\
+ C\rho^2\log\frac{1}{\rho}).
\end{aligned} \tag{3.13}
$$

Combining (3.4), (3.5), (3.6), (3.7), (3.10), (3.11) and (3.12) we deduce that

$$
\begin{aligned}
&\frac{1}{2}\int_{\partial B(a_j,\rho)\cap\Omega}\frac{1}{p}\frac{\partial\Phi_0}{\partial\nu}\Phi_0 \\
=&\sum_{i=1}^{m}\pi d_j\overline{d}_i R(b_j,b_i) + \pi\sum_{i\neq j} d_i\overline{d}_j p(b_i)\log|b_i-b_j| + 2\pi d_j\overline{d}_j p(b_j)\log\rho + X(\rho).
\end{aligned} \tag{3.14}
$$

where $X(\rho)=O(\rho\log\frac{1}{\rho})$ in the case I and $X(\rho)=\sum_{k\neq j}O(\frac{\rho}{|b_j-b_k|}+C\mid b_j-a_j\mid\rho\log\frac{1}{\rho}+C\rho^2\log\frac{1}{\rho})$ in the case II. Using (3.1), (3.2) and (3.14), we are led to the proof of Proposition 1.

We define $\tilde{h}_j(x)=h_j\times(h_j)_\tau(\frac{x-b_j}{\rho})$ for $x\in\partial B(b_j,\rho)$. The proof of the following proposition is a consequence of the proof of Theorem I.4 of [BBH2].

PROPOSITION 2. Let $\hat{u}_\rho^b$ be a minimizer for the problem (E_ρ^b). We have

$$
\int_{\Omega_\rho^b} p\mid\nabla\hat{u}_\rho^b\mid^2=\int_{\Omega_\rho^b}\frac{1}{p}\mid\nabla\hat{\Phi}_\rho^b\mid^2
$$

where $\hat{\Phi}_\rho^b$ is the solution of the linear problem

$$
\begin{cases}
\mathrm{div}(\frac{1}{p}\nabla\hat{\Phi}_\rho^b)=0 \text{ in } \Omega_\rho^b \\
\frac{1}{p}\frac{\partial\hat{\Phi}_\rho^b}{\partial\nu}=\frac{\tilde{h}_j}{\rho} \text{ on } \partial B(b_j,\rho)\cap\Omega, j=1,...,m \\
\frac{1}{p}\frac{\partial\hat{\Phi}_\rho^b}{\partial\nu}=h_0\times(h_0)_\tau \text{ on } \partial\Omega\setminus\cup_{j=l+1}^{m}B(b_j,\rho)
\end{cases} \tag{3.15}
$$

with the normalization condition $\int_{\partial\Omega}(h_0 \times (h_0)_\tau)\hat{\Phi}^b_\rho = 0$.

Now we extend the function p by a function still denoted p such that in the case I, for $j = l+1, ..., m$, $p(x) = p(r(x))$, where r is the reflection associated to the flat boundary $\partial\Omega \cap B(b_j, \beta)$, and such that in the both cases $p(x) = p_0$, for $\mid x \mid$ large enough. We define $\eta^{b_j}_\rho$ by

$$\begin{cases} \mathrm{div}(\frac{1}{p}\nabla\eta^{b_j}_\rho) = 0 \text{ in } \mathbf{R}^2\backslash\partial B(b_j,\rho) \\ \frac{1}{p}\frac{\partial\eta^{b_j}_\rho}{\partial\nu} = \frac{1}{\rho}(\frac{\tilde{h}_j}{\overline{d}_j} - 1) \text{ on } \partial B(b_j,\rho) \\ \eta^{b_j}_\rho \text{ bounded} \end{cases}$$

with the normalization condition $\int_{\partial B(b_j,\rho)} \tilde{h}_j\eta^{b_j}_\rho = 0$. In order to justify the existence of $\eta^{b_j}_\rho$ in $\mathbf{R}^2\backslash B(b_j,\rho)$, let us remark that by the inversion $\omega(z) = \eta^{b_j}_\rho(b_j + \frac{\rho}{z})$ (we use the complex notation), the system that defines $\eta^{b_j}_\rho$ is equivalent to

$$\begin{cases} \mathrm{div}(\frac{1}{\tilde{p}_j}\nabla\omega) = 0 \text{ in } B(0,1) \\ \frac{1}{\tilde{p}_j}\frac{\partial\omega}{\partial\nu} = -(\frac{\overline{h}_j}{\overline{d}_j} - 1) \text{ on } \partial B(0,1) \\ \int_{S^1}(h_j \times (h_j)_\tau)\omega = 0, \end{cases}$$

where $\tilde{p}_j(z) = p(b_j + \frac{\rho}{z})$ and $\overline{h}_j(z) = h_j \times (h_j)_\tau(z)$. Hence, the existence and the uniqueness of $\eta^{b_j}_\rho$ are assured by the existence of a unique ω, that follows from standard results. For $j = 1, ..., m$, we define $\tilde{\eta}_j$ by

$$\begin{cases} \Delta\tilde{\eta}_j = 0 \text{ in } \mathbf{R}^2\backslash S^1 \\ \frac{\partial\tilde{\eta}_j}{\partial\nu} = \frac{\tilde{h}_j}{\overline{d}_j} - 1 \text{ on } S^1 \\ \tilde{\eta}_j \text{ bounded} \\ \int_{\partial B(0,1)}\tilde{h}_j\tilde{\eta}_j = 0. \end{cases}$$

Let $\overline{\eta}^{b_j}_\rho(x) = \tilde{\eta}_j(\frac{x-b_j}{\rho})$, for $x \in \mathbf{R^2}\backslash B(b_j,\rho)$. We note that $\overline{\eta}^{b_j}_\rho$ is given by the explicit representation (given in [CM])

$$\overline{\eta}^{b_j}_\rho(x) = -\frac{1}{2\pi\rho}\int_{\partial B(b_j,\rho)} \log \mid x - z \mid^2 (\tilde{h}_j(z) - 1) + H$$

where H is a constant such that

$$\int_{\partial B(b_j,\rho)} \tilde{h}_j\overline{\eta}^{b_j}_\rho = 0.$$

We also use

$$\nabla\overline{\eta}^j_\rho(x) = -\frac{1}{\pi\rho}\int_{\partial B(b_j,\rho)} \frac{x-z}{\mid x-z \mid^2}(\tilde{h}_j(z)-1)$$

to infer that , for all $x \in \mathbf{R}^2\backslash B(b_j,\rho)$

$$\mid \overline{\eta}^{b_j}_\rho(x) - H \mid = O(\frac{\rho}{\mid x-b_j \mid -\rho}) \tag{3.16}$$

and

$$\mid \nabla\overline{\eta}^{b_j}_\rho(x) \mid = O(\frac{\rho}{(\mid x-b_j \mid^2 -\rho^2)^{\frac{1}{2}}} + \frac{\rho^2}{(\mid x-b_j \mid -\rho)^2}). \tag{3.17}$$

LEMMA 1. We have the following estimations, uniformly for $(h_1,...,h_m)$ in $\mathcal{H}^m_{\mathcal{M}}$ in the case I, for $x \in \mathbf{R}^2\backslash B(b_j,\rho)$

$$\mid \nabla\eta^{b_j}_\rho(x) \mid = O(\frac{\rho}{(\mid x-b_j \mid^2 -\rho^2)^{\frac{1}{2}}}) + \frac{\rho^2}{(\mid x-b_j \mid -\rho)^2}) + O(C\rho) \tag{3.18}$$

and for $x \in \partial\Omega \cap B(b_j,\beta)\backslash B(b_j,\rho)$

$$\mid \frac{\partial\eta^{b_j}_\rho}{\partial\nu}(x) \mid = O(\frac{\rho}{(\mid x-b_j \mid^2 -\rho^2)^{\frac{1}{2}}}) + O(C\rho). \tag{3.19}$$

In the case II we have for all $x \in \mathbf{R}^2\backslash B(b_j,\rho)$

$$\mid \nabla\eta^{b_j}_\rho(x) \mid = O(\frac{\rho}{(\mid x-b_j \mid^2 -\rho^2)^{\frac{1}{2}}}) + \frac{\rho^2}{(\mid x-b_j \mid -\rho)^2}) + O(C \mid b_j - a_j \mid + C\rho) \tag{3.20}$$

and

$$\mid \nabla\eta^{b_j}_\rho(x) \mid = O(\frac{\rho}{(\mid x-b_j \mid -\rho)^2}) + O(\frac{C\rho \mid b_j - a_j \mid + C\rho^2}{\mid x-b_j \mid -\rho}). \tag{3.21}$$

Proof. We note here $\omega(z) = \eta^{b_j}_\rho(b_j + \frac{\rho}{z})$, $z \in B(0,1)$, $\tilde{p}_{b_j}(z) = \tilde{p}(z) = p(b_j + \frac{\rho}{z})$ and $\overline{\omega}(z) = p(b_j)\tilde{\eta}_j(\frac{1}{z}) = p(b_j)\overline{\eta}^j_\rho(b_j + \frac{\rho}{z})$. We have

$$\left\{ \begin{array}{l} \text{div}\frac{1}{\tilde{p}}\nabla(\omega - \overline{\omega})(z) = -p(b_j)(\nabla\frac{1}{\tilde{p}}.\nabla\overline{\omega})(z) \text{ in } B(0,1) \\ \qquad \frac{1}{\tilde{p}}\frac{\partial(\omega - \overline{\omega})}{\partial\nu} = -(\frac{p(b_j)}{\tilde{p}} - 1)(\frac{h \times h_\tau}{\overline{d}} - 1) \text{ on } \partial B(0,1). \end{array} \right.$$

There exists $\alpha > 0$ such that

$$\begin{aligned}\nabla\frac{1}{p}(b_j+\frac{\rho}{z}) &= O(C\mid b_j-a_j+\frac{\rho}{z}\mid) \text{ for } \mid z\mid\geq\alpha \\ &= 0 \text{ for } \mid z\mid\leq\alpha\end{aligned}$$

and for any second derivative of $\frac{1}{p}$ denoted $D^2\frac{1}{p}$ we have

$$\begin{aligned}D^2\frac{1}{p}(b_j+\frac{\rho}{z}) &= O(C) \text{ for } \mid z\mid\geq\alpha \\ &= 0 \text{ for } \mid z\mid\leq\alpha.\end{aligned}$$

Thus

$$\begin{aligned}\mid\mid\nabla\frac{1}{\tilde{p}}.\nabla\overline{\omega}\mid\mid_{H^1(B(0,1)} &= O(C\rho^2) \text{ in the case I} \\ &= O(C\rho\delta+C\rho^2) \text{ in the case II.}\end{aligned}$$

On the other hand we get

$$\begin{aligned}\mid\mid(\frac{p(b_j)}{\tilde{p}}-1)(\frac{h\times h_\tau}{\overline{d}}-1)\mid\mid_{H^{\frac{3}{2}}(S^1)} &= O(C\rho^2) \text{ in the case I} \\ &= O(C\rho\delta+C\rho^2) \text{ in the case II .}\end{aligned}$$

We deduce from standard estimates that

$$\mid\mid\omega-\overline{\omega}\mid\mid_{H^3(B(0,1))}\leq M\mid\mid f\mid\mid_{H^1(B(0,1))}+M\mid\mid(\tilde{p}_{b_j}-p(b_j))(\frac{h\times h_\tau}{\overline{d}}-1)\mid\mid_{H^{\frac{3}{2}}(S^1)},$$

thus

$$\begin{aligned}\mid\mid\omega-\overline{\omega}\mid\mid_{H^3(B(0,1))} &= O(C\rho^2) \text{ in the case I} \\ &= O(C\rho\delta+C\rho^2) \text{ in the case II.}\end{aligned}\tag{3.22}$$

In particular we obtain

$$\begin{aligned}\mid\mid\nabla\eta_\rho^{b_j}-p(b_j)\nabla\overline{\eta}_\rho^{b_j}\mid\mid_{L^2(\Omega\backslash B(b_j,\rho))} &= O(C\rho^2) \text{ in the case I} \\ &= O(C\rho\delta+C\rho^2) \text{ in the case II.}\end{aligned}\tag{3.23}$$

By (3.22) we are led to

$$\begin{aligned}\mid\mid\nabla\eta_\rho^{b_j}-p(b_j)\nabla\overline{\eta}_\rho^{b_j}\mid\mid_{L^\infty(\Omega\backslash B(b_j,\rho))} &= O(C\rho) \text{ in the case I} \\ &= O(C\delta+C\rho) \text{ in the case II.}\end{aligned}\tag{3.24}$$

Now we use (3.17) and (3.24) to get (3.18) and (3.20). If $x\in\partial\Omega\cap B(b_j,\beta)\backslash B(b_j,\rho)$, for an appropriate choice of coordinates $z=(z_1,z_2)$ we set

$$\frac{\partial\overline{\eta}_\rho^{b_j}}{\partial\nu}(x)=\frac{1}{\pi\rho}\int_{\partial B(b_j,\rho)}\frac{z_2}{\mid x-z\mid^2}(\tilde{h}_j(z)-1).$$

We denote $z_2 = \rho \sin\theta$ and we use $\mid x - z \mid \geq \mid x - b_j \mid - \rho\cos\theta$ to get

$$\frac{\partial \overline{\eta}_\rho^{b_j}}{\partial \nu}(x) = O\left(\frac{\rho}{(\mid x - b_j \mid^2 - \rho^2)^{\frac{1}{2}}}\right),$$

thus, using (3.24), (3.19) is proved. In order to prove (3.21), we first remark that (3.22) leads to

$$\begin{aligned} \| \eta_\rho^{b_j} - p(b_j)\overline{\eta}_\rho^{b_j} \|_{L^\infty(\Omega\setminus B(b_j,\rho))} &= O(C\rho^2) \text{ in the case I} \\ &= O(C\rho\delta + C\rho^2) \text{ in the case II,} \end{aligned}$$

and this gives, in view of (3.16)

$$\begin{aligned} \mid \eta_\rho^{b_j}(x) - p(b_j)H \mid &= O\left(\frac{\rho}{\mid x - b_j \mid - \rho}\right) + O(C\rho^2) \text{ in the case I} \\ &= O\left(\frac{\rho}{\mid x - b_j \mid - \rho}\right) + O(C\rho\delta + C\rho^2) \text{ in the case II.} \end{aligned} \tag{3.25}$$

Now we let for $x \in \mathbf{R}^2 \setminus B(0,1)$, $v(x) = \eta_\rho^{b_j}(b_j + \rho x)$ and, for $y \in B(0,1)$, $\tilde{v}(y) = v(x + (\mid x \mid - 1)y)$. Standard estimates in $B(0,1)$ for the map $\tilde{v} - p(b_j)H$ give

$$\mid \nabla\tilde{v}(0) \mid \leq M \| \tilde{v} - p(b_j)H \|_{L^\infty(B(0,\frac{1}{2})},$$

thus

$$\mid \nabla v(x) \mid \leq \frac{M}{\mid x \mid - 1} . \| v - p(b_j)H \|_{L^\infty(B(x,\frac{1}{2}(|x|-1))} .$$

We set $z = b_j + \rho x$. We are led to

$$\mid \nabla \eta_\rho^{b_j}(z) \mid \leq \frac{M}{\mid z - b_j \mid - \rho} \| \eta_\rho^{b_j} - p(b_j)H \|_{L^\infty(B(z,\frac{1}{2}(|z-b_j|-\rho))}$$

and we use (3.25) to get (3.21).

We have the following Lemma

LEMMA 2. In the case I, that is if $b_j \in \overline{\Omega}$ and $p(x) = p_0 + C \mid x - b_j \mid^2 + o(\mid x - b_j \mid^2)$ we have respectively when $b_j \in \Omega$ and when $b_j \in \partial\Omega$, uniformly for $(h_1, ..., h_m) \in \mathcal{H}_{\mathcal{M}}^m$

$$\frac{1}{2}\int_{\Omega_\rho^b} \frac{1}{p} \mid \nabla \eta_\rho^{b_j} \mid^2 = \frac{p_0}{\overline{d}_j^2} W^1(h_j) + O(\rho^2) + O(C\rho^2 \log\frac{1}{\rho}),$$

and

$$\frac{1}{2}\int_{\Omega_\rho^b} \frac{1}{p} \mid \nabla \eta_\rho^{b_j} \mid^2 = \frac{p_0}{2\overline{d}_j^2} W^1(h_j) + O(\rho^2) + O(C\rho^2 \log\frac{1}{\rho}),$$

Now in the case II, that is if $b_j \in G$, $\mid b_j - a_j \mid \leq \delta$ and $p(x) = p_0 + C \mid x - a_j \mid^2 + o(\mid x - a_j \mid^2)$ we have uniformly for $(h_1, ..., h_m) \in \mathcal{H}^m_{\mathcal{M}}$

$$\frac{1}{2}\int_{\Omega^b_\rho} \frac{1}{p} \mid \nabla \eta^{b_j}_\rho \mid^2 = \frac{p^2(b_j)}{p_0 d_j^2} W^1(h_j) + O(C\rho^2 \log\frac{1}{\rho}) + O(C\rho\delta) + O(\frac{\rho^2}{\delta^2}).$$

Proof. We first remark that the $W^1(h_j)$, $j = 1, ..., m$ are bounded uniformly for $(h_1, ..., h_m) \in \mathcal{H}^m_{\mathcal{M}}$. Recall that $\overline{\eta}^{b_j}_\rho(x) = \tilde{\eta}_j(\frac{x-b_j}{\rho})$. We claim that, respectively for $b_j \in \partial G$ and for $b_j \in G$,

$$\begin{aligned} \frac{1}{2}\int_{\Omega^b_\rho} \frac{1}{p} \mid \nabla \overline{\eta}^{b_j}_\rho \mid^2 &= \frac{1}{2p_0 \overline{d}_j} W^1(h_j) + O(\rho^2) + O(C\rho^2 \log\frac{1}{\rho}) \\ &= \frac{1}{p_0 \overline{d}_j} W^1(h_j) + O(\rho^2) + O(C\rho^2 \log\frac{1}{\rho}). \end{aligned} \tag{3.26}$$

We only give the proof of the claim for $b_j \in \partial\Omega$, since the case $b_j \in \Omega$ remains to [CM]. Using the fact that $h_j(\overline{z}) = h_j(z)$ for $z \in S^1$ we transform $\overline{\eta}^j_\rho$, by the inversion $\Psi(z) = \overline{\eta}^j_\rho(b_j + \frac{\rho}{z})$, into the solution of the problem (2.6) and we transform $B(b_j, \beta) \cap \Omega \backslash B(b_j, \rho)$ into the half ring $(B_1-)\backslash B(0, \frac{\rho}{\beta})$, where $B_1- = \{(x_1, x_2) \in \mathbf{R}^2, x_1^2 + x_2^2 \leq 1, x_2 < 0\}$. The domain $\Omega\backslash B(b_j, \beta)$ is transformed into a domain D contained in $B(0, \frac{\rho}{\beta})$. We have, by (3.17)

$$\begin{aligned} &\frac{1}{2}\int_{\Omega^b_\rho} \frac{1}{p} \mid \nabla \overline{\eta}^{b_j}_\rho \mid^2 \leq \frac{1}{2p_0}\int_{\{x\in\Omega, \rho\leq|x-b_j|\leq\beta\}} \mid \nabla \overline{\eta}^{b_j}_\rho \mid^2 + \frac{1}{2p_0}\int_{\Omega\backslash B(b_j,\beta)} \mid \nabla \overline{\eta}^{b_j}_\rho \mid^2 \\ &\leq \frac{1}{2p_0}\int_{(B_1-)\backslash B(0,\frac{\rho}{\beta})} \mid \nabla\Psi \mid^2 + O(\frac{\rho^2}{\beta^2}). \end{aligned}$$

Since $h_j(\overline{z}) = h_j(z)$, we see that $\Psi(\overline{z}) = \Psi(z)$, thus

$$\frac{1}{2p_0}\int_{(B_1-)\backslash B(0,\frac{\rho}{\delta})} \mid \nabla\Psi \mid^2 = \frac{1}{4p_0}\int_{B_1\backslash B(0,\frac{\rho}{\beta})} \mid \nabla\Psi \mid^2 .$$

This gives, using the definition of $W^1(h_j)$,

$$\frac{1}{2}\int_{\Omega^b_\rho} \frac{1}{p} \mid \nabla \overline{\eta}^{b_j}_\rho \mid^2 \leq \frac{1}{2p_0 \overline{d}_j^2} W^1(h_j) + O(\rho^2).$$

Now

$$\begin{aligned} &\frac{1}{2}\int_{\Omega^b_\rho} \frac{1}{p} \mid \nabla \overline{\eta}^{b_j}_\rho \mid^2 \geq \frac{1}{2}\int_{\{x\in\Omega, \rho\leq|x-b_j|\leq\beta\}} \frac{1}{p} \mid \nabla \overline{\eta}^{b_j}_\rho \mid^2 \\ &\geq \frac{1}{2}\int_{(B_1-)\backslash B(0,\frac{\rho}{\beta})} \frac{1}{\tilde{p}} \mid \nabla\Psi \mid^2 \end{aligned}$$

where $\tilde{p}(z) = p(b_j + \frac{\rho}{z})$ By the symmetry of ω and $\tilde{p}$, we obtain

$$\frac{1}{2}\int_{(B_1-)\backslash B(0,\frac{\rho}{\beta})} \frac{1}{\tilde{p}} \mid \nabla\Psi \mid^2 = \frac{1}{4}\int_{B_1\backslash B(0,\frac{\rho}{\beta})} \frac{1}{\tilde{p}} \mid \nabla\Psi \mid^2 .$$

Since $\tilde{p}(z) \leq p_0 + C\frac{\rho^2}{|z|^2}$ for all $z \in B_1\backslash B(0,\frac{\rho}{\beta})$, we have

$$\frac{1}{2}\int_{\Omega_\rho^b} \frac{1}{p} \mid \nabla\overline{\eta}_\rho^{b_j} \mid^2 \geq \frac{1}{2p_0\overline{d}_j^2} W^1(h_j) + O(C\rho^2 \log\frac{\rho}{\beta}) + O(\rho^2),$$

and this gives the proof of the claim (3.26). We use the claims (3.23) and (3.26) to get the proof of the Lemma 2 in the case I. In the case II we denote $\omega^{b_j}(z) = \frac{1}{p(b_j)}\eta_\rho^{b_j}(b_j + \frac{\rho}{z})$, $z \in B(0,1)$ and we have, using (3.21),

$$\begin{aligned}\frac{1}{2}\int_{\Omega_\rho^b} \frac{1}{pp^2(b_j)} \mid \nabla\eta_\rho^{b_j} \mid^2 &\leq \frac{1}{2}\int_{B_1\backslash B(0,\frac{\rho}{\delta})} \frac{1}{\tilde{p}_{b_j}} \mid \nabla\omega^{b_j} \mid^2 + \frac{1}{2p_0^3}\int_{\Omega\backslash B(b_j,\delta)} \mid \nabla\eta_\rho^{b_j} \mid^2 \\ &\leq \frac{1}{2}\int_{B_1\backslash B(0,\frac{\rho}{\delta})} \frac{1}{\tilde{p}_{b_j}} \mid \nabla\omega^{b_j} \mid^2 + O(\frac{\rho^2}{\delta^2}),\end{aligned} \tag{3.27}$$

where $\tilde{p}_{b_j}(z) = p(b_j + \frac{\rho}{z})$, and

$$\frac{1}{2}\int_{\Omega_\rho^b} \frac{1}{pp^2(b_j)} \mid \nabla\eta_\rho^{b_j} \mid^2 \geq \frac{1}{2}\int_{B_1\backslash B(0,\frac{\rho}{\delta})} \frac{1}{\tilde{p}_{b_j}} \mid \nabla\omega^{b_j} \mid^2 . \tag{3.28}$$

Now, a_j being in Ω, we set $\overline{h}_j(x) = h_j \times h_{j\tau}(\frac{x-a_j}{\rho})$ for $x \in \partial B(a_j,\rho)$ and we consider the function $\eta_\rho^{a_j}$ defined by

$$\begin{cases} \operatorname{div}(\frac{1}{p}\nabla\eta_\rho^{a_j}) = 0 \text{ in } \mathbf{R}^2\backslash\partial B(a_j,\rho) \\ \frac{1}{p}\frac{\partial\eta_\rho^{a_j}}{\partial\nu} = \frac{1}{\rho}(\frac{\overline{h}_j}{\overline{d}_j} - 1) \text{ on } \partial B(a_j,\rho) \\ \eta_\rho^{a_j} \text{ bounded} \end{cases}$$

with the normalization condition $\int_{\partial B(a_j,\rho)} \overline{h}_j\eta_\rho^{a_j} = 0$. Using (3.27) and (3.28), we deduce

$$\begin{aligned}&\mid \int_{\Omega_\rho^b} \frac{1}{pp^2(b_j)} \mid \nabla\eta_\rho^{b_j} \mid^2 - \int_{\Omega_\rho^a} \frac{1}{pp_0^2} \mid \nabla\eta_\rho^{a_j} \mid^2 \mid \\ &= \mid \int_{B_1\backslash B(0,\frac{\rho}{\delta})} (\frac{1}{\tilde{p}_{b_j}} \mid \nabla\omega^{b_j} \mid^2 - \frac{1}{\tilde{p}_{a_j}} \mid \nabla\omega^{a_j} \mid^2) \mid + O(\frac{\rho^2}{\delta^2}) \\ &= \mid \int_{B_1} (\frac{1}{\tilde{p}_{b_j}} \mid \nabla\omega^{b_j} \mid^2 - \frac{1}{\tilde{p}_{a_j}} \mid \nabla\omega^{a_j} \mid^2) \mid + O(\frac{\rho^2}{\delta^2}).\end{aligned}$$

By the first part of the proof of Lemma 1, we have

$$\begin{aligned}&\| \omega^{b_j} - \omega^{a_j} \|_{H^2(B(0,1))} \leq \| \omega^{b_j} - \tilde{\omega} \|_{H^2(B(0,1))} + \| \omega^{a_j} - \tilde{\omega} \|_{H^2(B(0,1))}\\ &= O(C\rho \mid b_j - a_j \mid + C\rho^2).\end{aligned}$$

where $\tilde{\omega}(z) = \tilde{\eta}_j(\frac{1}{z})$. In particular we have

$$\| \nabla\omega^{b_j} - \nabla\omega^{a_j} \|_{L^2(B(0,1))} = O(C\rho \mid b_j - a_j \mid + C\rho^2).$$

Now

$$\begin{aligned}&\mid \frac{1}{2} \int_{B_1} (\frac{1}{\tilde{p}_{b_j}} \mid \nabla\omega^{b_j} \mid^2 - \frac{1}{\tilde{p}_{a_j}} \mid \nabla\omega^{a_j} \mid^2) \mid = \int_{B_1} \frac{1}{\tilde{p}_{b_j}} (\mid \nabla\omega^{b_j} \mid^2 - \mid \nabla\omega^{a_j} \mid^2)\\ &+ \int_{B_1} (\frac{1}{\tilde{p}_{b_j}} - \frac{1}{\tilde{p}_{a_j}}) (\mid \nabla\omega^{a_j} \mid^2 = O(C\rho \mid a_j - b_j \mid).\end{aligned}$$

Now we apply the result of the case I in the present Lemma and we see that

$$\frac{1}{2} \int_{\Omega_\rho^a} \frac{1}{p} \mid \nabla\eta_\rho^{a_j} \mid^2 = \frac{p_0}{d_j^2} W^1(h_j) + O(\rho^2) + O(C\rho^2 \log\frac{1}{\rho}),$$

and this gives the proof of Lemma 2.

Let $\eta_\rho^{b_j *}$ be a solution in $\mathbf{R}^2 \backslash \partial B(b_j, \rho)$ of

$$\begin{cases} \dfrac{\partial\eta_\rho^{b_j *}}{\partial x_1} = -\dfrac{1}{p} \dfrac{\partial\eta_\rho^{b_j}}{\partial x_2} \\ \dfrac{\partial\eta_\rho^{b_j *}}{\partial x_2} = \dfrac{1}{p} \dfrac{\partial\eta_\rho^{b_j}}{\partial x_1}. \end{cases}$$

The existence of $\eta_\rho^{b_j *}$ follows from the fact that $\int_{\partial B(b_j,\rho)} \frac{1}{p} \frac{\partial\eta_\rho^{b_j}}{\partial\nu} = 0$ and $\eta_\rho^{b_j *}$ verifies

$$\begin{cases} \mathrm{div}(p\nabla\eta_\rho^{b_j *}) = 0 \text{ in } \mathbf{R}^2 \backslash \partial B(b_j, \rho) \\ \dfrac{\partial\eta_\rho^{b_j *}}{\partial\tau} = \dfrac{1}{\rho} (\dfrac{\tilde{h}_j}{\bar{d}_j} - 1) \text{ on } \partial B(b_j, \rho). \end{cases}$$

By (3.20), for any compact set K in $\mathbf{R}^2 \backslash \{b_j\}$ we obtain, in the case II

$$\max_K \eta_\rho^{b_j *} - \min_K \eta_\rho^{b_j *} = O(\frac{\rho diam(K)}{dist(K, b_j)} + C\delta diam(K) + C\rho diam(K)). \tag{3.29}$$

Now we claim that in the case I

$$\max_{\partial\Omega \backslash \cup_{j=l+1}^m B(b_j,\rho)} \eta_\rho^{b_j *} - \min_{\partial\Omega \backslash \cup_{j=l+1}^m B(b_j,\rho)} \eta_\rho^{b_j *} = O(\rho \log\frac{1}{\rho}). \tag{3.30}$$

Indeed, letting $j = l+1, ..., m$ and $\{x_0, x_1\} = \partial\Omega \cap \partial B(b_j, \rho)$, the symmetry of h_j gives $\int_{x_0}^{x_1}(h_j \times (h_j)_\tau - 1) = 0$, that is $\eta_\rho^{b_j}{}^*(x_0) = \eta_\rho^{b_j}{}^*(x_1)$. Combining this fact with (3.19) we obtain (3.30). Now we define $\overline{\Phi}_\rho^b$ as the solution of the linear problem

$$\begin{cases} \operatorname{div}(\frac{1}{p}\nabla\overline{\Phi}_\rho^b) = 0 \text{ in } \Omega_\rho^b \\ \quad \frac{1}{p}\frac{\partial\overline{\Phi}_\rho^b}{\partial\nu} = \frac{\overline{d}_j}{\rho} \text{ on } \partial B(b_j, \rho) \cap \Omega, j = 1, ..., m \\ \quad \frac{1}{p}\frac{\partial\overline{\Phi}_\rho^b}{\partial\nu} = h_0 \times (h_0)_\tau \text{ on } \partial\Omega \backslash \cup_{j=l+1}^m B(b_j, \rho) \end{cases}$$

with the normalization condition $\int_{\partial\Omega}(h_0 \times (h_0)_\tau)\overline{\Phi}_\rho^b = 0$. The function $\overline{\Phi}_\rho^b$ is equal to the function $\hat{\Phi}_\rho^b$ in the particular case where $h_j(x) = (\frac{x}{|x|})^{\overline{d}_j}$. We write

$$\hat{\Phi}_\rho^b = \overline{\Phi}_\rho^b + \sum_{j=1}^m \overline{d}_j \eta_\rho^{b_j} + \eta_\rho^b, \tag{3.31}$$

thus η_ρ^b is the solution of

$$\begin{cases} \operatorname{div}(\frac{1}{p}\nabla\eta_\rho^b) = 0 \text{ in } \Omega_\rho^b \\ \quad \frac{1}{p}\frac{\partial\eta_\rho^b}{\partial\nu} = -\sum_{k\neq j}\overline{d}_k\frac{1}{p}\frac{\partial\eta_\rho^{b_k}}{\partial\nu} \text{ on } \partial B(b_j, \rho) \cap \Omega, j = 1, ..., m \\ \quad \frac{1}{p}\frac{\partial\eta_\rho^b}{\partial\nu} = -\sum_{k=1}^m\overline{d}_k\frac{1}{p}\frac{\partial\eta_\rho^{b_k}}{\partial\nu} \text{ on } \partial\Omega \backslash \cup_{j=l+1}^m B(b_j, \rho) \end{cases}$$

with the normalization condition $\int_{\partial\Omega}(h_0 \times (h_0)_\tau)\eta_\rho^b = -\sum_{j=1}^m \overline{d}_j \int_{\partial\Omega}(h_0 \times (h_0)_\tau)\eta_\rho^{b_j}$.

We have the following Lemma

LEMMA 3. In the case I

$$\int_{\Omega_\rho^b} | \nabla\eta_\rho^b |^2 = O(\rho^2 \log^3 \frac{1}{\rho})$$

and in the case II

$$\int_{\Omega_\rho^b} | \nabla\eta_\rho^b |^2 = O(\rho^2).$$

Proof. We have, for $j = 1, ..., l$

$$\int_{\partial B(b_j,\rho)} \frac{1}{p}\frac{\partial\eta_\rho^{b_k}}{\partial\nu} = \int_{B(b_j,\rho)} \operatorname{div}(\frac{1}{p}\nabla\eta_\rho^{b_k}) = 0,$$

and consequently

$$\int_{\gamma_j} \frac{1}{p}\frac{\partial \eta_\rho^b}{\partial \nu} = 0, \quad j = 0, 1, ..., l.$$

Thus we may define η_ρ^{b*}. We set in this proof $\eta_\rho = \eta_\rho^b$. First let us prove that we may choose η_ρ^* such that

$$\begin{aligned} | \, \eta_\rho^* \, |_{L^\infty(\Omega_\rho)} &= O(\rho \log \frac{1}{\rho}) \text{ in the case I} \\ &= O(\rho) \text{ in the case II.} \end{aligned} \tag{3.32}$$

We have

$$\int_{\gamma_j} p\frac{\partial \eta_\rho^*}{\partial \nu} = \int_{\gamma_j} \frac{\partial \eta_\rho}{\partial \tau} = 0, \quad j = 1, ..., l,$$

hence we are in position to apply Lemma I.4 of [BBH2] in order to infer

$$\max_{\Omega_\rho^b} \eta_\rho^* - \min_{\Omega_\rho^b} \eta_\rho^* \leq \sum_{j=1}^{l} (\max_{\gamma_j} \eta_\rho^* - \min_{\gamma_j} \eta_\rho^*) + \max_{\gamma_0} \eta_\rho^* - \min_{\gamma_0} \eta_\rho^*.$$

For $j = 1, ..., l$, there exists x and $y \in \gamma_j$ such that

$$\max_{\gamma_j} \eta_\rho^* - \min_{\gamma_j} \eta_\rho^* = \eta_\rho^*(x) - \eta_\rho^*(y)$$

We have

$$\max_{\gamma_j} \eta_\rho^* - \min_{\gamma_j} \eta_\rho^* = O(| \int_x^y \sum_{k \neq j} \frac{\partial \eta_\rho^{k*}(z)}{\partial \tau} |).$$

Using respectively (3.18) and (3.20), we get respectively in the cases I and II

$$\max_{\gamma_j} \eta_\rho^* - \min_{\gamma_j} \eta_\rho^* = O(\rho^2)$$

and

$$\max_{\gamma_j} \eta_\rho^* - \min_{\gamma_j} \eta_\rho^* = O(C\rho\delta + \frac{\rho^2}{\delta}).$$

Let x and y be such that $\max_{\gamma_0} \eta_\rho^* - \min_{\gamma_0} \eta_\rho^* = \eta_\rho^*(x) - \eta_\rho^*(y)$. Let us consider the case I. If x and y are in the same connex component of $\partial\Omega \backslash \cup_{j=l+1}^m B(b_j, \rho)$, we use (3.19) to get $\eta_\rho^*(x) - \eta_\rho^*(y) = O(\rho \log \frac{1}{\rho})$. If there exists j such that x and y are in $\partial B(b_j, \rho) \cap \Omega$, we conclude as in the previous case. Now, in the case II, we use (3.21) to get $\eta_\rho^*(x) - \eta_\rho^*(y) = O(\rho)$. Hence we have (3.32). Let us define $A_{2\rho,\rho}^j = B(b_j, 2\rho) \backslash B(b_j, \rho) \cap G$. Let us prove that in the both cases I and II we have

$$\begin{aligned} || \, \nabla \eta_\rho \, ||_{L^\infty(A_{2\rho,\rho}^j)} &= O(\log \frac{1}{\rho}) \text{ in the case I} \\ &= O(1) \text{ in the case II.} \end{aligned} \tag{3.33}$$

First we consider the case I and $j = l+1, ..., m$. Recall that we may assume that $\partial\Omega \cap B(b_j, \beta)$ is the axis $x_2 = 0$. We extend $\hat{\Phi}_\rho - \overline{\Phi}_\rho$ to a map still denoted $\hat{\Phi}_\rho - \overline{\Phi}_\rho$, defined in $B(b_j, \beta)\backslash B(b_j, \rho)$ by $(\hat{\Phi}_\rho - \overline{\Phi}_\rho)(r(x)) = (\hat{\Phi}_\rho - \overline{\Phi}_\rho)(x)$, r being the reflection associated to the axis $x_2 = 0$. Let

$$v(x) = (\hat{\Phi}_\rho - \overline{\Phi}_\rho - \overline{d}_j \eta_\rho^{b_j})(b_j + \rho x) = (\sum_{k \neq j} \overline{d}_k \eta_\rho^{b_k} + \eta_\rho)(b_j + \rho x).$$

Since we have $\frac{\partial(\hat{\Phi}_\rho - \overline{\Phi}_\rho)}{\partial \nu} = 0$ on $\partial\Omega \cap B(b_j, \beta)$, the map v is in $W^{2,p}(B(0, \frac{\beta}{\rho})\backslash B(0,1))$ for all $1 < p < \infty$ and verifies

$$\begin{cases} \operatorname{div}(\frac{1}{\tilde{p}} \nabla v) = 0 \text{ in } B(0, \frac{\beta}{\rho})\backslash B(0,1) \\ \frac{1}{\tilde{p}} \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial B(0,1), \end{cases}$$

where $\tilde{p}(x) = p(b_j + \rho x)$. Thanks to (3.18) we may choose $\eta_\rho^{b_k *}$ such that

$$|| \eta_\rho^{b_k *} ||_{L^\infty(A^j_{2\rho,\rho})} = O(\rho^2), \text{ for } k \neq j \tag{3.34}$$

and we may define $v^*(x) = \eta_\rho^* + \sum_{k \neq j} \eta_\rho^{b_k *}(b_j + \rho x)$. It verifies

$$\begin{cases} \operatorname{div}(\tilde{p} \nabla v^*) = 0 \text{ in } B(0, \frac{\beta}{\rho})\backslash B(0,1) \\ v^* = C(\rho) \text{ on } \partial B(0,1), \end{cases}$$

where $C(\rho)$ is a constant that verifies by (3.32) and (3.34) $| C(\rho) | = O(\rho \log \frac{1}{\rho})$. Standard elliptic estimates give

$$| \nabla v^* |_{L^\infty(B(0,2)\backslash B(0,1))} \leq M \, | v^* - C(\rho) |_{L^\infty(B(0,3)\backslash B(0,1))} \, .$$

The constant M is independent of ρ because $\tilde{p}$ is bounded uniformly in ρ. But we have $v(r(x)) = v(x)$, so, using (3.32) and (3.34) we obtain

$$| v^* |_{L^\infty(B(0,3)\backslash B(0,1))} = | \eta_\rho^* + \sum_{k \neq j} \eta_\rho^{b_k *} |_{L^\infty(B(b_j, 3\rho)\backslash B(b_j, \rho) \cap \Omega)} = O(\rho \log \frac{1}{\rho}).$$

We are led to

$$\rho \, | \nabla \eta_\rho^* |_{L^\infty(A^j_{2\rho,\rho})} = O(\rho) + | \sum_{k \neq j} \nabla \eta_\rho^{b_k *} |_{L^\infty(A^j_{2\rho,\rho})} = O(\rho \log \frac{1}{\rho}).$$

We have proved (3.33) in the case I, for $j = l+1, ..., m$. The proof of (3.33) in the case I with $j = 1, ..., l$ is the same, without the reflection r. In the case II, thanks to (3.29) we choose $\| \eta_\rho^{b_k *} \|_{L^\infty(A^j_{2\rho,\rho})} = O(\frac{\rho^2}{\delta} + C\rho\delta)$, and the same proof as above, without the reflection r, leads to (3.33). So we have (3.33) in the both cases. We claim now that we have in the case I, for $x \in \partial\Omega \backslash \cup_{j=l+1}^m B(b_j, \rho)$,

$$| \nabla \eta_\rho^*(x) | = O(\max_{j=l+1}^m \frac{\rho \log \frac{1}{\rho}}{| x - b_j | - \rho}) \tag{3.35}$$

and in the case II, for $x \in \partial\Omega$,

$$| \nabla \eta_\rho^*(x) | = O(\rho). \tag{3.36}$$

In order to prove (3.35) and (3.36), we define

$$u(x) = \eta_\rho + \sum_{j=1}^m \overline{d}_j \eta_\rho^{b_j}.$$

We may define $u^* = \eta_\rho^* + \sum_{j=1}^m \overline{d}_j \eta_\rho^{b_j *}$, and due to (3.30) and (3.32) we may choose, for all $x \in \partial\Omega \backslash \cup_{j=l+1}^m B(b_j, \rho)$, in the case I

$$| u^*(x) | = O(\rho \log \frac{1}{\rho}).$$

In the case II, for all $x \in \partial\Omega$ we may choose, due to (3.21) and (3.32)

$$| u^*(x) | = O(\rho).$$

In the case I , let $j = l+1, ..., m$, $x \in \partial\Omega \cap B(b_j, \beta) \backslash B(b_j, \rho)$, and $\lambda = | x - b_j | - \rho$. We define

$$\tilde{u}^*(x) = u^*(x + \lambda y) \quad y \in B_1 + .$$

We remark that $\frac{\partial u^*}{\partial \tau} = 0$ on $\partial\Omega \backslash \cup_{j=l+1}^m B(b_j, \rho)$, thus u^* is equal to a constant $C(\rho)$ in the connex component of $\partial\Omega \backslash \cup_{j=l+1}^m B(b_j, \rho)$ that contains x. We have $C(\rho) = O(\rho \log \frac{1}{\rho})$. The map $\tilde{u}^* - C(\rho)$ verifies

$$\begin{cases} \operatorname{div}(\tilde{p} \nabla (\tilde{u}^* - C(\rho))) = 0 \text{ in } B_1 + \\ \tilde{u}^* - C(\rho) = 0 \text{ on } \partial(B_1+) \cap (x_2 = 0). \end{cases}$$

Standard elliptic estimates in B_1+ give

$$| \nabla \tilde{u}^* |_{L^\infty(B(0,\frac{1}{2})+)} \leq M | \tilde{u}^* - C(\rho) |_{L^\infty(B_1+)}, \tag{3.37}$$

where the constant M is independent of ρ, since $\tilde{p}(y) = p(x + \rho y)$ is bounded uniformly on ρ. The proof of (3.35) follows directly from (3.37). Now in the case II we have $C(\rho) = O(\rho)$ and a similar proof, gives (3.36). We write

$$\int_{\Omega_\rho^b} | \nabla \eta_\rho |^2 = \int_{\Omega_{2\rho}^b} | \nabla \eta_\rho^* |^2 + \sum_{j=1}^m \int_{A^j_{2\rho,\rho}} | \nabla \eta_\rho^* |^2,$$

where

$$\Omega^b_{2\rho} = \Omega\backslash \cup_{j=1}^m B(b_j, 2\rho).$$

Thanks to (3.33), we get

$$\int_{A^j_{2\rho,\rho}} \mid \nabla\eta^*_\rho \mid^2 = O(\rho^2 \log^2 \frac{1}{\rho}) \text{ in the case I}$$
$$= O(\rho^2) \text{ in the case II.}$$

Now

$$\int_{\Omega^b_{2\rho}} p \mid \nabla\eta^*_\rho \mid^2 = \int_{\partial\Omega\backslash\cup_{j=1}^m B(b_j,2\rho)} p\frac{\partial\eta^*_\rho}{\partial\nu}\eta^*_\rho + \sum_{j=1}^m \int_{\partial B(b_j,2\rho)\cap\Omega} p\frac{\partial\eta^*_\rho}{\partial\nu}\eta^*_\rho,$$

By (3.32), (3.33),(3.35) and (3.36) this is $O(\rho^2 \log^3 \frac{1}{\rho})$ in the case I, and $O(\rho^2)$ in the case II. We have proved Lemma 3.

LEMMA 4. We have in the case I

$$\int_{\Omega^b_\rho} \mid \nabla\overline{\Phi}_\rho - \nabla\Phi_0 \mid^2 = O(\rho^2 \log \frac{1}{\rho})$$

and in the case II

$$\int_{\Omega^b_\rho} \mid \nabla\overline{\Phi}_\rho - \nabla\Phi_0 \mid^2 = O(\frac{\rho^2}{\delta^2}) + O(C^2\delta^2\rho^2 \log^2 \frac{1}{\rho}).$$

Proof. Let $u = \overline{\Phi}_\rho - \Phi_0$. It verifies

$$\begin{cases} \operatorname{div}(\frac{1}{p}\nabla u) = 0 \text{ in } \Omega^b_\rho \\ \frac{1}{p}\frac{\partial u}{\partial\nu} = 0 \text{ on } \partial\Omega\backslash \cup_{j=l+1}^m B(b_j,\rho) \\ \frac{1}{p}\frac{\partial u}{\partial\nu} = \frac{\overline{d}_j}{\rho} - \frac{1}{p}\frac{\partial\Phi_0}{\partial\nu} \text{ on } \partial B(b_j,\rho)\cap\Omega, j = 1,...,m. \end{cases}$$

We have

$$\int_{\gamma_j} \frac{1}{p}\frac{\partial u}{\partial\nu} = 0, \quad j = 1,...,l$$

so we may define u^*. Moreover, by (2.7) and (2.10), we have on $\partial B(b_j,\rho)\cap\Omega$, $j = 1,...,m$, respectively in the cases I and II

$$\mid \frac{1}{p}\frac{\partial u}{\partial\nu} \mid = \mid \frac{\overline{d}_j}{\rho}(1 - \frac{p(b_j)}{p}) - \frac{1}{p}\sum_k \overline{d}_k \frac{\partial R_k}{\partial\nu} - \sum_{k\neq j} \overline{d}_k \frac{p(b_k)}{p}\frac{\partial \log \mid x - b_k \mid}{\partial\nu} \mid = O(1)$$

and

$$= O(\frac{1}{\delta}) + O(C\delta \log \frac{1}{\rho}).$$

As in the proof of Lemma 3, Lemma I.4 of [BBH2] gives in the case I

$$\max_{\Omega_\rho^b} u^* - \min_{\Omega_\rho^b} u^* = O(\rho)$$

and in the case II

$$\max_{\Omega_\rho^b} u^* - \min_{\Omega_\rho^b} u^* = O(C\rho\delta\log\frac{1}{\rho} + \frac{\rho}{\delta})$$

and we choose respectively in the cases I and II

$$\begin{aligned} \mid u^* \mid_{L^\infty(\Omega_\rho^b)} &= O(\rho) \text{ and} \\ &= O(\frac{\rho}{\delta} + C\rho\delta\log\frac{1}{\rho}). \end{aligned} \tag{3.38}$$

The same proof as for (3.35) and (3.36) gives in the case I, for $x \in \partial\Omega \backslash \cup_{j=l+1}^m B(b_j, 2\rho)$

$$\mid \nabla u^*(x) \mid = O(\max_{j=l+1}^m \frac{\rho}{\mid x - b_j \mid - \rho}) \tag{3.39}$$

while in the case II, for $x \in \partial\Omega$,

$$\mid \nabla u^*(x) \mid = O(\frac{\rho}{\delta} + C\rho\delta\log\frac{1}{\rho}). \tag{3.40}$$

Now let us prove that in the case I we have for all $x \in \partial B(b_j, 2\rho) \cap \Omega$,

$$\mid \nabla u^*(x) \mid = O(1) \tag{3.41}$$

and in the case II we have for all $x \in \partial B(b_j, 2\rho)$,

$$\mid \nabla u^*(x) \mid = O(\frac{1}{\delta} + C\delta\log\frac{1}{\rho}). \tag{3.42}$$

In the case I let $j = l+1, ..., m$ and x_0 and x_1 be the two points of $\partial\Omega \cap \partial B(b_j, 2\rho)$ and

$$K = \partial B(b_j, 2\rho) \cap \Omega \backslash B(x_0, \frac{\rho}{2}) \backslash B(x_1, \frac{\rho}{2}).$$

Using the change of variable $x = b_j + \rho y$ and standard estimates we get

$$\mid \nabla u^* \mid_{L^\infty(K)} \leq M. \tag{3.43}$$

Then, x being in $\partial\Omega \cap \partial B(b_j, 2\rho)$ we let $\tilde{u}^*(x) = u^*(x + \rho y)$. We remark that u^* is equal to a constant $C(\rho)$ on $\partial\Omega \cap B(b_j, \delta) \backslash B(b_j, \rho)$, and that by (3.38) we have $\mid C(\rho) \mid \leq O(\rho)$. Thus $\tilde{u}^*$ verifies

$$\begin{cases} \operatorname{div}(\tilde{p}\nabla(\tilde{u}^* - C(\rho))) = 0 \text{ in } B_1+ \\ \tilde{u}^* - C(\rho) = 0 \text{ on } \partial(B_1+) \cap (x_2 = 0). \end{cases}$$

We use the following estimate

$$| \nabla \tilde{u}^* |_{L^\infty(B_{\frac{1}{2}}+)} \leq M \mid \tilde{u}^* - C(\rho) |_{L^\infty(B_1+)},$$

to get

$$| \nabla u^* |_{L^\infty(B(x_0,\frac{\rho}{2})\cap G)} \leq M. \tag{3.44}$$

The estimates (3.43) and (3.44) give (3.41). Now, in the case II, we use the change of variable $x = b_j + \rho y$ and standard estimates together with (3.38) to prove (3.42). Next we write, using (3.38)-(3.42),

$$\begin{aligned} \int_{\Omega^b_{2\rho}} p \mid \nabla u^* |^2 &= \int_{\partial\Omega\setminus\cup_{j=1}^m B(b_j,2\rho)} p\frac{\partial u^*}{\partial \nu} u^* - \sum_{j=1}^m \int_{\partial B(b_j,2\rho)\cap\Omega} p\frac{\partial u^*}{\partial \nu} u^* \\ &= O(\rho^2 \log \frac{1}{\rho}) \text{ in the case I,} \\ &= O(\frac{\rho}{\delta} + C\rho\delta \log \frac{1}{\rho})^2 \text{ in the case II.} \end{aligned} \tag{3.45}$$

In order to achieve the proof of Lemma 4, we prove that, respectively in the cases I and II

$$\begin{aligned} \int_{A^j_{2\rho,\rho}} | \nabla u |^2 &= O(\rho^2) \\ &\text{and} \\ &= O(\frac{\rho^2}{\delta^2} + C^2\rho^2\delta^2 \log^2 \frac{1}{\rho}). \end{aligned} \tag{3.46}$$

In the case I, let $j = l+1, ..., m$. We have by (3.37), (3.38) and (3.41)

$$\begin{cases} \operatorname{div}(p\nabla u^*) = 0 \text{ in } A^j_{2\rho,\rho} \\ | u^* | = O(\rho) \text{ on } \partial A^j_{2\rho,\rho} \\ | \frac{\partial u^*}{\partial \tau} | = O(1) \text{ on } \partial A^j_{2\rho,\rho}. \end{cases}$$

Thus, letting $f(x) = u^*(b_j + \rho x)$,

$$\begin{cases} \operatorname{div}(\tilde{p}\nabla f) = 0 \text{ in } A_{2,1}+ \\ | f | = O(\rho) \text{ on } \partial(A_{2,1}+) \\ | \frac{\partial f}{\partial \tau} | = O(\rho) \text{ on } \partial(A_{2,1}+), \end{cases}$$

where $A_{2,1}+ = B(0,2)\setminus B(0,1) \cap (x_2 > 0)$ and $\tilde{p}(x) = p(a_j + \rho x)$. We use an extension theorem, valid for domains with lipschitz boundary (see[G]) to get a map $v \in H^1(A_{2,1}+)$ having the same trace than f on $\partial(A_{2,1}+)$ and such that

$$| v |_{H^1(A_{2,1}+)} \leq M \mid f |_{H^{\frac{1}{2}}(\partial A_{2,1}+)} .$$

We have

$$\int_{A_{2,1}+} \tilde{p} \mid \nabla(f-v) \mid^2 = \int_{A_{2,1}+} -\tilde{p} \nabla v \nabla (f-v)$$

and thus

$$\mid \nabla f \mid_{L^2(A_{2,1}+)} \leq 2 \mid \nabla v \mid_{L^2(A_{2,1}+)} \leq M\rho,$$

that proves (3.46) in the case I, for $j = l+1, ..., m$. In the case I with $j = 1, ..., l$, the proof remains to [CM]. In the case II we have

$$\begin{cases} \operatorname{div}(\tilde{p}\nabla f) = 0 \text{ in } A_{2,1} \\ \mid f \mid = O(\frac{\rho}{\delta} + C\rho\delta \log \frac{1}{\rho}) \text{ on } \partial(A_{2,1}) \\ \mid \frac{\partial f}{\partial \tau} \mid = O(\frac{\rho}{\delta} + C\rho\delta \log \frac{1}{\rho}) \text{ on } \partial(A_{2,1}), \end{cases}$$

thus

$$\|\, f \,\|_{H^1(A_{2,1})} \leq M \,\|\, f \,\|_{H^{\frac{1}{2}}(\partial A_{2,1})} = O(\frac{\rho}{\delta} + C\rho\delta \log \frac{1}{\rho}).$$

We have proved (3.46) in the both cases. By (3.45) and (3.46) we have proved Lemma 4.

LEMMA 5. We have in the case I

$$\int_{\Omega_\rho^b} \frac{1}{p} \nabla \eta_\rho^{b_j} . \nabla \overline{\Phi}_\rho = O(\rho \log^2 \frac{1}{\rho}), \tag{3.47}$$

$$\int_{\Omega_\rho^b} \frac{1}{p} \nabla \eta_\rho^{b} . \nabla \overline{\Phi}_\rho = O(\rho \log^2 \frac{1}{\rho}) \tag{3.48}$$

and

$$\int_{\Omega_\rho^b} \frac{1}{p} \nabla \eta_\rho^{b_j} \nabla \eta_\rho^{b_k} = O(\rho \log \frac{1}{\rho}), \quad j \neq k, \tag{3.49}$$

while in the case II we have

$$\int_{\Omega_\rho^b} \frac{1}{p} \nabla \eta_\rho^{b_j} . \nabla \overline{\Phi}_\rho = O(\frac{\rho}{\delta} + C\rho\delta \log \frac{1}{\rho}), \tag{3.50}$$

$$\int_{\Omega_\rho^b} \frac{1}{p} \nabla \eta_\rho^{b} . \nabla \overline{\Phi}_\rho = O(C\rho\delta \log \frac{1}{\rho} + \rho + \frac{\rho^2}{\delta} \log \frac{1}{\rho}). \tag{3.51}$$

and

$$\int_{\Omega_\rho^b} \frac{1}{p} \nabla \eta_\rho^{b_j} \nabla \eta_\rho^{b_k} = O(\frac{\rho^2}{\delta^2} + \rho), \quad j \neq k. \tag{3.52}$$

Proof. We begin with the proof of (3.47) and (3.50). We denote $u = \overline{\Phi}_\rho - \Phi_0$. Lemma 4 and Cauchy-Schwarz inequality give

$$\begin{aligned}\int_{\Omega_\rho^b} \frac{1}{p} \nabla \eta_\rho^{b_j} . \nabla u &= O(\rho \log^{\frac{1}{2}} \frac{1}{\rho}) \text{ in the case I}\\ &= O(\frac{\rho}{\delta} + C\rho\delta \log \frac{1}{\rho}) \text{ in the case I.}\end{aligned}$$

Moreover

$$\int_{\Omega_\rho^b} \frac{1}{p} \nabla \eta_\rho^{b_j} . \nabla \overline{\Phi}_\rho = \int_{\Omega_\rho^b} \frac{1}{p} \nabla \eta_\rho^{b_j} . \nabla u + \int_{\Omega_\rho^b} \frac{1}{p} \nabla \eta_\rho^{b_j} . \nabla \Phi_0.$$

By (3.19), we have in the case I,

$$\int_{\partial\Omega \setminus \cup_{j=l+1}^m B(b_j,\rho)} \frac{1}{p} \frac{\partial \eta_\rho^{b_j}}{\partial \nu} \Phi_0 = O(\rho \int_{r=\rho}^{r=\beta} \frac{\mid \log r \mid}{(r^2-\rho^2)^{\frac{1}{2}}} dr) + O(\rho) = O(\rho \log^2 \frac{1}{\rho}). \tag{3.53}$$

In the case II we have by (3.21),

$$\int_{\partial\Omega} \frac{1}{p} \frac{\partial \eta_\rho^{b_k}}{\partial \nu} \Phi_0 = O(\rho). \tag{3.54}$$

For $k \neq j$, (3.18) and (3.20) give

$$\begin{aligned}\int_{\partial B(b_k,\rho)\cap\Omega} \frac{1}{p} \frac{\partial \eta_\rho^{b_j}}{\partial \nu} \Phi_0 &= O(\rho) \text{ in the case I}\\ &= O(\frac{\rho^2}{\delta} \log \frac{1}{\rho} + C\rho\delta \log \frac{1}{\rho}) \text{ in the case II.}\end{aligned}$$

Now we use the same notations as in the proof of Proposition 1 and (3.9) to get for $x \in \partial B(b_j,\rho) \cap \Omega$

$$\begin{aligned}\Phi_0(x) &= \log \rho + S_j(b_j) + O(\rho) \text{ in the case I}\\ &= \log \rho + S_j(b_j) + O(C\rho\delta \log \frac{1}{\rho} + \frac{\rho}{\delta}) \text{ in the case II}\end{aligned}$$

and we are led to

$$\begin{aligned}\int_{\partial B(b_j,\rho)\cap\Omega} \frac{1}{p} \frac{\partial \eta_\rho^{b_j}}{\partial \nu} \Phi_0 &= O(\rho) \text{ in the case I}\\ &= O(C\rho\delta \log \frac{1}{\rho} + \frac{\rho}{\delta}) \text{ in the case II.}\end{aligned}$$

We have proved (3.47) and (3.50).

Let us prove (3.48) and (3.51). We set

$$\int_{\Omega_\rho^b}\frac{1}{p}\nabla\eta_\rho.\nabla u=\int_{\partial\Omega\setminus\cup_{j=l+1}^m B(b_j,2\rho)}\frac{1}{p}\eta_\rho\frac{\partial u}{\partial\nu}+\sum_k\int_{\partial B(b_k,2\rho)\cap\Omega}\frac{1}{p}\eta_\rho\frac{\partial u}{\partial\nu}$$
$$+\sum_k\int_{A^k_{2\rho,\rho}}\frac{1}{p}\nabla\eta_\rho.\nabla u.$$

We use Lemmas 3 and 4 and (3.32), (3.39)-(3.42) to get in the case I

$$\int_{\Omega_\rho^b}\frac{1}{p}\nabla\eta_\rho.\nabla u=O(\rho^2\log^2\frac{1}{\rho})$$

and in the case II

$$\int_{\Omega_\rho^b}\frac{1}{p}\nabla\eta_\rho.\nabla u=O(\frac{\rho^2}{\delta}+C\rho^2\delta\log\frac{1}{\rho})).$$

Now

$$\int_{\Omega_\rho^b}\frac{1}{p}\nabla\eta_\rho.\nabla\Phi_0=\sum_k\int_{\partial\Omega\setminus\cup_i B(b_i,\rho)}\frac{1}{p}\frac{\partial\eta_\rho^{b_k}}{\partial\nu}\Phi_0+\sum_{j\neq k}\int_{\partial B(b_k,\rho)\cap\Omega}\frac{1}{p}\frac{\partial\eta_\rho^{b_j}}{\partial\nu}\Phi_0.$$

For $j\neq k$ we have by (3.18), (3.20), (2.7) and (2.9)

$$\begin{aligned}\int_{\partial B(b_k,\rho)\cap\Omega}\frac{1}{p}\frac{\partial\eta_\rho^{b_j}}{\partial\nu}\Phi_0&=O(\rho^2\log\frac{1}{\rho})\text{ in the case I}\\&=O(C\rho\delta\log\frac{1}{\rho})+O(\frac{\rho^2}{\delta}\log\frac{1}{\rho})\text{ in the case II.}\end{aligned}$$

We use (3.53) and (3.54) to complete the proof (3.48) and (3.51).

Let us prove (3.49) and (3.52). Let $k\neq j$.

$$\int_{\Omega_\rho^b}\frac{1}{p}\nabla\eta_\rho^{b_j}.\nabla\eta_\rho^{b_k}=\int_{\partial\Omega\setminus\cup_{i=l+1}^m B(b_i,\rho)}\frac{1}{p}\frac{\partial\eta_\rho^{b_j}}{\partial\nu}\eta_\rho^{b_k}-\sum_{i=1}^m\int_{\partial B(b_i,\rho)\cap\Omega}\frac{1}{p}\frac{\partial\eta_\rho^{b_j}}{\partial\nu}\eta_\rho^{b_k}.$$

For $i=j$ we get respectively in the cases I and II, in view of (3.18) and (3.21)

$$\begin{aligned}\int_{\partial B(b_i,\rho)\cap\Omega}\frac{1}{p}\frac{\partial\eta_\rho^{b_j}}{\partial\nu}\eta_\rho^{b_k}&=\int_{\partial B(b_i,\rho)\cap\Omega}\frac{1}{p}\frac{\partial\eta_\rho^{b_i}}{\partial\nu}(\eta_\rho^{b_k}-\eta_\rho^{b_k}(b_i))\\&=O(\rho^2)\\&=O(\frac{\rho^2}{\delta^2}).\end{aligned}$$

For $i \neq j$ we have, using (3.18) and (3.21), respectively in the cases I and II

$$\begin{aligned}\int_{\partial B(b_i,\rho)\cap\Omega} \frac{1}{p}\frac{\partial \eta_\rho^{b_j}}{\partial \nu}\eta_\rho^{b_k} &= O(\rho^2)\\ &= O(\frac{\rho^2}{\delta^2}).\end{aligned}$$

Moreover, in the case I we have by (3.19)

$$\int_{\partial\Omega\setminus\cup_{i=l+1}^m B(b_i,\rho)} \frac{1}{p}\frac{\partial \eta_\rho^{b_j}}{\partial \nu}\eta_\rho^{b_k} = O(\rho) + O(\int_{r=\beta}^{r=\rho} \frac{\rho}{(r^2-\rho^2)^{\frac{1}{2}}}) = O(\rho \log\frac{1}{\rho})$$

and in the case II, by (3.21)

$$\int_{\partial\Omega} \frac{1}{p}\frac{\partial \eta_\rho^{b_j}}{\partial \nu}\eta_\rho^{b_k} = O(\rho)$$

We have proved (3.49) and (3.52).

Proof of Theorem 4. We have, using (3.31)

$$\begin{aligned}\frac{1}{2}\int_{\Omega_\rho}\frac{1}{p} \mid \nabla\hat{\Phi}_\rho \mid^2 = \frac{1}{2}\int_{\Omega_\rho}\frac{1}{p}\mid \nabla\overline{\Phi}_\rho\mid^2 + \frac{1}{2}\int_{\Omega_\rho}\frac{1}{p}\mid\nabla\eta_\rho\mid^2 + \sum_j \overline{d}_j\int_{\Omega_\rho}\frac{1}{p}\nabla\overline{\Phi}_\rho.\nabla\eta_\rho^{b_j}\\ + \int_{\Omega_\rho}\frac{1}{p}\nabla\overline{\Phi}_\rho.\nabla\eta_\rho + \sum_j \overline{d}_j\int_{\Omega_\rho}\frac{1}{p}\nabla\eta_\rho^{b_j}.\nabla\eta_\rho\\ + \frac{1}{2}\sum_j \overline{d}_j^2\int_{\Omega_\rho}\frac{1}{p}\mid\nabla\eta_\rho^{b_j}\mid^2 + \frac{1}{2}\sum_{j\neq k}\overline{d}_j\overline{d}_k\int_{\Omega_\rho}\frac{1}{p}\nabla\eta_\rho^{b_j}.\nabla\eta_\rho^{b_k}.\end{aligned}$$

We use Lemma 5 and Lemma 3 to obtain

$$\frac{1}{2}\int_{\Omega_\rho}\frac{1}{p}\mid\nabla\hat{\Phi}_\rho\mid^2 = \frac{1}{2}\int_{\Omega_\rho}\frac{1}{p}\mid\nabla\overline{\Phi}_\rho\mid^2 + \frac{1}{2}\sum_{j=1}^m \overline{d}_j^2\int_{\Omega_\rho}\frac{1}{p}\mid\nabla\eta_\rho^{b_j}\mid^2 + X(\rho), \tag{3.55}$$

where $X(\rho) = O(\rho\log^2\frac{1}{\rho})$ in the case I, and

$$X(\rho) = O(C\rho\delta\log\frac{1}{\rho} + \frac{\rho}{\delta})$$

in the case II. Moreover we have

$$\begin{aligned}\int_{\Omega_\rho}\frac{1}{p}\mid\nabla\overline{\Phi}_\rho\mid^2 = \int_{\Omega_\rho}\frac{1}{p}\mid\nabla\Phi_0\mid^2 + \int_{\Omega_\rho}\frac{1}{p}\mid\nabla\overline{\Phi}_\rho - \nabla\Phi_0\mid^2\\ + 2\int_{\Omega_\rho}\frac{1}{p}\nabla(\overline{\Phi}_\rho - \nabla\Phi_0).\nabla\Phi_0\end{aligned}\tag{3.56}$$

and

$$\begin{aligned}\int_{\Omega_\rho}\frac{1}{p}\nabla(\overline{\Phi}_\rho-\Phi_0).\nabla\Phi_0&=\sum_i\int_{\partial B(b_i,\rho)\cap\Omega}\frac{1}{p}\Phi_0\frac{\partial(\overline{\Phi}_\rho-\Phi_0)}{\partial\nu}\\ &=\int_{\partial B(b_i,\rho)\cap\Omega}\Phi_0(\frac{p(b_i)}{p}\frac{\overline{d}_i}{\rho}-\frac{1}{p}\frac{\partial\Phi_0}{\partial\nu})+\int_{\partial B(b_i,\rho)\cap\Omega}\Phi_0(1-\frac{p(b_i)}{p})\frac{\overline{d}_i}{\rho}.\end{aligned}\tag{3.57}$$

We estimate separately the last two terms. Using the proof of Proposition 1 ((3.5), (3.6) and (3.11)) give

$$\begin{aligned}&\int_{\partial B(b_i,\rho)\cap G}\Phi_0(\frac{p(b_i)}{p}\frac{\overline{d}_i}{\rho}-\frac{1}{p}\frac{\partial\Phi_0}{\partial\nu})\\ &=-\int_{\partial B(b_i,\rho)\cap G}\frac{1}{p}\frac{\partial S_i}{\partial\nu}(S_i+\overline{d}_ip(b_i)\log\mid x-b_i\mid)=O(\rho\log\frac{1}{\rho})\text{ in the case I}\\ &=O(C\rho\delta\log\frac{1}{\rho}+\frac{\rho^2}{\delta^2})\text{ in the case II.}\end{aligned}\tag{3.58}$$

On the other hand we directly estimate, by the use of (3.8)

$$\begin{aligned}\int_{\partial B(b_i,\rho)\cap G}\Phi_0(1-\frac{p(b_i)}{p})\frac{\overline{d}_i}{\rho}&=O(\rho^2\log\frac{1}{\rho})\text{ in the case I}\\ &=O(C\rho\delta\log\frac{1}{\rho})\text{ in the case II.}\end{aligned}\tag{3.59}$$

Combining (3.57)-(3.59) we obtain

$$\begin{aligned}\int_{\Omega_\rho}\frac{1}{p}\nabla(\overline{\Phi}_\rho-\Phi_0).\nabla\Phi_0&=O(\rho\log\frac{1}{\rho})\text{ in the case I}\\ &=O(C\rho\delta\log\frac{1}{\rho}+\frac{\rho^2}{\delta^2})\text{ in the case I .}\end{aligned}\tag{3.60}$$

We conclude by (3.56), (3.60) and Lemma 4 that

$$\begin{aligned}\int_{\Omega_\rho}\frac{1}{p}\mid\nabla\overline{\Phi}_\rho\mid^2=\int_{\Omega_\rho}\frac{1}{p}\mid\nabla\Phi_0\mid^2+O(\rho\log\frac{1}{\rho})&\text{ in the case I}\\ =O(\frac{\rho^2}{\delta^2}+C\rho\delta\log\frac{1}{\rho})&\text{ in the case II.}\end{aligned}$$

We are led to

$$\frac{1}{2}\int_{\Omega_\rho}\frac{1}{p}\mid\nabla\hat{\Phi}_\rho\mid^2=\frac{1}{2}\int_{\Omega_\rho}\frac{1}{p}\mid\nabla\Phi_0\mid^2+\frac{1}{2}\sum_{j=1}^m\overline{d}_j^2\int_{\Omega_\rho}\frac{1}{p}\mid\nabla\eta_\rho^{b_j}\mid^2+X(\rho),$$

where in the case I $X(\rho) = O(\rho \log^2 \frac{1}{\rho})$ and in the case II $X(\rho) = O(C\rho\delta \log \frac{1}{\rho} + \frac{\rho}{\delta})$. We use Lemma 2 and Proposition 1 to prove Theorem 4.

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