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MINIMIZATION OF A GINZBURG-LANDAU TYPE ENERGY WITH POTENTIAL HAVING A ZERO OF INFINITE ORDER

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1. Introduction

Let G be a bounded and smooth, simply connected domain in $\mathbb{R}^2$ and let $g : \partial G \to S^1$ be a boundary condition of degree $\deg(g, \partial G) = d \geq 0$ (as we may assume without loss of generality). Consider a C^2 functional $J : \mathbb{R} \to [0, \infty)$ satisfying the following conditions:

(H_1) $J(0) = 0$ and $J(t) > 0$ on $(0, \infty)$,
(H_2) $J'(t) > 0$ on $(0, 1]$,
(H_3) there exists $\eta_0 > 0$ such that $J''(t) > 0$ on $(0, \eta_0)$.

For $\varepsilon > 0$ consider the energy functional

$$E_\varepsilon(u) = \int_G |\nabla u|^2 dx + \frac{1}{\varepsilon^2} \int_G J(1 - |u|^2)\, dx \tag{1.1}$$

over

$$H^1_g(G, \mathbb{C}) := \{u \in H^1(G, \mathbb{C}) : u = g \text{ on } \partial G\}. \tag{1.2}$$

It is easy to see that $\min_{u \in H^1_g(G,\mathbb{C})} E_\varepsilon(u)$ is achieved by some smooth u_ε which satisfies:

$$\begin{cases} -\Delta u_\varepsilon = \dfrac{1}{\varepsilon^2} j(1 - |u_\varepsilon|^2) u_\varepsilon & \text{in } G, \\ \quad u_\varepsilon = g & \text{on } \partial G, \end{cases} \tag{1.3}$$

where $j(t) := J'(t)$. The case $J(u) = (1 - |u|^2)^2$, corresponding to the Ginzburg-Landau (GL) energy, was studied by Bethuel, Brezis and Hélein [1, 2] (see also Struwe [5]), where it was shown that:

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(i) For a subsequence $\varepsilon_n \to 0$ we have,

$$u_{\varepsilon_n} \to u_* = e^{i\phi} \prod_{j=1}^{d} \frac{z - a_j}{|z - a_j|} \quad \text{in} \quad C^{1,\alpha}(\bar{G} \setminus \{a_1, ..., a_d\}),$$

where $a_1, ..., a_d$ are distinct points in G and ϕ is a smooth harmonic function determined by the requirement $u_* = g$ on ∂G.

(ii) $E_\varepsilon(u_\varepsilon) = 2\pi d |\log \varepsilon| + O(1)$ as $\varepsilon \to 0$.

The method of [1, 2, 5] can be adapted without difficulty to the case of J satisfying $(H_1) - (H_3)$ with a zero of *finite order* at $t = 0$. This applies for example to $J(t) = |t|^k$, for all $k \geq 2$. The main objective of the current paper is to treat the case of J with a zero of *infinite order* at $t = 0$, having in mind the examples

$$J_k(t) = \begin{cases} \exp(-1/t^k) & \text{for } t > 0, \\ 0 & \text{for } t \leq 0, \end{cases} \tag{1.4}$$

for any $k > 0$. It turns out that a convergence result, as in (i) above, holds for such J's as well. The main difference with respect to the usual GL-energy is in the energy asymptotics. For J with a zero of infinite order the "energy cost" of a degree-one vortex may be much less than the cost of $2\pi \log \frac{1}{\varepsilon}$ for the GL-functional (see (ii) above). In fact, we shall see that this cost equals

$$2\pi\Big(\log \frac{1}{\varepsilon} - \bar{I}(\frac{1}{\varepsilon})\Big),$$

where $\bar{I}(R)$ is a positive function satisfying $\bar{I}(R) = o(\log R)$ as $R \to \infty$, which is determined by the particular functional J. More precisely, the function $\bar{I}(R)$ satisfies

$$\bar{I}(R) = \frac{1}{2} \int_{1/R^2}^{j(\eta_0)} j^{-1}(t) \frac{dt}{t} + O(1), \quad \text{as } R \to \infty \quad \text{(see Lemma 2.4)}. \tag{1.5}$$

So for example, for J_1 in (1.4) we find $\bar{I}(R) = \frac{1}{2} \log \log(R) + O(1)$ (see Proposition 4.1 in the Appendix), and the asymptotics for the energies in this case read:

$$E_\varepsilon(u_\varepsilon) = 2\pi d\big(\log \frac{1}{\varepsilon} - \frac{1}{2} \log \log \frac{1}{\varepsilon}\big) + O(1).$$

Somewhat surprisingly, it turns out that we may have $\bar{I}(R) = O(1)$ also for J with a zero of infinite order, as is the case for $k \in (0, 1)$ in (1.4), see Proposition 4.1.

Our first main theorem describes the asymptotic behavior of the minimizers and their energies.

Theorem 1. *For each $\varepsilon > 0$, let u_ε be a minimizer for the energy E_ε over $H^1_g(G,\mathbb{C})$ with G, g (of degree $d \geq 0$) as above and J satisfying $(H_1)-(H_3)$. Then*

(i) For a subsequence $\varepsilon_n \to 0$ we have

$$u_{\varepsilon_n} \to u_* = e^{i\phi} \prod_{j=1}^{d} \Big(\frac{z-a_j}{|z-a_j|}\Big) \quad \text{in } C^{1,\alpha}(\bar{G} \setminus \{a_1, ..., a_d\}),$$

where $a_1, ..., a_d$ are distinct points in G and ϕ is a smooth harmonic function determined by the requirement $u_* = g$ on ∂G.

(ii) Setting, for $R > \frac{1}{\sqrt{j(\eta_0)}}$,

$$I_0(R) = \frac{1}{2}\int_{1/R^2}^{j(\eta_0)} j^{-1}(t)\frac{dt}{t},$$

we have

$$E_\varepsilon(u_\varepsilon) = 2\pi d\Big(\log\frac{1}{\varepsilon} - I_0\big(\frac{1}{\varepsilon}\big)\Big) + O(1). \tag{1.6}$$

We show in Lemma 2.4 below that the function I_0 satisfies $I_0(R) = o(\log R)$. This implies that the leading term in the energy is always of the order $|\log\varepsilon|$. It is easy to see that $I_0(R)$ is a positive, monotone increasing, concave function of $\log R$ (for large R). It is natural to ask whether every function with these properties can appear in the second-order term of the energy expansion, for some potential J. The answer to this "inverse problem" turns out to positive, as shown by our second theorem.

Theorem 2. *Let $h \in C^2[0,\infty)$ satisfy, for some $T > 0$,*

$$h'(t) > 0, \;\; h''(t) < 0, \quad \text{for } t \geq T > 0, \tag{1.7}$$

and

$$\lim_{t\to\infty} h'(t) = 0. \tag{1.8}$$

Then, there exists a functional J satisfying $(H_1)-(H_3)$, such that the minimizers $\{u_\varepsilon\}$ over $H^1_g(G,\mathbb{C})$, for E_ε defined by (1.1) and g of degree d as above, satisfy

$$E_\varepsilon(u_\varepsilon) = 2\pi d\Big(\log\frac{1}{\varepsilon} - h\big(\log\frac{1}{\varepsilon}\big)\Big) + O(1).$$

2. A study of an auxiliary optimization problem

Let us begin by explaining the main idea of the proof of Theorem 1 and by showing how it leads to a certain optimization problem which is the object of the current section. It is natural to estimate first the energy cost of a degree-one "vortex" in a disc, say the unit disc $B_1 = B_1(0)$. In the case of the Ginzburg-Landau energy, it is easy to guess the energy cost, by taking $v_\varepsilon(r^{i\theta}) = f_\varepsilon(r)e^{i\theta}$ with f_ε given by:

$$f_\varepsilon(r) = \begin{cases} \frac{r}{\varepsilon} & \text{for } 0 \le r < \varepsilon, \\ 1 & \text{for } \varepsilon \le r \le 1. \end{cases}$$

A simple computation gives

$$\int_{B_1} |\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1 - |v_\varepsilon|^2)^2 = 2\pi \log \frac{1}{\varepsilon} + O(1),$$

which turns out to be the optimal estimate, up to an additive constant, although the proof of this fact is far from trivial (see [2]). When looking for the right upper bound for the energy in the general case, we keep the formula $v_\varepsilon(r) = f_\varepsilon(r)e^{i\theta}$, and try to optimize over the function f_ε (since we do not know a priori what form it should take, for our particular J). What we can assume a priori on that function is that it satisfies

$$E_\varepsilon(v_\varepsilon, B_\varepsilon) = O(1), \tag{2.1}$$

and

$$\frac{1}{\varepsilon^2}\int_{B_1} J(1 - |v_\varepsilon|^2) = O(1). \tag{2.2}$$

Indeed, for a *minimizer* both (2.1) and (2.2) should hold, thanks to the estimates (3.1) and (3.2) that we shall verify below. Assuming then that f_ε is chosen in such a way that (2.1)–(2.2) are satisfied, we get for the energy of v_ε:

$$\begin{aligned} E_\varepsilon(v_\varepsilon) &= 2\pi \int_0^1 \Big((f_\varepsilon')^2 + \frac{f_\varepsilon^2}{r^2} + \frac{1}{\varepsilon^2} J(1 - f_\varepsilon^2)\Big)\, r dr \\ &= 2\pi \log \frac{1}{\varepsilon} - 2\pi \int_\varepsilon^1 \frac{1 - f_\varepsilon^2}{r}\, dr + \int_\varepsilon^1 (f_\varepsilon')^2\, r dr + O(1). \end{aligned} \tag{2.3}$$

In order to get minimal energy (up to an $O(1)$-term), we shall look for f_ε which *maximizes* the term $\int_\varepsilon^1 \frac{1-f_\varepsilon^2}{r}\, dr$ (representing the gain of energy with respect to the "usual" cost of $2\pi \log \frac{1}{\varepsilon}$) under the constraint $\int_\varepsilon^1 J(1 - f_\varepsilon^2)\, r dr \le C_0$. Here we did not take into account the contribution of the term $\int_\varepsilon^1 (f_\varepsilon')^2\, r dr$, but as we shall see below, this term is bounded for the solution of our optimization problem.

Rescaling by a factor of ε, we are led naturally to define the following quantity:

$$I(R, c) = \sup\Big\{ \int_1^R \frac{1 - f^2}{r} dr \ :\ \int_1^R J(1 - f^2) r\, dr \le c \Big\}, \tag{2.4}$$

for any $R > 1$ and $c > 0$.

Lemma 2.1. *For every $R > 1$ and $c > 0$, there exists a maximizer $f_0 = f_0^{(R)}$ in (2.4) satisfying $0 \le f_0(r) \le 1$, for all r, such that $f_0(r)$ is nondecreasing. Moreover, if $r_0 = r_0(c)$ is defined by the equation*

$$c = J(1)\Big(\frac{r_0^2 - 1}{2}\Big), \tag{2.5}$$

then there exists $\tilde{r}_0 = \tilde{r}_0(c, R) \in [1, r_0]$ such that

$$f_0(r) \begin{cases} = 0 & \text{if } r \in [1, R] \text{ and } r < \tilde{r}_0, \\ > 0 & \text{if } r > \tilde{r}_0. \end{cases} \tag{2.6}$$

Furthermore,

$$\int_1^R J(1 - f_0^2) r\, dr = c, \quad \text{for } R > r_0, \tag{2.7}$$

and

$$j(1 - f_0^2(r)) = \frac{1}{\lambda r^2}, \quad r > \tilde{r}_0, \tag{2.8}$$

for some $\lambda = \lambda(R, c) > 0$.

Proof. We may consider only admissible f satisfying $0 \le f \le 1$. Indeed, note that replacing f by $|f|$ does not change the value of the integrals appearing in the target functional and in the constraint functional. Furthermore, if the set $\{r : f(r) > 1\}$ has positive measure, then replacing $f(r)$ by 1 on this set would strictly increase the value of the target functional, $\int_1^R \frac{1-f^2}{r}\, dr$, without violating the constraint.

Moreover, we may assume that f is nondecreasing by applying Schwarz symmetrization to the function $f(x) = f(|x|)$ (with respect to the Lebesgue measure on $\mathbb{R}^2$, setting first $f \equiv 0$ on $B_1(0)$). Then, using Helly's selection principle we obtain the convergence almost everywhere of a subsequence of a maximizing sequence $\{f_n\}$ to a limit f_0 which is a maximizer. Clearly, for $R \le r_0$ the maximizer is $f_0 \equiv 0$. On the other hand, for $r > r_0$ we must have $f_0(r) > 0$. This implies the existence of $\tilde{r}_0$ satisfying (2.6).

In order to prove (2.7), assume by negation that the inequality

$$\int_1^R J(1 - f_0^2) r\, dr < c$$

holds for some $R > r_0$. We can then choose a small enough interval where f_0 is positive, and redefine f_0 to be zero there, thus increasing the value of the target functional, without violating the constraint

$$\int_1^R J(1-f_0^2) r\,dr \le c.$$

This is a contradiction. Finally, note that the maximizer f_0 satisfies the following Euler equation associated to (2.4):

$$f_0 j(1-f_0^2(r)) = \frac{f_0}{\lambda r^2}, \quad r\in[1,R], \tag{2.9}$$

where λ is a Lagrange multiplier. For $r>\tilde r_0$ we may divide both sides of (2.9) by $f_0(r)$ to obtain (2.8). □

Lemma 2.2. *There exist two constants* $0<a(c)<b(c)$ *such that*

$$a(c)\le\lambda\le b(c),\ \ R\ge r_0+1.$$

Proof. First, applying (2.8) for $r=r_0$ yields

$$\frac{1}{\lambda} = r_0^2 j(1-f_0^2(r_0)) \le r_0^2 \max_{0\le t\le 1} j(t) \implies \lambda\ge a(c). \tag{2.10}$$

In order to prove the upper bound for λ we distinguish two cases:
(i) $1-f_0^2(\tilde r_0)\ge\eta_0$,
(ii) $1-f_0^2(\tilde r_0)<\eta_0$,
where we denoted $f_0(\tilde r_0)=\lim_{r\searrow\tilde r_0^+} f_0(r)$.

In case (i) we simply have

$$j(1-f_0^2(\tilde r_0)) \ge \min_{t\in[\eta_0,1]} j(t) := \alpha_0 > 0$$

(by (H_2)). Using the last estimate in conjunction with (2.8) yields

$$\lambda \le \frac{1}{\alpha_0\tilde r_0^2} \le \frac{1}{\alpha_0}.$$

In case (ii) we argue as follows. Since by (H_3) the function j is strictly monotone increasing on $[0,\eta_0]$, it has an inverse function that we denote by $\gamma=j^{-1}$. We may rewrite then (2.8) as

$$1-f_0^2(r) = \gamma\Big(\frac{1}{\lambda r^2}\Big), \quad r\in(\tilde r_0,R]. \tag{2.11}$$

Using the constraint (2.7) we obtain

$$c = \int_{\tilde r_0}^R J\Big(\gamma\big(\frac{1}{\lambda r^2}\big)\Big)\,r dr + \frac{\tilde r_0^2-1}{2} J(1). \tag{2.12}$$

We use the right-hand side of (2.12) to define a function of two variables

$$F(\lambda,t) = \int_t^R J\Big(\gamma\big(\frac{1}{\lambda r^2}\big)\Big)\,r dr + \frac{t^2-1}{2} J(1).$$

A simple computation gives

$$
\begin{aligned}
F_\lambda(\lambda,t) &= -\int_t^R j\Big(\gamma\big(\frac{1}{\lambda r^2}\big)\Big)\gamma'\big(\frac{1}{\lambda r^2}\big)\,\frac{dr}{\lambda^2 r}\\
&= -\int_t^R \gamma'\big(\frac{1}{\lambda r^2}\big)\,\frac{dr}{\lambda^3 r^3} = -\frac{1}{2\lambda^2}\Big(\gamma\big(\frac{1}{\lambda t^2}\big)-\gamma\big(\frac{1}{\lambda R^2}\big)\Big) < 0, \quad (2.13)
\end{aligned}
$$

provided that $t < R$. In particular, λ is uniquely determined by (2.12) as a C^1-function $\lambda = l(t)$ in a neighborhood of $t = \tilde{r}_0$. Note also that

$$
F_t(\lambda,t) = -J\Big(\gamma\big(\frac{1}{\lambda t^2}\big)\Big)t + tJ(1) = t\big(J(1) - J(1-f_0^2(t))\big). \quad (2.14)
$$

Next, using (2.11) we get that

$$
I(R,c) = \log\tilde{r}_0 + \int_{\tilde{r}_0}^R \gamma\big(\frac{1}{l(\tilde{r}_0)r^2}\big)\,\frac{dr}{r} := H(\tilde{r}_0). \quad (2.15)
$$

Since $t = \tilde{r}_0$ is a maximum point for the function $H(t)$, we must have

$$
H'(\tilde{r}_0) = 0. \quad (2.16)
$$

Using (2.16) and (2.13)–(2.14) we obtain

$$
\begin{aligned}
0 = H'(\tilde{r}_0) &= \frac{1}{\tilde{r}_0} - \gamma\Big(\frac{1}{l(\tilde{r}_0)\tilde{r}_0^2}\Big)\frac{1}{\tilde{r}_0} - \frac{l'(\tilde{r}_0)}{l^2(\tilde{r}_0)}\int_{\tilde{r}_0}^R \gamma'\Big(\frac{1}{l(\tilde{r}_0)r^2}\Big)\,\frac{dr}{r^3}\\
&\le \frac{1}{\tilde{r}_0} + l'(\tilde{r}_0)l(\tilde{r}_0)F_\lambda(l(\tilde{r}_0),\tilde{r}_0) = \frac{1}{\tilde{r}_0} - l(\tilde{r}_0)F_{\tilde{r}_0}(l(\tilde{r}_0),\tilde{r}_0) \quad (2.17)\\
&= \frac{1}{\tilde{r}_0} - l(\tilde{r}_0)\tilde{r}_0\big(J(1) - J(1-f_0^2(\tilde{r}_0))\big).
\end{aligned}
$$

From (2.17) it follows (using (ii) and (H_2)) that

$$
\lambda = l(\tilde{r}_0) \le \frac{1}{J(1) - J(\eta_0)},
$$

and the result follows in this case as well. □

Remark 2.1. *The proof of Lemma 2.2 actually shows that the bounds for λ are uniform for c lying in a bounded interval.*

Lemma 2.3. *For every $c > 0$ there exists a constant $C = C(c)$ such that for every $0 < c_1, c_2 \le c$ we have*

$$
|I(R,c_1) - I(R,c_2)| \le C,\ \forall R \ge 1.
$$

Proof. Let $f_1^{(R)}$ and $f_2^{(R)}$ denote, respectively, the maximizers for $I(R,c_1)$ and $I(R,c_2)$. By Lemma 2.1 we have

$$j(1-f_i^2(r)) = \frac{1}{\lambda_i r^2}, \quad i=1,2,\ r \geq \max(r_0(c_1), r_0(c_2)),$$

for the Lagrange multipliers $\lambda_1 = \lambda_1(R) = \lambda(R,c_1)$ and $\lambda_2 = \lambda_2(R) = \lambda(R,c_2)$. Recall that by Lemma 2.2 we have

$$0 < a \leq \lambda_1^{(R)},\ \lambda_2^{(R)} \leq b < \infty, \quad \forall R \geq \max(r_0(c_1), r_0(c_2)) + 1,$$

for some positive constants a and b. By Lemma 2.1, Lemma 2.2 and Remark 2.1 it follows that there exists $R_0 = R_0(c)$ such that $1 - f_i^2(r) \leq \eta_0/2$, $i=1,2$, for $r \geq R_0$. Therefore, for $i=1,2$ we have

$$\begin{aligned} I(R,c_i) = \int_1^R \frac{1-f_i^2}{r}\,dr &= \int_{R_0}^R j^{-1}\big(\frac{1}{\lambda_i r^2}\big)\,\frac{dr}{r} + O(1) \\ &= \frac{1}{2}\int_{\frac{1}{\lambda_i R^2}}^{\frac{1}{\lambda_i R_0^2}} j^{-1}(t)\,\frac{dt}{t} + O(1). \end{aligned} \tag{2.18}$$

Using $\max_{[0,j(\eta_0/2)]} j^{-1}(t) \leq C_1$, for some constant C_1, we obtain from (2.18),

$$\begin{aligned} |I(R,c_1) - I(R,c_2)| &\leq \Big| \int_{\frac{1}{\lambda_1 R^2}}^{\frac{1}{\lambda_2 R^2}} j^{-1}(t)\,\frac{dt}{t} \Big| + O(1) \\ &\leq C_1 \big|\log\big(\frac{\lambda_1}{\lambda_2}\big)\big| + O(1) \leq C_1 \log\big(\frac{b}{a}\big) + O(1). \qquad \square \end{aligned}$$

In view of Lemma 2.3 it is natural to set:

$$I(R) := I(R,1). \tag{2.19}$$

For any fixed $c_0 > 0$ we have then:

$$|I(R,c) - I(R)| \leq C(c_0), \quad \forall c \leq c_0,\ \forall R \geq 1. \tag{2.20}$$

Next we prove, by the method of proof of Lemma 2.3, an explicit estimate for $I(R)$. In the sequel we shall denote by f_0 a maximizer for $I(R) = I(R,1)$ as given by Lemma 2.1.

Lemma 2.4. *We have*

$$I(R) = \frac{1}{2}\int_{\frac{1}{R^2}}^{j(\eta_0)} j^{-1}(t)\,\frac{dt}{t} + O(1), \ \ \forall R > \frac{1}{\sqrt{j(\eta_0)}}. \tag{2.21}$$

In particular,

$$\lim_{R\to\infty} \frac{I(R)}{\log R} = 0. \tag{2.22}$$

Proof. By Lemma 2.1 we have $j(1-f_0^2(r)) = \frac{1}{\lambda r^2}$ for $r > r_0(1)$ and by Lemma 2.2 we have

$$\lambda = \lambda(R) \in [a,b], \quad \text{for } R \geq r_0(1)+1, \tag{2.23}$$

for some two positive constants a and b. Using hypothesis (H_3) we conclude that

$$1 - f_0^2(r) = j^{-1}\Big(\frac{1}{\lambda r^2}\Big), \quad \text{for } R \geq r \geq \mu_0 := \max\Big(r_0(1), \frac{1}{\sqrt{aj(\eta_0)}}\Big). \tag{2.24}$$

It follows that

$$I(R) = \int_{\mu_0}^{R} j^{-1}\Big(\frac{1}{\lambda r^2}\Big)\frac{dr}{r} + O(1) = \frac{1}{2}\int_{\frac{1}{\lambda R^2}}^{j(\eta_0)} j^{-1}(t)\frac{dt}{t} + O(1).$$

In order to get (2.21) it suffices to notice that

$$\begin{aligned}\Big|\int_{\frac{1}{\lambda R^2}}^{j(\eta_0)} j^{-1}(t)\frac{dt}{t} - \int_{\frac{1}{R^2}}^{j(\eta_0)} j^{-1}(t)\frac{dt}{t}\Big| &\leq \Big|\int_{\frac{1}{R^2}}^{\frac{1}{\lambda R^2}} j^{-1}(t)\frac{dt}{t}\Big| \\ &\leq C|\log\lambda| \leq C\max(|\log b|, |\log a|) = O(1).\end{aligned}$$

Finally we note that (2.22) follows easily from (2.21) since $j^{-1}(0) = 0$. □

The next lemma provides an estimate that we shall use in the proof of the upper bound for the energy in subsection 3.2.

Lemma 2.5. *We have*

$$\int_{\mu_0}^{R} (f_0')^2\, r dr \leq C, \quad \forall R > \mu_0, \tag{2.25}$$

for a as in (2.23) *and μ_0 as defined in* (2.24).

Proof. Differentiating the equality (2.24) yields for $r \geq \mu_0$,

$$-2f_0 f_0' = (j^{-1})'\Big(\frac{1}{\lambda r^2}\Big)\cdot\Big(-\frac{2}{\lambda r^3}\Big),$$

which implies

$$f_0'(r) \leq C(j^{-1})'\Big(\frac{1}{br^2}\Big)\cdot\frac{1}{r^3},$$

with b given by (2.23). Therefore, denoting by C different positive constants, we get

$$\int_{\mu_0}^{R}(f_0')^2\, rdr \leq C\int_{\mu_0}^{R}\Big[(j^{-1})'\Big(\frac{1}{br^2}\Big)\Big]^2\frac{dr}{r^5} = C\int_{\frac{1}{bR^2}}^{\frac{1}{b\mu_0^2}}\big[(j^{-1})'(\alpha)\big]^2\alpha\, d\alpha$$

$$= C\int_{\frac{1}{bR^2}}^{\frac{1}{b\mu_0^2}} \frac{\alpha d\alpha}{\big(j'(j^{-1}(\alpha))\big)^2} = C\int_{j^{-1}(\frac{1}{bR^2})}^{j^{-1}(\frac{1}{b\mu_0^2})} \frac{j(\beta)}{j'(\beta)}\,d\beta. \tag{2.26}$$

It is elementary to verify that

$$\lim_{\beta\to 0^+}\frac{j(\beta)}{j'(\beta)} = 0. \tag{2.27}$$

Indeed, if $J''(0) = j'(0) > 0$, then

$$\lim_{\beta\to 0^+}\frac{j(\beta)}{j'(\beta)} = \lim_{\beta\to 0^+}\frac{J'(\beta)}{J''(\beta)} = 0,$$

since $J'(0) = 0$ by (H_1), while if $J''(0) = 0$ then by L'hôpital's rule

$$\lim_{\beta\to 0^+}\frac{j(\beta)}{j'(\beta)} = \lim_{\beta\to 0^+}\frac{J'(\beta)}{J''(\beta)} = \lim_{\beta\to 0^+}\frac{J(\beta)}{J'(\beta)} = 0,$$

since by convexity $J(\beta) = \int_0^\beta J'(s)\,ds \le \beta J'(\beta)$ for $\beta \le \eta_0$. Therefore, (2.25) follows from (2.26) and (2.27). □

We next study a similar functional to that of (2.4). It will serve in the proof of the lower bound of the energy in subsection 3.3. For any $R > 1$ and $c > 0$ set

$$\widetilde{I}(R,c) = \sup\Big\{\int_1^R \Big(\frac{1-f^2}{r} + 4\frac{(1-f^2)^2}{r}\Big)dr : \int_1^R J(1-f^2)r\,dr = c\Big\}. \tag{2.28}$$

Lemma 2.6. *There exists a constant $C = C(c)$ such that*

$$|\widetilde{I}(R,c) - I(R,c)| \le C, \quad \forall R \ge 1. \tag{2.29}$$

Proof. The existence of a maximizer $\tilde f_0 = \tilde f_0^{(R)}$, which is a nondecreasing function on $[1,R]$ satisfying $0 \le \tilde f_0 \le 1$, follows as in the proof of Lemma 2.1. Moreover, there exists $\tilde r_0' \in [1, r_0]$ (with r_0 given by (2.5)) such that $\tilde f_0(r) > 0$ for $r > \tilde r_0'$ and $\tilde f_0(r) = 0$ for $r < \tilde r_0'$. The Euler-Lagrange equation for $\tilde f_0$ reads

$$-\frac{2\tilde f_0}{r} - \frac{16\tilde f_0(1-\tilde f_0^2)}{r} = \lambda r j(1-\tilde f_0^2)(-2\tilde f_0), \quad 1 \le r \le R.$$

Thus,

$$\frac{j(1-\tilde f_0^2)}{9-8\tilde f_0^2} = \frac{1}{\lambda r^2}, \quad \tilde r_0' < r \le R. \tag{2.30}$$

Setting $\tilde{j}(t) = \frac{j(t)}{8t+1}$, we may rewrite (2.30) as

$$\tilde{j}(1 - \tilde{f}_0^2) = \frac{1}{\lambda r^2}, \quad \tilde{r}_0' < r \leq R.$$

We claim that there exists $\tilde{\eta}_0 > 0$ such that $\tilde{j}$ is strictly monotone increasing on $[0, \tilde{\eta}_0]$. Indeed, this is a direct consequence of

$$\Big(\frac{j(t)}{8t+1}\Big)' = \frac{(8t+1)j'(t) - 8j(t)}{(8t+1)^2}$$

and (2.27). We can then repeat the argument of Lemma 2.2, with $\tilde{j}$ and $\tilde{\eta}_0$ taking the place of j and η_0, respectively, to conclude the existence of $\tilde{a}(c), \tilde{b}(c)$ such that $0 < \tilde{a}(c) \leq \lambda \leq \tilde{b}(c) < \infty$. We can then apply the argument of Lemma 2.3 to deduce (2.29). Indeed, we only need to notice that $\tilde{j}(t) = j(t) + o(t)$ implies that $\tilde{j}^{-1}(t) = j^{-1}(t) + o(t)$. □

By using the above arguments we also obtain the following result.

Lemma 2.7. *For every $c_0, \alpha > 0$ there exists a constant $C_1(c_0, \alpha)$ such that*

$$\begin{cases} |I(\alpha R, c) - I(R)| \leq C_1(c, \alpha) \\ |\widetilde{I}(\alpha R, c) - I(R)| \leq C_1(c, \alpha) \end{cases} \tag{2.31}$$

for $R > \max(1, \frac{1}{\alpha})$ and $c \in (0, c_0]$.

3. Proof of the main results

In this section we shall give the proof of our main results Theorem 1 and Theorem 2. We begin with some basic estimates for the minimizer u_ε, which follow as in the case of the GL-energy (see [1, 2]).

3.1. Some basic estimates for u_ε.

The next lemma provides L^∞-estimates for u_ε and its gradient.

Lemma 3.1. *Any solution u_ε of (1.3) satisfies:*

$$\|u_\varepsilon\|_{L^\infty(G)} \leq 1 \quad \text{and} \quad \|\nabla u_\varepsilon\|_{L^\infty(G)} \leq \frac{C}{\varepsilon}. \tag{3.1}$$

Proof. The first estimate follows easily from the observation that replacing $u_\varepsilon(x)$ by $u_\varepsilon(x)/|u_\varepsilon(x)|$ on the set $\{x \in G : |u_\varepsilon(x)| > 1\}$ strictly decreases the energy if the latter set has a positive measure. The second estimate in (3.1) follows from a simple rescaling argument and standard elliptic estimates as in [1, 5]. □

In the case of a starshaped G the following Pohozaev identity holds for u_ε (actually it is valid for any solution of problem (1.3)). The proof is identical to the one for the GL-energy in [2], so we omit it.

Lemma 3.2. *If G is starshaped then*

$$\frac{1}{\varepsilon^2}\int_G J(1-|u_\varepsilon|^2) \le C_0,\ \forall \varepsilon > 0. \tag{3.2}$$

We shall show later that the assumption of starshapeness of the domain can be dropped, by applying an argument of del Pino and Felmer [4].

3.2. **The upper bound for the energy.** This subsection is devoted to the proof of the following proposition which provides the upper bound assertion of Theorem 1. Recall that u_ε is a minimizer for E_ε over $H^1_g(G,\mathbb{C})$. We assume without loss of generality that $d \ge 0$.

Proposition 3.1. *We have*

$$E_\varepsilon(u_\varepsilon) \le 2\pi d\Big(\log\Big(\frac{1}{\varepsilon}\Big) - I\Big(\frac{1}{\varepsilon}\Big)\Big) + O(1),\ \forall \varepsilon > 0. \tag{3.3}$$

Proof. We treat only the case $d > 0$ since the case $d = 0$ is trivial. We shall define $U_\varepsilon \in H^1_g(G,\mathbb{C})$ for which $E_\varepsilon(U_\varepsilon)$ satisfies the bound (3.3). Fix d distinct points $b_1,\dots,b_d \in G$, and let ρ satisfy

$$0 < \rho < \frac{1}{4}\min\big(\min_{i\ne j}|b_i - b_j|, \min_i \operatorname{dist}(b_i,\partial G)\big).$$

Fix any $k \in \{1,\dots,d\}$. Let $f_0(r)$ be a maximizer for $I(\frac{\rho}{\varepsilon})$ as given by Lemma 2.1, see (2.19), and let θ_k denote a polar coordinate around b_k. Next we define U_ε on $B_{2\rho}(b_k)$ as follows:

$$U_\varepsilon(x) = \begin{cases} \frac{|x-b_k|}{\mu_0\varepsilon} f_0(\mu_0) e^{i\theta_k} & \text{on } B_{\mu_0\varepsilon}(b_k),\\ f_0(\frac{|x-b_k|}{\varepsilon}) e^{i\theta_k} & \text{on } B_\rho(b_k)\setminus B_{\mu_0\varepsilon}(b_k),\\ \Big(f_0(\frac{\rho}{\varepsilon}) + \big(\frac{|x-b_k|-\rho}{\rho}\big)\big(1 - f_0(\frac{\rho}{\varepsilon})\big)\Big) e^{i\theta_k} & \text{on } B_{2\rho}(b_k)\setminus B_\rho(b_k). \end{cases}$$

We claim that

$$E_\varepsilon(U_\varepsilon, B_{2\rho}(b_k)) = 2\pi\Big(\log\Big(\frac{1}{\varepsilon}\Big) - I\Big(\frac{1}{\varepsilon}\Big)\Big) + O(1). \tag{3.4}$$

Clearly,

$$E_\varepsilon(U_\varepsilon, B_{\mu_0\varepsilon}(b_k)) = O(1). \tag{3.5}$$

Next, by the same computation as in (2.3) we obtain

$$E_\varepsilon(U_\varepsilon, B_\rho(b_k)\setminus B_{\mu_0\varepsilon}(b_k)) = 2\pi\int_{\mu_0}^{\rho/\varepsilon}\Big((f_0')^2 + \frac{f_0^2}{r^2} + J(1-f_0^2)\Big)\, r dr$$

$$= 2\pi \log\Big(\frac{\rho}{\mu_0\varepsilon}\Big) - 2\pi\int_{\mu_0}^{\rho/\varepsilon}\frac{1-f_0^2}{r}\,dr + 2\pi\int_{\mu_0}^{\rho/\varepsilon}(f_0')^2\,r dr + O(1)$$
$$:= A_1 - A_2 + A_3 + O(1), \quad (3.6)$$

where we used the constraint satisfied by f_0. Clearly, $A_1 = 2\pi\log(1/\varepsilon) + O(1)$. By the definition of f_0 and Lemma 2.7 we get that $A_2 = 2\pi I(1/\varepsilon) + O(1)$. Moreover, $A_3 = O(1)$ thanks to (2.25). Therefore, (3.4) will follow from (3.5) and (3.6) once we show that

$$E_\varepsilon\big(U_\varepsilon, B_{2\rho}(b_k)\backslash B_\rho(b_k)\big) = O(1). \quad (3.7)$$

In order to verify (3.7) we write, on $B_{2\rho}(b_k)\backslash B_\rho(b_k)$,

$$U_\varepsilon(b_k + re^{i\theta_k}) = z(r)e^{i\theta_k} \quad \text{with} \quad z(r) = f_0(\frac{\rho}{\varepsilon}) + \big(\frac{r-\rho}{\rho}\big)\big(1 - f_0(\frac{\rho}{\varepsilon})\big)$$

and compute

$$\int_{B_{2\rho}(b_k)\backslash B_\rho(b_k)} |\nabla U_\varepsilon|^2\,dx = \int_{B_{2\rho}(b_k)\backslash B_\rho(b_k)} z^2|\nabla\theta_k|^2 + 2\pi\int_\rho^{2\rho}(z')^2\,r dr$$
$$= O(1) + 2\pi\Big(\frac{1-f_0(\rho/\varepsilon)}{\rho}\Big)^2\int_\rho^{2\rho} r\,dr = O(1). \quad (3.8)$$

As for the second term of the energy, we obtain, using again the inequality $J(t) \le tj(t)$, (2.8) and Lemma 2.2

$$\frac{1}{\varepsilon^2}\int_{B_{2\rho}(b_k)\backslash B_\rho(b_k)} J(1-|U_\varepsilon|^2) \le \frac{C}{\varepsilon^2}\int_{B_{2\rho}(b_k)\backslash B_\rho(b_k)} j(1-|U_\varepsilon|^2) \quad (3.9)$$
$$\le \frac{C}{\varepsilon^2} j\big(1 - f_0^2(\frac{\rho}{\varepsilon})\big) = \frac{C}{\varepsilon^2}\frac{1}{\lambda\cdot(\frac{\rho}{\varepsilon})^2} \le C. \quad (3.10)$$

Therefore, (3.7) follows from (3.8)–(3.9).

We use the same construction on each of the annuli $B_{2\rho}(b_k)$, $k = 1, \dots, d$, and finally on $G\backslash \cup_{k=1}^d B_{2\rho}(b_k)$ we set $U_\varepsilon = w$ where w is a fixed smooth S^1-valued map which is equal to $e^{i\theta_k}$ on each $\partial B_{2\rho}(b_k)$ and to g on ∂G. The result follows since clearly $E_\varepsilon(u_\varepsilon) \le E_\varepsilon(U_\varepsilon)$. □

3.3. **The lower bound of the energy and the convergence result.** The following simple lemma, which is a variant of [3, Theorem 3], is a basic tool in the proof of the lower bound.

Lemma 3.3. *Let $u \in C^1(S^1, \mathbb{C})$ satisfy $|u(x)| \ge \frac{1}{2}$ on S^1 and deg$(u/|u|) = d$. Then,*

$$\int_{S^1}\Big(|u'|^2 + d^2(1-|u|^2) + 4d^2(1-|u|^2)^2\Big)\,d\tau \ge 2\pi d^2. \quad (3.11)$$

Proof. We may write $u = \rho e^{i(d\theta+\psi)}$ with $\psi \in C^1(S^1, \mathbb{R})$. We have

$$|u'|^2 = (\rho')^2 + \rho^2\big(d^2 + 2d\psi' + (\psi')^2\big).$$

Integrating over S^1, using the equality

$$\int_{S^1} \rho^2\psi' = \int_{S^1} (\rho^2 - 1)\psi'$$

and the Cauchy-Schwarz inequality gives

$$\int_{S^1} |u'|^2 + d^2(1-\rho^2) \geq 2\pi d^2 + \int_{S^1} 2d(\rho^2-1)\psi' + \rho^2(\psi')^2$$

$$\geq 2\pi d^2 - \int_{S^1} \Big(4d^2(\rho^2-1)^2 + \frac{1}{4}(\psi')^2 - \rho^2(\psi')^2\Big) \geq 2\pi d^2 - \int_{S^1} 4d^2(\rho^2-1)^2,$$

and (3.11) follows. □

An immediate consequence is a lower bound on an annulus.

Proposition 3.2. *Let A_{R_1,R_2} denote the annulus $\{R_1 < |x| < R_2\}$ and let $u \in C^1(A_{R_1,R_2}, \mathbb{C}) \cap C(\overline{A_{R_1,R_2}}, \mathbb{C})$ satisfy:*

$$deg(u, \partial B_{R_j}(0)) = d,\ j = 1, 2, \tag{3.12}$$

$$\frac{1}{2} \leq |u| \leq 1 \quad on\ A_{R_1,R_2}, \tag{3.13}$$

and

$$\frac{1}{R_1^2}\int_{A_{R_1,R_2}} J(1-|u|^2) \leq c_0, \tag{3.14}$$

for some constant c_0. Then, there is a constant c_1, depending only on c_0, such that

$$\int_{A_{R_1,R_2}} |\nabla u|^2 \geq 2\pi d^2\Big(\log\big(\frac{R_2}{R_1}\big) - I\big(\frac{R_2}{R_1}\big)\Big) - d^2 c_1. \tag{3.15}$$

Proof. The rescaled version of (3.11) reads (with $u' := \frac{\partial u}{\partial \tau}$)

$$\int_{\partial B_r(0)} \Big(|u'|^2 + \frac{d^2}{r^2}(1-|u|^2) + 4\frac{d^2}{r^2}(1-|u|^2)^2\Big)\, d\tau \geq 2\pi \frac{d^2}{r}. \tag{3.16}$$

Integrating (3.16) over $r \in (R_1, R_2)$ yields

$$\int_{A_{R_1,R_2}} |\nabla u|^2 \geq 2\pi d^2 \log\big(\frac{R_2}{R_1}\big) - d^2\int_{A_{R_1,R_2}} \Big(\frac{1-|u|^2}{|x|^2} + \frac{4(1-|u|^2)^2}{|x|^2}\Big)\, dx. \tag{3.17}$$

Consider the rescaled map $\tilde{u}(x) = u(R_1 x)$ on the annulus

$$A_{1,R_2/R_1} = \{1 < |x| < R_2/R_1\}.$$

Let $\tilde{f}$ denote the symmetric nondecreasing rearrangement of $\tilde{u}$ with respect to Lebesgue measure on $\mathbb{R}^2$. We have

$$\int_{A_{R_1,R_2}}\Big(\frac{1-|u|^2}{|x|^2}+\frac{4(1-|u|^2)^2}{|x|^2}\Big)dx=\int_{A_{1,R_2/R_1}}\Big(\frac{1-|\tilde{u}|^2}{|x|^2}+\frac{4(1-|\tilde{u}|^2)^2}{|x|^2}\Big)dx$$

$$\leq 2\pi\int_1^{R_2/R_1}\Big(\frac{1-|\tilde{f}|^2}{r}+\frac{4(1-|\tilde{f}|^2)^2}{r}\Big)\,dr. \tag{3.18}$$

We have also

$$\int_{A_{1,R_2/R_1}}J(1-|\tilde{f}|^2)\,dx=\int_{A_{1,R_2/R_1}}J(1-|\tilde{u}|^2)\,dx=\frac{1}{R_1^2}\int_{A_{R_1,R_2}}J(1-|u|^2)\,dx\leq c_0, \tag{3.19}$$

by (3.14). Therefore, from (3.18)–(3.19) and Lemma 2.7 we conclude that

$$\int_{A_{R_1,R_2}}\Big(\frac{1-|u|^2}{|x|^2}+\frac{4(1-|u|^2)^2}{|x|^2}\Big)\,dx\leq\tilde{I}(R_2/R_1,c_0)\leq I(R_2/R_1)+c_1,$$

with c_1 depending only on c_0, which combined with (3.17) clearly implies (3.15). □

We next use Proposition 3.2 in order to establish a lower bound on a more general perforated domain. The proof uses a variant of an argument of Struwe [5].

Proposition 3.3. *Let $x_1,\ldots,x_m$ be m points in $B_\sigma(0)$ satisfying*

$$|x_i-x_j|\geq 4\delta,\ \forall i\neq j\quad\textit{and}\quad |x_i|<\frac{\sigma}{4},\ \forall i,$$

with $\delta\leq\sigma/32$. Set $\Omega=B_\sigma(0)\setminus\bigcup_{j=1}^m B_\delta(x_j)$ and let u be a C^1-map from Ω into $\mathbb{C}$, which is continuous on $\partial\Omega$, satisfying

$$\frac{1}{2}\leq|u|\leq 1\ \textit{in}\ \Omega\quad\textit{and}\quad \mathrm{deg}(u,\partial B_\sigma(x_j))=d_j,\ \forall j,$$

and

$$\frac{1}{\delta^2}\int_\Omega J(1-|u|^2)\leq K.$$

Then, denoting $d=\sum_{j=1}^m d_j$, we have

$$\int_\Omega|\nabla u|^2\geq 2\pi|d|\Big(\log\frac{\sigma}{\delta}-I\big(\frac{\sigma}{\delta}\big)\Big)-C, \tag{3.20}$$

with $C=C(K,m,\sum_{j=1}^m|d_j|)$.

Proof. We use induction on m. The case $m = 1$ follows from Proposition 3.2. Suppose we are given m such points and that the assertion holds for any smaller number of points satisfying the above conditions. Put

$$d_j^{(1)} = d_j, x_j^{(1)} = x_j,\ \forall j \quad \text{and} \quad R^{(1)} = \delta, m^{(1)} = m \text{ and } J^{(1)} = \{1,\dots,m\}.$$

Set $r^{(1)} = \frac{1}{2}\min_{i\neq j}|x_i^{(1)} - x_j^{(1)}|$ and $A_j^{(1)} = B(x_j^{(1)}, r^{(1)}) \setminus B(x_j^{(1)}, R^{(1)})$, $\forall j$. By Proposition 3.2 we have

$$\begin{aligned}\int_{\bigcup_{j=1}^m A_j^{(1)}} |\nabla u|^2 &\geq 2\pi \sum_{j=1}^m \big(d_j^{(1)}\big)^2 \Big(\log\big(\frac{r^{(1)}}{R^{(1)}}\big) - I\big(\frac{r^{(1)}}{R^{(1)}}\big)\Big) - C \\ &\geq 2\pi d \Big(\log\big(\frac{r^{(1)}}{R^{(1)}}\big) - I\big(\frac{r^{(1)}}{R^{(1)}}\big)\Big) - C.\end{aligned} \tag{3.21}$$

Next, define $R^{(2)}$ as the minimal number $R \in [r^{(1)}, (3/4)\sigma]$ for which there exists a subset $J^{(2)} \subset J^{(1)}$ such that:

$$\bigcup_{j=1}^m B_{R^{(1)}}(x_j^{(1)}) \subset \bigcup_{i\in J^{(2)}} B_{R/4}(x_i^{(1)}) \ \text{ and } \ |x_{i_1}^{(1)} - x_{i_2}^{(1)}| \geq 4R,\ i_1 \neq i_2 \text{ in } J^{(2)}. \tag{3.22}$$

If no such $R^{(2)}$ exists, then necessarily $r^{(1)} \geq \alpha\sigma$ for some $\alpha = \alpha(m) > 0$ and the result follows from (3.21). So assume in the sequel that $R^{(2)}$ does exist. In that case we have

$$R^{(2)} \leq \beta r^{(1)}, \quad \text{for some constant } \beta = \beta(m).$$

We relabel the points $\{x_i^{(1)}\}_{i\in J^{(2)}}$ by $\{x_j^{(2)}\}_{j=1}^{m^{(2)}}$ with $m^{(2)} = |J^{(2)}|$. If $R^{(2)} \geq \sigma/8$, then we stop the construction with $k = 2$. Otherwise, we continue and define

$$r^{(2)} = \begin{cases} \frac{1}{2}\min_{i\neq j}|x_i^{(2)} - x_j^{(2)}| & \text{if } |J^{(2)}| \geq 2, \\ \frac{2\sigma}{3} & \text{if } |J^{(2)}| = 1. \end{cases} \tag{3.23}$$

Continuing the above construction yields

$$R^{(1)} < r^{(1)} < R^{(2)} < r^{(2)} < \cdots < R^{(k-1)} < r^{(k-1)} < R^{(k)} \leq \infty,$$

where k is the first index for which, either $R^{(k)}$ exists and satisfies $R^{(k)} \geq \sigma/8$, or $R^{(k)}$ does not exist, and then we set $R^{(k)} = \infty$. At each stage $R^{(l)}$ is chosen to satisfy an analogous condition to (3.22), and similarly, $r^{(l)}$ is chosen according to an analogous condition to (3.23). Note that we have

$$r^{(k-1)} \geq \alpha\sigma, \quad \text{for some } \alpha = \alpha(m) > 0. \tag{3.24}$$

For each l denote the resulting set of points by $\{x_j^{(l)}\}_{j=1}^{m^{(l)}}$. The corresponding degrees are

$$d_j^{(l)} = \deg\big(u, \partial B_{R^{(l)}}(x_j^{(l)})\big), \quad j = 1, \dots, m^{(l)},\ l = 1, \dots, k-1.$$

Note that

$$\sum_{j=1}^{m^{(l)}} d_j^{(l)} = d,\ l = 1, \dots, k-1. \tag{3.25}$$

Using the induction hypothesis together with (3.24), (3.25) and (2.31) yields

$$\begin{aligned}\int_\Omega |\nabla u|^2 &\geq \sum_{j=1}^{m^{(k-1)}} \int_{B_{r^{(k-1)}}(x_j^{(k-1)})} |\nabla u|^2 \\ &\geq 2\pi \sum_{j=1}^{m^{(k-1)}} |d_j^{(k-1)}| \Big(\log\big(\frac{r^{(k-1)}}{\delta}\big) - I\big(\frac{r^{(k-1)}}{\delta}\big)\Big) - C \\ &\geq 2\pi |d| \Big(\log\big(\frac{\sigma}{\delta}\big) - I\big(\frac{\sigma}{\delta}\big)\Big) - C. \qquad \square\end{aligned} \tag{3.26}$$

3.4. **Proof of Theorem 1.** Let $R > 0$ be large enough so that $G \subset B_R(0)$. Fix any smooth map $U : \overline{B_R(0)} \setminus G \to S^1$ such that $U|_{\partial G} = g$ and let $\tilde{g} = U|_{\partial B_R(0)}$ (which has necessarily degree d). Denote for each ε by $\tilde{u}_\varepsilon$ a minimizer for E_ε over $H^1_{\tilde{g}}(B_R(0), \mathbb{C})$. Clearly,

$$\begin{aligned}&\int_{B_R(0)} |\nabla \tilde{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2} J(1 - |\tilde{u}_\varepsilon|^2) \\ &\leq \int_G |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2} J(1 - |u_\varepsilon|^2) + \int_{B_R(0)\setminus G} |\nabla U|^2 = E_\varepsilon(u_\varepsilon) + C.\end{aligned} \tag{3.27}$$

Since $B_R(0)$ is star-shaped, we know from Lemma 3.2 that the estimate (3.2) holds for $\tilde{u}_\varepsilon$. Since (3.1) also holds for $\tilde{u}_\varepsilon$ we may apply the argument of [2] to obtain the existence of an integer N and of a real $\lambda > 0$, such that for each ε there exists a collection of discs $\{B_{\lambda\varepsilon}(x_i^\varepsilon)\}_{i=1}^{N_\varepsilon}$, with $N_\epsilon \leq N$, such that

$$\widetilde{S}_\varepsilon := \{x \in B_R \,:\, |u_\varepsilon(x)| < \frac{1}{2}\} \subset \bigcup_{i=1}^{N_\varepsilon} B_{\lambda\varepsilon}(x_i^\varepsilon),$$

and

$$|x_i^\varepsilon - x_j^\varepsilon| \geq 8\lambda\varepsilon, \quad \forall i, j, \text{ such that } i \neq j.$$

We fix $R_1 > 4R$ and consider an S^1-valued extension U_1 of U to $\overline{B_{R_1}(0)} \setminus B_R(0)$, of class C^1. This induces an extension of each $\tilde{u}_\varepsilon$ to $\overline{B_{R_1}(0)}$ (only a constant is added to its energy). Applying Proposition 3.3 yields

$$E_\varepsilon(\tilde{u}_\varepsilon, B_{R_1}(0)) \geq 2\pi d\Big(\log\frac{R_1}{\lambda\varepsilon} - I\big(\frac{R_1}{\lambda\varepsilon}\big)\Big) - C.$$

It follows that

$$\begin{aligned} E_\varepsilon(u_\varepsilon, G) &\geq E_\varepsilon(\tilde{u}_\varepsilon, B_R(0)) - C \geq E_\varepsilon(\tilde{u}_\varepsilon, B_{R_1}(0)) - C \\ &\geq 2\pi d\Big(\log\frac{1}{\varepsilon} - I\big(\frac{1}{\varepsilon}\big)\Big) - C. \end{aligned} \tag{3.28}$$

The energy estimate (1.6) is an immediate consequence of (3.28) and the upper bound (3.3).

An argument of del Pino and Felmer [4] can now be used to show that (3.2) holds without the assumption on the starshapeness of G. In fact, applying (3.28) for 2ε instead of ε yields

$$\begin{aligned} \int_G |\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}J(1-|u_\varepsilon|^2) &\geq \int_G |\nabla u_{2\varepsilon}|^2 + \frac{1}{4\varepsilon^2}J(1-|u_{2\varepsilon}|^2) \\ &\geq 2\pi d\Big(\log\frac{1}{2\varepsilon} - I\big(\frac{1}{2\varepsilon}\big)\Big) - C. \end{aligned} \tag{3.29}$$

On the other hand, by the upper bound (3.3),

$$\int_G |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2}J(1-|u_\varepsilon|^2)^2 \leq 2\pi d\Big(\log\frac{1}{\varepsilon} - I\big(\frac{1}{\varepsilon}\big)\Big) + C. \tag{3.30}$$

Subtracting (3.29) from (3.30) yields the result (3.2).

Having the estimate (3.2) on our hands, we can now follow the bad-discs construction of [2] and complete the convergence assertion of Theorem 1. Since the arguments are identical to those of [2], we omit the details.

3.5. **The inverse problem (proof of Theorem 2).** In view of Theorem 1, it is enough to find a C^2–functional J, which is strictly convex on some interval $(0, \eta_0)$, such that $j = J'$ satisfies

$$\frac{1}{2}\int_{\frac{1}{R^2}}^{\eta_0} j^{-1}(t)\frac{dt}{t} = h(\log R).$$

Setting $\delta = \frac{1}{R^2}$, we obtain the equivalent condition

$$\frac{1}{2}\int_{\delta}^{\eta_0} j^{-1}(t)\frac{dt}{t} = h(\frac{1}{2}\log\frac{1}{\delta}).$$

Differentiating with respect to δ gives

$$-\frac{1}{2\delta}j^{-1}(\delta) = h'(\frac{1}{2}\log\frac{1}{\delta})\cdot(-\frac{1}{2\delta}).$$

Therefore,

$$j^{-1}(\delta) = h'\big(\frac{1}{2}\log\frac{1}{\delta}\big). \tag{3.31}$$

In order to be able to recover j, and then J, from (3.31), we shall verify that the right-hand side of (3.31) is a strictly increasing function, for δ small enough. Indeed,

$$\frac{d}{d\delta}\Big(h'\big(\frac{1}{2}\log\frac{1}{\delta}\big)\Big) = -\big(\frac{1}{2\delta}\big)h''\big(\frac{1}{2}\log\frac{1}{\delta}\big) > 0 \quad \text{for } \delta \le e^{-2T},$$

since $h''(t) < 0$ for $t \ge T$ by (1.7). Note also that $j(0) = j^{-1}(0) = 0$ by (1.8). Therefore, we have constructed J such that $j = J'$ satisfies (3.31) on an interval $[0, \eta_0]$, with $\eta_0 = e^{-2T}$, so that (H_3) is satisfied. Finally, it is easy to see that J can be extended to all of $\mathbb{R}$ as a C^2-functional satisfying $(H_1) - (H_2)$.

4. Appendix

In this Appendix we compute the energy cost of a degree one vortex for the functionals J_k, $k > 0$, that were defined in (1.4). In view of Theorem 1 it suffices to compute for each $k > 0$:

$$I_{0,k}(R) := \frac{1}{2}\int_{1/R^2}^{j_k(\eta_k)} j_k^{-1}(t)\frac{dt}{t}, \tag{4.1}$$

with $j_k = J_k'$ and $\eta_k = \big(\frac{k}{k+1}\big)^{1/k}$ (a simple computation shows that $J_k'' > 0$ on $(0, \eta_k)$).

Proposition 4.1. *As R goes to infinity, we have:*

$$I_{0,k}(R) = \begin{cases} O(1), & 0 < k < 1, \\ \frac{1}{2}\log\log R + O(1), & k = 1, \\ 2^{-\frac{1}{k}}\frac{k}{k-1}\big(\log(R)\big)^{\frac{k-1}{k}} + O(1), & k > 1. \end{cases} \tag{4.2}$$

Proof. The change of variable $s = j_k^{-1}(t)$ gives

$$I_{0,k}(R) = \frac{1}{2}\int_{j_k^{-1}(1/R^2)}^{\eta_k} s\frac{j_k'(s)}{j_k(s)}\,ds = \frac{1}{2}\int_{j_k^{-1}(1/R^2)}^{\eta_k}\Big(\frac{k}{s^k} - (k+1)\Big)\,ds. \tag{4.3}$$

If $k < 1$, then it follows immediately from (4.3) that $I_{0,k}(R) = O(1)$.

For $k > 1$ we obtain from (4.3) that

$$I_{0,k}(R) = \frac{k}{2(k-1)}\Big((j_k)^{-1}\big(\frac{1}{R^2}\big)\Big)^{1-k} + O(1). \tag{4.4}$$

Set $\alpha = \alpha(R) = j_k^{-1}\big(\frac{1}{R^2}\big)$. Since $j_k(\alpha) = \big(\frac{k}{\alpha^{k+1}}\big)\exp(-1/\alpha^k)$, we have

$$\frac{1}{R^2} = \big(\frac{k}{\alpha^{k+1}}\big)\exp(-1/\alpha^k).$$

Taking the logarithm of both sides gives

$$-2\log R = \log k - (k+1)\log\alpha - \frac{1}{\alpha^k}, \quad \text{for } k > 0. \tag{4.5}$$

By (4.5) we have $\lim_{R\to\infty} 2\alpha^k \log R = 1$, which we plug into (4.4) to obtain the case $k > 1$ in (4.2).

Finally, if $k = 1$, then by (4.3) we have

$$I_{0,1}(R) = \frac{1}{2}\int_{j_1^{-1}(1/R^2)}^{\eta_1} (\tfrac{1}{s}-2)\,ds = -\tfrac{1}{2}\log\Big(j_1^{-1}\big(\tfrac{1}{R^2}\big)\Big) + O(1) = -\tfrac{1}{2}\log\alpha + O(1), \tag{4.6}$$

with $\alpha = j_1^{-1}\big(\frac{1}{R^2}\big)$, as above. In our case (4.5) gives $\lim_{R\to\infty} 2\alpha\log R = 1$, which implies that $\log\alpha = \log(\frac{1}{2\log R}) + o(1)$. Plugging it in (4.6) gives the result (4.2) for $k = 1$. □

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