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Remarks on solutions of a fourth-order problem

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Abstract

In this paper, we study the two following minimization problems:

$$S_0(q,\varphi)=\inf_{u\in H_0^2(\Omega),\,\|u+\varphi\|_q=1}\int_\Omega|\Delta u|^2\qquad\text{and}\qquad S_\theta(q,\varphi)=\inf_{u\in H_\theta^2(\Omega),\,\|u+\varphi\|_q=1}\int_\Omega|\Delta u|^2.$$

We prove that for a class of maps φ, we have $S_\theta(q,\varphi)<S_0(q,\varphi)$ and for another class, we have $S_\theta(q,\varphi)=S_0(q,\varphi)$.

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1. Introduction

Let us consider the following two minimization problems:

$$S_\theta(q,\varphi)=\inf_{u\in H_\theta^2(\Omega),\,\|u+\varphi\|_q=1}\int_\Omega|\Delta u|^2 \tag{I_θ}$$

and

$$S_0(q,\varphi)=\inf_{u\in H_0^2(\Omega),\,\|u+\varphi\|_q=1}\int_\Omega|\Delta u|^2, \tag{I_0}$$

where Ω is a domain in $\mathbb{R}^N$, $N\geq 3$ if $1\leq q<q_c=\frac{2N}{N-4}$ and $N\geq 5$ if $q=q_c$. The function φ is given in $C(\Omega)\cap L^q(\Omega)$ and $H_\theta^2(\Omega)=H^2(\Omega)\cap H_0^1(\Omega)$.

Recall that q_c is the limiting Sobolev exponent in the imbedding $H_0^2(\Omega)\hookrightarrow L^r(\Omega)$, $1\leq r\leq q_c$. Note (see Van der Vorst [5] and [6]) that $S_\theta(q_c,0)=S_0(q_c,0)=S$ is the best Sobolev constant and S is not achieved.

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In [3] it is shown that if φ is not zero, then the infima $S_\theta(q,\varphi)$ (resp. $S_0(q,\varphi)$) are achieved by u_θ (resp. u_0), which satisfy respectively the following Euler–Lagrange equations:

$$\begin{cases} -\Delta^2 u_\theta = \Lambda_\theta |u_\theta+\varphi|^{q-2}(u_\theta+\varphi) & \text{in} \quad \Omega, \\ \Delta u_\theta = u_\theta = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{E_θ}$$

and

$$\begin{cases} -\Delta^2 u_0 = \Lambda_0 |u_0+\varphi|^{q-2}(u_0+\varphi) & \text{in} \quad \Omega, \\ \dfrac{\partial u_0}{\partial \nu} = u_0 = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{E_0}$$

where Λ_θ (resp. Λ_0) is the Lagrange multiplier associated to u_θ (resp. u_0).

The interest in this type of equations comes from the fact that it resembles some geometrical equations involving the Paneitz operator, which is a fourth-order conformally covariant elliptic operator (see [4]).

Since $H_0^2(\Omega) \subset H^2(\Omega) \cap H_0^1(\Omega)$, we have $S_\theta(q,\varphi) \le S_0(q,\varphi)$. It is natural to wonder if $S_\theta(q,\varphi) < S_0(q,\varphi)$ and if the infimum on $H^2(\Omega) \cap H_0^1(\Omega)$ is achieved by a function of $H_0^2(\Omega)$.

Remark 1. The signs of the Lagrange multipliers depend on φ. We have (see [3]),
if $\|\varphi\|_q < 1$ then $\Lambda_0 > 0$ and $\Lambda_\theta > 0$,
if $\|\varphi\|_q > 1$ then $\Lambda_\theta < 0$ and $\Lambda_0 < 0$.

Our main result is

Theorem. *Suppose that $q \in [2, q_c]$ and that φ is not identically 0.*

(i) *If $\|\varphi\|_q < 1$ and φ has a constant sign on Ω, then every minimizer of (I_θ) is not in $H_0^2(\Omega)$ and we have $S_\theta(q,\varphi) < S_0(q,\varphi)$.*

(ii) *Let $(H_0^2(\Omega))^\perp$ be the orthogonal of $H_0^2(\Omega)$ in the space $H_\theta^2(\Omega)$. If φ is in $(H_0^2(\Omega))^\perp$, then every minimizer of (I_θ) is not in $H_0^2(\Omega)$ and we have $S_\theta(q,\varphi) < S_0(q,\varphi)$.*

(iii) *For $q \ge 2$, if $\|\varphi\|_q > 1$ and φ is in $H_0^2(\Omega)$, then $S_\theta(q,\varphi) = S_0(q,\varphi)$.*

We do not know if $S_\theta(q,\varphi) = S_0(q,\varphi)$ in the other cases.

Proof of (i). Suppose that $\varphi \ge 0$ and φ is not identically 0. We will adapt the argument of [6] to our situation. Let u_θ be any minimizer of (I_θ). We argue by contradiction. We suppose that u_θ is in $H_0^2(\Omega)$.

Let v be the solution of the following problem:

$$\begin{cases} -\Delta v = |\Delta u_\theta| & \text{in} \quad \Omega, \\ v = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1}$$

We have

$$\begin{cases} -\Delta(v-u_\theta) \ge 0 & \text{in} \quad \Omega, \\ v - u_\theta = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

and

$$\begin{cases} -\Delta(v+u_\theta) \ge 0 & \text{in} \quad \Omega, \\ v + u_\theta = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3}$$

We deduce from the maximum principle applied to (2) and (3) that either $v > |u_\theta|$ in Ω or $v = -u_\theta$ or $v = u_\theta$. We use (1) to see that in both cases $v = u_\theta$ and $v = -u_\theta$ the function Δu_θ has a constant sign. This fact together with $u_\theta = \frac{\partial u_\theta}{\partial \nu} = 0$ on $\partial\Omega$ and with the maximum principle lead to $u_\theta = 0$ in Ω, that is false. Thus we have $v > |u_\theta|$ in Ω. Using this inequality and the fact that $\varphi \ge 0$, we obtain $u_\theta + \varphi < v + \varphi$ in Ω and $-u_\theta - \varphi < v + \varphi$ in Ω; thus $|u_\theta + \varphi| < |v + \varphi|$ in Ω and consequently we have

$$\int_\Omega |v+\varphi|^q > 1.$$

Now, let us consider the function $f(t) = \int_\Omega |tv + \varphi|^q$ for $t \in [0, 1]$. Since f is continuous, $f(0) < 1$ and $f(1) > 1$, there is $s \in [0, 1]$ such that $f(s) = 1$. Then we have

$$\int_\Omega |\Delta u_\theta|^2 \leq s^2 \int_\Omega \Delta v\,|^2,$$

that contradicts the definition of u_θ. The first part of the theorem is proved.

Proof of (ii). Let us distinguish two cases.

Case 1. Let us suppose that $\|\varphi\|_q > 1$ and that φ is in $(H_0^2(\Omega))\perp$. Let u_θ be any solution of (I_θ). Multiplying (E_θ) by u_θ and integrating by parts, we obtain

$$\int_\Omega |\Delta u_\theta|^2 + \int_\Omega \Delta u_\theta \cdot \Delta\varphi = \Lambda_\theta.$$

Since $\|\varphi\|_q > 1$, we have $\Lambda_\theta < 0$, and thus $\int_\Omega \Delta u_\theta \cdot \Delta\varphi < 0$, which implies that u_θ is not in $H_0^2(\Omega)$, and then we conclude that $S_\theta(q,\varphi) < S_0(q,\varphi)$.

Case 2. If $\|\varphi\|_q < 1$, let us suppose first that $q = q_c$. For $a \in \Omega$, set $u_{a,\varepsilon} = \zeta U_{a,\varepsilon}$, where ζ is a smooth function, such that ζ is equal to 1 near a and $U_{a,\varepsilon}(x) = (\frac{\varepsilon}{\varepsilon^2+|x-a|^2})^{\frac{N-4}{2}}$. We have, see [3], that $\int_\Omega |u_{a,\varepsilon}|^q = B + o(1)$, and $\int_\Omega |\Delta u_{a,\varepsilon}|^2 = A + o(1)$, such that $\frac{A}{B^{\frac{2}{q}}} = S$.

Since $\|\varphi\|_q < 1$, there exists $c_\varepsilon > 0$ such that $\|\varphi + c_\varepsilon u_{a,\varepsilon}\|_q = 1$. Using the Brezis-Lieb identity (see [1]) we obtain

$$c_\varepsilon^q = B\left[1 - \int_\Omega |\varphi|^q\right] + o(1),$$

where $o(1)$ tends to 0 as ε tends to 0. Direct computations give

$$S_\theta(q,\varphi) \leq c_\varepsilon^2 \int_\Omega |\Delta u_{a,\varepsilon}|^2 = S\left[1 - \int_\Omega |\varphi|^q\right]^{\frac{2}{q}} + o(1).$$

As ε tends to 0, we find

$$S_\theta(q,\varphi) \leq S\left(1 - \int_\Omega |\varphi|^q\right)^{\frac{2}{q}}. \tag{4}$$

On the other hand, multiplying (E_θ) by $(u_\theta + \varphi)$ and integrating yields

$$S_\theta(q,\varphi) = \Lambda_\theta \int_\Omega |u_\theta + \varphi|^{q-1}(u_\theta + \varphi)u_\theta,$$

and therefore the Hölder inequality gives

$$S_\theta(q,\varphi) \leq \Lambda_\theta \|u_\theta\|_q. \tag{5}$$

Using (5) and the Sobolev inequality we find that

$$S_\theta(q,\varphi) \leq \Lambda_\theta \left(\frac{1}{S}\int_\Omega |\Delta u_\theta|^2\right)^{\frac{1}{2}}. \tag{6}$$

Combining (4) and (6) we see that

$$S_\theta(q,\varphi) \leq \frac{\Lambda_\theta}{S^{\frac{1}{2}}} S^{\frac{1}{2}} \left[1 - \int_\Omega |\varphi|^q\right]^{\frac{1}{q}}. \tag{7}$$

Now, multiplying (E_θ) by $(u_\theta + \varphi)$ and integrating we obtain

$$\int_\Omega \Delta u_\theta . \Delta\varphi = \Lambda_\theta - S_\theta(q,\varphi). \tag{8}$$

Finally, combining (7) and (8) we are led to

$$\int_\Omega \Delta u_\theta . \Delta\varphi \geq \Lambda_\theta [1 - (1 - \int_\Omega |\varphi|^q)^{\frac{1}{q}}] > 0.$$

This means that u_θ is not in $H_0^2(\Omega)$.

The case where $q < q_c$ can be obtained as this last case.

Remark 2. The same argument as below shows that if φ is in $H_0^2(\Omega)$, then every solution of (I_θ) is not in the orthogonal of $H_0^2(\Omega)$. The second part of the theorem is proved.

Proof of (iii). Let φ be in $H_0^2(\Omega)$. We remark first that for $\|\varphi\|_q \geq 1$, we have

$$S_\theta(q,\varphi) = \inf_{\substack{u \in H_\theta^2(\Omega) \\ \|u+\varphi\|_q \leq 1}} \int_\Omega |\Delta u|^2. \tag{9}$$

We have a convex problem. We are going to use a method of duality. We refer to [2] for this proof. For all $p \in L^2(\Omega)$, let us define

$$\beta_\theta = \sup_{\substack{u \in H_\theta^2(\Omega) \\ \|u+\varphi\|_q \leq 1}} \int_\Omega p\Delta u \qquad \text{and} \qquad \beta_0 = \sup_{\substack{u \in H_0^2(\Omega) \\ \|u+\varphi\|_q \leq 1}} \int_\Omega p\Delta u.$$

We have

$$\beta_\theta = \sup_{\substack{v \in H_\theta^2(\Omega) \\ \|v\|_q \leq 1}} \int_\Omega p\Delta v - \int_\Omega p\Delta\varphi.$$

Let us prove that we have for every $p \in L^2(\Omega)$

$$\beta_\theta = \beta_0. \tag{10}$$

First, we remark that β_θ and β_0 are finite. This follows from the Hölder inequality $|\int_\Omega p\Delta v| \leq \|p\|_2 \|\Delta v\|_2$, together with (9). We deduce that the linear operator

$$\begin{aligned} L : H_\theta^2(\Omega) &\to \mathbb{R} \\ v &\to \int_\Omega p\Delta v \end{aligned}$$

is continuous for the $L^q(\Omega)$ topology. Thus there exists $\tilde{p} \in L^{\frac{q}{q-1}}(\Omega)$ such that for all $v \in H_\theta^2(\Omega)$ we have $L(v) = \int_\Omega \tilde{p}v$. We deduce that

$$\beta_\theta = \sup_{\substack{v \in H_\theta^2(\Omega) \\ \|v\|_q \leq 1}} \int_\Omega \tilde{p}v - \int_\Omega p\Delta\varphi; \qquad \beta_0 = \sup_{\substack{v \in H_0^2(\Omega) \\ \|v\|_q \leq 1}} \int_\Omega \tilde{p}v - \int_\Omega p\Delta\varphi. \tag{11}$$

On the other hand, it is easy to prove that, for all $\tilde{p} \in L^{\frac{q}{q-1}}(\Omega)$, we have

$$\sup_{\substack{v \in H_\theta^2(\Omega) \\ \|v\|_q \leq 1}} \int_\Omega \tilde{p}v = \sup_{\substack{v \in L^q(\Omega) \\ \|v\|_q \leq 1}} \int_\Omega \tilde{p}v = \sup_{\substack{v \in H_0^2(\Omega) \\ \|v\|_q \leq 1}} \int_\Omega \tilde{p}v = \|\tilde{p}\|_{\frac{q}{q-1}}. \tag{12}$$

Thus we have proved (10). Now, in the case where $\|\varphi\|_q \geq 1$, let us prove that

$$\frac{1}{2} S_\theta(q,\varphi) = \sup_{p \in L^2(\Omega)} \left\{ -\frac{1}{2}\int_\Omega |p|^2 - \sup_{\substack{u \in H_\theta^2(\Omega) \\ \|u+\varphi\|_q \leq 1}} \int_\Omega p\Delta u \right\} \tag{13}$$

and

$$\frac{1}{2}S_0(q,\varphi)=\sup_{p\in L^2(\Omega)}\left\{-\frac{1}{2}\int_\Omega |p|^2-\sup_{\substack{u\in H_0^2(\Omega)\\ \|u+\varphi\|_q\le 1}}\int_\Omega p\Delta u\right\}. \tag{14}$$

Let us define, for $p\in L^2(\Omega)$ and for $u\in H_\theta^2(\Omega)$,

$$L(u,p)=-\frac{1}{2}\int_\Omega |p|^2-\int_\Omega (\Delta u)p.$$

We can see easily that

$$\sup_{p\in L^2(\Omega)} L(u,p)=\frac{1}{2}\int_\Omega |\Delta u|^2. \tag{15}$$

By (9) and (15) we have

$$\frac{1}{2}S_\theta(q,\varphi)=\inf_{\substack{u\in H_\theta^2(\Omega)\\ \|u+\varphi\|_q\le 1}}\ \sup_{p\in L^2(\Omega)} L(u,p). \tag{P}$$

Let us prove that $(P)=(P^*)$ where (P^*) is the dual problem of (P), that is

$$\sup_{p\in L^2(\Omega)}\ \inf_{\substack{u\in H_\theta^2(\Omega)\\ \|u+\varphi\|_q\le 1}} L(u,p). \tag{P^*}$$

Let us define

$$A=\{u\in H_\theta^2(\Omega);\ \|u+\varphi\|_q\le 1\},$$

let u_θ be a minimizer that realizes $S_\theta(q,\varphi)$ and let $p_\theta=-\Delta u_\theta$. By (15), we have

$$L(u_\theta,p)\le \sup_{p\in L^2(\Omega)} L(u_\theta,p)=\frac{1}{2}S_\theta(q,\varphi)\qquad \text{for all } p\in L^2(\Omega). \tag{16}$$

Now, for all $u\in A$ we have

$$L(u,p_\theta)\ge -\frac{1}{2}\int_\Omega |p_\theta|^2-\sup_{u\in A}\int_\Omega (\Delta u)p_\theta,$$

that gives, using $\Delta u_\theta=0$ on $\partial\Omega$,

$$L(u,p_\theta)\ge -\frac{1}{2}\int_\Omega |p_\theta|^2-\sup_{u\in A}\int_\Omega u(\Delta p_\theta).$$

Thus we have, using (11),

$$L(u,p_\theta)\ge -\frac{1}{2}\int_\Omega |p_\theta|^2-\|\Delta p_\theta\|_{\frac{q}{q-1}}+\int_\Omega p_\theta\Delta\varphi. \tag{17}$$

But the Euler equation (E_θ) for u_θ gives

$$\|\Delta^2 u_\theta\|_{\frac{q}{q-1}}=|\Lambda_\theta|. \tag{18}$$

On the other hand, multiplying the Euler equation (E_θ) by $u_\theta+\varphi$, we obtain

$$\Lambda_\theta=\int_\Omega |\Delta u_\theta|^2+\int_\Omega \Delta u_\theta\Delta\varphi. \tag{19}$$

We know that $\Lambda_\theta<0$, thus (18) and (19) give

$$-\|\Delta p_\theta\|_{\frac{q}{q-1}}+\int_\Omega p_\theta\Delta\varphi=S_\theta(q,\varphi). \tag{20}$$

Now we obtain by (20) and (17) that

$$L(u, p_\theta) \geq \frac{1}{2} S_\theta(q, \varphi) \qquad \text{for all } u \in A. \tag{21}$$

It is classical (see [2]) that (21) and (16) infer that $(P)= (P^*)$.

The same proof remains valid for $\frac{1}{2}S_0(q, \varphi)$ instead of $\frac{1}{2}S_\theta(q, \varphi)$, thus we have proved (13) and (14). Now let us use (10) in order to conclude that $S_\theta(q, \varphi) = S_0(q, \varphi)$. This ends the proof of the theorem. □

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