Generalized Riemann minimal surfaces examples in three-dimensional manifolds products

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Abstract

In this paper, we construct and classify minimal surfaces foliated by horizontal constant curvature curves in $M \times \mathbb{R}$, where M is $\mathbb{H}^2, \mathbb{R}^2$ or $\mathbb{S}^2$. The main tool is the existence of a so called "Shiffman" Jacobi field which characterize the property to be foliated in circles in these product manifolds.

1 Introduction

In this paper, we are interested in minimal surfaces properly embedded in the product space $M \times \mathbb{R}$, where M is a complete Riemannian surface with constant curvature c_0. The main examples are $M = \mathbb{H}^2, \mathbb{R}^2, \mathbb{S}^2$. When $c_0 = 0$, this is the theory of periodic (singly, doubly, and triply) minimal surfaces in $\mathbb{R}^3$ and has been well developed; see [8], [9] ,[10] , [11]. For general M, the theory was initiated by Rosenberg in [18] and developped in [15] and [12], [13]. They found a rich family of examples like helicoids, catenoids and onduloids (surfaces of genus zero). Solving Plateau problems, they construct higher topological type examples inspired by the classical theory in $\mathbb{R}^3$.

Examples are so numerous that we intend to classify some of them. This paper is devoted to the annuli minimal surfaces properly embedded in product spaces and transverse to $M \times \{t\}$ for every $t \in \mathbb{R}$. We classify and construct all examples foliated by constant curvature horizontal curves, a two parameter family in each $M \times \mathbb{R}$. They are all simply periodic i.e. properly embedded in the quotient space $M \times \mathbb{R}/T$, where T is a vertical translation or screw motion except catenoids in $\mathbb{H}^2 \times \mathbb{R}$.

The main point is to present a unified point of view. In particular our computations, parametrizations are transversal to each product space and contains the classical theory of $\mathbb{R}^3$.

In particular Riemann [17] has construct and classified minimal surface examples foliated by straight lines and circles in horizontal planes of $\mathbb{R}^3$. He construct a family of minimal annuli with an infinite number of parallel flat ends distributed in a periodic way along the vertical. We generalize this construction to the case where the ambient space is $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ (section 3).

In section 2, we generalize the beautiful work of M. Shiffman ([21]) to find a Jacobi field derived from the derivative of the horizontal curves curvature.

Such a Jacobi field has been used and explained by Y. Fang [4] to characterize Riemann's examples in $\mathbb{R}^3$ as the unique properly embedded compact annuli bounded by two circles. Y. Fang and F. Wei

[5] extend this uniqueness result to the case where the minimal annulus is bounded by two straight lines or circle with finite total curvature less than 12π.

The Shiffman's jacobi field is an important ingredient in the study of the uniqueness conjecture of the Riemann example. In theorem 2.3, in the spirit of the work of Y. Fang [4], we prove an uniqueness result for compact annuli A having a Jacobi operator L of Index less or equal than 1:

Main Theorem *Let A be a compact minimal annulus embedded in $M \times \mathbb{R}$, where M is $\mathbb{H}^2, \mathbb{R}^2$ or $\mathbb{S}^2$, bounded by two curves of constant curvature in $M \times \{t_1\}$ and $M \times \{t_2\}$. If A has* $\text{Index}(L) \leq 1$ *then A is foliated by circles or geodesics i.e., $A \cap (M \times \{t\})$ is a curve with constant curvature k_g for all $t \in [t_1, t_2]$.*

In section 3 we are inspired by the work of Abresch [1] on constant mean curvature tori in $\mathbb{R}^3$ to represent our examples by periodic elliptic functions. Then we construct and classify a two parameter family of minimal surfaces foliated by constant curvature curves in horizontal section. In particular we find a parametrization of the classical Riemann's example which has been used to understand the Jacobi operator in a forthcoming paper [14]. In forthcoming paper, Ricardo Sa Earp and Eric Toubiana [19] have construct other minimal examples invariant by screw motion and B. Daniel [3] has explained the Gauss-Codazzi equations. In this last work he found interesting formula and geometric properties of some examples describe in this paper.

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2 Minimal annulus transverse to horizontal planes

We consider $X = (F, h) : A \subset \mathbb{R}^2 \to M \times \mathbb{R}$ a minimal surface embedded in a product space, transverse to $M \times \{t\}$ for every $t \in \mathbb{R}$. M is a Riemannian complete two-manifold with metric g and Gauss curvature K_M. Assume that M is isometrically embedded in $\mathbb{R}^k$, for k large enough. Now $F : A \longrightarrow M \subset \mathbb{R}^k$ have coordinates $F = (F_1, F_2, ..., F_k)$ in $\mathbb{R}^k$. By definition (see Lawson [7]) the mean curvature vector in $\mathbb{R}^k$ is $\vec{H} = (\triangle X)^{T_X M \times \mathbb{R}} = ((\triangle F)^{T_F M}, \triangle x_3) = 0$. Then $F : A \longrightarrow M$ is a harmonic map between A and the Riemannian surface M, while $h : A \longrightarrow \mathbb{R}$ is a proper real harmonic function (see [18]).

In first, we parametrize conformally the surface by the vertical coordinate. Since $dh \neq 0$ (the surface is transverse to $M \times \{t\}$ by hypothesis), the holomorphic map $i(h + ih^*) : A \to \mathbb{C}$ ($h^*$is the conjugate of h) gives conformal parameters $z = x + iy$ of A. In the following we will assume that $X(z) = (F(z), y)$ is a conformal embedding of A with the metric $ds^2 = \lambda(z)|dz|^2)$ induced by X.

If $(U, \rho(u)|du|^2)$ is a local parametrization of M, the harmonic map equation in the complex coordinate $u = u_1 + iu_2$ of M (see [20], page 8) is

$$F_{z\bar{z}} + (\log \rho)_u F_z F_{\bar{z}} = 0 \tag{2.1}$$

where $2F_z = F_x - iF_y$. Since $X = (F, y)$ is a conformal immersion, we have $|F_x|_g^2 = |F_y|_g^2 + 1 = ch^2\omega$ and $\langle F_x, F_y \rangle_g = 0$. Then the holomorphic quadratic Hopf differential is

$$Q_F = \phi(z)(dz)^2 = \frac{1}{4}\left(|F_x|_g^2 - |F_y|_g^2 + 2i\langle F_x, F_y \rangle_g\right) = \frac{1}{4}(dz)^2.$$

It is a well known fact (see [20] page 9) that harmonic maps fulfill the Böchner formula:

$$\frac{1}{\lambda}\triangle_0 \log\frac{|F_z|}{|F_{\bar z}|} = -2K_M J(F) \tag{2.2}$$

where $J(F) = \frac{\rho}{\lambda}\left(|F_z|^2 - |F_{\bar z}|^2\right)$ is the Jacobian of F with $|F_z|^2 = F_z\overline{F_z}$.

Now we will prove in the following that if we write the metric as $ds^2 = \lambda|dz|^2 = ch^2\omega|dz|^2$, the Böchner formula is the *sinh*-gordon equation for the function ω (see proposition 2.1). It will give us some structure equation that we use to study extrinsec properties of the surfaces.

We consider the projection Π of A on $M\times\{0\}$ and we consider the level curve $\gamma_h = A\cap(M\times\{y = h\}) = F(x,h)$ and $\gamma_{\rm v} = \Pi(\{x = {\rm v}\}) = F({\rm v}, y)$. We derive general formulae for the geodesic curvature in M of γ_h and $\gamma_{\rm v}$ (at points where $\gamma_{\rm v}$ is not a singular curve) in function of ω:

Proposition 2.1 *Let us define the real function $\omega : A \longrightarrow \mathbb{R}$ by $ds^2 = \lambda|dz|^2 = ch^2\omega|dz|^2$. Then ω is a solution of the following structure equation:*

$$\triangle_0\omega + K_M sh\omega ch\omega = 0 \tag{2.3}$$

where $\triangle_0\omega = \omega_{xx} + \omega_{yy}$. The geodesic curvature in M of the horizontal level curve γ_h and $\gamma_{\rm v}$ are given by

$$k_g(\gamma_h) = \frac{-\omega_y}{ch\omega} \text{ and (for } \omega\neq 0\text{) } k_g(\gamma_{\rm v}) = \frac{\omega_x}{ch\omega}\coth\omega. \tag{2.4}$$

Proof. Since $X = (F, y)$ is a conformal immersion, we have $|F_x|_g^2 = |F_y|_g^2 + 1 = ch^2\omega$ and $\langle F_x, F_y\rangle_g = 0$. Now let us consider $(U, \rho(u)|du|^2)$ a local parametrization of M. We define the local function ψ as the argument of F_x:

$$F_x = \frac{1}{\sqrt{\rho}}ch\omega e^{i\psi} \text{ and } F_y = \frac{i}{\sqrt{\rho}}sh\omega e^{i\psi}.$$

From Böchner formula (2.2) we have:

$$\frac{1}{\lambda}\triangle_0\log\frac{|F_z|}{|F_{\bar z}|} = -2K_M\frac{\rho}{\lambda}|F_z||F_{\bar z}|\left(\frac{|F_z|}{|F_{\bar z}|} - \frac{|F_{\bar z}|}{|F_z|}\right) \tag{2.5}$$

But by a direct computation with $\rho|F_x|^2 = ch^2\omega$ and $\rho|F_y|^2 = sh^2\omega$ we derive:

$$|F_z|^2|F_{\bar z}|^2 = \frac{1}{16}\left(\left(|F_x|^2 - |F_y|^2\right)^2 + 4\langle F_x, F_y\rangle^2\right) = \frac{1}{16\rho^2}.$$

Now from (2.5) with $2\sqrt{\rho}|F_z| = e^{\omega}$ and $2\sqrt{\rho}|F_{\bar z}| = e^{-\omega}$, we derive the *sinh*-Gordon equation (2.3)

$$\triangle_0\omega = -\frac{1}{2}K_M sh2\omega = -K_M sh\omega ch\omega.$$

We consider γ_h and $\gamma_{\rm v}$ the curves parametrized in $(U, \rho(u)|du|^2)$ with tangent vector F_x and F_y respectively. If k_g is the curvature of a curve γ in $(U, \rho(u)|du|^2)$ and k_e is the Euclidean curvature in $(U, |du|^2)$, we get by conformal change of the metric:

$$k_g = \frac{k_e}{\sqrt{\rho}} - \frac{\left\langle \nabla\sqrt{\rho}, n\right\rangle}{\rho}$$

where n is the Euclidean normal to the curve γ. In particular $n = (-sin\psi, cos\psi)$ for the curve γ_h (n is along F_y). If s denotes the arclength of γ_h, we have

$$k_e(\gamma_h) = \psi_s = \frac{\psi_x\sqrt{\rho}}{ch\omega}$$

and

$$\frac{\left\langle \nabla\sqrt{\rho}, n\right\rangle}{\rho} = \frac{\left\langle \nabla\log\sqrt{\rho}, n\right\rangle}{\sqrt{\rho}} = \frac{1}{2\sqrt{\rho}}\left(cos\psi(\log\rho)_{u_2} - sin\psi(\log\rho)_{u_1}\right).$$

The tangent vector of γ_{v} is F_y which is zero at points where $\omega = 0$. If s denote the arclength of γ_{v}, we have

$$k_e(\gamma_{\mathrm{v}}) = \psi_s = \frac{\psi_y\sqrt{\rho}}{sh\omega}$$

and with $n = (-cos\psi, -sin\psi)$

$$\frac{\left\langle \nabla\sqrt{\rho}, n\right\rangle}{\rho} = \frac{\left\langle \nabla\log\sqrt{\rho}, n\right\rangle}{\sqrt{\rho}} = -\frac{1}{2\sqrt{\rho}}\left(cos\psi(\log\rho)_{u_1} + sin\psi(\log\rho)_{u_2}\right).$$

In summary we have

$$k_g(\gamma_h) = \frac{\psi_x}{ch\omega} - \frac{1}{2\sqrt{\rho}}\left(cos\psi(\log\rho)_{u_2} - sin\psi(\log\rho)_{u_1}\right)$$

and

$$k_g(\gamma_{\mathrm{v}}) = \frac{\psi_y}{sh\omega} + \frac{1}{2\sqrt{\rho}}\left(cos\psi(\log\rho)_{u_1} + sin\psi(\log\rho)_{u_2}\right).$$

Now let us compute ψ_x as a function of ω_y and ψ_y as a function of ω_x. In complex coordinate z

$$F_z = \frac{e^{\omega+i\psi}}{2\sqrt{\rho}} \text{ and } F_{\bar{z}} = \frac{e^{-\omega+i\psi}}{2\sqrt{\rho}}.$$

Placing these expressions in the harmonic equation (2.1) we derive that

$$(-\omega + i\psi)_z = -\sqrt{\rho}\left(\frac{1}{\sqrt{\rho}}\right)_z - (\log\rho)_u F_z.$$

Now note that

$$\begin{aligned} -\sqrt{\rho}\left(\frac{1}{\sqrt{\rho}}\right)_z &= \frac{1}{2}(\log\rho)_z \\ &= \frac{1}{2}\left((\log\rho)_u F_z + (\log\rho)_{\bar{u}}\bar{F}_z\right) \end{aligned}$$

where $2(\log\rho)_u = (\log\rho)_{u_1} - i(\log\rho)_{u_2}$ and $\bar{F}_z = \dfrac{1}{2\sqrt{\rho}}e^{-\omega-i\psi}$. Collecting these equations we obtain:

$$(-\omega + i\psi)_z = \frac{1}{2}(\log \rho)_{\bar{u}} \bar{F}_z - \frac{1}{2}(\log \rho)_u F_z.$$

The real and imaginary parts give

$$\psi_x + \omega_y = \frac{ch\omega}{2\sqrt{\rho}} \left(cos\psi(\log \rho)_{u_2} - sin\psi(\log \rho)_{u_1}\right) \tag{2.6}$$

$$\psi_y - \omega_x = \frac{-sh\omega}{2\sqrt{\rho}} \left(cos\psi(\log \rho)_{u_1} + sin\psi(\log \rho)_{u_2}\right) \tag{2.7}$$

Insert this last expression in the curvature expression:

$$k_g(\gamma_h) = \frac{\psi_x}{ch\omega} - \frac{1}{2\sqrt{\rho}} \left(cos\psi(\log \rho)_{u_2} - sin\psi(\log \rho)_{u_1}\right) = \frac{-\omega_y}{ch\omega}$$

$$k_g(\gamma_{\mathrm{v}}) = \frac{\psi_y}{sh\omega} + \frac{1}{2\sqrt{\rho}} \left(cos\psi(\log \rho)_{u_1} + sin\psi(\log \rho)_{u_2}\right) = \frac{\omega_x}{sh\omega} = \frac{\omega_x}{ch\omega} coth\omega.$$

□

In the rest of this section we will consider only the geodesic curvature of γ_h that we will denote by k_g. Now we generalize a result of Shiffmann [21]. He proved in 1956, that $\sqrt{\lambda}(k_g)_x$ is a Jacobi field. In particular if u is zero on A, then the horizontal curves are of constant curvature.

Theorem 2.1 *Let A be a minimal surface embedded in a product space $M \times \mathbb{R}$ with $K_M = c_0$ a constant, and assume A transverse to every section $M \times \{t\}$. Then the function $u = -ch\omega(k_g)_x$ is a Jacobi field i.e. u is solution of the elliptic equation:*

$$Lu = \triangle_g u + Ric(N)u + |dN|^2 u = 0.$$

where $Ric(N)$ is the Ricci curvature of the two planes tangent to A, $|dN|$ is the norm of the second fundamental form and $\triangle_g = \frac{1}{\lambda}\triangle_0$.

Proof. Since $k_g(\gamma_h) = \dfrac{-\omega_y}{ch\omega}$ we have $u = \omega_{xy} - th\omega\omega_x\omega_y$. We establish by a straighforward computation that

$$\triangle_0 u = u_{xx} + u_{yy} = -\left(c_0 + 2\frac{|\nabla\omega|^2}{ch^2\omega}\right) u \tag{2.8}$$

which is $\lambda Lu = 0$. To prove (2.8) we compute $Ric(N)$. Let (e_1, e_2, e_3) be an oriented orthonormal frame in $M \times \mathbb{R}$. Then if $K(e_i, e_j)$ denotes the sectional curvature of the two plane (e_i, e_j) in $M \times \mathbb{R}$ and $S = K(e_1, e_2) + K(e_1, e_3) + K(e_2, e_3) = K_M = c_0$ is the scalar curvature, we have the well-known formula

$$Ric(N) + |dN|^2 = S + K(X_x, X_y) - 2K_g.$$

Now let us compute $K(X_x, X_y)$ the sectional curvature of the tangent plane T_pA :

$$K(X_x, X_y) = \frac{\langle R(F_x, F_y + e_3)F_x, F_y + e_3\rangle}{|X_x|^2|X_y|^2 - \langle X_x, X_y\rangle} = c_0 \frac{|F_x|^2|F_y|^2}{|F_x|^2(|F_y|^2 + 1)} = c_0 - \frac{c_0}{\lambda}.$$

We plug $\lambda = ch^2\omega$ in the expression of the Gauss curvature:

$$K_g = -\frac{1}{2\lambda}\triangle_0 \log\lambda = -\frac{4}{ch^2\omega}(\log ch\omega)_{z\bar{z}} = c_0 th\omega^2 - \left(\frac{\omega_x^2 + \omega_y^2}{ch\omega^4}\right).$$

Finally, we justify that (2.8) is $\lambda Lu = 0$ by:

$$Ric(N) + |dN|^2 = 2c_0 - \frac{c_0}{ch^2\omega} - 2c_0 th\omega^2 + 2\frac{|\nabla\omega|^2}{ch^4\omega} = \frac{c_0}{ch^2\omega} + 2\frac{|\nabla\omega|^2}{ch^4\omega}.$$

Now we prove the formula (2.8)

$$\triangle_0 u = (\triangle_0\omega)_{xy} - (\triangle_0 th\omega)\omega_x\omega_y - th\omega(\triangle_0\omega_x\omega_y) - 2(th\omega)_x(\omega_x\omega_y)_x - 2(th\omega)_y(\omega_x\omega_y)_y$$

Using $\triangle_0\omega + c_0 sh\omega ch\omega = 0$, we have

$$(\triangle_0\omega)_{xy} = \left(\frac{-c_0}{2}sh2\omega\right)_{xy} = -c_0\omega_{xy}ch2\omega - 2c_0\omega_x\omega_y sh2\omega$$

$$\triangle_0\omega_x\omega_y = -c_0\omega_{xy}sh2\omega - 2c_0\omega_x\omega_y ch2\omega$$

$$\triangle_0 th\omega = \left(-c_0 - 2\frac{|\nabla\omega|^2}{ch^2\omega}\right) th\omega$$

$$2(th\omega)_x(\omega_x\omega_y)_x + 2(th\omega)_y(\omega_x\omega_y)_y = \frac{2\omega_{xy}|\nabla\omega|^2 - 2c_0\omega_x\omega_y sh\omega ch\omega}{ch^2\omega}$$

Then

$$\begin{aligned}\triangle_0 u = \; & -c_0(ch2\omega - th\omega sh2\omega)\omega_{xy} - 2\frac{|\nabla\omega|^2}{ch^2\omega}\omega_{xy} \\ & -2c_0\omega_x\omega_y(sh2\omega - th\omega ch2\omega) - 2c_0 th\omega\omega_x\omega_y \\ & + \left(c_0 + 2\frac{|\nabla\omega|^2}{ch^2\omega}\right) th\omega\omega_x\omega_y\end{aligned}$$

Since $ch2\omega - th\omega sh2\omega = 1$ and $th\omega ch2\omega - sh2\omega = -th\omega$ we proved (2.8). □

Now with this Jacobi fields we derive some global result on annuli embedded in $M \times \mathbb{R}$. First we generalize a theorem of M. Shiffman [21]:

Theorem 2.2 *Let A be a compact minimal annulus immersed in $M \times \mathbb{R}$ with $K_M = c_0 \leq 0$. If A is bounded by two curves Γ_1 and Γ_2 with positive geodesic curvature in $M \times \{t_1\}$ and $M \times \{t_2\}$ then A is foliated by horizontal curves of positive curvature i.e., $A \cap (M \times \{t\})$ is a curve with curvature $k_g > 0$.*

Remark 2.1 *In the case where $K_M = c_0 > 0$, this result is false. We can consider compact part of onduloids in $\mathbb{S}^2 \times \mathbb{R}$ (see section 4.1) which give annulus bounded by two circles of positive curvature, and containing some geodesics and negative curvature curves in its interior.*

Proof. It is a consequence of maximun principle and the proposition 2.1, in the linearized *sinh*-Gordon equation:

$$\begin{cases} \triangle_0 \omega_y + K_M \omega_y \cosh 2\omega = 0 & \text{on} \quad A \\ \omega_y < 0 & \text{on} \quad \partial A = \Gamma_1 \cup \Gamma_2 \end{cases}$$

□

Now we generalize geometric characterisation of M. Shiffman [21] and Y. Fang [4] for annulus with low index bounded by constant curvature curves:

Theorem 2.3 *Let A be a compact minimal annulus embedded in $M \times \mathbb{R}$ with $K_M = c_0$. We assume that the boundary $\partial A = \Gamma_1 \cup \Gamma_2$ are curves with constant geodesic curvature in $M \times \{t_1\}$ and $M \times \{t_2\}$ i.e., $u = 0$ on ∂A.*

If $M = \mathbb{H}^2, \mathbb{R}^2$ or $\mathbb{S}^2$ and A has Index$(L) \leq 1$,*then u is identically zero and A is foliated by horizontal curves of constant curvature in M.*

In the case where M is not simply connected, the result is true with additional hypothesis that Index$(L) = 0$ *(A is stable).*

Proof. We refer the work of Y. Fang [4] for details. By the four vertex theorem, u has four zeros on each horizontal Jordan curve of a simply connected space (see S. B. Jackson [6]) and then u has at least four nodal domains on the annulus A. Then u is an eigenfunction corresponding to the third eigenvalue and then Index$(L) \geq 2$, a contradiction. In the case of a general riemannian surface, u may have only two zeros and then the annulus have Index$(L) \geq 1$ if u is not identically zero. □

3 The Gauss-Codazzi equation of generalized Riemann examples

In this section we construct the family of Riemann examples in $M \times \mathbb{R}$, with $K_M = c_0$ a constant. We classify all examples foliated by one constant curvature curves in the horizontal plane. These surfaces are annuli or simply connected surfaces transverse to each horizontal plane. We describe the space moduli of these surfaces in terms of elliptic functions.

We parametrize these surfaces by the third coordinate and with the notation of the previous section, the embedding $X = (F, y) : \widetilde{A} = \{(x, y) \in]\alpha_1, \alpha_2[\times]\beta_1, \beta_2[\} \longrightarrow M \times \mathbb{R}$ is minimal with $F : \widetilde{A} \longrightarrow M$ harmonic. Here $\widetilde{A}$ is the universal covering of A and $\alpha_1, \alpha_2, \beta_1, \beta_2$ can be infinite. We describe the space of these surfaces in terms of elliptic functions.

Let $\omega : \widetilde{A} \longrightarrow \mathbb{R}$ be a function defined by $ds^2 = ch^2\omega |dz|^2$. When A is transverse to each horizontal plane $\omega \neq \infty$ on A and ω is solution of the system:

$$(I) \begin{cases} \triangle_0 \omega + K_M sh\omega ch\omega = 0 \\ \omega_{xy} - th\omega \omega_x \omega_y = 0. \end{cases}$$

From proposition 2.1, the first equation reflects the Gauss equation of M and the second equation states that the curvature of each level curve is constant. In case $K_M = 1$, this system has been studied by Abresch [1] to classify constant mean curvature tori in $\mathbb{R}^3$ with planar large lines of curvature (the second equation is the torsion of a large line of curvature of C.M.C. surfaces).

To construct examples, we apply Abresch's technique in theorem 3.4 to solve the system (I) on the whole plane $\mathbb{R}^2$. We will represent the space of these examples in a two parameter family. When solutions ω are periodic in the variable x and $\omega \neq \infty$, we can expect an annulus by closing periods of the immersion. The harmonic map has to be periodic in x and the immersion X is well defined on $A = \{(x, y) \in \mathbb{R}/(x_0\mathbb{R}) \times \mathbb{R}\}$ (see section 4).

Solutions can take infinite values and then it will define the domain of $\widetilde{A}$ where the solutions are well defined. When $c_0 < 0$, the condition $\omega \neq \infty$ is valid only in domains homeomorphic to a strip, a disk or the plane with a countable set of disks removed. In particular there are helicoidal surfaces embedded in $\mathbb{H}^2 \times \mathbb{R}$ defined on a strip. The set where ω takes infinite values represents a curve in the boundary at infinity $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$.

Using these solutions ω, we use Gauss-Codazzi equation to construct a harmonic map $F : \widetilde{A} \to M$ in theorem 3.5. It remains to study the period problem and the geometry of the family in section 4.

Theorem 3.4 *Let $\omega : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a real-analytic solution of the system (I), with $K_M = c_0$ a given constant. We define the function f, g in function of ω by*

$$f = \frac{-\omega_x}{ch\omega} \text{ and } g = \frac{-\omega_y}{ch\omega} \tag{3.9}$$

Then the real functions $x \to f(x)$ and $y \to g(y)$ of one variable solves the following system:

$$\begin{aligned} -(f_x)^2 &= f^4 + (c_0 + a)f^2 + c & & \\ -f_{xx} &= 2f^3 + (c_0 + a)f & & \text{with } c, d \in \mathbb{R}, a = \frac{c-d}{c_0} \text{ if } c_0 \neq 0 \\ -(g_y)^2 &= g^4 + (c_0 - a)g^2 + d & & \text{with } c = d \text{ and } a \in \mathbb{R} \text{ if } c_0 = 0 \\ -g_{yy} &= 2g^3 + (c_0 - a)g & & \end{aligned}$$

Reciprocally, we can recover the solution ω from functions of f and g. In the case where $c_0+f^2+g^2$ is not identically zero we have

$$sh\omega = (c_0 + f^2 + g^2)^{-1}(f_x + g_y) = (f_x - g_y)^{-1}(g^2 - f^2 - a) \tag{3.10}$$

In the case where $c_0 + f^2 + g^2 \equiv 0$ on $\mathbb{R}^2$, the function $f := \alpha$ and $g := \beta$ are constant and the solutions are given by

$$sh\omega = -\tan(\alpha x + \beta y). \tag{3.11}$$

When the curvature $K_M = c_0 \leq 0$, the solution ω may have infinite values. If we define $D = \{(x, y) \in \mathbb{R}^2; \omega = \infty\}$ we have, in the case of equation (3.10)

$$\begin{aligned} D &= \{(x, y) \in \mathbb{R}^2; f^2 + g^2 + c_0 = 0 \text{ and } f_x + g_y \neq 0\} \text{ for } c \neq 0, d \neq 0 \\ D &= \{(x, y) \in \mathbb{R}^2; f^2 + c_0 = 0\} \text{ for } d \neq 0 \\ D &= \{(x, y) \in \mathbb{R}^2; g^2 + c_0 = 0\} \text{ for } c \neq 0. \end{aligned}$$

and in the case where ω is given by equation (3.11):

$$D = \{(x,y) \in \mathbb{R}^2; \alpha x + \beta y = \frac{k\pi}{2}, k \in \mathbb{Z}\} \tag{3.12}$$

When the curvature $K_M = c_0 = 0$, the set D is a countable set of isolate points ($D = \{(x,y) \in \mathbb{R}^2; f(x) = g(y) = 0\}$).

When the curvature $K_M = c_0 > 0$, there is a solution of the system if and only if $c \leq 0$ and $d \leq 0$ and ω is periodic and definite on the whole plane $\mathbb{R}^2$.

Proof. We apply Abresch's technique with $K_M = c_0$ a given constant. Let ω be a solution of (I). We work at the point where $c_0 + f^2 + g^2 \neq 0$. The second equation of (I) leads to a separation of the variables:

$$ch\omega f_y = th\omega\omega_x\omega_y - \omega_{xy} = ch\omega g_x = 0.$$

Then ω solves the system (I), if and only if f and g are in one variable and satisfy the equation

$$f_x + g_y = \frac{-\triangle_0\omega}{ch\omega} + th\omega\frac{\omega_x^2 + \omega_y^2}{ch\omega} = (c_0 + f^2 + g^2)sh\omega \tag{3.13}$$

Now we integrate f, g. By the derivative in (3.10) and substituing:

$$\frac{f_{xx}}{c_0 + f^2 + g^2} - f\frac{f_x^2 - g_y^2}{(c_0 + f^2 + g^2)^2} = -f \tag{3.14}$$

$$\frac{g_{yy}}{c_0 + f^2 + g^2} - g\frac{g_y^2 - f_x^2}{(c_0 + f^2 + g^2)^2} = -g \tag{3.15}$$

Multiplying (3.14) with $2f_x$ and integrating with respect to x, we obtain for a constant $k(y)$:

$$\frac{f_x^2 - g_y^2}{c_0 + f^2 + g^2} = -f^2 + k(y).$$

Multiplying the above equation with $(c_0 + f^2 + g^2)$ and f respectively, and adding them up:

$$f_{xx} = -2f^3 - (c_0 + g^2(y) - k(y))f.$$

Since f does not depend on y, we can pick any of the values of $c_0 + g^2 + k(t)$ for $\bar{c}$ and we get the formulae (the similar computation holds for g):

$$-f_{xx} = 2f^3 + \bar{c}f \tag{3.16}$$

$$-g_{yy} = 2g^3 + \bar{d}g \tag{3.17}$$

These equations have first integrals

$$-(f_x)^2 = f^4 + \bar{c}f^2 + c \tag{3.18}$$

$$-(g_y)^2 = g^4 + \bar{d}g^2 + d. \tag{3.19}$$

Now if we consider some real function f, g which satisfies equations (3.17) and (3.19), we get a function ω by the equation (3.10). Now ω is defined and solve the system (I) if and only if f and g can be expressed as in (3.9). By the derivative in (3.10), one can prove that (3.9) is equivalent to (3.14). We plug (3.17) and (3.19) in (3.14) to get that ω is a solution of (3.13) if and only if:

$$f(c_0^2 - \bar{c}c_0 + c - d) + fg^2(2c_0 - \bar{c} - \bar{d}) = 0 \tag{3.20}$$

$$g(c_0^2 - \bar{d}c_0 + d - c) + f^2g(2c_0 - \bar{d} - \bar{c}) = 0 \tag{3.21}$$

Then for $c_0 \neq 0$, if $f \neq 0$ and $g \neq 0$, we deduce from (3.20), $\bar{c} = c_0 + \frac{c-d}{c_0} = c_0 + a$ and $\bar{d} = 2c_0 - \bar{c} = c_0 - a$.

If $f \equiv 0$ and $g \neq 0$, we have $c = 0$ and from (3.21) we derive $\bar{d} = c_0 + \frac{d}{c_0} = c_0 - a$ while if $g \equiv 0$ and $f \neq 0$, $d = 0$ and $\bar{c} = c_0 + \frac{c}{c_0} = c_0 + a$.

When $c_0 = 0$, if $f \neq 0$, $g \neq 0$ we have $c = d$ and $\bar{c} = -\bar{d}$; if $f = 0$ or $g = 0$ then $c = d = 0$.

We note that all our computations are valid at points where $c_0 + f^2 + g^2 \neq 0$, but f and g are real functions definite on $\mathbb{R}$. At a point where $c_0 + f^2 + g^2 = 0$, one can consider the limiting value of ω. It depends on $f_x + g_y$. We note that

$$f_x^2 - g_y^2 = (c_0 + g^2 + f^2)(g^2 - f^2 - a)$$

and ω is well defined if $f_x + g_y = 0$ and $f_x - g_y \neq 0$ (i.e. $f_x = -g_y \neq 0$). Then we can define ω by continuity at this point.

In the case where $f_x = g_y = 0$ and $c_0 + f^2 + g^2 = 0$ at one point, we have $f_{xx} = g_{yy} = 0$ by derivative in (3.13) and then f and g are constant by unique continuation theorem. Now $c_0 + f^2 + g^2$ is identically zero on the domain. In this case f and g are solution of the system with $f^2 = \frac{1+c-d}{2} := \alpha^2$ and $g^2 = \frac{1+d-c}{2} := \beta^2$ and the additional condition $(1 + d - c)^2 = 4d$. In this case, one can integrate directly the solutions and then

$$sh\omega = -\tan(\alpha x + \beta y).$$

Then $D = \{(x, y) \in \mathbb{R}^2; c_0 + f^2 + g^2 = 0 \text{ and } f_x + g_y \neq 0\}$. When $c_0 < 0$, $\mathbb{R}^2 - D$ gives us different connected components.

In the case where $d = 0$, i.e. $g \equiv 0$ and $-(f_x)^2 = (f^2 - c)(f^2 + c_0)$. Then

$$sh\omega = \frac{f_x}{f^2 + c_0} = \frac{f^2 - c}{-f_x} \to \infty \text{ as } f^2 \to -c_0 \text{ (and then } f_x \to 0)$$

which proves $\omega = \infty$ on $D = \{(x, y) \in \mathbb{R}^2; f^2 + c_0 = 0\}$. The same is true for $c = 0$.

When $c_0 \geq 0$, we have the real valued function f and g if and only if $c \leq 0$ and $d \leq 0$. To see that, notice that values of f^2 and g^2 are between distincts zeroes of $X^2 + (c_0 + a)X + c$ and $Y^2 + (c_0 - a)Y + d$ respectively. Assume $c > 0$, then $0 \leq (c_0 + a)^2 - 4c < (c_0 + a)^2$. In the case $(c_0 + a) > 0$ we find $-(c_0 + a) - \sqrt{(c_0 + a)^2 - 4c} \leq 2f^2 \leq -(c_0 + a) + \sqrt{(c_0 + a)^2 - 4c} < 0$ and if $(c_0 + a) < 0$, we have $c_0 - a > 2c_0 > 0$ and $0 < c < d$, then $-(c_0 - a) - \sqrt{(c_0 - a)^2 - 4c} \leq 2g^2 \leq -(c_0 - a) + \sqrt{(c_0 - a)^2 - 4c} < 0$. This contradicts the fact that we have real valued functions, then $c \leq 0$ and $d \leq 0$. As we will see in the following this is not true when $c_0 < 0$. □

We use Gauss-Codazzi equations to integrate solutions of system (I).

Theorem 3.5 *Let ω be a solution of the system (I) on a simply connected domain Ω, then there exists a minimal isometric embedding of $(\Omega, ds^2 = ch^2\omega|dz|^2)$ in $M(c_0) \times \mathbb{R}$ foliated by constant curvature curves at each horizontal level.*

Proof. Let ω be a solution of the system (I). When $c_0 > 0$, ω is defined on the whole plane. For $c_0 = 1$, it is a well-known fact that the first equation is the Gauss condition of local existence of a constant mean curvature surface $H = 1/2$ in $\mathbb{R}^3$ (it is a sinh-Gordon equation, see [1]). Since Ω is simply connected, there is $H : \Omega \to \mathbb{R}^3$, a C.M.C immersion, and its Gauss map $F : \Omega \to \mathbb{S}^2$ is the harmonic map associated with ω (see [1]).

In the case $c_0 = -1$, one can use the same construction explained in a work of Tom Y.H. Wan [22] or K. Akutagawa and S. Nishikawa [2]. The system (I) gives us a Gauss equation to construct a space-like surface of constant mean curvature in the Minkowski space $M^{2,1}$ with Hopf map $Q = \frac{1}{4}(dz)^2$. The unit normal vector to this surface in $M^{2,1}$ is a harmonic map $F : \Omega \to \mathbb{H}^2$ associated with the solution ω.

For $c_0 \neq 0$, we use some dilatation. Consider $\widetilde{\omega} = \omega\left(\frac{x}{\sqrt{|c_0|}}, \frac{y}{\sqrt{|c_0|}}\right)$. Then $\widetilde{\omega}$ is solution of $\triangle_0\widetilde{\omega} = \frac{c_0}{|c_0|} sh\widetilde{\omega}ch\widetilde{\omega}$. Then we find a harmonic map $\widetilde{F} : \Omega(\sqrt{|c_0|}) \to M(\pm 1)$ with $\Omega(\sqrt{|c_0|}) = \left\{(x, y) \in \mathbb{R}^2; \left(\frac{x}{\sqrt{|c_0|}}, \frac{y}{\sqrt{|c_0|}}\right) \in \Omega\right\}$. Now with a dilatation, we have the harmonic map

$$F = |c_0|\widetilde{\phi}(\sqrt{|c_0|}x, \sqrt{|c_0|}y) : \Omega \to M(c_0)$$

which corresponds to our system (I).

Immersions are given by $X = (F, y)$ on Ω. The second equation in (I) states that these examples are foliated by constant curvature curves at each horizontal level (see proposition 2.1).

For minimal surfaces in $\mathbb{R}^3$, we construct a surface by considering the Weierstrass data $g = -ie^{\omega+i\psi}$ and $\eta = -idz$ ($\omega + i\psi$ is holomorphic).

$$2X(z) = Re\int_z ((g^{-1} - g)\eta, i(g^{-1} + g)\eta, 2\eta).$$

□

Now we try to describe some geometric properties of these families of surfaces. Let ω be a solution of (I) on $\widetilde{A}$ described in theorem 3.5, then $X = (F, y)$ is a minimal surface foliated by horizontal curves of constant curvature. By proposition 2.1, $g(y)$ is the curvature of $A \cap M \times \{y\}$ and $-f(x_0)coth\omega$ is the geodesic curvature of the projection of $X(x_0, y)$ on $M \times \{0\}$. We will consider $c_0 = +1, 0, -1$ as generic cases (other cases come from dilatation). Now we describe the geometry of examples in different space forms.

4 The geometry of generalized Riemann examples

4.1 Minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$

Theorem 4.6 *The space of minimal surfaces of genus zero embedded in $\mathbb{S}^2 \times \mathbb{R}$ and foliated by horizontal constant curvature curves is a two parameter family parametrized by $\mathcal{M} = \{(c, d) \in \mathbb{R}^2; c \leq 0, d \leq 0\}$. All examples are annulus, periodic in the vertical direction.*

a) $c = 0$, $d \in \mathbb{R}_-$ *is a family of rotational surfaces described by Pedrosa, Ritore [16] and Rosenberg [18] . The curvature of the horizontal curves are oscillating between two values of opposite sign.*

b) $d = 0$, $c \in \mathbb{R}_-$ *is a family of helicoid. The horizontal constant curves are geodesics passing by two antipodal points (the axis).*

c) $(c, d) \in (\mathbb{R}^*_-)^2$ *is a two parameter family of Riemann type surfaces. These annuli are foliated by circles with radius oscillating between two oposite values and center located on a given geodesic.*

d) $(c, d) = (0, 0)$ *is a vertical flat annulus foliated by a great circle.*

Proof. In the case where $K_M = c_0 > 0$ (i.e. $M = \mathbb{S}^2$ up to a homothety in $\mathbb{R}^4$), functions f, g are described as in Abresch's paper. By theorem [1], the functions f and g are both periodic, oscillating around zero. The zeros of $X^2 + (c_0 + a)X + c$ and $Y^2 + (c_0 - a)Y + d$ are of opposite sign and then $f^2 \in [0, X_+], g^2 \in [0, Y_+])$. Let us assume that $f(0) = 0$ and $g(0) = 0$, $f_x(0) = \alpha \geq 0$, $g_y(0) = \beta \geq 0$ with $c = -\alpha^2$ and $d = -\beta^2$. By the proposition 2.1, g is the curvature of horizontal curves (does not depend on x) while $-f(\mathrm{v})coth\omega$ corresponds to the curvature of the curve γ_{v} obtained by projection of $X(\mathrm{v}, y)$ on $\mathbb{S}^2 \times \{0\}$. The tangent vector of this curve is F_y and $\langle F_y, F_x \rangle = 0$, then if (v, y) is chosen such that $f(\mathrm{v}) = 0$ and $\omega(\mathrm{v}, y) \neq 0$, the curve γ_{v} is a geodesic orthogonal to each horizontal level curve. When $f(\mathrm{v}) = 0$ and $\omega(\mathrm{v}, y) = 0$, the tangent vector $F_y = 0$ and the corresponding curve is a vertical straight line.

Since constant curvature curves are periodics on $\mathbb{S}^2$, the immersion of $\widetilde{A}$ is the covering of a minimal annulus embedded in $\mathbb{S}^2 \times \mathbb{R}$.

When $d = 0$ and $c \leq 0$, the horizontal curves are geodesics i.e. great circles in $\mathbb{S}^2$. The function f_x has two zeroes x_0 and x_1. Then $sh\omega(x_0, y) = sh\omega(x_1, y) = 0$ which corresponds to two vertical axes at antipodal points $F_y(x_0, y) = F_y(x_1, y) = 0$. This is the helicoidal family described in [18].

When $c = 0$ and $d \leq 0$, the horizontal curves have constant geodesic curvature oscillating between d and $-d$. The function $f = 0$ and $\omega \neq 0$ if $g \neq 0$. Then the center of each horizontal circle is at the same point. It is a rotational invariant surface. It is the onduloid family of Pedrosa-Ritore [16], described in [18].

The tangent vector of this curve is F_y and $\langle F_y, F_x \rangle = 0$, then if v is chosen such that $f(\mathrm{v}) = 0$, the curve γ_{v} is a geodesic orthogonal to each horizontal level circle and then the center of this circle is on this geodesic.

The other surfaces are those of Riemann example type in the following sense. Since $f(0) = 0$, the curve γ_h is a geodesic orthogonal to each horizontal level circle and then the center of these circles are on this geodesic. □

4.2 Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$

Theorem 4.7 *The space of minimal surfaces of genus zero embedded in* $\mathbb{H}^2 \times \mathbb{R}$ *and foliated by horizontal constant curvature curves is a two parameter family parametrized by:*

$$\mathcal{M} = \{(c, d) \in \mathbb{R}^2; (1 + c - d)^2 \geq 4c \text{ and } c - 1 \leq d \leq c + 1\} \cup \{c \leq 0\} \cup \{d \leq 0\}.$$

(In figure 1, we represent $\mathcal{M} = \mathbb{R}^2 - (A_1 \cup A_2 \cup A_3)$*) In first we describe some special one parameter family of* $\mathcal{M}$*:*

1) *The curve* $\Gamma = \{(c,d) \in \mathbb{R}^2; (1+c-d)^2 = 4c$ *and* $c-1 \leq d \leq c+1\}$ *parametrize surfaces of helicoidal type where the horizontal curves have constant curvature curves constant* $k = g = c$*. When* $c = 0$ *it is a helicoid. When* $d = 0$*, the surface is an annulus foliated by horocycles.*

2) $d = 0$*. The surfaces are foliated by horizontal geodesics.*

   a) $c > 0$ *parametrize the helicoidal family.*

   b) $c < 0$ *The surfaces are global graph on* $\mathbb{H}^2$ *and can be assimilate to some "oblique" plane which interpolate the horizontal plane and the vertical one.*

3) $c = 0$*. The surfaces are bounded in the third component. They are catenoids and graph.*

   a) $d > 1$*. The surfaces are rotational annuli related to catenoidal examples. They are described in the work of Nelli and Rosenberg [15].*

   b) $0 < d \leq 1$*. The examples are catenoids foliated by equidistant curves* $k_g < 1$ *in* $\mathbb{H}^2$*. In an Euclidean way, the surface is homeomorphic to a part of a catenoid described above (3-a), intersecting a solid cylinder with axis translated by a horizontal translation, in such a way that every horizontal circles intersect the boundary of the cylinder.*

   c) $d < 1$*. The surface are global graph on* $\mathbb{H}^2$*, foliated by equidistant curves.*

Now we describe the regions of $\mathcal{M}$ *bounded by curves described in 1),2), and 3) (see figure 1).*

4)*The region 1 (*$c < 0$ *and* $d > 0$*). The surfaces are annulus with two non horizontal boundary curve at infinity. These annulus contains two horocycles in some horizontal section. They are parametrized on a region homeomorphic to a strip (see figure 2). The figure 3 represent an example.*

5)*The region 2 (*$c > 0$ *and* $d < 0$*). The surfaces are "ondulated" helicoid. They are parametrized by a vertical strip and it is periodic in the third component (see figure 4). In a period there is two horizontal geodesic in the surface. Between these two horizontal sections, the surface is foliated by equidistant* $k < 1$*. It is ondulated surface in the sense that the curvature is changing of sign after crossing a geodesic.*

6)*The region 3 (*$c > 0$ *and* $d > 0$*). The surface are "blowed" helicoid. They are parametrized by a strip (see figure 6) but the curvature of horizontal curves is never zero.*

7)*The region 4 (*$c < 0$ *and* $d < 0$*). The surfaces are Riemann type examples. They are parametrized conformally by a cylinder minus a countable set of disk (see figure 7). They have a vertical plane of symmetry and the boundary set of curves at infinity is a disjoint set of circles in the cylinder (see figure 8).*

Remark 4.2 *In a forthcoming preprint B. Daniel have explicit formula of the surfaces describe in 1), 2) and 3) ([3]).*

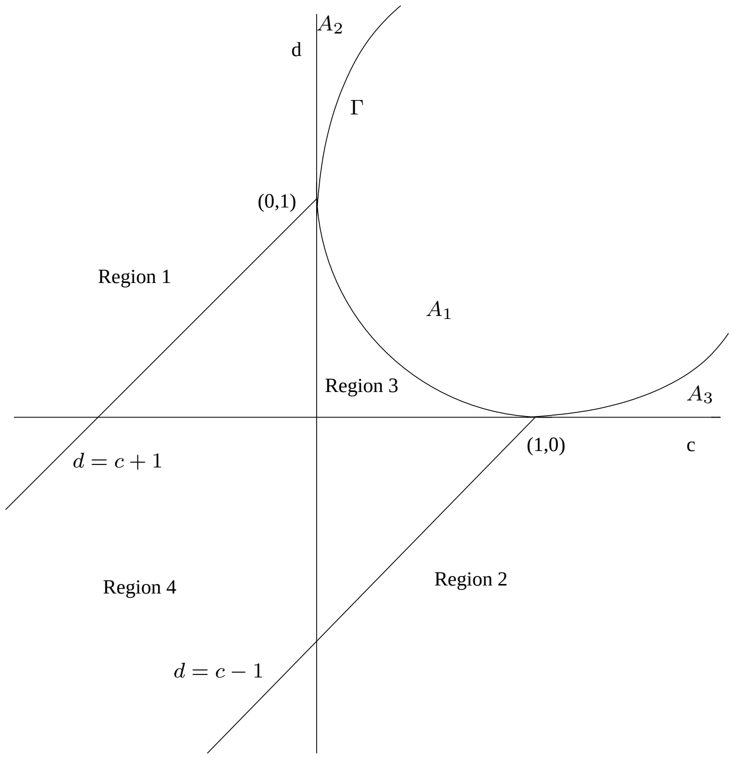


Figure 1: Two parameter family

Proof. In the case where $K_M = -1$ (we can consider $\mathbb{H}^2$ as the universal covering of M), the family is quite important. The existence of solutions f and g depends on $P(X) = X^2 - (1-a)X + c = X^2 - (1+c-d)X + c$ and $Q(y) = Y^2 - (1+a)Y + d = Y^2 - (1+d-c)Y + d$. Note that P, Q have same discriminant

$$\Delta = (1+c-d)^2 - 4c = (1+d-c)^2 - 4d.$$

Then roots of $P(X) = 0$ are

$$X_+ = \frac{1}{2}(1+c-d+\sqrt{\Delta}) \text{ and } X_- = \frac{1}{2}(1+c-d-\sqrt{\Delta})$$

and those of $Q(Y) = 0$ are:

$$Y_+ = \frac{1}{2}(1+d-c+\sqrt{\Delta}) \text{ and } Y_- = \frac{1}{2}(1+d-c-\sqrt{\Delta}).$$

Since $P(f^2) = -(f_x)^2$ and $Q(g^2) = -(g_y)^2$, the functions f and g exist if and only if $\Delta \geq 0$ and $X_+ \geq 0$, $Y_+ \geq 0$. We parametrize our family of surfaces in the plane (c, d) in Figure 1.

When $\Delta = 0$, f^2 and g^2 are constant, then $f_x = g_y = 0$ and ω is given by Theorem 3.4, equation (3.11). We can see that $\Delta < 0$ if and only if $(1-\sqrt{c})^2 < d < (1+\sqrt{c})^2$ which define the region A_1 in figure 1 bounded by the curve Γ passing by $(0,1)$ and $(1,0)$.

Moreover one can see that when $d > c+1$, $c > 0$ and $\Delta > 0$ (region A_2), we have $1+c-d \leq -\sqrt{\Delta} < 0$ and then $X_+ < 0$. In the case $c > d+1$, $d > 0$ and $\Delta > 0$ (region A_3), we have $Y_+ < 0$. Then the space $\mathcal{M} = \{(c,d) \in \mathbb{R}^2; \Delta \geq 0, X_+ \geq 0, Y_+ \geq 0\} = \mathbb{R}^2 - A_1 \cup A_2 \cup A_3$.

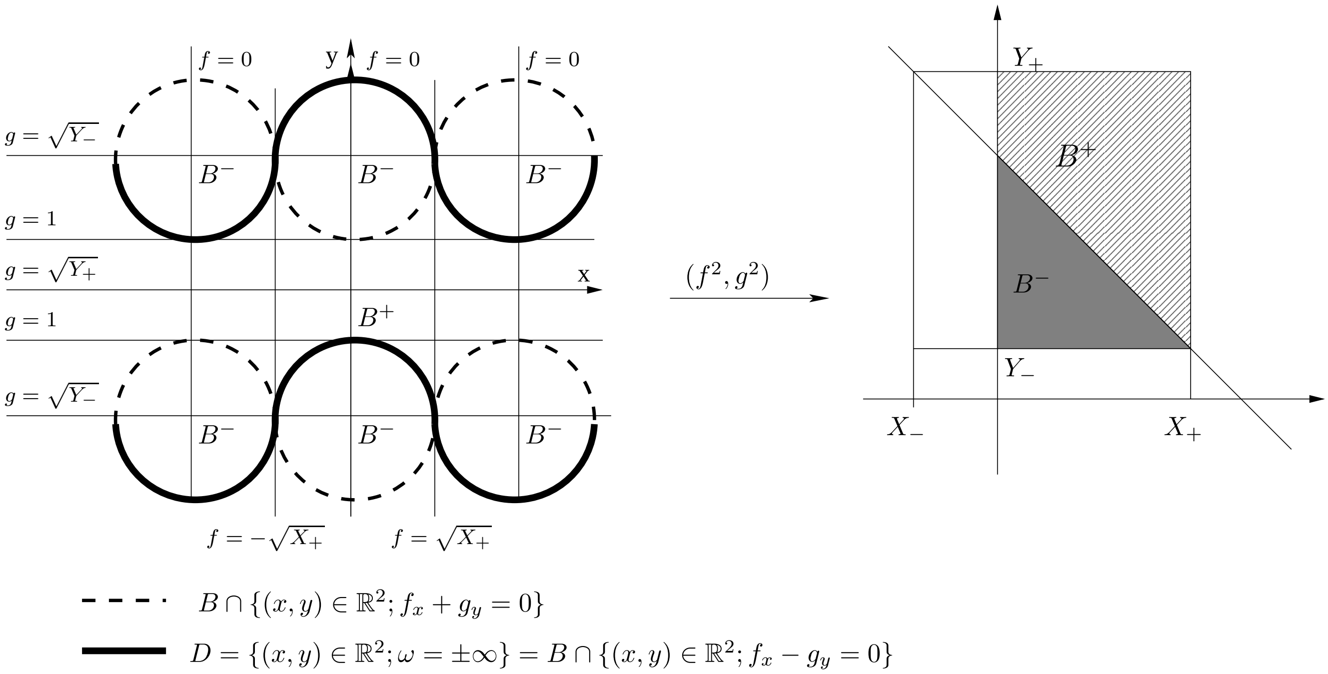


Figure 2: The annulus family

The important phenomenon here is that $(f^2 + g^2 - 1)$ can be zero. To help us in the following we define

$$B = \{(x, y) \in R^2; f^2 + g^2 = 1\}$$
$$B^- = \{(x, y) \in R^2; f^2 + g^2 < 1\}$$
$$B^+ = \{(x, y) \in R^2; f^2 + g^2 > 1\}.$$

Now the set $D = B \cap \{(x, y) \in R^2; f_x + g_y \neq 0\}$ separates $\mathbb{R}^2$ in connected components where our surfaces are defined.

The region 1 ($c < 0$ and $d > 0$) contains some annulus bounded in the third component. The region 2 ($c > 0$ and $d < 0$) and region 3 ($c > 0$ and $d > 0$) are homeomorphic to simply connected strips embedded in the cylinder. The region 4 ($c < 0$ and $d < 0$) contains Riemann minimal surfaces example type, $\mathbb{R}^2 - D$ is the plane with a countable set of disks removed.

First we classify the examples depending only on one parameter, x or y.

1) The curve Γ. In first, we are looking for the set of parameter where $\Delta = 0$ and $c - 1 \leq d \leq c + 1$. On Γ, the discriminant $\Delta = 0$ and $X_+ = X_-$, $Y_+ = Y_-$, then $f_x \equiv g_y \equiv 0$ and $f^2 + g^2 \equiv 1$. We are in the case where ω is given by formula (3.11). The surface is foliated by curves of constant geodesic curvature $k_g = g = \sqrt{\frac{1+d-c}{2}} := \alpha \leq 1$. By (3.11), $\omega \neq \infty$ if and only if $-\frac{\pi}{2} < \alpha x + \beta y < \frac{\pi}{2}$. Then ω is defined on a strip and the straight line $\alpha x + \beta y = 0$ define an axis since $\omega = 0$. After an isometry, we can assume that the axis project on the origin of the Poincare disk model. The horizontal vector F_x has the argument ψ which by (2.6),(2.7) and $\omega(-\frac{\beta}{\alpha} y, y) = 0$ has derivative $\psi_y = \frac{1}{\alpha}$ (with $\rho(0) = 1$ and $\rho_{u_1}(0) = \rho_{u_2}(0) = 0$ in formula (2.6),(2.7)). The horizontal curves are turning with constant speed. The case $c = 0$ is parametrized on a horizontal strip. The third component is bounded and

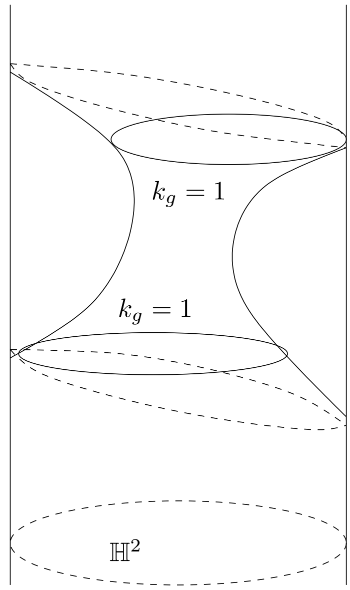


Figure 3: The annulus family

every horizontal curves are all horocycles ($k_g = 1$). The horizontal section $\{y = 0\}$ is a plane of symmetry of this annulus.

2) The Helicoid and planar family $d = 0$. First, we classify the family of surfaces foliated by geodesics at each horizontal level i.e. $g = 0$ identically and $d = 0$. In this case we have $-(f_x)^2 = f^4 - (1+c)f^2 + c = (f^2 - c)(f^2 - 1)$. The case $c = 0$ and $d = 0$, represent a geodesic vertical plane $\gamma \times \mathbb{R}$ (a geodesic product of $\mathbb{R}$ and $\omega = 0$). We note that for $c \neq 0$

$$sh\omega = \frac{f_x^2}{f_x(f^2-1)} = \frac{-(f^2-c)}{f_x} \longrightarrow \pm\infty \text{ as } f^2 \to 1.$$

Then the surface is defined on the vertical strip D bounded by the set $D = \{(x,y) \in \mathbb{R}^2; x = a_0 \text{ and } x = a_1\}$ (with $f^2(a_0) = f^2(a_1) = 1$). Each component of $\mathbb{R}^2 - \{f^2 = 1\}$ gives rise to the same surface.

2-a) The case $c > 0$. The case $c = 1$ is given in 1. If $c > 1$, we have $B^- = \emptyset$ and the surface is parametrized on $B^+ = \{(x,y) \in \mathbb{R}^2; a_0 < x < a_1\}$. If $0 < c < 1$, the set $B^+ = \emptyset$ and the surface is parametrized on $B^- = \{(x,y) \in \mathbb{R}^2; a_0 < x < a_1\}$. When $f^2(x_0) = c$, we have $f_x = 0$ and $sh\omega = 0$ describes a vertical axis. We can assume that $X(x_0, y)$ project on the origin in the Poincare unit disc model of $\mathbb{H}^2$. For the other values of f, the projection is circles of curvature greater than 1 in the horizontal plane. These curves describe helicoidal movement in the cylinder model of $\mathbb{H}^2 \times \mathbb{R}$. Note that horizontal geodesics (radius of the disc) turn with constant speed. The horizontal vector F_x has the argument ψ which by (2.6),(2.7) and $\omega(x_0, y) = 0$ has the derivative $\psi_y(x_0, y) = \omega_x(x_0, y) = -f(x_0)$ (a constant speed of rotation).

2-b)The case $c < 0$. We have $-(f_x) = (f^2 - c)(f^2 - 1)$ and then $f^2 \in [0,1]$ i.e. $f \in [-1,1]$ and the surface is parametrized by B^- ($B^+ = \emptyset$). We have f_x of constant sign. We are looking for $k_g(\gamma) = -f(x)\coth\omega$. When $f(x_0) = 0$, we have $sh\omega = \pm\sqrt{-c} \neq 0$ and $k_g(\gamma_v) = -f(x_0)coth\omega = 0$

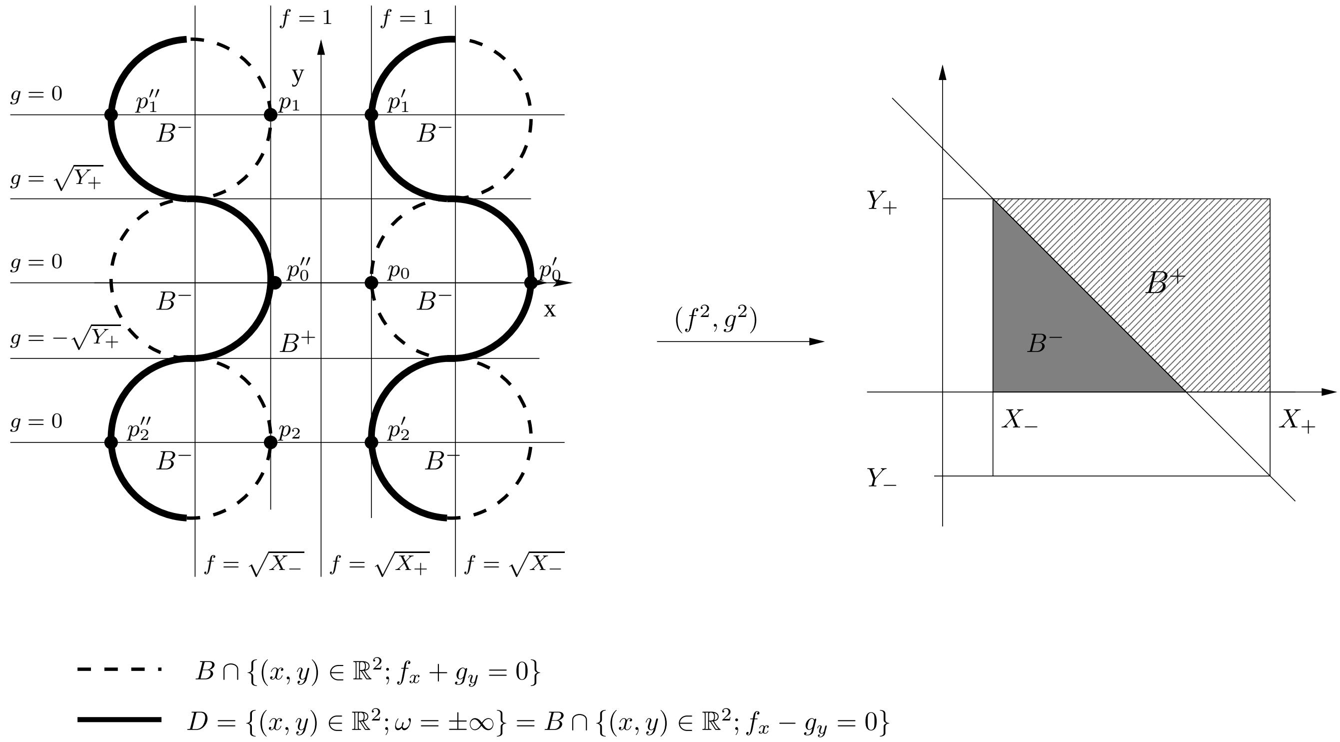


Figure 4: The helicoidal family of the region 2

but $|F_y|\neq 0$. Then the curve $X(x_0,y)$ projects on a geodesic of the disc which contains the center of the disc by assumption. The horizontal curves are geodesics crossing $X(x_0,y)$ in an orthogonal way (since $\langle F_x,F_y\rangle_g=0$). These surfaces are type of oblique planes in $\mathbb{H}^2\times\mathbb{R}$. When $c\to 0$ these planes converge to a vertical plane (a geodesic product with $\mathbb{R}$) and when $c\to-\infty$, f_x takes large values and these surfaces converge to horizontal section.

3) The catenoid family $c=0$. Now we classify the family of rotational type. The vertical curves $X(x_0,y)$ project on geodesics in the plane i.e.$f=0$ identically and $c=0$. In this case we have $-(g_y)^2=g^4-(1+d)g^2+d=(g^2-d)(g^2-1)$. It is a conjugate situation of the preceding cases exchanging the geometric interpretation between the horizontal and vertical path. Surfaces are well defined on a horizontal strip bounded by the set $D=\{(x,y)\in\mathbb{R}^2; y=b_0 \text{ and } y=b_1\}$ (with $g^2(b_0)=g^2(b_1)=1$).

3-a)The case $d>1$. The set D is two straight lines. A is a horizontal strip but the image is an annulus foliated infinitely and having the third coordinate y bounded in $\mathbb{H}^2\times\mathbb{R}$ (each level curve has a curvature strictly greater than one). The horizontal curves have curvature $k_g(\gamma_h)=g(y_0)>1$ constant and they are periodic in $x\in\mathbb{R}$. The curve γ_h are circles with curvatures greater than one. Theses surfaces are rotationally invariant catenoids described in [15]. They are bounded by two parallel horizontal circles at infinity in the cylinder model of $\mathbb{H}^2\times\mathbb{R}$.

3-b) The case $0<d\leq 1$. A is a strip, but the horizontal curve has a constant curvature less than the value one. Then the horizontal curves are not compact, they are equidistant curves. The third coordinate y is bounded. When $g^2(y_0)=c$, we have $\omega(y_0)=0$ and the tangent plane of A is vertical

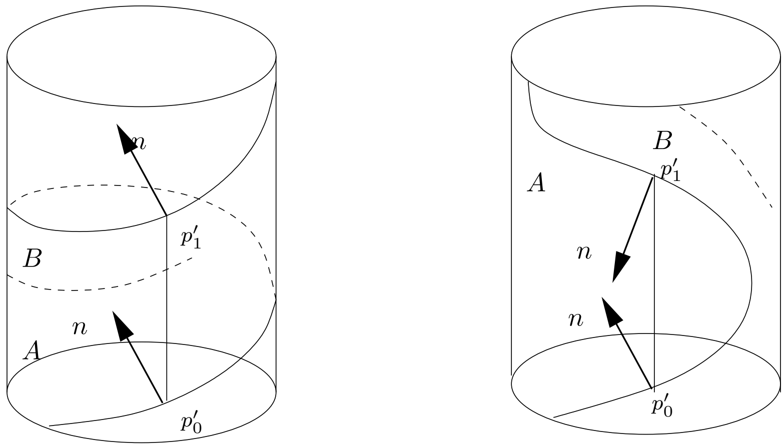


Figure 5: The ondulated helicoid at infinity

along this curve of curvature $k_g = \sqrt{c} < 1$. We can assume that $L = \mathbb{H}^2 \times \{0\}$ is a plane of symmetry. The equidistant curves are deforming and disappearing at infinity. But since $sh\omega = (1+g^2)^{-1} g_y$ has no change of sign above the plane of symmetry, the horizontal horocycles are contained in the non convex side of $(L \cap A) \times \mathbb{R}$. These surfaces converge to a geodesic vertical plane when $c \to 0$.

3-c) The case $d < 0$. In this case, $g \in [-1, 1]$. Then the horizontal level curves are not compact. There is y_0 (assume $y_0 = 0$ by vertical translation) such that $g(y_0) = 0$. The corresponding level curve is a geodesic passing by the center of the disc by hyperbolic translation in the cylinder. But on this curve ω is never zero and the tangent plane is never vertical to this curve. The surface is a global graph on $\mathbb{H}^2$ with third component y bounded and foliated by equidistant curves.

4) The annulus family. We consider the case where $c < 0$ and $d > 0$. We localize the behavior of f^2 and g^2 in the figure 2. The important fact is that $Y_- > 0$ ($\sqrt{\Delta} < 1 + d - c$). Then g^2 oscillates between Y_- and Y_+. The set B is represented by the straight line $X + Y = 1$ in the plane (X, Y) of figure 2. We notice the interesting property

$$X_- + Y_+ = 1 \text{ and } X_+ + Y_- = 1.$$

The function $f^2 \in [0, X_+]$. When $d - c \geq 1$ then $2X_- = 1 + c - d - \sqrt{\Delta} < 0$. When $d - c \leq 1$ we have $\sqrt{\Delta} > 1 + c - d$ which is $X_- < 0$. Then the set B^+ contains the horizontal strip $\{(x, y) \in \mathbb{R}^2; 1 < g^2\}$. On this strip, the horizontal curves have curvature greater than one and then they are infinite covering of circles. The strip covers the annulus with the period of the function f. The horizontal curve $g^2 = 1$ is a horocycle having one point at infinity, parametrized by one point of the set D. The set B^- is a countable set of disks (see figure 2), each of them tangents to two other ones. Since f is an oscillating function between $-\sqrt{X_+}$ and $\sqrt{X_+}$, f_x has alternatively positive and negative sign and one can see that $f_x + g_y = 0$ or $f_x - g_y = 0$ on the half boundary of each disk B^-. Then D is a set of disconnected curves homeomorphic to $\mathbb{R}$ that disconnect $\mathbb{R}^2$ in connected components homeomorphic to strips. The set D is represented in figure 2.

On each period, there is x_0 and x_1 with $f(x_0) = f(x_1) = 0$. The curve $X(x_0, y)$ has the same point at infinity with one of the horocycle ($g^2 = 1$), and intersects in an orthogonal way the other horocycle. Then $X(x_0, y)$ projects on a geodesic having the same points at infinity as the horocycles.

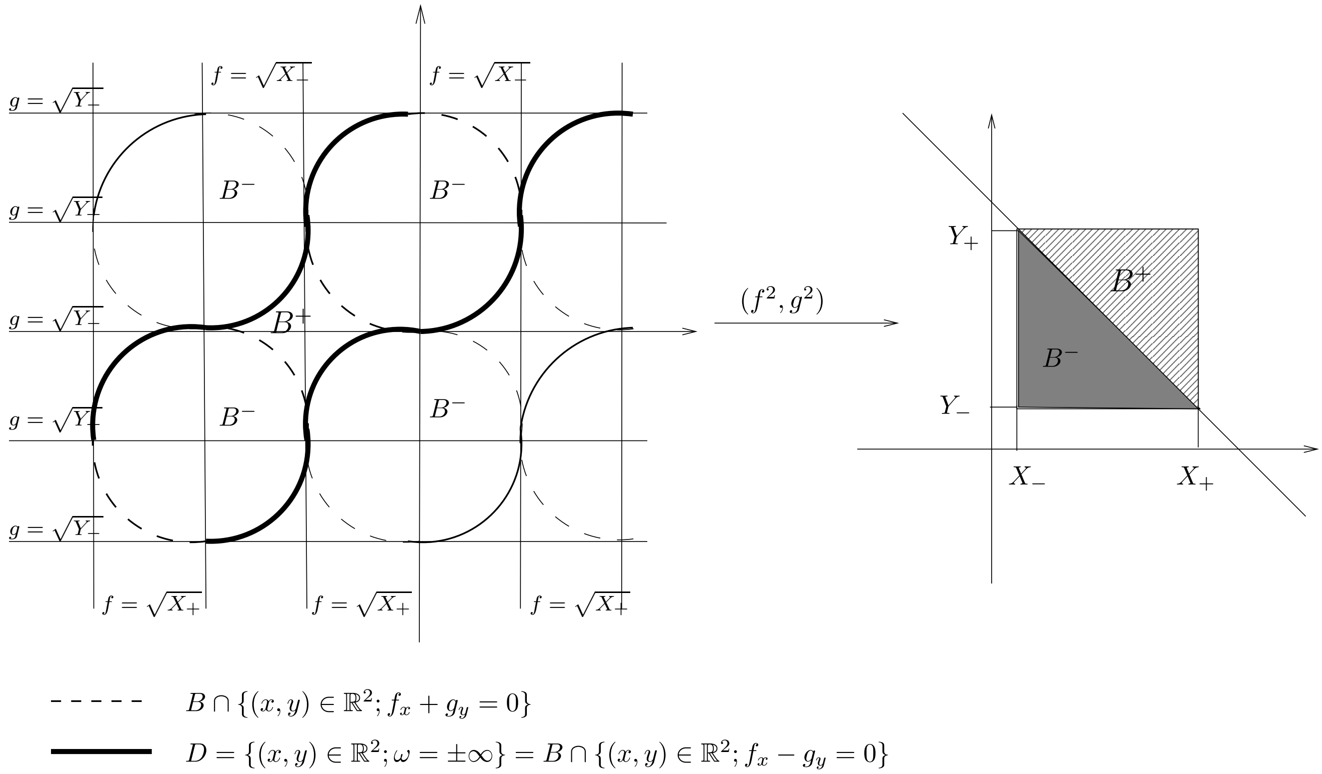


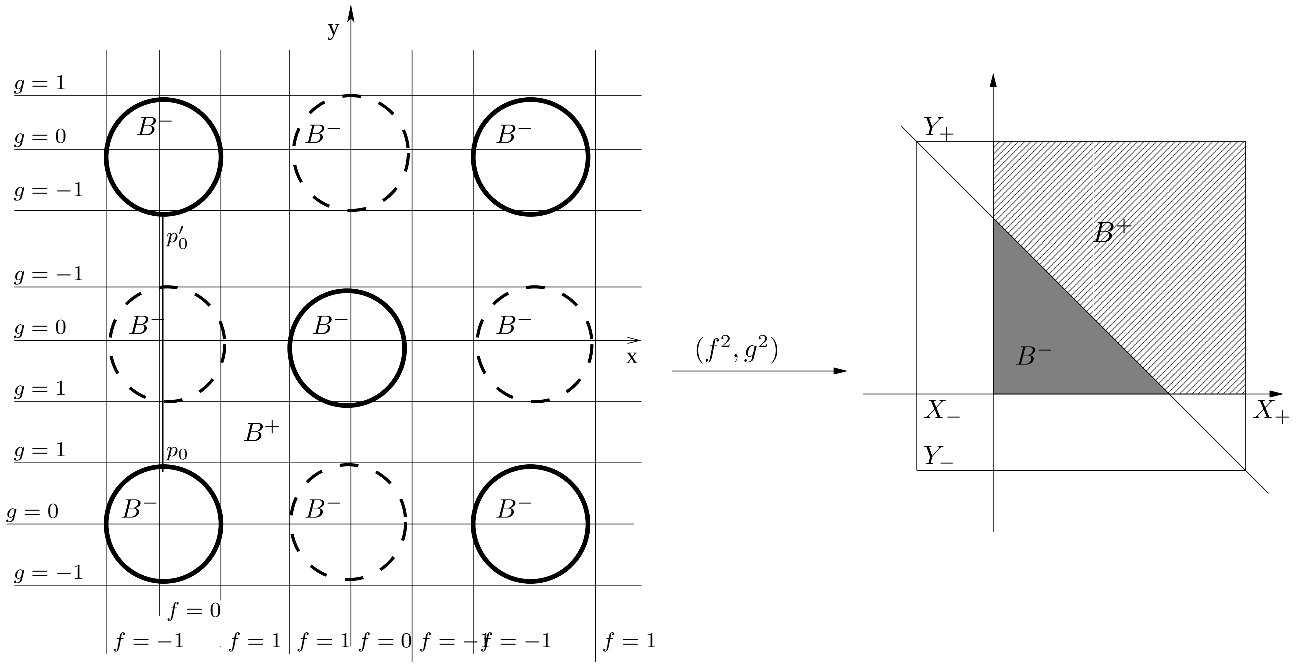
Figure 6: The helicoidal family of the region 3

We have to determine now if the end points of the curve $X(x_0, y)$ project on the same point or on the two end points of the geodesic. This behavior depends of the sign of ω which determines the vertical component of the Gauss map. If ω has no change of sign, the curve $X(x_0, y)$ is a graph on the geodesic and then $X(x_0, y)$ projects on two different points at infinity. If ω is positive and then negative, $X(x_0, y)$ projects on a half geodesic and projects on the same point at infinity (like a catenoid). When $X(x_0, y)$ have same points at infinity as one horocycle H_1, the curves $X(x_1, y)$ and H_1 are orthogonal at their intersection point. Then $X(x_0, y)$ and $X(x_1, y)$ project on the same geodesic γ. The vertical plane $\gamma \times \mathbb{R}$ is a plane of symmetry.

If $\omega = 0$, then $f_x + g_y = 0$. Recall

$$f_x^2 - g_y^2 = (g^2 + f^2 - 1)(g^2 - f^2 + c - d).$$

Since $f = 0$ on $X(x_0, y)$, we are looking points where $g^2 = d - c > 0$. Since $g^2 \in [Y_-, Y_+]$ with $Y_+ = \frac{1}{2}(1 + d - c + \sqrt{\Delta})$, we have $Y_+ - (d - c) = \frac{1}{2}(1 + c - d + \sqrt{\Delta}) = X_+ > 0$ and $Y_- - (d-c) = \frac{1}{2}(1+c-d-\sqrt{\Delta}) = X_- < 0$. Assume $d - c \neq 1$. We have two zeroes of $f_x^2 - g_y^2$ which are in B^- or B^+. Since g is oscillating, g_y changes sign at these two points while f_x is constant. Then at one of these points $f_x + g_y = 0$ and at the other $f_x - g_y = 0$. There is only one of these points where $\omega = 0$. By analysing the limit at infinity of $sh\omega = \dfrac{f_x + g_y}{f^2 + g^2 - 1}$ at the neighborhood of D we can see that ω changes sign. The analysis is similar in the case $d = c + 1$. This led us expect the behavior of an annulus as in figure 3, having two S^1 at infinity not homologuous to zero in the cylinder's boundary of $\mathbb{H}^2 \times \mathbb{R}$.



– – – – $B \cap \{(x,y) \in \mathbb{R}^2; f_x + g_y = 0\}$

$D = \{(x,y) \in \mathbb{R}^2; \omega = \pm\infty\} = B \cap \{(x,y) \in \mathbb{R}^2; f_x - g_y = 0\}$

Figure 7: The Riemann family

5) The ondulated Helicoidal family. We consider the case where $c > 0$ and $d < 0$. We localize the behavior of f^2 and g^2 in figure 4. The important fact is that $X_- > 0$ ($\sqrt{\Delta} < 1 + c - d$). Then f^2 oscillates between X_- and X_+. The set B is represented by the straight line $X + Y = 1$ in the plane (X, Y) of figure 4. We note the interesting property

$$X_- + Y_+ = 1 \text{ and } X_+ + Y_- = 1.$$

The function $g^2 \in [0, Y_+]$ in this case. When $c - d \geq 1$ then $2Y_- = 1 + d - c - \sqrt{\Delta} < 0$. When $c - d \leq 1$ we have $\sqrt{\Delta} > 1 + d - c$ which is $Y_- < 0$. Then the set B^+ contains the vertical strip $\{(x,y) \in \mathbb{R}^2; 1 \leq f^2 \leq Y_+\}$. Each horizontal curve has a curvature less than one (since $g^2 \leq Y_+ < 1$). The surface is simply connected. Curves $\{g = 0\}$ are geodesics having same points at infinity. To see that, we note that the vertical curve $\{f = 1\}$ projects on curves γ_v ending by p'_1 and p'_2, points at infinity in $\mathbb{H}^2$ of two geodesics. The curve γ_{v} has a curvature $|k_g(\gamma_{\mathrm{v}})| = |coth\omega| > 1$ and is crossing a geodesic in an orthogonal way at a point p_0. By the maximum principle γ_{v} is contained in the convex part of a horocycle passing this point p_0 in an orthogonal way to the geodesic. Then γ_{v}, the horocycle and the geodesic passing p_0 have the same point at infinity $p'_0 = p'_1 = p'_2$. The same holds for $p''_0 = p''_1 = p''_2$. This proves that surfaces have a vertical period. Geodesics are line of symmetry of the surface.

The horizontal curves between two geodesics have a curvature less than one. Then the surface has two connected components diffeomorphic to $\mathbb{R}$ at infinity. They are separating the cylinder $S^1 \times \mathbb{R}$ in two connected components A and B. We consider n, an Euclidean unit normal vector on these

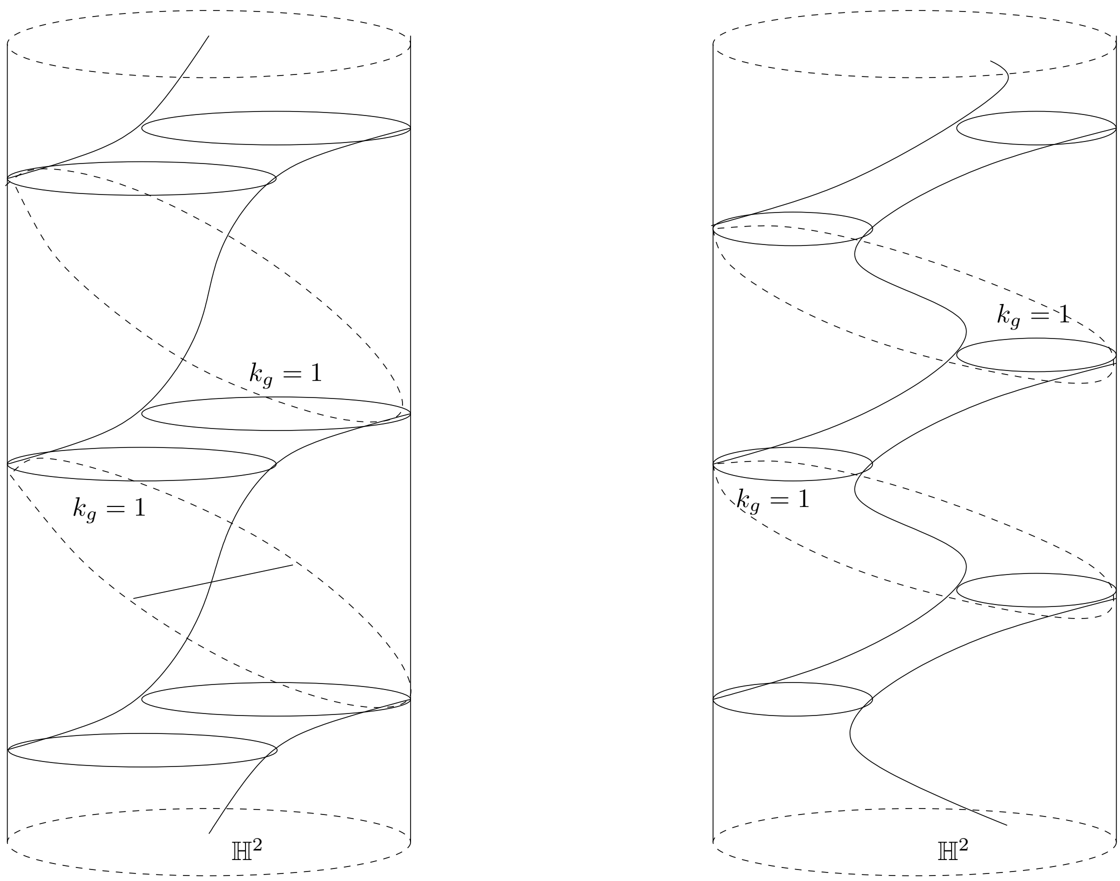


Figure 8: Two examples of type Riemann in $\mathbb{H}^2 \times \mathbb{R}$

curves pointing to one of the connected components (say A). If n is pointing up at p_1' and p_0' then the curve at infinity is spiraling, i.e. the projection of the curve on $S^1 \times \{0\}$ is not homologuous to zero. If n is pointing in the opposite direction at p_0' and p_1', the projection is homologuous to zero (see figure 5) and the surface will be homologuous to a vertical plane $\gamma \times \mathbb{R}$.

In $\mathbb{H}^2$, we have $\phi_y = \frac{sh\omega}{\sqrt{\rho}} e^{i\psi}$ and $|\phi_y|_\rho^2 = |sh\omega|^2 \to \infty$ when the curve is going to infinity. Then the tangent plane is becoming horizontal and the unit normal vector to the surface is pointing up or down. In fact, if (N_h, N_{v}) is the unit normal vector with N_h the horizontal component in $\mathbb{H}^2$ and $N_{\mathrm{v}} = th\omega$ the vertical component, the sign of ω will determine if N is pointing down or up.

By construction (the surface is embedded), we have $\langle N, n \rangle \geq 0$ at infinity, then the sign of the limit in D will tell us if we are spiraling or not on the cylinder.

One can see that $\omega = +\infty$ on one component and $-\infty$ in the other one, by analysing the limit of $sh\omega = \dfrac{f_x + g_y}{f^2 + g^2 - 1}$ at the neighborhood of each component of D. Then curves at infinity are spiraling and then we are describing an ondulated helicoid.

6) We consider the case where $c > 0$ and $d > 0$. We have

$$\sqrt{\Delta} > |1 + c - d| \text{ and } \sqrt{\Delta} > |1 + d - c|$$

then $X_- > 0$ and $Y_- > 0$. The set B^+ contains no strip in this case and we have two connected components homeomorphic to $\mathbb{R}$ in D (see figure 6). As in the preceding case we can see that $\omega = +\infty$

on one component and $\omega = -\infty$ on the other one. The curves are spiraling at infinity in a periodic way (ω is periodic and so is the argument of ϕ_x). The horizontal curves have curvature less than one at each level section but there is no horizontal geodesic in the surface. It is a blowed helicoid.

7) The Riemann family. We consider the case where $c < 0$ and $d < 0$. We localize the behavior of f^2 and g^2 in the figure 7. We have $X_- < 0$ ($\sqrt{\Delta} > |1 + c - d|$) and $Y_- < 0(\sqrt{\Delta} > |1 + d - c|)$. Then $f^2 \in [0, X_+]$ and $g^2 \in [0, Y_+]$. The set B is represented by $X + Y = 1$ and the inverse image of B, disconnect the plane in one non compact component and a countable set of disks. The set B^+ contains the vertical strip $\{(x, y) \in \mathbb{R}^2; 1 \leq f^2 \leq Y_+\}$ and horizontal strip $\{(x, y) \in \mathbb{R}^2; 1 \leq g^2 \leq X_+\}$. The sign of f_x and g_y gives the behavior of D (see figure 7) on the period. The horizontal strip $\{(x, y) \in \mathbb{R}^2; 1 \leq g^2 \leq X_+\}$ gives us an annulus bounded by two horocycles $k_g(\gamma_h) = g = \pm 1$ as in the annulus case.

Each vertical curve $\{f = 0\}$ has many connected components with end points on D. These curves project on geodesics in the horizontal section. We will prove that these curves project on only one geodesic γ i.e. they are contained in exactly one vertical flat plane ($\gamma \times \mathbb{R}$) of symmetry of the surface.

First we prove that each connected component of $\{f = 0\}$ projects on the whole geodesic γ. A first indication is the sign of $sh\omega = \dfrac{f_x + g_y}{f^2 + g^2 - 1}$ which does not change at p_0 and p_0' (see 7). The normal vector is pointing up (or down) at the two end points. Recall

$$f_x^2 - g_y^2 = (g^2 + f^2 - 1)(g^2 - f^2 + c - d).$$

If $g^2 \neq 1$ and $f = 0$, we have $f_x + g_y = 0$ iff $g^2 = d - c$. If $g^2 = 1$ and $f = 0$ we have $sh\omega = \dfrac{f_x + g_y}{g^2 - 1} = \dfrac{g^2 + c - d}{f_x - g_y} = 0$ iff $g^2 = d - c$. Then if $d - c < 0$, ω has constant sign on the vertical connected component of $\{f = 0\}$ which is a graph on the geodesic (see figure 8). If $c = d$, then $\omega = 0$ at one point $q_1 = \{f = 0\} \cap \{g = 0\}$ but ω has a constant sign on the curve. If $d - c > 0$, $g^2 = d - c < Y^+$ at two points and the sign of the vertical component of the normal changes twice (see figure 8). However ω has the same sign in the neighborhood of its end points. It is projecting in a non injective way on the whole geodesic γ.

□

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