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Higher genus Riemann minimal surfaces

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1 Introduction

B. Riemann [21] has constructed a one parameter family of non congruent singly periodic minimal surfaces which are foliated by circles (or straight lines). Each member of this family is a periodic embedded minimal surface in $\mathbb{R}^3$ with infinitely many parallel ends.

Even though the classification of genus zero, embedded minimal surfaces is not yet complete, W.H. Meeks, J. Perez and A. Ros [15–17] have made progress concerning the question of the uniqueness of the Riemann's examples in the class of genus zero embedded minimal surfaces that have an infinite number of ends. They conjecture in [16] that every embedded minimal surface of finite genus and with infinite number of ends is asymptotic (away from a compact piece) to some *middle planar ends* and to two halves of a Riemann's surface, which are referred to as the *limit ends*.

In this paper we construct such surfaces. More precisely, we have the:

Theorem 1.1. *Given* $k = 1, \dots, 37$*, there exists a one parameter family of properly embedded minimal surfaces of genus* k *with two limit ends asymptotic to halves of a Riemann's surface.*

We briefly describe the idea behind the proof since this will give further information about the surfaces constructed. In 1981, C. Costa [2,3] found a genus one, complete, properly embedded minimal surface with three ends. The top end and the bottom end of this surface are (up to a vertical translation) respectively asymptotic to the top end and bottom end of a catenoid of vertical axis while the third middle end is asymptotic to a horizontal plane situated between the two catenoidal ends. Later, D. Hoffman

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and W.H. Meeks [7,8] have generalized this example and found, for every genus $k \geq 2$, a complete, properly embedded minimal surface with two catenoidal ends and one planar end. This surface will be referred to as the "genus k Costa–Hoffman–Meeks surface".

Minimal surfaces belonging to Riemann's family, once they are normalized so that their planar ends are asymptotic to horizontal planes are distance 1 from each other, depend on a parameter (which is related to the value of the horizontal flux). As this parameter tends to 0, the members of this family can be understood as infinitely many horizontal planes at distance 1 from each other, such that two consecutive planes are connected together by slightly bent catenoids that have been scaled down by a small factor.

In our construction, the key idea is to replace one of these *slightly bent* catenoids by the genus k Costa–Hoffman–Meeks minimal surface. In doing so, we insert an additional planar end between two consecutive ends of the Riemman's surface we stated with. It turns out that this construction is successful provided one is able to bend the upper and lower end of the genus k Costa–Hoffman–Meeks surface, keeping the middle end asymptotic to the horizontal plane. The moduli space theory for minimal surfaces with catenoidal ends and a nondegeneracy result by S. Nayatani [18,19], allows us to prove that the bending of the ends of the genus k Costa–Hoffman–Meeks surface is possible precisely for any genus $1 \leq k \leq 37$.

The paper is organized as follows: In Sect. 2, we give a description of the Costa–Hoffman–Meeks minimal surfaces and we proceed with the deformation of the top and bottom ends of such surfaces. In Sect. 3, we describe an isothermal parametrization of a Riemann's surface, we also obtain an important expansion of pieces of Riemann's surfaces as the flux becomes vertical. Section 4 is devoted to the study of the mapping properties of the Jacobi operator about half of a Riemann's surface, as its flux becomes vertical. In Sect. 5, we apply the implicit function theorem to perturb a half of a Riemann's surface, we obtain an infinite dimensional family of minimal surfaces that have one boundary and that are asymptotic to the half of the Riemann's surface we started with. In Sect. 6, we perturb the Costa–Hoffman–Meeks surface using again the implicit function theorem, we again obtain an infinite dimensional family of minimal surfaces that have two boundaries and one horizontal end. In the last section, we explain how the boundary data of the minimal surfaces constructed in Sect. 5 and Sect. 6 can be chosen so that the union of these forms a smooth minimal surface with fixed genus $1 \leq k \leq 37$ and two limit ends.

2 The genus k Costa–Hoffman–Meeks minimal surface

C. Costa [2,3] and later on D. Hoffman and W.H. Meeks [7,8] have shown that, for each $k \geq 1$, there exists M_k a genus k, properly embedded, complete, minimal surface with three ends. After a suitable rotation and translation, M_k can be assumed to enjoy the following properties:

(i) The surface M_k has one planar end E_m asymptotic to the $x_3 = 0$ plane, one top end E_t asymptotic to the upper end of a catenoid with x_3-axis of revolution embedded in the half space $x_3 > 0$ and one bottom end E_b asymptotic to the lower end of a catenoid with x_3-axis of revolution, embedded in the half space $x_3 < 0$. In particular, the planar end E_m lies between the two catenoidal ends.

(ii) The surface M_k is invariant under the action of the rotation of angle $\frac{2\pi}{k+1}$ about the x_3-axis, it is also invariant under the action of the symmetry with respect to the $x_2 = 0$ plane. Finally, it is invariant under the action of the composition of the rotation of angle $\frac{\pi}{k+1}$ about the x_3-axis and the symmetry with respect to the $x_3 = 0$ plane.

(iii) The surface M_k intersects the $x_3 = 0$ plane in $k + 1$ straight lines, which intersect at equal angles $\frac{\pi}{k+1}$ at the origin. The intersection of M_k with the plane $x_3 = const\ (\neq 0)$ is a single Jordan curve. The intersection of M_k with the upper half space $x_3 > 0$ (resp. with the lower half space $x_3 < 0$) is topologically an open annulus.

The surface M_k will be referred to as the *genus k Costa–Hoffman–Meeks surface*. Observe that, when k is even the surface M_k is also invariant under the action of the rotation of angle π about the x_2-axis.

The main purpose of this section is to explain how the genus k Costa–Hoffman–Meeks surface M_k can be deformed into a smooth one parameter family of minimal surfaces $M_k(\xi)$, for $\xi \in (-\xi_0, \xi_0)$ and $\xi_0 > 0$ small enough, that are not embedded anymore, are invariant under the action of the symmetry with respect to the $x_2 = 0$ plane, have one horizontal end asymptotic to the $x_3 = 0$ plane and have two catenoidal type ends that are (up to some translation) respectively asymptotic to the upper end and the lower end of a catenoid whose axis of revolution is directed by $\sin\xi\, e_1 + \cos\xi\, e_3$. The construction of $M_k(\xi)$ will be a simple consequence of the moduli space theory as described in [11,20] or [9]. It also relies on a nondegeneracy assumption that is known to be true when $k \leq 37$, thanks to result of S. Nayatani [18,19].

We start by a local description of the surface M_k near its ends and in particular we describe coordinates that will be used to define some weighted spaces of functions on M_k.

The middle end of M_k. The planar end E_m of the surface M_k can be parameterized by

$$X_m : x \in \bar{B}_{2r_0} \longmapsto \left(\frac{x}{|x|^2}, u_m(x) \right) \in \mathbb{R}^3$$

where $\bar{B}_r \subset \mathbb{R}^2$ denotes the closed ball of radius $r > 0$ centered at the origin and where the function u_m tends to 0 as x tends to 0. This reflects the fact that the middle end of M_k is asymptotic to the horizontal plane.

Recall that, for surfaces parameterized by

$$x \longmapsto \left(\frac{x}{|x|^2}, u(x) \right) \in \mathbb{R}^3$$

the minimal surface equation reads

$$|x|^4 \operatorname{div} \left(\frac{\nabla u}{(1 + |x|^4 \, |\nabla u|^2)^{1/2}} \right) = 0. \tag{1}$$

The function u_m is (by definition) a solution of this equation and it turns out that u_m, which is *a priori* only defined in $B_{2r_0} - \{0\}$, can be extended continuously at the origin (and in fact as a $\mathcal{C}^{2,\alpha}$ function, using Weierstrass representation on finite total curvature ends as in the paper of J. Perez and A. Ros [20]). Observe that we have $u_m(x) = \mathcal{O}_{\mathcal{C}^{2,\alpha}_b}(|x|)$ near 0 since u_m tends to 0 at the origin. The expression $\mathcal{O}_{\mathcal{C}^{\ell,\alpha}_b}(f(x))$ denotes a function that, together with its partial derivatives of order less than or equal to $\ell + \alpha$ with respect to the vector fields $r\,\partial_r$ and ∂_θ, is bounded by a constant times f where r, θ are polar coordinates in the plane. However, given the fact that M_k is symmetric with respect to the rotation of vertical axis and angle $\frac{2\pi}{k+1}$, we claim that this estimate can be improved into

$$u_m(x) = \mathcal{O}_{\mathcal{C}^{2,\alpha}_b}(|x|^{k+1})$$

near 0. Indeed, since u_m solves (1) we have

$$\Delta u_m = \frac{1}{2} \, \frac{\nabla u_m \cdot \nabla\big(|x|^4 \, |\nabla u_m|^2\big)}{1 + |x|^4 \, |\nabla u_m|^2}. \tag{2}$$

It is then easy to check that u_m as a nice polyhomogeneous expansion (i.e. an asymptotic expansion in terms of $|x|^i \, (-\log|x|)^j$) the leading term in the expansion of u_m is necessarily a harmonic function. Given the symmetries of our surface, this harmonic function has to be invariant under the action of a rotation of angle $\frac{2\pi}{k+1}$, the action of the symmetry with respect to $x_2 = 0$ line and has to tend to 0 at the origin. Therefore, in polar coordinates, this leading term has to be collinear to the function $(r, \theta) \longrightarrow r^{j(k+1)} \cos(j\,(k+1)\,\theta)$, for some $j \geq 1$. This directly implies the claim. Furthermore, inserting this information into the right hand side of (2) we find that

$$|\Delta u_m| \leq c\,|x|^{3k+3}$$

so that u_m can be decomposed as

$$u_m(x) = \sum_{j=1}^{3} u_m^{(j)} \, r^{j(k+1)} \, \cos(j(k+1)\theta) + v_m(x)$$

where $v_m(x) = \mathcal{O}_{\mathcal{C}_b^\infty}(|x|^j)$ for all $j < 3k+5$. In particular this implies that

$$u_m \in \mathcal{C}^{3k+4,\alpha}(\bar{B}_{2r_0})$$

for all $\alpha \in (0, 1)$.

The top end of M_k. We now turn to the description of the top end of M_k (the description of the bottom end will follow at once using the invariance of the surface M_k by the symmetries that are described in (ii)). As already mentioned, the top end is asymptotic to a catenoid with vertical axis of revolution. We use

$$X_c(s, \theta) := (\cosh s\ \cos\theta, \cosh s\ \sin\theta, s) \in \mathbb{R}^3$$

as a parametrization of the *standard catenoid* C with x_3-axis of revolution. The unit normal vector field about C is chosen to be

$$n_c(s, \theta) := \frac{1}{\cosh s}\,(\cos\theta, \sin\theta, -\sinh s).$$

Up to some dilation, we can assume that the top end E_t of the surface M_k is asymptotic to some translated copy of the catenoid parameterized by X_c in the vertical direction. Therefore, E_t can be parameterized by

$$X_t := X_c + w_t\, n_c + \sigma_t\, e_3$$

for $(s, \theta) \in (s_0 - 1, \infty) \times S^1$, where the function w_t tends to 0 as s tends to ∞ (and in fact tends exponentially fast to 0 as s tends to ∞) and $\sigma_t \in \mathbb{R}$. Again, w_t tends to 0 as s tends to ∞, reflecting the fact that the end E_t is asymptotic to the standard catenoid translated by $\sigma_t\, e_3$.

We recall that the surface parameterized by $X := X_c + w\, n_c$ is minimal if and only if the function w satisfies the minimal surface equation which, for normal graphs over the standard catenoid, can be expanded in powers of w (and its partial derivatives) as

$$\begin{aligned}\frac{1}{\cosh^2 s}\Bigg(\left(\partial_s^2 + \partial_\theta^2 + \frac{2}{\cosh^2 s}\right) w + Q_2\left(\frac{w}{\cosh s}\right)\\ + \cosh s\, Q_3\left(\frac{w}{\cosh s}\right)\Bigg) = 0.\end{aligned} \tag{3}$$

Here Q_2 and Q_3 are nonlinear second order differential operators that satisfy

$$\begin{aligned}&\|Q_j(v_2) - Q_j(v_1)\|_{\mathcal{C}^{0,\alpha}([s,s+1]\times S^1)}\\ &\quad\le c\left(\sup_{i=1,2}\|v_i\|_{\mathcal{C}^{2,\alpha}([s,s+1]\times S^1)}\right)^{j-1}\|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}([s,s+1]\times S^1)}\end{aligned} \tag{4}$$

for all $s \in \mathbb{R}$ and all v_1, v_2 such that $\|v_i\|_{\mathcal{C}^{2,\alpha}([s,s+1]\times S^1)} \le 1$. The important fact is that, in this estimate, the constant $c > 0$ does not depend on s. The proof of this expansion can be easily adapted from the proof of the corresponding expansion for higher dimensional catenoids which is provided in [4], a complete proof is given in the Appendix A for the reader's convenience.

Since w_t is a solution of (3), we can write

$$\left(\partial_s^2 + \partial_\theta^2\right) w_t = -\frac{2}{\cosh^2 s}\, w_t - Q_2\left(\frac{w_t}{\cosh s}\right) - \cosh s\, Q_3\left(\frac{w_t}{\cosh s}\right).$$

Given the fact that M_k is invariant under the action of the rotation of vertical axis and angle $\frac{2\pi}{k+1}$ and also the symmetry with respect to the plane $x_2 = 0$, we claim that

$$w_t = \mathcal{O}_{\mathcal{C}^{2,\alpha}_b}(e^{-(k+1)s}),$$

where the expression $\mathcal{O}_{\mathcal{C}^{\ell,\alpha}_b}(f(s,\theta))$ denotes a function that, together with its partial derivatives of order less than or equal to $\ell + \alpha$ with respect to the vector fields ∂_s and ∂_θ, is bounded by a constant times f. Indeed, it is again easy to check that w_t as a nice polyhomogeneous expansion (i.e. an asymptotic expansion in terms of $e^{-js}\, t^i$) and the leading term in the expansion of w_t is harmonic (on the cylinder $\mathbb{R} \times S^1$) and invariant under the action of the rotation on S^1 by the angle $\frac{2\pi}{k+1}$ and the action of the symmetry $\theta \longmapsto -\theta$, hence it has to be proportional to the functions $(s,\theta) \longrightarrow e^{-j\,(k+1)\,s} \cos(j\,(k+1)\,\theta)$, for some $j \geq 1$. This completes the proof of the claim.

The bottom end of M_k. Similarly, we define X_b to parameterize the bottom end E_b of the surface M_k so that

$$X_b := X_c - w_b\, n_c - \sigma_b\, e_3$$

for $(s,\theta) \in (-\infty, 1 - s_0) \times S^1$, where the function w_b tends to 0 as s tends to $-\infty$ and $\sigma_b \in \mathbb{R}$. Again, w_b tends to 0 exponentially as s tends to $-\infty$, reflecting the fact that the end E_b is asymptotic to the standard catenoid translated by $-\sigma_b\, e_3$. Granted the symmetries of the surface M_k, there is an obvious relation between X_t and X_b. Indeed, starting from the parametrization of E_t which we compose by a rotation of angle $\frac{\pi}{k+1}$ about the x_3-axis and a symmetry with respect to the $x_3 = 0$ one finds a parametrization of E_b. This implies that $\sigma_b = \sigma_t$ and also that

$$w_b(s,\theta) = -w_t\left(-s, \theta - \frac{\pi}{k+1}\right).$$

Weighted spaces. For all $r < 2r_0$ and $s > s_0 - 1$, we define

$$M_k(s,r) := M_k - \big(X_t((s,\infty) \times S^1) \cup X_b((-\infty,-s) \times S^1) \cup X_m(B_r)\big). \tag{5}$$

The parametrizations of the three ends of M_k induce a decomposition of M_k into slightly overlapping components as follows: a compact piece $M_k(s_0+1, r_0/2)$ and three noncompact pieces $X_t((s_0,\infty)\times S^1)$, $X_b((-\infty,-s_0)\times S^1)$ and $X_m(\bar{B}_{r_0})$. We are now in a position to define weighted spaces of functions on M_k.

Definition 2.1. *Given $\ell\in\mathbb{N}$, $\alpha\in(0,1)$ and $\delta\in\mathbb{R}$, the space $\mathcal{C}^{\ell,\alpha}_\delta(M_k)$ is defined to be the space of functions in $\mathcal{C}^{\ell,\alpha}_{loc}(M_k)$ for which the following norm is finite*

$$\|w\|_{\mathcal{C}^{\ell,\alpha}_\delta(M_k)} := \|w\|_{\mathcal{C}^{\ell,\alpha}(M_k(s_0+1,r_0/2))} + \|w\circ X_m\|_{\mathcal{C}^{\ell,\alpha}(\bar{B}_{r_0})} + \|w\circ X_t\|_{\mathcal{C}^{\ell,\alpha}_\delta([s_0,\infty)\times S^1)} + \|w\circ X_b\|_{\mathcal{C}^{\ell,\alpha}_\delta((-\infty,-s_0]\times S^1)}$$

where

$$\|f\|_{\mathcal{C}^{\ell,\alpha}_\delta([s_0,\infty)\times S^1)} := \sup_{s\geq s_0}\left(e^{-\delta s}\,\|f\|_{\mathcal{C}^{\ell,\alpha}([s,s+1]\times S^1)}\right)$$
$$\|f\|_{\mathcal{C}^{\ell,\alpha}_\delta((-\infty,-s_0]\times S^1)} := \sup_{s\leq -s_0}\left(e^{\delta s}\,\|f\|_{\mathcal{C}^{\ell,\alpha}([s-1,s]\times S^1)}\right)$$

and that are invariant under the action of the symmetry with respect to the $x_2=0$ plane, i.e. $w(p)=w(\bar{p})$ for all $p\in M_k$, where $\bar{p}:=(x_1,-x_2,x_3)$ if $p=(x_1,x_2,x_3)$.

Observe that the weight parameter δ only enters in the definition of these spaces to describe the behavior of the functions at the top and bottom ends E_t and E_b but there is no weight parameter that describes the behavior of the functions at the middle end E_m. In fact, the parameterization of the middle end E_m by X_m amounts to compactify this end and the weighted spaces we consider are weighted spaces of functions defined on a 2 ended surface. Later on, when we will perturb the surface M_k using a normal graph for a function in $\mathcal{C}^{2,\alpha}_\delta(M_k)$ the middle end E_m will just be translated in the vertical direction since it will be parameterized by

$$x\longmapsto\left(\frac{x}{|x|^2},u(x)\right)$$

for some function $u\in\mathcal{C}^{2,\alpha}(\bar{B}_{r_0})$.

In the above parametrization of the end E_t (resp. E_b), the volume form $dvol_{M_k}$ can be written as

$$dvol_{M_k}=\gamma_t\,ds\,d\theta\quad(\text{resp. } dvol_{M_k}=\gamma_b\,ds\,d\theta)$$

and on the end E_m, the volume form can be written as

$$dvol_{M_k}=\gamma_m\,dx_1\,dx_2.$$

This allows one to define globally on M_k a smooth function

$$\gamma : M_k\longrightarrow[0,\infty)$$

that is identically equal to 1 on $M_k(s_0-1,2r_0)$ and equal to γ_t (resp. γ_b, γ_m) on the end E_t (resp. E_b, E_m). Observe that, on $X_t((s_0,\infty)\times S^1)$ and on

$X_b((-\infty, s_0) \times S^1)$ we have

$$\gamma \circ X_t(s,\theta) \sim \cosh^2 s \quad \text{and} \quad \gamma \circ X_b(s,\theta) \sim \cosh^2 s.$$

Finally on $X_m(B_{r_0})$, we have

$$\gamma \circ X_m(x) \sim |x|^{-4}.$$

The Jacobi operator about M_k. The Jacobi operator about M_k is defined by

$$\mathbb{L}_{M_k} := \Delta_{M_k} + |A_{M_k}|^2$$

where $|A_{M_k}|$ is the norm of the second fundamental form on M_k. Granted the above defined spaces, one can check that:

$$\begin{aligned} \mathcal{L}_\delta : \mathcal{C}^{2,\alpha}_\delta(M_k) &\longrightarrow \mathcal{C}^{0,\alpha}_\delta(M_k) \\ w &\longmapsto \gamma\, \mathbb{L}_{M_k}(w) \end{aligned}$$

is a bounded linear operator. The subscript δ is meant to keep track of the weighted space over which the Jacobi operator is acting. Observe that, the function γ is here to counterbalance the effect of the conformal factor $\frac{1}{\sqrt{|g_{M_k}|}}$ in the expression of the Laplacian in the coordinates we use to parameterize the ends of the surface M_k. This is precisely what is needed to have the operator defined from the space $\mathcal{C}^{2,\alpha}_\delta(M_k)$ into the target space $\mathcal{C}^{0,\alpha}_\delta(M_k)$.

To have a better grasp of what is going on, let us linearize the nonlinear Equation (3) at $w = 0$ we get the expression of the Jacobi operator about the standard catenoid

$$\mathbb{L}_C := \frac{1}{\cosh^2 s}\left(\partial_s^2 + \partial_\theta^2 + \frac{2}{\cosh^2 s}\right).$$

Observe that, the operator $\cosh^2 s\, \mathbb{L}_C$ maps the space $(\cosh s)^\delta\, \mathcal{C}^{2,\alpha}((s_0,\infty) \times S^1)$ into the space $(\cosh s)^\delta\, \mathcal{C}^{0,\alpha}((s_0,\infty) \times S^1)$.

Similarly, if we linearize the nonlinear Equation (1) at $u = 0$, we obtain the expression of the Jacobi operator about the plane

$$\mathbb{L}_{\mathbb{R}^2} := |x|^4\, \Delta.$$

Again, the operator $|x|^{-4}\, \mathbb{L}_{\mathbb{R}^2} = \Delta$ clearly maps the space $\mathcal{C}^{2,\alpha}(\bar{B}_{r_0})$ into the space $\mathcal{C}^{0,\alpha}(\bar{B}_{r_0})$. Now, the function γ plays, for the ends of the surface M_k, the role played by the function $\cosh^2 s$ for the ends of the standard catenoid and the role played by the function $|x|^{-4}$ for the plane. Since the Jacobi operator about M_k is asymptotic to $\mathbb{L}_{\mathbb{R}^2}$ at E_m and is asymptotic to $\mathbb{L}_C$ at E_t and E_b, we conclude that the operator $\mathcal{L}_\delta$ maps $\mathcal{C}^{2,\alpha}_\delta(M_k)$ into $\mathcal{C}^{0,\alpha}_\delta(M_k)$.

This being understood, we now recall the notion of nondegeneracy [11] that is classically used in this context:

Definition 2.2. *The surface M_k is said to be nondegenerate if $\mathcal{L}_\delta$ is injective for all $\delta < -1$.*

The mapping properties of the operator $\mathcal{L}_\delta$ depends crucially on the choice of δ. To see this, we define the space $L^2_\delta(M_k)$ as the space of functions $u \in L^2_{loc}(M_k)$ for which the following norm is finite

$$\|u\|_{L^2_\delta(M_k)} := \left(\int_{M_k} u^2 \, W^{-2\delta} \, \gamma^{-1} \, dvol_{M_k} \right)^{1/2}$$

where the smooth function

$$W : M_k \longrightarrow (0, \infty)$$

is defined so that $W \circ X_t(s, \theta) = W \circ X_b(s, \theta) = \cosh s$, for $s \geq s_0$ and $W \circ X_m(x) \equiv 1$, for $x \in \bar{B}_{r_0}$.

We define the unbounded operator

$$\begin{aligned} A_\delta : L^2_\delta(M_k) &\longrightarrow L^2_\delta(M_k) \\ u &\longmapsto \gamma \, \mathbb{L}_{M_k} \, u. \end{aligned}$$

This operator has closed graph and dense domain

$$\mathrm{Dom}\, A_\delta = \left\{ u \in L^2_\delta(M_k) : \gamma \, \mathbb{L}_{M_k} u \in L^2_\delta(M_k) \right\}.$$

The indicial roots of the operator $\gamma \, \mathbb{L}_{M_k}$ at the ends E_t (resp. E_b) are the real numbers δ for which there exists a function $\theta \longmapsto v(\theta)$ such that

$$\gamma \, \mathbb{L}_{M_k} \, (e^{\delta s} \, v(\theta)) = \mathcal{O}_{L^\infty}(e^{\delta' s})$$

on E_t (resp. E_b) for some $\delta' < \delta$. The ends E_t and E_b being asymptotic to the ends of a catenoid, the set of indicial roots of $\gamma \, \mathbb{L}_{M_k}$ at the ends E_t and E_b is equal to the set of indicial roots of the operator $\cosh^2 s \, \mathbb{L}_C$. This later is easily seen to be equal to $\mathbb{Z}$ since

$$\gamma \, \mathbb{L}_C \, (e^{\pm j s} \, \cos(j \, \theta)) = \mathcal{O}_{L^\infty}(e^{(\pm j - 2) s})$$

at $+\infty$.

It follows from the general theory [14] that A_δ has closed range and is Fredholm provided δ is not an indicial root, that is $\delta \notin \mathbb{Z}$.

We use the scalar product

$$\langle u, v \rangle := \int_{M_k} u \, v \, \gamma^{-1} \, dvol_{M_k}$$

to identify $(L^2_\delta(M_k))^*$ with $L^2_{-\delta}(M_k)$. With this identification in mind, it is possible to identify A^*_δ, the adjoint of A_δ, with $A_{-\delta}$. It then follows from the theory of unbounded operators with closed range that [1]

$$(A_\delta \text{ is injective}) \quad \Leftrightarrow \quad (A_{-\delta} \text{ is surjective})$$

provided $\delta \notin \mathbb{Z}$. Next, elliptic regularity theory [12] together with a relative index formula, implies the:

Proposition 2.1. *Assume that M_k is nondegenerate and $\delta \in (1, 2)$. Then the operator $\mathcal{L}_\delta$ is surjective. Moreover the kernel of $\mathcal{L}_\delta$ is* 4*-dimensional.*

One should keep in mind that, in the definition of the weighted spaces, we have imposed the invariance under some symmetry and that, in addition, the definition of the function spaces we have implies that we are implicitly working on a 2 ended surface, not a 3 ended surface! This is reflected by the fact that the weight δ only affects the behavior of a function in $\mathcal{C}^{\ell,\alpha}_\delta(M_k)$ or $L^2_\delta(M_k)$ at the top and bottom ends but does not play any role at the middle end. This explains why, in the above Proposition, the dimension of the kernel is only equal to 4 and not equal to $9 (= 3\times$the number of ends) as is usually the case when one is working with a 3 ended surface and when no symmetries are imposed. Moreover, if s denote a symmetry and a function $f = f \circ s$, then the solution of $\mathcal{L} u = f$ satisfy $u \circ s = u$.

If we were working with a 3 ended surface (namely without compactification of the middle end), then when performing a deformation of the surface M_k, this would imply geometrically that we are ready to rotate, translate and dilate any of the 3 ends of the surface M_k and also, we are ready to change the type of the middle end from a planar type end into a catenoidal type end (the planar end being understood as a special type of catenoidal end with 0 vertical flux). By contrast, since we have compactified the middle end, when performing a deformation of the surface M_k, this means geometrically that we are ready to rotate, translate and dilate the top and bottom ends of the surface M_k but we do not want to rotate the middle end nor do we want to change its type from a planar type end into a catenoidal type end.

Jacobi fields. Recall that a smooth one parameter group of isometries (or dilations) containing the identity generates a Jacobi field i.e. a solution of the homogeneous problem $\mathbb{L}_{M_k} w = 0$ by taking the scalar product between the vector field generating the group of isometries (or dilations) and the normal vector field to the surface M_k. Using this procedure, we now define four of these Jacobi fields and we also provide their asymptotic expansion at the ends of M_k. Let n denote a unit normal vector field on M_k (for example, we agree that the orientation is chosen so that $n \sim e_3$ at E_m).

The group of vertical translations generated by the Killing vector field $\Xi \equiv e_3$ gives rise to the Jacobi field

$$\Phi^{0,+}(p) := n(p) \cdot e_3.$$

The fact that the ends E_t and E_b of M_k are normal graphs over the end of a catenoid for some function that decays exponentially, implies readily that the following expansions hold

$$\begin{aligned} \Phi^{0,+}(X_t(s,\theta)) &= -\tanh s + \mathcal{O}_{\mathcal{C}^{2,\alpha}_b}((\cosh s)^{-k-2}) \quad \text{on } E_t \\ \Phi^{0,+}(X_b(s,\theta)) &= \tanh s + \mathcal{O}_{\mathcal{C}^{2,\alpha}_b}((\cosh s)^{-k-2}) \quad \text{on } E_b \end{aligned} \tag{6}$$

while $\Phi^{0,+}(X_m(x)) = 1 + \mathcal{O}_{\mathcal{C}^{2,\alpha}_b}(|x|^{2k+4})$ on E_m.

The vector field $\Xi(p) = p$ that is associated to the one parameter group of dilation generates a Jacobi field

$$\Phi^{0,-}(p) := n(p) \cdot p.$$

The asymptotic expansion of this Jacobi field is given by

$$\begin{aligned} \Phi^{0,-}(X_t(s,\theta)) &= 1 - s\ \tanh s + \mathcal{O}_{\mathcal{C}_b^{2,\alpha}}((\cosh s)^{-k-1}) \quad \text{on } E_t \\ \Phi^{0,-}(X_b(s,\theta)) &= s\ \tanh s - 1 + \mathcal{O}_{\mathcal{C}_b^{2,\alpha}}((\cosh s)^{-k-1}) \quad \text{on } E_b \end{aligned} \tag{7}$$

while $\Phi^{0,-}(X_m(x)) = \mathcal{O}_{\mathcal{C}_b^{2,\alpha}}(|x|^{k+1})$ on E_m.

The Killing vector field $\Xi \equiv e_1$ that generates the group of translations along the x_1-axis is associated to the Jacobi field

$$\Phi^{1,+}(p) := n(p) \cdot e_1.$$

We have the asymptotic expansions of this Jacobi field at the ends that are given by

$$\begin{aligned} \Phi^{1,+}(X_t(s,\theta)) &= \frac{1}{\cosh s} \cos\theta + \mathcal{O}_{\mathcal{C}_b^{2,\alpha}}((\cosh s)^{-k-2}) \quad \text{on } E_t \\ \Phi^{1,+}(X_b(s,\theta)) &= -\frac{1}{\cosh s} \cos\theta + \mathcal{O}_{\mathcal{C}_b^{2,\alpha}}((\cosh s)^{-k-2}) \quad \text{on } E_b \end{aligned} \tag{8}$$

while $\Phi^{1,+}(X_m(x)) = \mathcal{O}_{\mathcal{C}_b^{2,\alpha}}(|x|^{k+2})$ on E_m.

Finally, we denote by

$$\Phi^{1,-}(p) := n(p) \cdot (e_2 \times p)$$

the Jacobi field associated to the Killing vector field $\Xi(p) = e_2 \times p$ that generates the group of rotations about the x_2-axis. Observe that

$$\begin{aligned} \Phi^{1,-}(X_t(s,\theta)) &= \left(\frac{s}{\cosh s} + \sinh s\right) \cos\theta + \mathcal{O}_{\mathcal{C}_b^{2,\alpha}}((\cosh s)^{-k-1}) \ \text{ on } E_t \\ \Phi^{1,-}(X_m(s,\theta)) &= -\left(\frac{s}{\cosh s} + \sinh s\right) \cos\theta + \mathcal{O}_{\mathcal{C}_b^{2,\alpha}}((\cosh s)^{-k-1}) \ \text{ on } E_b \end{aligned} \tag{9}$$

while $\Phi^{1,-}(X_m(x)) = \frac{x_1}{|x|^2} + \mathcal{O}_{\mathcal{C}_b^{2,\alpha}}(|x|^{2k+3})$ on E_m.

All these globally defined Jacobi fields are invariant under the action of the symmetry with respect to the $x_2 = 0$ plane. There are in addition three other Jacobi fields that are not invariant under this symmetry, namely the Jacobi field associated to the group of translation along the x_2-axis, the Jacobi field corresponding to the one parameter group of rotations about the x_1-axis and the Jacobi field corresponding to the one parameter group of rotations about the x_3-axis.

With these notations, we define the deficiency space

$$\mathcal{D} := \mathrm{Span}\{\chi_t\, \Phi^{j,\pm},\, \chi_b\, \Phi^{j,\pm} : j = 0, 1\}$$

where χ_t is a cutoff function that is identically equal to 1 on $X_t((s_0+1, \infty) \times S^1)$, identically equal to 0 on $M_k - X_t((s_0, \infty) \times S^1)$ and that is invariant under the action of the symmetry with respect to the $x_2 = 0$ plane. Also, we agree that

$$\chi_b(\cdot) := \chi_t(-\,\cdot).$$

Clearly

$$\begin{aligned} \tilde{\mathcal{L}}_\delta : \mathcal{C}^{2,\alpha}_\delta(M_k) \oplus \mathcal{D} &\longrightarrow \mathcal{C}^{0,\alpha}_\delta(M_k) \\ w &\longmapsto \gamma\, \mathbb{L}_{M_k}(w) \end{aligned}$$

is a bounded linear operator.

The linear decomposition Lemma proved in [11] for constant mean curvature surfaces (see also [9] for minimal hypersurfaces) can be adapted to our situation and we get the:

Proposition 2.2. *Assume that M_k is nondegenerate and that $\delta \in (-2, -1)$. Then the operator $\tilde{\mathcal{L}}_\delta$ is surjective and has a kernel of dimension* 4.

Moduli space theory. We are interested in $\mathcal{M}$, the space of all minimal surfaces (not necessarily embedded) that are close to M_k, have 2 catenoidal ends, one horizontal planar end and that are invariant under the action of the symmetry with respect to the $x_2 = 0$ plane. When the catenoidal ends have no vertical axis, the surface is not embedded. The moduli space theory developed in [11] for constant mean curvature surfaces or in [9] for minimal hypersurfaces can be adapted to our framework and as a corollary of Proposition 2.2, we conclude that, close to M_k, the space $\mathcal{M}$ is a smooth manifold of dimension 4, provided M_k is nondegenerate. In addition, the elements of the kernel of $\tilde{\mathcal{L}}_\delta$ span the tangent space to $\mathcal{M}$. Therefore, in order to understand the space $\mathcal{M}$ in a neighborhood of M_k, we just need to understand the elements that span the kernel of $\tilde{\mathcal{L}}_\delta$ and this will provide a natural set of parameters that describe $\mathcal{M}$ in a neighborhood of M_k.

From now on, we assume that $\delta \in (-2, -1)$ and that M_k is nondegenerate. It should be clear that the functions $\Phi^{0,\pm}$ and $\Phi^{1,+}$ belong to $\mathcal{C}^{2,\alpha}_\delta(M_k) \oplus \mathcal{D}$ and hence we already know explicitly 3 linearly independent elements of the kernel of $\tilde{\mathcal{L}}_\delta$. However, $\Phi^{1,-}$ fails to belong to the kernel of $\tilde{\mathcal{L}}_\delta$ since this function is not bounded on E_m (and in fact blows up like $|x|^{-1}$ as x tends to 0). Thus, we are left to understand the behavior of the fourth Jacobi field that belongs $\mathcal{C}^{2,\alpha}_\delta(M_k) \oplus \mathcal{D}$, this is the contain of the:

Lemma 2.1. *Assume that all bounded Jacobi fields on M_k that are invariant with respect to the $x_2 = 0$ plane are linear combination of $\Phi^{0,\pm}$ and $\Phi^{1,+}$. We fix $\delta \in (-2, -1)$. Then there exists $\Phi \in \mathcal{C}^{2,\alpha}_\delta(M_k) \oplus \mathcal{D}$ that belongs to* *the kernel of* $\tilde{\mathcal{L}}_\delta$ *but does not belong to* $\mathrm{Span}\{\Phi^{0,\pm}, \Phi^{1,+}\}$ *and that can be expanded as*

$$\Phi(X_t(s,\theta)) = \Phi^{1,-}(X_t(s,\theta)) + c_t\, \Phi^{1,+}(X_t(s,\theta)) + \mathcal{O}_{\mathcal{C}^{2,\alpha}_b}((\cosh s)^\delta)$$

on E_t *and*

$$\Phi(X_b(s,\theta)) = \Phi^{1,-}(X_b(s,\theta)) + c_b\, \Phi^{1,+}(X_b(s,\theta)) + \mathcal{O}_{\mathcal{C}^{2,\alpha}_b}((\cosh s)^\delta)$$

on E_b.

Proof. Under the assumption of the result, the surface M_k is nondegenerate. Dimension count shows that there exists $\Phi \in \mathcal{C}^{2,\alpha}_\delta(M_k) \oplus \mathcal{D}$ that belongs to the kernel of $\tilde{\mathcal{L}}_\delta$ but does not belong to $\mathrm{Span}\{\Phi^{0,\pm}, \Phi^{1,+}\}$. Without loss of generality (i.e. taking suitable linear combination of Φ with $\Phi^{0,\pm}$ and $\Phi^{1,+}$) we can assume that the expansion of Φ at E_t is given by

$$\Phi(X_t(s,\theta)) = a_t\, \Phi^{1,-}(X_t(s,\theta)) + \mathcal{O}_{\mathcal{C}^{2,\alpha}_b}((\cosh s)^\delta)$$

for some $a_t \in \mathbb{R}$ and that the expansion of Φ at E_b is given by

$$\begin{aligned}\Phi(X_b(s,\theta)) = a_b\, \Phi^{1,-}(X_b(s,\theta)) + b_b\, \Phi^{1,+} + c_b\, \Phi^{0,+}(X_b(s,\theta))\\ + d_b\, \Phi^{0,-}(X_b(s,\theta)) + \mathcal{O}_{\mathcal{C}^{2,\alpha}_b}((\cosh s)^\delta)\end{aligned}$$

for $a_b, b_b, c_b, d_b \in \mathbb{R}$.

Given a function Ψ defined on M_k, we set

$$\Omega(\Psi) := \lim_{s\to\infty} \lim_{r\to 0} \int_{M_k(s,r)} (\Phi\, \mathbb{L}_{M_k} \Psi - \Psi\, \mathbb{L}_{M_k} \Phi)\, dvol_{M_k}$$

where we recall that $M_k(s,r)$ has been defined in (5).

Since $\mathbb{L}_{M_k}\Phi = 0$ and $\mathbb{L}_{M_k}\, \Phi^{0,+} = 0$, we can use the divergence theorem together with the expansions (6)–(7) to get

$$0 = \Omega(\Phi^{0,+}) = 2\pi\, d_b.$$

Similarly, using the fact that $\mathbb{L}_{M_k}\, \Phi^{0,-} = 0$ together with the expansions (6)–(7), we get

$$0 = \Omega(\Phi^{0,-}) = -2\pi\, c_b.$$

Finally, using the fact that $\mathbb{L}_{M_k}\, \Phi^{1,+} = 0$ together with the expansions (8)–(9), we have

$$0 = \Omega(\Phi^{1,+}) = 2\pi\, (a_b - a_t).$$

Therefore, we conclude that $c_b = d_b = 0$ and also that $a_b = a_t$. Now, if we had $a_t = 0$, then we would also have $a_b = 0$ and hence we would conclude that Φ is a bounded Jacobi field that decays exponentially at E_t.

By assumptions a bounded Jacobi field (that shares the invariance with respect to the $x_2 = 0$ plane) is a linear combination of $\Phi^{0,\pm}$ and $\Phi^{1,+}$ and hence does not decay at E_t unless it is identically equal to 0. This is clearly a contradiction since we have assumed that $\Phi \neq 0$. Therefore, we conclude that $a_t \neq 0$. In other words, there exists an element of the kernel of $\tilde{\mathcal{L}}_\delta$ which is asymptotic at E_t (and in fact also at E_b) to the Jacobi field associated to the rotation of the catenoidal ends of M_k. □

Applying the implicit function theorem as in [11] (see also [9]), we see that this fourth Jacobi field is integrable. This yields the:

Theorem 2.1. *Assume that all bounded Jacobi fields on M_k that are invariant with respect to the $x_2 = 0$ plane are linear combination of $\Phi^{0,\pm}$ and $\Phi^{1,+}$. Then, there exists $\xi_0 > 0$ and a smooth one parameter family of minimal surfaces $(M_k(\xi))_\xi$ in $\mathcal{M}$, for $\xi \in (-\xi_0, \xi_0)$, such that $M_k(0) = M_k$ and the upper (resp. lower) catenoidal end of $M_k(\xi)$ is, up to a translation along its axis, asymptotic to the upper (resp. lower) end of the* standard *catenoid whose axis of revolution is directed by* $\sin\xi\, e_1 + \cos\xi\, e_3$.

If R_ξ denotes the rotation of angle ξ about the x_2-axis, the upper end $E_t(\xi)$ of $M_k(\xi)$ can be parameterized by

$$X_{t,\xi} = R_\xi\,(X_c + w_{t,\xi}\, N_c) + \sigma_{t,\xi}\, e_3 + \varsigma_\xi\, e_1 \tag{10}$$

and the lower end $E_b(\xi)$ of $M_k(\xi)$ can then be parameterized by

$$X_{b,\xi} = R_\xi\,(X_c - w_{b,\xi}\, N_c) - \sigma_{b,\xi}\, e_3 - \varsigma_\xi\, e_1 \tag{11}$$

where

$$\begin{aligned} &|\sigma_{t,\xi} - \sigma_t| + |\sigma_{b,\xi} - \sigma_b| + |\varsigma_\xi| + \|w_{t,\xi} - w_t\|_{\mathcal{C}^{2,\alpha}_{-2}([s_0,+\infty)\times S^1)} \\ &+ \|w_{b,\xi} - w_b\|_{\mathcal{C}^{2,\alpha}_{-2}((-\infty,-s_0]\times S^1)} \leq c\,|\xi| \end{aligned}$$

for $|\xi| \leq \xi_0$.

Proof. The application of the implicit function theorem, as in [11] implies that the Jacobi field Φ in Lemma 2.1 is integrable and hence yields the existence of a one parameter family of minimal surfaces $(M_k(\xi))_\xi$, for $\xi \in (-\xi_0, \xi_0)$, such that $M_k(0) = M_k$. Moreover $M_k(\xi)$ has a horizontal planar end and two catenoidal type ends. The upper (resp. lower) catenoidal end of $M_k(\xi)$ is asymptotic to the upper (resp. lower) end of a catenoid whose axis of revolution is directed by $\sin\xi\, e_1 + \cos\xi\, e_3$.

Observe that $M_k(\xi)$ is well defined up to a translation in the $x_2 = 0$ plane and up to a dilation. Therefore, we can require that the upper end of $M_k(\xi)$ is asymptotic to a translated and rotated version of the *standard* catenoid and also require that the middle end $E_m(\xi)$ is asymptotic to the $x_3 = 0$ plane. Then application of the flux formula [10] shows that the lower end of $M_k(\xi)$ is, up to a translation, asymptotic to the lower end of the same standard catenoid.

If R_ξ denotes the rotation of angle ξ about the x_2-axis, the upper end $E_t(\xi)$ of $M_k(\xi)$ can be parameterized by

$$X_{t,\xi} = R_\xi\,(X_c + w_{t,\xi}\, n_c) + \sigma_{t,\xi}\, e_3 + \varsigma_{t,\xi}\, e_1$$

where the function $w_{t,\xi}$ and $\sigma_{t,\xi},\ \varsigma_{t,\xi} \in \mathbb{R}$ depend smoothly on ξ and satisfy $w_{t,0} = w_t$, $\sigma_{t,0} = \sigma_t$ and $\varsigma_{t,0} = 0$. More precisely, if follows from the application of the implicit function theorem that

$$|\sigma_{t,\xi} - \sigma_t| + |\varsigma_{t,\xi}| + \|w_{t,\xi} - w_t\|_{\mathcal{C}^{2,\alpha}_{-2}([s_0,+\infty)\times S^1)} \leq c\,|\xi|$$

for $|\xi| \leq \xi_0$. The lower end $E_b(\xi)$ of $M_k(\xi)$ can then be parameterized by

$$X_{b,\xi} = R_\xi\,(X_c - w_{b,\xi}\, n_c) - \sigma_{b,\xi}\, e_3 - \varsigma_{b,\xi}\, e_1$$

where the function $w_{b,\xi}$ and $\sigma_{b,\xi},\ \varsigma_{b,\xi} \in \mathbb{R}$ depend smoothly on ξ and satisfy $w_{b,0} = w_b$, $\sigma_{b,0} = \sigma_b$ and $\varsigma_{b,0} = 0$. Again, we also have

$$|\sigma_{b,\xi} - \sigma_b| + |\varsigma_{b,\xi}| + \|w_{b,\xi} - w_b\|_{\mathcal{C}^{2,\alpha}_{-2}((-\infty,-s_0]\times S^1)} \leq c\,|\xi|$$

for $|\xi| \leq \xi_0$. Finally, up to now the surface $M_k(\xi)$ is defined up to a translation along the x_1-axis but we can help allay this confusion by requiring that

$$\varsigma_{b,\xi} = \varsigma_{t,\xi}.$$

Indeed, translation of the surface along e_1 by υ changes $\varsigma_{b,\xi}$ and $\varsigma_{t,\xi}$ into $\varsigma_{b,\xi} + \upsilon$ and $\varsigma_{t,\xi} - \upsilon$. So it is enough to choose

$$\upsilon = \frac{1}{2}\,(\varsigma_{b,\xi} - \varsigma_{t,\xi}).$$

We set $\varsigma_\xi := \varsigma_{b,\xi} = \varsigma_{t,\xi}$. This completes the proof of the result. □

Observe that, when k is even, the surface M_k is symmetric with respect to the rotation of angle π about the x_2-axis and one can prove, even if we will not need this information, that the surfaces $M_k(\xi)$ can be defined so that it is also invariant under the same symmetry.

On each $M_k(\xi)$ one can define weighted spaces as in Definition 2.1 and also define the corresponding notion of nondegeneracy. The Jacobi operator about $M_k(\xi)$ will be denoted by $\mathbb{L}_{M_k(\xi)}$. We define

$$\begin{aligned} \mathcal{L}_{\xi,\delta} : \mathcal{C}^{2,\alpha}_\delta(M_k(\xi)) &\longrightarrow \mathcal{C}^{0,\alpha}_\delta(M_k(\xi)) \\ w &\longmapsto \gamma_\xi\, \mathbb{L}_{M_k(\xi)}\,(w) \end{aligned}$$

where γ_ξ is the natural extension of the function γ to $M_k(\xi)$. It is easy to check, reducing ξ_0 if this is necessary, that all the surfaces $M_k(\xi)$ are nondegenerate and hence we have the:

Proposition 2.3. *Assume that all bounded Jacobi fields on M_k that are invariant with respect to the $x_2 = 0$ plane are linear combination of $\Phi^{0,\pm}$ and $\Phi^{1,+}$ and choose $\delta \in (1, 2)$. Then (reducing ξ_0 if this is necessary) the operator $\mathcal{L}_{\xi,\delta}$ is surjective and has a kernel of dimension 4. Moreover, there exists $G_{\xi,\delta}$ a right inverse for $\mathcal{L}_{\xi,\delta}$ that depends smoothly on ξ and in particular whose norm is bounded uniformly as $|\xi| < \xi_0$.*

For example, a right inverse $G_{\xi,\delta}$ that depends smoothly on ξ can be obtained by a simple perturbation argument starting from G_δ a right inverse for $\mathcal{L}_\delta$, and reducing ξ_0 if this is necessary.

The purpose of the next Lemma is to write a portion of the upper and lower ends of the surface $M_k(\xi)$ as vertical graphs over the horizontal plane. It is clear that the ends E_t and E_b of M_k can be written, at least away from a compact set, as vertical graphs over the horizontal plane $x_3 = 0$. This fact is not true anymore for the ends $E_t(\xi)$ and $E_b(\xi)$ of $M_k(\xi)$, when $\xi \neq 0$. Nevertheless this property will remain true for the piece of $M_k(\xi)$ we will be interested in. To be more specific, we fix $\varepsilon > 0$ small enough and consider the surface $M_k(\xi)$ for

$$\xi = \frac{1}{2}\,\varepsilon.$$

On the top and bottom ends of $M_k(\varepsilon/2)$, the annulus we are interested in corresponds to the parameter $s \sim s_\varepsilon$ in the parametrization given in (10) and (11), where

$$s_\varepsilon := -\frac{1}{2}\log\varepsilon + \log 2.$$

We also define

$$r_\varepsilon := \frac{1}{2}\,\varepsilon^{-1/2}$$

so that $s_\varepsilon = \log(2\,r_\varepsilon)$. We have the:

Lemma 2.2. *There exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, an annular part of $E_t(\varepsilon/2)$ (resp. $E_b(\varepsilon/2)$) in $M_k(\varepsilon/2)$ can be written as vertical graphs over the annulus $\bar{B}_{2r_\varepsilon} - B_{r_\varepsilon/2}$ in the horizontal plane for the functions*

$$\bar{U}^0_{t,\varepsilon}(r,\tilde\theta) = \quad \sigma_{t,\varepsilon/2} + \ln(2r) - \tfrac{\varepsilon}{2}\, r\, \cos\tilde\theta + \mathcal{O}_{\mathcal{C}^\infty_b}(\varepsilon)$$

$$(\textit{resp.}\quad \bar{U}^0_{b,\varepsilon}(r,\tilde\theta) = -\,\sigma_{b,\varepsilon/2} - \ln(2r) - \tfrac{\varepsilon}{2}\, r\, \cos\tilde\theta + \mathcal{O}_{\mathcal{C}^\infty_b}(\varepsilon)).$$

Here $(r,\tilde\theta)$ are polar coordinates in the $x_3 = 0$ plane. The functions $\mathcal{O}_{\mathcal{C}^\infty_b}(\varepsilon)$ are defined in the annulus $\bar{B}_{2r_\varepsilon} - B_{r_\varepsilon/2}$ and have partial derivatives with respect to the vector fields $r\,\partial_r$ and $\partial_{\tilde\theta}$ (or equivalently with respect to the vector fields ∂_t and ∂_θ) bounded by a constant (independent of ε) times ε.

Proof. The proof involves elementary though quite tiresome computations. We give some details of the derivation of the first estimate. When $s \sim s_\varepsilon$ and $\theta \in S^1$, we write

$$X_{t,\xi}(s,\theta) = (r\ \cos\tilde\theta, r\ \sin\tilde\theta, \bar U^0_{t,\varepsilon}(r,\tilde\theta))$$

for some new variables $r := r(s,\theta)$ and $\tilde\theta := \tilde\theta(s,\theta) \in S^1$.

We have the explicit formula

$$\begin{aligned} r\ \cos\tilde\theta &= \cos(\varepsilon/2)\left(\cosh s + \frac{w_{t,\varepsilon/2}}{\cosh s}\right)\cos\theta + \varsigma_{\varepsilon/2} \\ &\quad + \sin(\varepsilon/2)\,(s - w_{t,\varepsilon/2}\ \tanh s) \\ r\ \sin\tilde\theta &= \sin(\varepsilon/2)\left(\cosh s + \frac{w_{t,\varepsilon/2}}{\cosh s}\right)\sin\theta. \end{aligned} \tag{12}$$

This system gives implicitly $s(r,\tilde\theta)$ and $\theta(r,\tilde\theta)$. Then the function $\bar U^0_{t,\varepsilon}$ is given by

$$\begin{aligned} \bar U^0_{t,\varepsilon}(r,\tilde\theta) = &- \sin(\varepsilon/2)\ \left(\cosh s + \frac{w_{t,\varepsilon/2}}{\cosh s}\right)\ \cos\theta + \sigma_{t,\varepsilon/2} \\ &+ \cos(\varepsilon/2)\,(s - w_{t,\varepsilon/2}\ \tanh s) \end{aligned} \tag{13}$$

where on the right hand side $s = s(r,\tilde\theta)$ and $\theta = \theta(r,\tilde\theta)$.

Taking the sum of the square of both identities in (12), we find $r \sim r_\varepsilon$ when $s \sim s_\varepsilon$. We also get $\cosh^2 s = r^2 + \mathcal{O}_{\mathcal{C}^\infty_b}(\varepsilon^{1/2}\ \log 1/\varepsilon)$, from which it follows that

$$s = \log(2r) + \mathcal{O}_{\mathcal{C}^\infty_b}(\varepsilon) \tag{14}$$

we also get from the first identity in (12)

$$\cosh s\ \cos\tilde\theta = r\cos\theta + \mathcal{O}_{\mathcal{C}^\infty_b}(\varepsilon^{1/2}\ \log 1/\varepsilon).$$

The expansion of $\bar U^0_{t,\varepsilon}$ then follows by inserting these information in the right hand side of (13). □

We end this section by recalling the result of S. Nayatani [18, 19]:

Theorem 2.2. *(S. Nayatani* [18, 19]*) Assume that* $k \leq 37$*, then any bounded Jacobi field on* M_k *is a linear combination of* $n \cdot e_j$*, for* $j = 1, 2, 3$*, and* $n \cdot (p \times e_3)$.

According to this result, when $k \leq 37$, the only Jacobi fields that are bounded and invariant with respect to the action of $p \to \bar p$, are linear combinations of $n \cdot e_1$ and $n \cdot e_3$.

Corollary 2.1. *Assume that* $k \leq 37$*, then all bounded Jacobi fields on* M_k *that are invariant with respect to the* $x_2 = 0$ *plane are linear combination of* $\Phi^{0,\pm}$ *and* $\Phi^{1,+}$.

3 Riemann minimal surface

B. Riemann [21] has discovered a one parameter family of periodic minimal surfaces embedded in $\mathbb{R}^3$ which are foliated by circles (and straight lines). Each element of this family has infinitely many planar ends, is topologically a cylinder $\mathbb{R} \times S^1$ and in fact is conformal to the cylinder $\mathbb{R} \times S^1$ with infinitely many points removed in a periodic way, each of these points corresponds to one of the planar ends of the surface. In this section we obtain various parameterizations of Riemann's surfaces, the first parameterization will be used to obtain a nice expansion of part of Riemann's surfaces as their flux is almost vertical. The second parameterization is an isothermal parameterization that will be used for the study of the Jacobi operator about any Riemann's surface.

Circle foliation of Riemann's surfaces. Recall that (up to some dilation and some rigid motion), we can parameterize a fundamental piece of a Riemann's surface by

$$(t, \theta) \longmapsto (m(t) + \rho(t) \cos\theta, \rho(t) \sin\theta, t) \tag{15}$$

where $t \in (-t_\varepsilon, t_\varepsilon)$, $\theta \in S^1$ and where the functions m and ρ are determined by

$$(\partial_t \rho)^2 + 1 = \alpha\, \rho^2 + \varepsilon^2 \rho^4$$

and

$$\partial_t m = \varepsilon \rho^2$$

for some constant $\alpha \in \mathbb{R}$ and $\varepsilon > 0$. We will only be interested in part of the space of Riemann's minimal surface that correspond to $\alpha = 1$ so that we henceforth assume that the functions ρ and m satisfy

$$(\partial_t \rho)^2 + 1 = \rho^2 + \varepsilon^2 \rho^4 \tag{16}$$

and

$$\partial_t m = \varepsilon \rho^2.$$

Here $\varepsilon \in (0, \infty)$ parameterizes the family of Riemann's surfaces. We shall normalize the solutions of these ordinary differential equations by asking that

$$\rho_0 := \rho(0) > 0, \quad \partial_t \rho(0) = 0$$

and

$$m(0) = 0,$$

and naturally, ρ is a nonconstant smooth solution of (16). In particular, ρ_0 is the unique positive solution of

$$\rho_0^2 + \varepsilon^2 \rho_0^4 = 1.$$

Even though ρ and m both depend on ε, we shall not make this dependence explicit in the notation. It is easy to check that the functions ρ and m blow up in finite time $t_\varepsilon < \infty$ and that

$$\ell_\varepsilon := \lim_{t\to t_\varepsilon} (m(t) - \rho(t))$$

exists. The Riemann surface Σ_ε, corresponding to the parameter $\varepsilon \in (0, \infty)$ is then obtained by translation of the fundamental piece by $2\,(\ell_\varepsilon\, e_1 + t_\varepsilon\, e_3)\,\mathbb{Z}$.

In the next Lemma, we give precise expansions of the functions ρ and m when $t \in (-t_\varepsilon, t_\varepsilon)$.

Lemma 3.1. *For $\varepsilon > 0$ small enough, we have*

$$\rho(t) = \cosh t + \mathcal{O}_{\mathcal{C}^\infty_b}(\varepsilon^2 \cosh^3 t) \tag{17}$$

and

$$m(t) = \varepsilon \left(\frac{1}{2}\, t + \frac{1}{4}\, \sinh(2t) \right) + \mathcal{O}_{\mathcal{C}^\infty_b}(\varepsilon^3 \cosh^4 t)$$

when $t \in [-t_\varepsilon + 1, t_\varepsilon - 1]$. Here $\mathcal{O}_{\mathcal{C}^\infty_b}(f(t))$ is a function that together with its derivatives with respect to t is bounded by a constant times f.

Proof. We define a positive function $t \longmapsto v(t)$ such that $\rho(t) = \rho_0 \cosh v(t)$ and $v(0) = 0$. It follows from (16) that v is a solution of

$$(\partial_t v)^2 = 1 + \varepsilon^2 \rho_0^2\, (1 + \cosh^2 v).$$

Now, as long as $t \leq v(t) \leq t + c$ (where $c > 0$ is some fixed constant), we can estimate $(\partial_t v)^2 = 1 + \mathcal{O}_{\mathcal{C}^\infty_b}(\varepsilon^2\, \cosh^2 t)$ and hence we conclude that $v(t) = t + \mathcal{O}_{\mathcal{C}^\infty_b}(\varepsilon^2 \cosh^2 t)$. We remark *a posteriori* that $v(t) \leq t + c$ holds for $t \in [0, t_\varepsilon - 1]$ provided $c > 0$ has been fixed large enough. The first estimate then follows at once from the fact that $\rho_0 = 1 + \mathcal{O}(\varepsilon^2)$. The second estimate follows directly from

$$\partial_t m = \varepsilon \rho^2$$

once the first estimate has been established. □

The purpose of the next Lemma is to write the pieces of Σ_ε at height $t \sim s_\varepsilon = -\frac{1}{2} \log \varepsilon + \log 2$ (resp. at height $t \sim -s_\varepsilon$) as a vertical graph over the horizontal plane for some function t_t (resp. t_b). But before doing so we first dilate the surface Σ_ε by some factor $(1 + \lambda)$ and we next translate this dilated surface along the x_1-axis by ς, so that the fundamental piece of this surface is now parameterized by

$$(t, \theta) \longmapsto (\varsigma + (1 + \lambda)\, (m(t) + \rho(t)\, \cos \theta),\, (1 + \lambda)\, \rho(t)\, \sin \theta,\, (1 + \lambda)\, t).$$

We consider the change of coordinates:

$$(r \cos \tilde\theta, r\, \sin \tilde\theta) = (\varsigma + (1 + \lambda)\, (m(t) + \rho(t)\, \cos \theta),\, (1 + \lambda)\, \rho(t) \sin \theta) \tag{18}$$

where as before, $(r, \tilde{\theta})$ are polar coordinates in the $x_3 = 0$ plane. Obviously this change of coordinates is not valid everywhere but we are only interested in the range $t \sim \pm s_\varepsilon$ where the change of coordinates holds. Hence, when $t \sim s_\varepsilon$ (resp. $t \sim -s_\varepsilon$) we will be able to write

$$(1+\lambda)\, t = U_t^0(r, \tilde{\theta}) \quad \big(\text{resp. } (1+\lambda)\, t = U_b^0(r, \tilde{\theta})\big).$$

In the following Lemma, we give expansions of the functions U_t^0 and U_b^0.

Lemma 3.2. *Assume that* $|\varsigma| \leq 1$ *and also assume that* $|\lambda| \leq \frac{1}{2}$. *Then the following expansion holds*

$$U_t^0(r, \tilde{\theta}) = (1+\lambda)\, \log\left(\frac{2r}{1+\lambda}\right) - \left(\frac{\varepsilon}{2}\, r + (1+\lambda)\, \frac{\varsigma}{r}\right) \cos\tilde{\theta} + \mathcal{O}_{\mathcal{C}_b^\infty}(\varepsilon) \tag{19}$$

and

$$U_b^0(r, \tilde{\theta}) = -(1+\lambda)\, \log\left(\frac{2r}{1+\lambda}\right) - \left(\frac{\varepsilon}{2}\, r - (1+\lambda)\, \frac{\varsigma}{r}\right) \cos\tilde{\theta} + \mathcal{O}_{\mathcal{C}_b^\infty}(\varepsilon) \tag{20}$$

in $\bar{B}_{2r_\varepsilon} - B_{r_\varepsilon/2}$ *(with* $s_\varepsilon = \log(2r_\varepsilon)$*). Here the functions* $\mathcal{O}_{\mathcal{C}_b^\infty}(\varepsilon)$ *are that are defined in the annulus* $\bar{B}_{2r_\varepsilon} - B_{r_\varepsilon/2}$ *and that are bounded, together with their derivatives with respect to the vector fields* $r\, \partial_r$ *and* $\partial_{\tilde{\theta}}$ *(or equivalently,* ∂_t *and* ∂_θ*), by a constant (independent of* ε*) times* ε*. In addition all these estimates hold uniformly in* σ *and* λ*, provided* $|\varsigma| \leq 1$ *and* $|\lambda| \leq \frac{1}{2}$.

Proof. The proof of the result makes use of the result of Lemma 3.1. (In Lemma 3.6 we prove $s_\varepsilon < t_\varepsilon - 1$.) We give a few details for the proof of the first expansion.

When $t \sim s_\varepsilon$, the expansions of Lemma 3.1, yield

$$\rho = \frac{1}{2}\, e^t + \mathcal{O}_{\mathcal{C}_b^\infty}(\varepsilon^{1/2})$$

and

$$m = \frac{\varepsilon}{8}\, e^{2t} + \mathcal{O}_{\mathcal{C}_b^\infty}(\varepsilon\, \log 1/\varepsilon). \tag{$*$}$$

Hence we get the relation

$$m - \frac{\varepsilon}{2}\, \rho^2 = \mathcal{O}_{\mathcal{C}_b^\infty}(\varepsilon\, \log 1/\varepsilon). \tag{21}$$

We now perform the change of variable given in (18) and write

$$(t, y) = K_\varepsilon(r, \tilde{\theta}).$$

Since $t \sim s_\varepsilon$ and $\rho \sim r \sim r_\varepsilon$, we get from (18), (*)

$$r^2 - 2(\varsigma + (1+\lambda)\, m)\, r\, \cos\tilde{\theta} = (1+\lambda)^2 \rho^2 + \mathcal{O}_{\mathcal{C}_b^\infty}(1).$$

In particular, using (21) we find

$$m = \frac{\varepsilon}{2(1+\lambda)^2}\, r^2 + \mathcal{O}_{\mathcal{C}_b^\infty}(\varepsilon \log 1/\varepsilon).$$

Inserting this into the previous expansion we conclude that

$$r^2 - 2\left(\varsigma + \frac{\varepsilon\, r^2}{2(1+\lambda)}\right) r\, \cos\tilde{\theta} = (1+\lambda)^2 \rho^2 + \mathcal{O}_{\mathcal{C}_b^\infty}(1).$$

Now, use again the result of Lemma 3.1 to get

$$\frac{1}{4}\,(1+\lambda)^2 e^{2t} = r^2 - 2\left(\varsigma + \frac{\varepsilon\, r^2}{2(1+\lambda)}\right) r\, \cos\tilde{\theta} + \mathcal{O}_{\mathcal{C}_b^\infty}(1).$$

This provides the expansion of $t = t(r, \tilde{\theta})$ as

$$t = \log\left(\frac{2r}{1+\lambda}\right) - \left(\frac{\varsigma}{r} + \frac{\varepsilon\, r}{2\,(1+\lambda)}\right) \cos\tilde{\theta} + \mathcal{O}_{\mathcal{C}_b^\infty}(\varepsilon). \qquad (22)$$

The first expansion follows from the fact that $U_t^0 = (1+\lambda)\, t$. □

Isothermal parameterizations of Riemann's surfaces. An isothermal parametrization of Riemann's surfaces had already been considered by M. Shiffman [22] and has been generalized by L. Hauswirth in [5]. Given the above, almost explicit, parametrization in order to define this isothermal parametrization, it turns out that it is enough to look for a function $(t, y) \longmapsto \psi(t, y)$ such that

$$X_\varepsilon(t, y) := (m(t) + \rho(t)\, \cos(\psi(t, y)), \rho(t)\, \sin(\psi(t, y)), t) \qquad (23)$$

is a conformal parametrization. This leads to the first order differential equation

$$\partial_t \psi = \varepsilon\, \rho\, \sin\psi \qquad (24)$$

that comes from the requirement that $\partial_t X_\varepsilon \cdot \partial_y X_\varepsilon = 0$ and also

$$(\partial_y \psi)^2 = 1 + \varepsilon^2 \rho^2 (1 + \cos^2\psi) + 2\,\varepsilon\, \partial_t \rho\, \cos\psi \qquad (25)$$

that comes from the requirement that $|\partial_t X_\varepsilon|^2 = |\partial_y X_\varepsilon|^2$. One checks easily (differentiating (24) with respect to y and (25) with respect to t) that the integrability condition $\partial_y\,(\partial_t \psi) = \partial_t\,(\partial_y \psi)$ is fulfilled, and hence the function ψ is well defined.

Observe that $|\partial_y X_\varepsilon|^2 = |\partial_t X_\varepsilon|^2 \geq 1$ and hence we can define the real valued function ω so that

$$\begin{aligned}\partial_t X_\varepsilon &= (\sinh\omega \, \cos\psi, \sinh\omega \, \sin\psi, 1) \quad \text{and} \\ \partial_y X_\varepsilon &= (-\cosh\omega \, \sin\psi, \cosh\omega \, \cos\psi, 0).\end{aligned}$$

In particular

$$\cosh\omega = \rho \, \partial_y \psi. \tag{26}$$

With these notations, the first fundamental form about the surface parameterized by X_ε reads

$$ds^2 = \cosh^2\omega \, (dt \otimes dt + dy \otimes dy)$$

and, if we define the normal vector field by

$$N_\varepsilon := \frac{1}{\cosh\omega} \, (\cos\psi, \sin\psi, -\sinh\omega), \tag{27}$$

the second fundamental form about the surface parameterized by X_ε is then given by

$$h = \partial_t\omega \, dt \otimes dt - \partial_y\psi \, dy \otimes dy - \partial_t\psi \, (dt \otimes dy + dy \otimes dt).$$

Observe that $\partial_y(\partial_t X) = \partial_t(\partial_y X)$ and hence

$$\partial_y\omega = -\partial_t\psi \quad \text{and} \quad \partial_t\omega = \partial_y\psi. \tag{28}$$

In particular, this shows that both ω and ψ are harmonic functions. It will be convenient to define

$$a := \frac{\partial_t\psi}{\cosh\omega} \quad \text{and} \quad b := \frac{\partial_y\psi}{\cosh\omega}. \tag{29}$$

With these notations, we have the:

Lemma 3.3. *In the parameterization* (23)*, the Jacobi operator about Riemann's surface* Σ_ε *is given by*

$$\mathbb{L}_{\Sigma_\varepsilon} = \frac{1}{\cosh^2\omega} \, \left(\partial_t^2 + \partial_y^2 + 2\,(a^2 + b^2)\right)$$

where the functions a *only depends on* y*, the function* b *only depends on* t *and are solutions of*

$$(\partial_y a)^2 + a^4 = \varepsilon^2 - a^2 \quad \textit{and} \quad (\partial_t b)^2 + b^4 = \varepsilon^2 + b^2. \tag{30}$$

Proof. Observe that it follows from (26) that $b = \frac{1}{\rho}$ and hence the equation satisfied by b reads

$$(\partial_t b)^2 + b^4 = \varepsilon^2 + b^2. \tag{31}$$

Moreover, $\partial_y b = 0$ since ρ, and hence b, does not depend on y.

It should be clear that (29) together with (28) yields $\partial_t a + \partial_y b = 0$ and hence the function a does not depend on t. It remains to find the ordinary differential equation satisfied by a. We have

$$\partial_y(\cosh \omega\, a) - \partial_t(\cosh \omega\, b) = 0.$$

Hence

$$(a^2 + b^2) \sinh \omega = \partial_y a - \partial_t b. \tag{32}$$

Taking the derivative of (32) with respect to y and using the fact that the function b does not depend on y, we get

$$\partial_y^2 a = -a\,(a^2 + b^2) + \frac{a}{a^2 + b^2} \left((\partial_y a)^2 - (\partial_t b)^2\right). \tag{33}$$

We have used (32) to eliminate $\sinh \omega$ and $\cosh^2 \omega = 1 + \sinh^2 \omega$. In other words

$$\partial_y \left(\frac{(\partial_y a)^2 - (\partial_t b)^2}{a^2 + b^2} + a^2 \right) = 0.$$

Hence, we conclude that the function

$$(t,\, y) \longmapsto \frac{(\partial_y a)^2 - (\partial_t b)^2}{a^2 + b^2} + a^2$$

only depends on t. Taking this information into account in (33) and using the fact that the function b only depends on t, we conclude that

$$\partial_y^2 a + 2\, a^3 + \alpha\, a = 0$$

where α is a function which only depends on t, and hence the function α has to be constant. Therefore, we get after integration

$$(\partial_y a)^2 + a^4 + \alpha\, a^2 - \beta = 0$$

for some constants α and β. The values of α and β can be determined. Indeed, inserting these into (33) and using (31), we conclude that $\alpha = 1$ and $\beta = \varepsilon^2$.

$$(\partial_y a)^2 + a^4 + a^2 - \varepsilon^2 = 0$$

This completes the proof of the result. □

The functions $y \longmapsto a(y)$ and $t \longmapsto b(t)$ are defined up to some translation in the y or t variables. In particular, we can require that a (resp. b) takes its maximal value at $y = 0$ (resp. $t = 0$). Observe that the function a is periodic, we will denote by $2\pi\tau_\varepsilon$ its least period. We extend the function b^2 to be a $2\,t_\varepsilon$-periodic function.

The next Lemma is a simple consequence of the proof of Lemma 3.1.

Lemma 3.4. *The following estimate holds*

$$b(t)\,\cosh t \;\leq \frac{1}{\rho_0}$$

for all $t \in [-t_\varepsilon, t_\varepsilon]$.

Proof. This follows from the proof of Lemma 3.1. Indeed, keeping the notations of the proof of Lemma 3.1, we can write $\rho(t) = \rho_0 \cosh v(t)$ and, since $\partial_t v \geq 1$ and $v(0) = 0$, we always have $v(t) \geq t$. Therefore, $\rho(t) \geq \rho_0 \cosh t$. This complete the proof of the result. □

Since ρ_0 converges to 1 as ε tends to 0 and using the fact that b is even, this yields a uniform upper bound for b as ε tends to 0. We now study the behavior of the functions a and b as ε tends to 0. We have the:

Lemma 3.5. *As ε tends to 0, the sequence of functions $(b)_{\varepsilon>0}$ converges uniformly on compacts to the function*

$$t \longmapsto \frac{1}{\cosh t}.$$

and the sequence of functions $(\varepsilon^{-1}\, a)_{\varepsilon>0}$ converges uniformly on compacts to the function

$$y \longmapsto \cos y.$$

Proof. It is easy to check that, as ε tends to 0 the functions $\varepsilon^{-1}\, a$, b and their derivatives remain uniformly bounded. Indeed, we have on the one hand

$$(\partial_y a)^2 \leq \varepsilon^2 \quad \text{and} \quad 2a^2 \leq \sqrt{1+4\varepsilon^2} - 1 \tag{34}$$

and on the other hand

$$(\partial_t b)^2 \leq \varepsilon^2 + \frac{1}{4} \quad \text{and} \quad 2b^2 \leq 1 + \sqrt{1+4\varepsilon^2}. \tag{35}$$

Passing to the limit in (30), a simple application of Ascoli–Arzela's Theorem implies that the sequence of functions $(b)_\varepsilon$ converges uniformly on compact to $\bar{b}$ solution of

$$(\partial_t \bar{b})^2 + \bar{b}^4 = \bar{b}^2$$

with $\bar{b}(0) = 1$ while the sequence of functions $(\varepsilon^{-1}\, a)_\varepsilon$ converges uniformly on compacts to $\bar{a}$ solution of

$$(\partial_t \bar{a})^2 + \bar{a}^2 = 1$$

with $\bar{a}(0) = 1$. The result then follows at once. □

The last result of this section is concerned with the estimate of $2\pi\tau_\varepsilon$, the least period of the function a, and also t_ε, the existence time of the functions ρ and b.

Lemma 3.6. *As* ε *tends to* 0*, the following estimate hold*

$$t_\varepsilon = -\log \varepsilon + \mathcal{O}(1).$$

Moreover

$$\frac{1}{\sqrt{1+4\varepsilon^2}} \leq \tau_\varepsilon^2 \leq 1. \tag{36}$$

Proof. Concerning the estimate of t_ε, we have the formula

$$t_\varepsilon = \int_0^{\zeta_\varepsilon} \frac{1}{\sqrt{1+\zeta^2 - \varepsilon^2\, \zeta^4}}\, d\zeta$$

where $0 < \zeta_\varepsilon$ is the largest root of $\varepsilon^2\, \zeta^4 = \zeta^2 + 1$. Using this, it is easy to check that $t_\varepsilon = -\log \varepsilon + \mathcal{O}(1)$ as ε tends to 0.

In order to estimate τ_ε, we write $a(y) = a_0\, \cos v$ where $a_0 > 0$ is defined by $1 + 2\, a_0^2 = \sqrt{1+4\varepsilon^2}$. Using (34), we get

$$(\partial_y v)^2 = 1 + a_0^2\, (1 + \cos^2 v)$$

from which it follows that $1 \leq (\partial_y v)^2 \leq 1 + 2\, a_0^2$. Integration of these inequalities from 0 to $2\pi\tau_\varepsilon$ yields the required inequalities since $v(2\pi\tau_\varepsilon) = 2\pi$. □

Remark 3.1. The sequence of functions $\theta \longmapsto \varepsilon^{-1}\, a(\tau_\varepsilon\, \theta)$ converges uniformly to $\theta \longmapsto \cos\theta$ as ε tends to 0. This follows from the analysis in the proof of Lemma 3.5 together with the fact that τ_ε tends to 1 as ε tends to 0.

4 The Jacobi operator about Riemann's surfaces

We keep the notations of the previous section. Recall that the Jacobi operator about Riemann's surface Σ_ε is given by

$$\mathbb{L}_{\Sigma_\varepsilon} := \frac{1}{\cosh^2 \omega}\, \left(\partial_t^2 + \partial_y^2 + 2\,(a^2 + b^2)\right). \tag{37}$$

Instead of studying the mapping properties of this operator, it turns out that it is simpler to study the operator

$$L_\varepsilon := \partial_t^2 + \partial_y^2 + 2\,(a^2 + b^2). \tag{38}$$

We define, for all $\varepsilon \geq 0$ the operator

$$D_\varepsilon := \partial_y^2 + 2\,a^2$$

which acts on functions of y which are $2\pi\,\tau_\varepsilon$ periodic and even. Recall that $2\pi\,\tau_\varepsilon$ is the least period of the function a that depends on ε. The operator D_ε is clearly elliptic and self adjoint, hence has discrete spectrum $(\lambda_i)_{i\geq 0}$. Since we only consider even functions, each eigenvalue is simple and we can arrange the eigenvalues so that $\lambda_i < \lambda_{i+1}$. The corresponding eigenfunctions are denoted by f_i and are normalized so that

$$\int_0^{2\pi\tau_\varepsilon} f_i^2\,dy = 1.$$

Even though we have not made this explicit in the notations, the eigendata of D_ε do depend on ε since a does. Since the function a tends uniformly to 0, it is easy to check that, as ε tends to 0, the eigendata of D_ε tend to the eigendata of ∂_y^2 (defined on even functions on S^1). In particular, λ_j converge to j^2 and f_j converges to $\cos(j\,\cdot)$. We will also need the:

Lemma 4.1. *The following estimate holds*

$$\lambda_i \geq i^2 + 1 - \sqrt{1+4\varepsilon^2}. \tag{39}$$

Proof. Recall the variational characterization of the eigenvalues

$$\lambda_i = \sup_{\operatorname{codim} E = i} \left(\inf_{f\in E,\, \|f\|_{L^2}=1} \int_0^{2\pi\tau_\varepsilon} \left((\partial_y f)^2 - 2\,a^2 f^2 \right) dy \right).$$

The assertion follows from

$$\begin{aligned} \lambda_i &\geq \inf_{f\in E,\, \|f\|_{L^2}=1} \int_0^{2\pi\tau_\varepsilon} \left((\partial_y f)^2 - 2\,a^2 f^2 \right) dy \\ &\geq \inf_{f\in E,\, \|f\|_{L^2}=1} \int_0^{2\pi\tau_\varepsilon} \left((\partial_y f)^2 \right) dy - 2\,\sup a^2 \end{aligned}$$

with $E \subset H^1$ being the L^2 orthogonal of

$$\operatorname{Span}\{\cos(j\cdot/\tau_\varepsilon) : j = 0, \ldots, i-1\}$$

together with the fact that $\tau_\varepsilon \leq 1$ and $0 \leq 2a^2 \leq \sqrt{1+4\varepsilon^2} - 1$. □

Let $S^1(\tau)$ denote the circle of radius τ. The previous estimate immediately implies the following injectivity result:

Lemma 4.2. *Assume that* $\varepsilon \in (0, \sqrt{\frac{3}{4}})$ *and* $-\infty < t_0 < t_1 \leq +\infty$. *Let* v *be a solution of*

$$L_\varepsilon \, v = 0$$

on $(t_0, t_1) \times S^1(\tau_\varepsilon)$ *with* $v(t_0, \cdot) = 0$. *Assume that* $v(t_1, \cdot) = 0$ *when* $t_1 < \infty$ *or that* $|v(t, y)| \leq e^{-\mu t}$ *for some* $\mu > 0$ *when* $t_1 = +\infty$. *Further assume that*

$$\int_0^{2\pi\tau_\varepsilon} v(t, y) \, f_i(y) \, dy = 0$$

for all $t \in (t_0, t_1)$ *and* $i = 0, 1$. *Then* $v \equiv 0$.

Proof. For each $\varepsilon > 0$, the family $\{f_i\}_{i \in \mathbb{N}}$ is a Hilbert basis of the space of L^2-integrable functions that are even and $2\pi\tau_\varepsilon$-periodic. We consider the eigenfunction decomposition of a function $(t, y) \longmapsto v(t, y)$, that is $2\pi\tau_\varepsilon$-periodic and even in the y variable,

$$v(t, y) = \sum_{i=0}^{\infty} v_i(t) \, f_i(y).$$

This decomposition induces a decomposition of the operator L_ε into the sequence of ordinary differential operators

$$L_{\varepsilon,i} := \partial_y^2 + 2\, b^2 - \lambda_i,$$

where we recall that b and λ_i both depend on ε. It follows from the result of Lemma 4.1 and from (35) that

$$2\, b^2 - \lambda_i \leq 2\sqrt{1 + 4\,\varepsilon^2} - i^2. \tag{40}$$

Therefore, $2\, b^2 - \lambda_i < 0$ when $\varepsilon \in (0, \sqrt{\frac{3}{4}})$ and $i \geq 2$. The result then follows from the maximum principle. □

We now define weighted Hölder spaces that will be useful for the understanding the mapping properties of the operator L_ε as the parameter ε tends to 0.

Definition 4.1. *Given* $\ell \in \mathbb{N}$, $\alpha \in (0, 1)$, $\mu \in \mathbb{R}$ *and a closed interval* $I \subset \mathbb{R}$, *we define the space* $\mathcal{C}^{\ell,\alpha}_\mu(I \times S^1(\tau))$ *to be the space of functions* $u \in \mathcal{C}^{\ell,\alpha}_{loc}(I \times S^1(\tau))$ *for which the following norm*

$$\|u\|_{\mathcal{C}^{\ell,\alpha}_\mu} := \|e^{-\mu t} \, u\|_{\mathcal{C}^{\ell,\alpha}(I \times S^1(\tau))},$$

is finite.

It should be obvious that

$$\begin{aligned}\mathcal{C}^{2,\alpha}_{\mu}\big([t_0,\infty)\times S^1(\tau_\varepsilon)\big) &\longrightarrow \mathcal{C}^{0,\alpha}_{\mu}\big([t_0,\infty)\times S^1(\tau_\varepsilon)\big)\\ w &\longmapsto L_\varepsilon\, w\end{aligned}$$

for any $\mu \in \mathbb{R}$ and $t_0 \in \mathbb{R}$. We prove that, provided the parameter μ is suitably chosen, there exists a right inverse for L_ε whose norm is uniformly bounded as ε tends to 0 and independently of $t_0 \in \mathbb{R}$. This is the content of the following:

Proposition 4.1. *Fix $\mu \in (-2,-1)$. Then, there exists $\varepsilon_0 > 0$ and, for all $\varepsilon \in (0,\varepsilon_0)$, for all $t_0 \in \mathbb{R}$, there exists an operator*

$$G_{\varepsilon,t_0} : \mathcal{C}^{0,\alpha}_{\mu}\big([t_0,\infty)\times S^1(\tau_\varepsilon)\big) \longrightarrow \mathcal{C}^{2,\alpha}_{\mu}\big([t_0,\infty)\times S^1(\tau_\varepsilon)\big),$$

such that for all $g \in \mathcal{C}^{0,\alpha}_{\mu}([t_0,\infty)\times S^1(\tau_\varepsilon))$, the function $v := G_{\varepsilon,t_0}(g)$ solves

$$\begin{cases} L_\varepsilon\, v = g & \text{in } [t_0,\infty)\times S^1(\tau_\varepsilon)\\ v \in \operatorname{Span}\{f_0, f_1\} & \text{on } \{t_0\}\times S^1(\tau_\varepsilon).\end{cases}$$

Moreover,

$$||G_{\varepsilon,t_0}(g)||_{\mathcal{C}^{2,\alpha}_{\mu}} \leq c\, \|g\|_{\mathcal{C}^{0,\alpha}_{\mu}},$$

for some constant $c > 0$ which is independent of $\varepsilon \in (0,\varepsilon_0)$ and also independent of $t_0 \in \mathbb{R}$.

Proof. We decompose the function g into

$$g = g_0\, f_0 + g_1\, f_1 + \bar g$$

where $\bar g(t,\cdot)$ is L^2 orthogonal to f_0 and f_1 for each t. For the sake of simplicity in the notations, we shall not mention the parameter τ_ε and write S^1 instead of $S^1(\tau_\varepsilon)$. Observe that (36) implies that τ_ε tends to 1 as ε tends to 0.

Step 1. We show that, for each $t_1 > t_0 + 1$ it is possible to solve

$$L_\varepsilon \bar v = \bar g,$$

on $S^1 \times [t_0, t_1]$ with $\bar v(t_0,\cdot) = \bar v(t_1,\cdot) = 0$. This just follows from the result of Lemma 4.2 which states that, restricted to the set of functions L^2 orthogonal to f_0 and f_1 for each t, the operator L_ε is injective.

We claim that, provided ε is chosen small enough, there exists a constant $c > 0$ such that

$$\sup_{[t_0,t_1]\times S^1} e^{-\mu t}\, |\bar v| \leq c \sup_{[t_0,t_1]\times S^1} e^{-\mu t}\, |\bar g|.$$

The proof of this fact is by contradiction. If this were false, there would exist a sequence $(\varepsilon_n)_n$ tending to 0, sequences $(t_{0,n})_n$ and $(t_{1,n})_n$ such that $t_{0,n} + 1 \leq t_{1,n}$, a sequence of functions $(\bar{g}_n)_n$ and a sequence of solutions $(\bar{v}_n)_n$ of $L_\varepsilon \, \bar{v}_\varepsilon = \bar{g}_\varepsilon$ with 0 boundary data, such that

$$\sup_{[t_{0,n}, t_{1,n}] \times S^1} e^{-\mu t} \, |\bar{g}_n| = 1$$

and

$$\lim_{n \to +\infty} A_n = +\infty$$

where

$$A_n := \sup_{[t_{0,n}, t_{1,n}] \times S^1} e^{-\mu t} \, |\bar{v}_n|.$$

We denote by $(t_n, y_n) \in [t_{0,n}, t_{1,n}] \times S^1$ a point where A_n is achieved. Observe that $a^2 + b^2$ is uniformly bounded as ε_n tends to 0 and hence, elliptic estimates imply that

$$\sup e^{-\mu t} |\nabla \bar{v}_n| \leq c \, (1 + A_n) \tag{41}$$

and, since each $\bar{v}_n$ vanishes on the boundaries of $[t_{0,n}, t_{1,n}] \times S^1$, this in turn implies that the sequences $(t_{0,n} - t_n)_n$ and $(t_{1,n} - t_n)_n$ remain bounded away from 0.

We define the function $\tilde{v}_n$ by

$$\tilde{v}_n(t, y) = \frac{1}{A_n} \, e^{-\mu t_n} \, \bar{v}_n(t_n + t, y).$$

Without loss of generality, we can assume that the sequence $(t_{0,n} - t_n)_n$ (resp. $(t_{1,n} - t_n)_n$) converges to $\bar{t}_0 \in ([-\infty, 0)$ (resp. to $\bar{t}_1 \in (0, +\infty]$). We denote by $I = (\bar{t}_0, \bar{t}_1)$.

Up to a subsequence, we can assume without loss of generality that the sequence of functions $(\tilde{v}_n)_n$ converges on compacts to a nontrivial function $\tilde{v}$ defined on $I \times S^1(1)$ (recall that τ_ε converges to 1 as ε tends to 0). This follows from Ascoli–Arzela Theorem, once it is observed that the sequence of functions $(\tilde{v}_n)_n$ is uniformly bounded (by $t \longmapsto e^{\mu t}$) and, by elliptic regularity theory, the sequence of functions $(\nabla \tilde{v}_n)_n$ is also uniformly bounded (by a constant times $t \longmapsto e^{\mu t}$). We now derive some properties of the limit function $\tilde{v}$. These properties are all inherited from similar properties which hold for the functions $\bar{v}_n$.

First, $\tilde{v}(t, \cdot)$ is L^2 orthogonal to the constant function and the function $y \to \cos y$ for each $t \in I$. Next, $\tilde{v}$ is equal to 0 on $\{\bar{t}_0\} \times S^1$ and on $\{\bar{t}_1\} \times S^1$ if either $\bar{t}_0 > -\infty$ or $\bar{t}_1 < +\infty$. Also

$$\sup_{I \times S^1} e^{-\mu t} \, |\tilde{v}| = 1. \tag{42}$$

Finally, using Lemma 3.5, we find that $\tilde{v}$ is either a solution of

$$\left(\partial_t^2 + \partial_y^2 + \frac{2}{\cosh^2(\cdot + \tilde{t}\,)}\right) \tilde{v} = 0 \tag{43}$$

for some $\tilde{t} \in \mathbb{R}$ or is a solution of

$$\left(\partial_t^2 + \partial_y^2\right) \tilde{v} = 0$$

in $\mathbb{R} \times S^1$. Indeed, according to the result of Lemma 3.5, as n tends to ∞, the sequence ε_n tends to 0 and hence the sequence of functions $a(\cdot + t_n)$ converges uniformly to 0. Next, again according to the result of Lemma 3.5, as n tends to ∞ and up to a subsequence, the sequence of functions $b(\cdot + t_n)$ converges either to 0 or to $\frac{1}{\cosh(\cdot + \tilde{t}\,)}$, for some $\tilde{t} \in \mathbb{R}$.

To reach a contradiction we consider the eigenfunction decomposition of $\tilde{v}$

$$\tilde{v}(t, y) = \sum_{j=2}^{\infty} a_j(t) \cos(jy).$$

When $\bar{t}_0 = -\infty$, observe that the function a_j is either blowing up like $t \longmapsto e^{-jt}$ or decaying like $y \longmapsto e^{jt}$. The choice of $\mu \in (-2, -1)$ implies that a_j decays exponentially at $-\infty$. Similar considerations show that a_j also decays exponentially at $+\infty$ when $\bar{t}_1 = +\infty$. Multiplying the Equation (43) by $a_j \cos(j\,\cdot)$ and integrating by parts over I (all integrations are justified because a_j decays exponentially at both $\pm\infty$ if either $\bar{t}_0 = -\infty$ or $\bar{t}_1 = +\infty$), we get either

$$\int_{-\infty}^{+\infty} \left(|\partial_t a_j|^2 + j^2\, a_j^2\right) dt = \int_{-\infty}^{+\infty} \frac{2}{\cosh^2(t + \tilde{t}\,)}\, a_j^2\, dt$$

or

$$\int_{-\infty}^{+\infty} \left(|\partial_t a_j|^2 + j^2\, a_j^2\right) dt = 0.$$

In either case, we obtain $a_j = 0$ which clearly contradicts (42).

Since we have reached a contradiction, the proof of the claim is complete. Once the claim is proven, we can use once more elliptic estimates and Ascoli–Arzela Theorem to pass to the limit as t_1 tends to $+\infty$ in a sequence of solutions that are defined on $[t_0, t_1] \times S^1$. This proves the existence of a solution of

$$L_\varepsilon \bar{v} = \bar{g}$$

which is defined in $[t_0, +\infty) \times S^1$ and that satisfies $\bar{v}(t_0, \cdot) = 0$. In addition, we know that

$$\sup_{[t_0, +\infty) \times S^1} e^{-\mu t}\, |\bar{v}| \leq c \sup_{[t_0, +\infty) \times S^1} e^{-\mu t}\, |\bar{g}|.$$

Using a last time elliptic estimates, we complete the proof of the result in the case where the eigenfunction decomposition of g does not involve f_0 or f_1. Uniqueness of the solution in this case follows at once from Lemma 4.2.

Step 2. Now we consider the case where the function g is collinear to f_0 and f_1, namely

$$g(t, y) = g_0(t)\, f_0(y) + g_1(t)\, f_1(y).$$

We extend the function g to be equal to 0 when $t < t_0$, keeping the same notation. Given $t_1 > t_0$, we consider the equation

$$\left(\partial_t^2 + 2\, b^2 - \lambda_j\right) v_j = g_j$$

in $(-\infty, t_1)$ with boundary data $v_j(t_1) = \partial_t v_j(t_1) = 0$. The existence of v_j is standard. We claim that

$$\sup_{(-\infty,t_1)} e^{-\mu t}\, |v_j| \leq c\, \sup_{\mathbb{R}} e^{-\mu t}\, |g_j|$$

for some constant that does not depend on t_1, provided ε is chosen small enough. As before, we argue by contradiction. Assume that the claim is not true, there would exist a sequence $(\varepsilon_n)_n$ tending to 0, sequences $(t_{0,n})_n$ and $(t_{1,n})_n$ such that $t_{0,n} \leq t_{1,n}$, a sequence of functions $(g_{j,n})_n$ and a sequence $(v_{j,n})_n$ of solutions of $(\partial_t^2 + 2\, b^2 - \lambda_j)\, v_{j,n} = g_{j,n}$, with $g_{j,n}(t_{1,n}) = \partial g_{j,n}(t_{1,n}) = 0$, such that

$$\sup_{(-\infty,t_{1,n}]\times S^1} e^{-\mu t}\, |g_{j,n}| = 1$$

and

$$\lim_{n\to+\infty} A_n = +\infty$$

where

$$A_n := \sup_{(-\infty,y_{1,n}]\times S^1} e^{-\mu t}\, |v_{j,n}|.$$

We denote by $(t_n, y_n) \in (-\infty, t_{1,n}) \times S^1$ a point where A_n is achieved. Observe that, the solution $v_{j,n}$ is a linear combination of the two solutions of the homogeneous problem $L_{\varepsilon,j}\, w = 0$ and these are known to be at most linearly growing thanks to the explicit knowledge of the Jacobi fields. Indeed, solutions of the homogeneous problem

$$\left(\partial_t^2 + \partial_y^2 + 2\,(a^2 + b^2)\right) w = 0 \tag{44}$$

that are collinear to f_0 are linear combinations of $N_\varepsilon \cdot e_3$, the Jacobi field associated to vertical translation, and $N_\varepsilon \cdot X_\varepsilon$, the Jacobi field associated to dilations, and these at most grow linearly as t tends to $\pm\infty$ (recall that X_ε and N_ε have been defined in (23) and (27)). While, solutions of (44) that are collinear to f_1 are linear combinations of $N_\varepsilon \times e_1$, the Jacobi field associated to translation along the x_1-axis, and $N_\varepsilon \cdot (X_\varepsilon \times e_2)$, the Jacobi field associated to the rotation about the x_2-axis. Again, these at most grow linearly as t tends to $\pm\infty$. Hence the above supremum is achieved.

We define the function $\tilde{v}_{j,n}$ by

$$\tilde{v}_{j,n}(t, y) = \frac{1}{A_n} e^{-\mu t_n} v_{j,n}(t_n + t, y).$$

As in the first step, one shows that the sequence $(t_{1,n} - t_n)_n$ remains bounded away from 0.

Without loss of generality, we can assume that the sequence $(t_{1,n} - t_n)_n$ converges to $\bar{t}_1 \in (0, +\infty]$. We set $I := (-\infty, \bar{t}_1)$.

As in Step 1, we can also assume, without loss of generality that the sequence of functions $(\tilde{v}_{j,n})_n$ converges on compacts to a nontrivial function $\tilde{v}_j$ defined on $I \times S^1$. We now derive some properties of the limit function $\tilde{v}_j$.

We have

$$\sup_{I \times S^1} e^{-\mu t} |\tilde{v}_j| = 1. \tag{45}$$

Finally, $\tilde{v}_j$ is either a solution of

$$\left(\partial_t^2 - j^2 + \frac{2}{\cosh^2(\cdot + \tilde{t})} \right) \tilde{v}_j = 0 \tag{46}$$

for some $\tilde{t} \in \mathbb{R}$ or is a solution of

$$\left(\partial_t^2 - j^2 \right) \tilde{v}_j = 0. \tag{47}$$

Finally, when $\bar{t}_1 < +\infty$, we also have $\tilde{v}_j(\bar{t}_1, \cdot) = \partial_t \tilde{v}_j(\bar{t}_1, \cdot) = 0$. Since we are now dealing with second ordinary differential equations, we immediately conclude that $\tilde{v}_j \equiv 0$ in this case and this is already a contradiction with (45).

To reach a contradiction in the case where $\bar{t}_1 = +\infty$, observe that, when $j = 0$, the solutions of (46) are linear combinations of the functions

$$t \longmapsto \tanh(t + \tilde{t}) \quad \text{and} \quad t \longmapsto (t + \tilde{t}) \tanh(t + \tilde{t}) - 1$$

and when $j = 1$ they are linear combinations of the following functions

$$t \longmapsto \frac{1}{\cosh(t + \tilde{t})} \quad \text{and} \quad t \longmapsto \frac{(t + \tilde{t})}{\cosh(t + \tilde{t})} + \sinh(t + \tilde{t}).$$

Similarly, when $j = 0$ the solutions of (47) are linear combinations of the functions

$$t \longmapsto 1 \quad \text{and} \quad t \longmapsto t$$

and when $j = 1$ they are linear combinations of the functions

$$t \longmapsto e^{t} \quad \text{and} \quad t \longmapsto e^{-t}.$$

Now, when $\bar{t}_1 = +\infty$, observe that all the solutions of these two equations, when $j = 0, 1$ are explicitly known and that none of them satisfies (45) since we have chosen $\mu \in (-2, -1)$. Again a contradiction with (45).

Since we have reached a contradiction, the proof of the claim is complete. Once the claim is proven, we pass to the limit as t_1 tends to $+\infty$ in a sequence of solutions that are defined on $[t_0, t_1] \times S^1$. This proves the existence of a solution of

$$L_{\varepsilon,j}\, v_j = g_j$$

that is defined in $[t_0, +\infty) \times S^1$. In addition, we know that

$$\sup_{[t_0,+\infty)\times S^1} e^{-\mu t}\, |v_j| \leq c \sup_{[t_0,+\infty)\times S^1} e^{-\mu t}\, |g_j|.$$

The estimates for the derivatives follow from standard elliptic estimates. Uniqueness follows at once from the fact that no nontrivial solution of (44) that is a linear combination of f_0 and f_1 decays exponentially at ∞. □

The following result is standard and left to the reader (a proof can be found in [4]).

Lemma 4.3. *There exists $c > 0$ and, for all $\varphi \in \mathcal{C}^{2,\alpha}(S^1)$, with φ orthogonal to 1 and $\theta \longmapsto \cos\theta$ in the L^2-sense and is an even function of $\theta \in S^1$ there exists a unique function $H_\varphi \in \mathcal{C}^{2,\alpha}_{-2}([0, +\infty) \times S^1)$ solution of*

$$\begin{cases} \left(\partial_t^2 + \partial_\theta^2\right) H_\varphi = 0 & \text{in } [0, +\infty) \times S^1 \\ H_\varphi = \varphi & \text{on } \{0\} \times S^1 \end{cases}$$

satisfying

$$\|H_\varphi\|_{\mathcal{C}^{2,\alpha}_{-2}} \leq c\, \|\varphi\|_{\mathcal{C}^{2,\alpha}}.$$

To have a grasp on the reason why the result should be true, just decompose the boundary data φ in Fourier series

$$\varphi(\theta) = \sum_{j\geq 2} \varphi_j \cos(j\theta)$$

and observe that H_φ is given explicitly by

$$H_\varphi(t, \theta) = \sum_{j\geq 2} \varphi_j\, e^{-jt} \cos(j\theta).$$

From a formal point of view, all coefficients in H_φ decay at least like e^{-2t} at ∞.

5 An infinite dimensional family of minimal surfaces that are close to half of a Riemann's surface

In this section we are interested in minimal surfaces that are close to half of a Riemann's surface and have prescribed boundary. We will only consider surfaces that are normal graphs over Riemann's surface and hence are parameterized by

$$Z_{\varepsilon,u} := X_\varepsilon + u\, N_\varepsilon$$

where X_ε and N_ε have been defined in (23) and (27). In the next Proposition, we give the expansion of the mean curvature operator for this surface in terms of the function u and its partial derivatives.

Proposition 5.1. *The surface parameterized by $Z_{\varepsilon,u} := X_\varepsilon + u\, N_\varepsilon$ is minimal if and only if the function u is a solution of*

$$L_\varepsilon\, u = (\cosh\omega)^2\, Q_\varepsilon\left(\frac{u}{\cosh\omega}, \frac{\nabla u}{\cosh\omega}, \frac{\nabla^2 u}{\cosh\omega}\right)$$

where L_ε is the operator that has already been defined in (38) *and the nonlinear operator Q_ε satisfies*

$$\begin{aligned} &|Q_\varepsilon(v_2) - Q_\varepsilon(v_1)|_{\mathcal{C}^{0,\alpha}([t,t+1]\times S^1(\tau_\varepsilon))} \\ &\leq c \sup_{i=1,2} |v_i|_{\mathcal{C}^{2,\alpha}([t,t+1]\times S^1(\tau_\varepsilon))}\, |v_2 - v_1|_{\mathcal{C}^{2,\alpha}([t,t+1]\times S^1(\tau_\varepsilon))} \end{aligned}$$

for all v_1, v_2 such that $|v_i|_{\mathcal{C}^{2,\alpha}([t,t+1]\times S^1(\tau_\varepsilon))} \leq 1$. Here the constant $c > 0$ does not depend on $t \in \mathbb{R}$, nor on $\varepsilon \in (0,1)$.

Proof. We omit the indices ε and u for the sake of simplicity in the notations. We have

$$\partial_t Z = \partial_t X + \partial_t u N + u \partial_t N \quad \text{and} \quad \partial_y Z = \partial_y X + \partial_y u N + u \partial_y N.$$

A simple computation shows that the coefficients of g_u, the first fundamental form of the surface parameterized by Z_u, are given by

$$\begin{aligned} |\partial_t Z|^2 &= \cosh^2\omega - 2\, b \cosh\omega\, u + (\partial_t u)^2 + (a^2 + b^2)\, u^2 \\ \partial_t Z \cdot \partial_y Z &= 2\, a \cosh\omega\, u + \partial_t u\, \partial_y u \\ |\partial_y Z|^2 &= \cosh^2\omega + 2\, b \cosh\omega\, u + (\partial_y u)^2 + (a^2 + b^2)\, u^2. \end{aligned}$$

Collecting these, we have the expansion of the determinant of g_u

$$\begin{aligned} |g_u| = \cosh^4\omega \Bigg(1 + \frac{1}{\cosh^2\omega}\left(|\partial_t u|^2 + |\partial_y u|^2 - 2\,(a^2+b^2)\, u^2\right) \\ + P_3\left(\frac{u}{\cosh\omega}, \frac{\nabla u}{\cosh\omega}\right) + P_4\left(\frac{u}{\cosh\omega}, \frac{\nabla u}{\cosh\omega}\right)\Bigg) \end{aligned}$$

where P_i has coefficients that are bounded independently of $\varepsilon \in (0, 1)$ and is a homogeneous polynomial of degree i. Here we have implicitly used the fact that the functions a and b, as well as their derivatives, are uniformly bounded when $\varepsilon \in (0, 1)$.

We consider the area energy

$$A(u) := \int \sqrt{|g_u|}\, dt\, dy$$

and the surface parameterized by Z will be minimal if and only if the first variation of A vanishes. This can be written as

$$2\, D_u A(v) = \int \frac{1}{\sqrt{|g_u|}} D_u |g_u|\, (v)\, dt\, dy.$$

Observe that

$$\begin{aligned} \frac{1}{\sqrt{|g_u|}}\, D_u |g_u|\, (v) = 2 \left(\partial_t u\, \partial_t v + \partial_y u\, \partial_y v - 2\,(a^2 + b^2)\, u\, v \right) \\ + \cosh \omega\, Q \left(\frac{u}{\cosh \omega}, \frac{\nabla u}{\cosh \omega} \right) v \\ + \cosh \omega\, \tilde{Q} \left(\frac{u}{\cosh \omega}, \frac{\nabla u}{\cosh \omega} \right) \partial_t v \\ + \cosh \omega\, \hat{Q} \left(\frac{u}{\cosh \omega}, \frac{\nabla u}{\cosh \omega} \right) \partial_y v \end{aligned} \tag{48}$$

where the operator Q, $\tilde{Q}$ and $\hat{Q}$ enjoy properties similar to the one enjoyed by Q_ε in the statement of the result.

The result then follows at once provided one notices that

$$\begin{aligned} |\partial_t \cosh \omega| + \left|\partial_y \cosh \omega\right| &\leq (|\omega_t| + |\omega_y|)\, \cosh \omega \\ &\leq (|a| + |b|)\, \cosh^2 \omega \leq c\, \cosh^2 \omega, \end{aligned} \tag{49}$$

for some constant $c > 0$ which does not depend on $\varepsilon \in (0, 1)$. This explains the $\cosh^2 \omega$ in front of the nonlinearity Q_ε whereas a quick inspection of (48) would have only suggested a $\cosh \omega$ term. □

We consider the surface Σ_ε parameterized by X_ε that we first dilate by a factor $(1 + \lambda)$ and then translate by $\frac{\varsigma}{1+\lambda}$ along the x_1-axis and by $(1 + \lambda)\, \log(1 + \lambda) + \sigma$ along the x_3-axis. This surface, that will be referred to as $\Sigma_\varepsilon(\lambda, \sigma, \varsigma)$, can be parameterized by

$$\begin{aligned} Y_{\varepsilon,\lambda,\sigma,\varsigma}(t, y) := (1 + \lambda)\, X_\varepsilon(t, y) + \frac{\varsigma}{1 + \lambda}\, e_1 \\ + ((1 + \lambda)\, \log(1 + \lambda) + \sigma)\, e_3. \end{aligned} \tag{50}$$

From now on, we assume that the parameters λ and ς are chosen to satisfy

$$|\lambda| + \varepsilon^{1/2}\, |\varsigma| \leq \kappa\, \varepsilon$$

for some constant $\kappa > 0$, that will be fixed later on.

Recall that we have defined

$$r_\varepsilon := \frac{1}{2}\, \varepsilon^{-1/2} \quad \text{and} \quad s_\varepsilon := -\frac{1}{2}\, \log \varepsilon + \log 2,$$

and, using the result of Lemma 3.2, we see that part of the surface $\Sigma_\varepsilon(\lambda, \sigma, \varsigma)$ (basically the one at height $t \sim s_\varepsilon$ is a vertical graph over the annulus $B_{2\, r_\varepsilon} - B_{r_\varepsilon/2}$ in the $x_3 = 0$ plane for the function

$$\tilde{U}_t^0(r, \theta) := U_t^0(r, \theta) + (1+\lambda)\, \log(1+\lambda) + \sigma$$

and we have the expansion

$$\tilde{U}_t^0(r, \theta) = (1+\lambda)\, \log(2r) + \sigma - \frac{\varepsilon}{2}\, r\, \cos\theta - \frac{\varsigma}{r}\, \cos\theta + \mathcal{O}_{\mathcal{C}_b^\infty}(\varepsilon) \quad (51)$$

that follows from (19). We now truncate the surface $\Sigma_\varepsilon(\lambda, \sigma, \varsigma)$. To this aim, we consider the curve $\Gamma_\varepsilon(\lambda, \sigma, \varsigma)$ that is the vertical graph over the circle of radius $r = r_\varepsilon$ in the x_3-plane, by the function $\tilde{U}_t^0$ defined above. Then $\Sigma_\varepsilon(\lambda, \sigma, \varsigma) - \Gamma_\varepsilon(\lambda, \sigma, \varsigma)$ has two connected components and the one that lies above $\Gamma_\varepsilon(\lambda, \sigma, \varsigma)$ is the truncated surface we are interested in, and we will denote it by $\Sigma_\varepsilon^t(\lambda, \sigma, \varsigma)$.

We are interested in normal graphs over the surface $\Sigma_\varepsilon^t(\lambda, \sigma, \varsigma)$ that are minimal surfaces and are asymptotic to $\Sigma_\varepsilon^t(\lambda, \sigma, \varsigma)$. Thanks to Proposition 5.1, we can state that the surface parameterized by

$$(1+\lambda)\, (X_\varepsilon + u\, N_\varepsilon) + \frac{\varsigma}{1+\lambda}\, e_1 + ((1+\lambda)\, \log(1+\lambda) + \sigma)\, e_3$$

is minimal, if and only if the function u is a solution of

$$L_\varepsilon\, u = (1+\lambda)^{-1} \times \cosh^2 \omega Q_\varepsilon \left((1+\lambda) \frac{u}{\cosh\omega},\, (1+\lambda)\, \frac{\nabla u}{\cosh\omega},\, (1+\lambda)\, \frac{\nabla^2 u}{\cosh\omega} \right) \quad (52)$$

where the operator L_ε is the one defined in (38). The fact that the perturbed surface is asymptotic to the nonperturbed one can then be translated into the fact that the function u tends to 0 at ∞.

Two modifications are required. First, even though the surface $\Sigma_\varepsilon^t(\lambda, \sigma, \varsigma)$ can be parameterized by (50), its boundary does not correspond to the curve $t = s_\varepsilon$. In fact, it follows from (22) in the proof of Lemma 3.2 that the boundary of the surface can be parameterized (in the (t, y) variables) as the curve

$$y \longmapsto \hat{t}_\varepsilon(y)$$

and, using (22), we get the estimate

$$|\hat{t}_\varepsilon(\cdot) - s_\varepsilon| \leq c\, \varepsilon^{1/2} \tag{53}$$

for some constant $c > 0$ independent of $\varepsilon \in (0, 1)$. We modify the parametrization given in (50) so that the part of $\Sigma_\varepsilon^t(\lambda, \sigma, \varsigma)$ corresponding to $t \geq s_\varepsilon + \log 2$ is still parameterized by (50), but, in a collar neighborhood of the boundary of this surface, the curves corresponding to $t = const$ in the parameterization given by (50) also correspond to the vertical graphs of curves $r = const$ by the function $\bar{U}_t^0$. To get an explicit formula, recall that we have defined in the proof of Lemma 3.2 the change of variables

$$(t, y) = K_\varepsilon(r, \theta).$$

Now we define

$$K_\varepsilon^0(r, \theta) := (\log(2r), \tau_\varepsilon\, \theta).$$

Then we define the change of variables

$$q(t, y) := \chi(t - s_\varepsilon)\, K_\varepsilon \circ \left(K_\varepsilon^0\right)^{-1}(t, y) + (1 - \chi(t - s_\varepsilon))\, (t, y)$$

where χ is a cutoff function identically equal to 1 on $(-\infty, \frac{1}{4} \log 2)$ and identically equal to 0 on $(\frac{3}{4} \log 2, \infty)$. It is easy to check, using the arguments in the proof of Lemma 3.2 that

$$\|q - Id\|_{\mathcal{C}_b^{2,\alpha}} \leq c\, \varepsilon^{1/2}$$

and in addition $q(s_\varepsilon, y) = (s_\varepsilon, y)$ and $q(t, y) = (t, y)$ for $t \geq \tilde{t}_\varepsilon + \log 2$. This being understood, we define the now parametrization of $\Sigma_\varepsilon^t(\lambda, \sigma, \varsigma)$ by

$$\tilde{Y}_{\varepsilon,\lambda,\sigma,\varsigma}(t, y) = Y_{\varepsilon,\lambda,\sigma,\varsigma}(q(t, y)).$$

Observe that, for all $t \in [s_\varepsilon, s_\varepsilon + \log 2]$ and $y \in S^1(\tau_\varepsilon)$, we have by construction,

$$\tilde{Y}_{\varepsilon,\lambda,\sigma,\varsigma}(t, y) = \left(r \cos\theta, r \sin\theta, \bar{U}_\varepsilon^0(r, \theta)\right)$$

where $r = \frac{1}{2} \log t$ and $\theta = y/\tau_\varepsilon$.

The second modification we need to do is concerned with the normal vector field about $\Sigma_\varepsilon^t(\lambda, \sigma, \varsigma)$ since we would like this vector field to be vertical in a collar neighborhood of the boundary of this surface. This can be achieved by modifying the normal vector field into a transverse vector field $\tilde{N}_\varepsilon$ that agrees with the unit normal vector field N_ε for all $t \geq s_\varepsilon + \log 2$ and that agrees with e_3 for all $t \in [s_\varepsilon, s_\varepsilon + \frac{1}{2} \log 2]$.

We consider a graph over this surface for some function u, using the modified vector field $\tilde{N}_\varepsilon$. This graph will be minimal if and only if the function u is a solution of some nonlinear elliptic equation that is not exactly equal to (52) because of the above two modifications. Indeed, starting from (52) and taking into account the effects of the change of parametrization we have done as well as the change in the vector field N_ε into $\tilde{N}_\varepsilon$, we see that the minimal surface equation now reads

$$L_\varepsilon\, u = \tilde{L}_\varepsilon\, u + \cosh^2\omega\, \tilde{Q}_\varepsilon \left(\frac{u}{\cosh\omega}, \frac{\nabla u}{\cosh\omega}, \frac{\nabla^2 u}{\cosh\omega} \right). \tag{54}$$

The nonlinear operator $\tilde{Q}_\varepsilon$ enjoys the same properties as Q_ε in Proposition 5.1. We will write for short

$$\hat{Q}_\varepsilon(u) := \tilde{Q}_\varepsilon \left(\frac{u}{\cosh\omega}, \frac{\nabla u}{\cosh\omega}, \frac{\nabla^2 u}{\cosh\omega} \right).$$

Observe that $\tilde{Q}_\varepsilon$ is explicitly given by

$$\tilde{Q}_\varepsilon(\cdot) = (1+\lambda)^{-1}\, Q_\varepsilon((1+\lambda)\cdot)$$

when $t \geq s_\varepsilon + \log 2$.

The operator $\tilde{L}_\varepsilon$ is a linear second order operator the partial derivatives of whose coefficients (computed with respect to the vector fields ∂_t and ∂_y) are supported in $[s_\varepsilon, s_\varepsilon + \log 2] \times S^1(\tau_\varepsilon)$ and are bounded by a constant times $\varepsilon^{1/2}$. Let us briefly comment on the estimate of the coefficients of $\tilde{L}_\varepsilon$. Since

$$\tilde{N}_\varepsilon \cdot N_\varepsilon = 1 + \mathcal{O}_{\mathcal{C}^{2,\alpha}_b}(\varepsilon)$$

when $t \in [s_\varepsilon, s_\varepsilon + \log 2]$, the result of Appendix B, shows that the change of vector field induces a linear operator whose coefficients are bounded by a constant times ε. Next, using (53), we see that the change of parametrization induces linear operator whose coefficients are bounded by a constant times $\varepsilon^{1/2}$. The estimate of the coefficients of $\tilde{L}_\varepsilon$ follows from these considerations.

Now, assume that we are given a function $\varphi \in \mathcal{C}^{2,\alpha}(S^1)$ that is even with respect to θ, L^2-orthogonal to 1 and $\theta \longmapsto \cos\theta$ and that satisfies

$$\|\varphi\|_{\mathcal{C}^{2,\alpha}} \leq \kappa\, \varepsilon. \tag{55}$$

We set

$$w_\varphi(\cdot,\cdot) := H_\varphi(\cdot - s_\varepsilon, \cdot/\tau_\varepsilon).$$

In order to solve (54), we choose

$$\mu \in (-2, -1)$$

and look for u of the form

$$u = w_\varphi + v$$

where $v \in \mathcal{C}^{2,\alpha}_\mu((s_\varepsilon, \infty) \times S^1(\tau_\varepsilon))$. Using the result of Proposition 4.1, we can rephrase this problem as a fixed point problem

$$v = S(\varphi; v) \tag{56}$$

where the nonlinear mapping S (which depends on ε and λ) is defined by

$$S(\varphi; v) := G_{\varepsilon, s_\varepsilon}\big(\tilde{L}_\varepsilon(w_\varphi + v) - L_\varepsilon\, w_\varphi + \cosh^2\omega\, \hat{Q}_\varepsilon(w_\varphi + v)\big)$$

where the operator $G_{\varepsilon, s_\varepsilon}$ is the one defined in Proposition 4.1. The existence of a fixed point of (56) is an easy consequence of the following technical:

Lemma 5.1. *There exist constants $c_\kappa > 0$ and $\varepsilon_\kappa > 0$, such that*

$$\|S(\varphi; 0)\|_{\mathcal{C}^{2,\alpha}_\mu} \leq c_\kappa\, (\varepsilon^{(3+\mu)/2} + \varepsilon^{4+2\mu}) \tag{57}$$

and, for all $\varepsilon \in (0, \varepsilon_\kappa)$

$$\|S(\varphi; v_2) - S(\varphi; v_1)\|_{\mathcal{C}^{2,\alpha}_\mu} \leq \frac{1}{2}\, \|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}_\mu}$$

and

$$\|S(\varphi_2; v) - S(\varphi_1; v)\|_{\mathcal{C}^{2,\alpha}_\mu} \leq c_\kappa\, (\varepsilon^{(1+\mu)/2} + \varepsilon^{3+2\mu})\, \|\varphi_2 - \varphi_1\|_{\mathcal{C}^{2,\alpha}}$$

provided the norms of $v, v_1, v_2 \in \mathcal{C}^{2,\alpha}_\mu([s_\varepsilon, \infty) \times S^1(\tau_\varepsilon))$ is bounded by $2\, c_\kappa (\varepsilon^{(3+\mu)/2} + \varepsilon^{4+2\mu})$ and $\varphi, \varphi_1, \varphi_2$ are even, L^2-orthogonal to 1 and $\theta \longmapsto \cos\theta$ and satisfy (55).

Proof. Using the properties of w_φ given in Lemma 4.3 together with the properties of $\tilde{L}_\varepsilon$, we immediately get

$$\|\tilde{L}_\varepsilon(w_\varphi)\|_{\mathcal{C}^{0,\alpha}_\mu} \leq c_\kappa\, \varepsilon^{(3+\mu)/2}.$$

Next, we use the fact that

$$L_\varepsilon\, w_\varphi = \left(\frac{1}{\tau_\varepsilon^2} - 1\right)\, \partial_\theta^2 w_\varphi + 2\,(a^2 + b^2)\, w_\varphi.$$

We have proved in (34) that $a^2 \leq \varepsilon^2$ and it follows from (36) in Lemma 3.6 that $|\frac{1}{\tau_\varepsilon^2} - 1| \leq c\, \varepsilon^2$ for ε small enough. Furthermore, b^2 is a $2\, t_\varepsilon$ periodic, even function and, thanks to Lemma 3.4, we know that $b^2 \leq c\, (\cosh t)^{-2}$ for all $t \in [0, t_\varepsilon]$ and some constant $c > 0$ independent of ε small enough. Therefore, we conclude (with little work) that

$$\|L_\varepsilon\, w_\varphi\|_{\mathcal{C}^{0,\alpha}_\mu} \leq c_\kappa\, (\varepsilon^{2+\mu/2} + \varepsilon^{4+2\mu}).$$

Observe that the expression on the right hand side comes essentially from the evaluation of $L_\varepsilon w_\varphi$ in the weighted norm, when $t \sim t_\varepsilon/2 \sim s_\varepsilon$ and $t \sim 2\, t_\varepsilon \sim 4\, s_\varepsilon$.

The last term in $S(\varphi; 0)$ can be estimated by

$$\left\| \cosh^2 \omega\, \hat{Q}_\varepsilon(w_\varphi) \right\|_{\mathcal{C}^{0,\alpha}_\mu} \leq c_\kappa\, \varepsilon^{2+\mu/2}.$$

The only difficulty in estimating this term comes from the estimate of the Hölder derivatives that will involve the Hölder derivative of $\frac{1}{\cosh \omega}$. To this aim, one uses (49) that implies that the first order partial derivatives of $\frac{1}{\cosh \omega}$ are bounded.

This completes the proof of the first estimate. The other estimates follow from similar considerations and is left to the reader. □

The previous Lemma shows that, provided ε is chosen small enough, the nonlinear mapping $S(\varphi; \cdot)$ is a contraction mapping from the ball of radius $2\, c_\kappa (\varepsilon^{(3+\mu)/2} + \varepsilon^{4+2\mu})$ in $\mathcal{C}^{2,\alpha}_\mu([s_\varepsilon, \infty) \times S^1(\tau_\varepsilon))$ into itself. Consequently $S(\varphi; \cdot)$ has a unique fixed point v_φ in this ball. This function v_φ provides a minimal surface $\Sigma^t_\varepsilon(\lambda, \sigma, \varsigma, \varphi)$ that is the image of $[s_\varepsilon, \infty) \times S^1(\tau_\varepsilon)$ by

$$\begin{aligned}\tilde{Z}_{\varepsilon,\lambda,\sigma,\varsigma,\varphi}(t, y) &:= (1+\lambda)\, (X_\varepsilon + (w_\varphi + v_\varphi)\, \tilde{N}_\varepsilon)(q(t, y)) \\ &\quad + \frac{\varsigma}{1+\lambda}\, e_1 + ((1+\lambda)\, \log(1+\lambda) + \sigma)\, e_3.\end{aligned}$$

This surface is asymptotic to half of the Riemann surface $\Sigma^t_\varepsilon(\lambda, \sigma, \varsigma)$. Observe that, thanks to the expansion (51), a collar neighborhood of the boundary of this surface is a vertical graph over the annulus $\bar{B}_{2 r_\varepsilon} - B_{r_\varepsilon}$ for some function U_t that can be expanded as

$$\begin{aligned}U_t(r, \theta) = (1+\lambda)\, \log(2r) + \sigma - \frac{\varepsilon}{2}\, r\, \cos\theta - \frac{\varsigma}{r}\, \cos\theta \\ + H_\varphi(\log(2r) - s_\varepsilon, \theta) + V_t(r, \theta).\end{aligned}$$

In this parametrization, the boundary of the surface corresponds to $r = r_\varepsilon$. Here the function $V_t = V_t(\varepsilon, \lambda, \varsigma, \varphi)$ depends nonlinearly on λ, φ and satisfies the following estimates

$$\| V_t(\varepsilon, \lambda, \varsigma, \varphi)(r_\varepsilon\, \cdot) \|_{\mathcal{C}^{2,\alpha}(\bar{B}_2 - B_1)} \leq c\, \varepsilon$$

and

$$\begin{aligned}&\| V_t(\varepsilon, \lambda, \varsigma, \varphi)(r_\varepsilon \cdot) - V_t(\varepsilon, \lambda, \varsigma, \varphi')(r_\varepsilon \cdot) \|_{\mathcal{C}^{2,\alpha}(\bar{B}_2 - B_1)} \\ &\leq c\, (\varepsilon^{1/2} + \varepsilon^{3+3\mu/2})\, \|\varphi - \varphi'\|_{\mathcal{C}^{2,\alpha}}\end{aligned} \tag{58}$$

where the constant $c > 0$ does not depend on ε or κ and c_κ only depends on κ but not on ε. The first estimate comes from the estimate on v_φ together with (51) while the second estimate is a consequence of the estimates in Lemma 5.1.

A similar analysis can be performed starting from the lower end of Riemann's surface, to obtain a minimal surface $\Sigma_\varepsilon^b(\lambda, \sigma, \varsigma, \varphi)$ that is asymptotic to a half of a Riemann's surface and that, near its boundary is a vertical graph over the annulus $\bar{B}_{2\, r_\varepsilon} - B_{r_\varepsilon}$ for some function U_b that can be expanded as

$$\begin{aligned} U_b(r,\theta) = &-(1+\lambda)\,\log(2r) - \sigma - \frac{\varepsilon}{2}\, r\, \cos\theta \\ &+ \frac{\varsigma}{r}\, \cos\theta + H_\varphi(\log(2r) - s_\varepsilon, \theta) + V_b(r,\theta) \end{aligned}$$

in which case the boundary of the surface corresponds to $r = r_\varepsilon$. The function V_b satisfies exactly the inequalities satisfied by the function V_t.

Remark 5.1. Equivalently one can define this second surface by applying a rotation of angle π about the x_2-axis to the surface $\Sigma_\varepsilon^t(\lambda, \sigma, \varsigma, \bar{\varphi})$, with $\bar{\varphi}(\cdot) := -\varphi(\cdot + \pi)$.

6 An infinite dimensional family of minimal surfaces which are close to M_k

We perform an analysis close to the one performed in the previous section, starting this time from the minimal surface $M_k(\xi)$ defined in Sect. 2 with

$$\xi = \frac{\varepsilon}{2}.$$

Recall that the surface $M_k(\varepsilon/2)$ has two ends $E_t(\varepsilon/2)$ and $E_b(\varepsilon/2)$ that can be parameterized as in (10) and (11). Also recall that, according to the result of Lemma 2.2, a portion of these ends can be written as a graph over the $x_3 = 0$ plane for functions $\bar{U}^0_{t,\varepsilon/2}$ and $\bar{U}^0_{b,\varepsilon/2}$ which are defined in the annulus $\bar{B}_{2r_\varepsilon} - B_{r_\varepsilon/2}$.

As in the previous section, we modify the parametrization of the end $E_t(\varepsilon/2)$ when $s \in [s_\varepsilon - \log 2, s_\varepsilon + \log 2]$, so that, when $r \in [3r_\varepsilon/4, 3\, r_\varepsilon/2]$ the curve corresponding to the image of

$$\theta \longmapsto \left(r\, \cos\theta, r\, \sin\theta, \bar{U}^0_{t,\varepsilon/2}(r,\theta)\right)$$

corresponds to the curve $s = \log(2r)$. The effect of this change of parametrization can be quantitatively estimated using (14) in the proof of Lemma 2.2. We perform a similar task for the parametrization of $E_b(\varepsilon/2)$ so that, when $r \in [3\, r_\varepsilon/4, 3\, r_\varepsilon/2]$ the curve corresponding to the image of

$$\theta \longmapsto \left(r\, \cos\theta, r\, \sin\theta, \bar{U}^0_{b,\varepsilon/2}(r,\theta)\right)$$

corresponds to the curve $s = -\log(2r)$.

This being understood and again as in the previous section, we modify the unit normal vector field on $M_k(\varepsilon/2)$ to produce a transverse unit vector field $\tilde{n}_\varepsilon$ that coincides with the normal vector field n_ε on $M_k(\varepsilon/2)$, is equal to e_3 on the graph over $\bar{B}_{3r_\varepsilon/2} - B_{3r_\varepsilon/4}$ of the functions $\bar{U}^0_{t,\varepsilon/2}$ and $\bar{U}^0_{b,\varepsilon/2}$ and interpolate smoothly in between the different definitions of $\tilde{n}_\varepsilon$ in different subsets of $M_k(\varepsilon/2)$.

The graph of the function u, using the vector field $\tilde{n}_\varepsilon$, will be a minimal surface if and only if u is a solution of a second order nonlinear elliptic equation of the form

$$\mathbb{L}_{M_k(\varepsilon/2)}\, u = \tilde{L}_\varepsilon\, u + Q_\varepsilon\,(u)$$

where $\mathbb{L}_{M_k(\varepsilon/2)}$ is the Jacobi operator about $M_k(\varepsilon/2)$, Q_ε is a nonlinear second order differential operator which collects all the nonlinear terms and $\tilde{L}_\varepsilon$ is a linear operator which takes into account the change of parametrization and the change of the normal vector field n_ε into $\tilde{n}_\varepsilon$, which are described above. Now, we can be more precise at the ends $E_t(\varepsilon/2)$ and $E_b(\varepsilon/2)$. For example at $E_t(\frac{\varepsilon}{2})$, and granted the above parametrization, the nonlinear operator Q_ε can be expanded as

$$Q_\varepsilon(u) = \frac{1}{\cosh^2 s}\left(Q_{2,\varepsilon}\left(\frac{u}{\cosh s}\right) + \cosh s\; Q_{3,\varepsilon}\left(\frac{u}{\cosh s}\right)\right)$$

where $Q_{2,\varepsilon}$ and $Q_{3,\varepsilon}$ are nonlinear second order differential operators which satisfy (4), uniformly in ε.

The operator $\tilde{L}_\varepsilon$ is a linear operator which is supported in $[s_\varepsilon - \log 2, s_\varepsilon + \log 2] \times S^1$ and has coefficients which are bounded by a constant times ε^2, uniformly in ε. In this estimate, the first ε comes from the conformal factor $(\cosh s)^{-2}$ and the second ε comes from the modification in the parametrization whose effect can be evaluated using (14) and also from the modification of the vector field since

$$n_\varepsilon \cdot \tilde{n}_\varepsilon = 1 + \mathcal{O}_{\mathcal{C}^{2,\alpha}_b}(\varepsilon)$$

as in the previous section.

Finally, observe that, in $E_t(\varepsilon/2)$, the difference

$$(\cosh s)^4\left(\mathbb{L}_{M_k(\varepsilon/2)} - \frac{1}{\cosh^2 s}\left(\partial_s^2 + \partial_\theta^2\right)\right)$$

is a second order differential operator in ∂_s and ∂_θ whose coefficients as well as their partial derivatives with respect to the vector fields ∂_s and ∂_θ, are bounded by a constant times ε. This follows easily from the expansion provided in (3) together with the fact that $w_{t,\varepsilon/2}$ decays at least like $(\cosh s)^{-2}$ on $E_t(\varepsilon/2)$.

Now, assume that we are given two functions $\varphi_t, \varphi_b \in \mathcal{C}^{2,\alpha}(S^1)$ which is even with respect to θ and L^2 orthogonal to 1 and $\theta \longmapsto \cos\theta$ and satisfy

$$\|\varphi_t\|_{\mathcal{C}^{2,\alpha}} + \|\varphi_b\|_{\mathcal{C}^{2,\alpha}} \leq \kappa\,\varepsilon. \tag{59}$$

We set $\Phi := (\varphi_t, \varphi_b)$ and we define w_Φ to be the function which is equal to $\chi_t\, H_{\varphi_t}(\cdot - t_\varepsilon, \cdot)$ on the image of $X_{t,\varepsilon/2}$ where χ_t is a cutoff function equal to 0 for $s \leq s_0 + 1$ and identically equal to 1 for $s \geq s_0 + 2$, and is equal to $\chi_b \, H_{\varphi_b}(\cdot + t_\varepsilon, \cdot)$ on the image of $X_{b,\varepsilon/2}$ where χ_b is a cutoff function equal to 0 for $s \geq -s_0 - 1$ and identically equal to 1 for $s \leq -s_0 - 2$.

We define $M_k^T(\varepsilon/2)$ to be equal to $M_k(\varepsilon/2)$ with the image of $(s_\varepsilon, +\infty) \times S^1$ by $X_{t,\varepsilon/2}$ and the image of $(-\infty, -s_\varepsilon) \times S^1$ by $X_{b,\varepsilon/2}$ excised. We would like to solve the equation

$$\gamma_{\varepsilon/2} \, \mathbb{L}_{M_k(\varepsilon/2)}(w_\Phi + v) = \gamma_{\varepsilon/2} \, (\tilde{L}_\varepsilon(w_\Phi + v) + Q_\varepsilon(w_\Phi + v))$$

on $M_k^T(\varepsilon/2)$, so that the graph of $w_\Phi + v$ will be a minimal surface. Here $\gamma_{\varepsilon/2}$ is the weight function on $M_k(\varepsilon/2)$ that has been defined just before Proposition 2.3.

We choose

$$\delta \in (1, 2)$$

and use the result of Proposition 2.3 so that we can rephrase the above problem as a fixed point problem

$$v = T(\Phi; v) \tag{60}$$

where

$$T(\Phi; v) = G_{\varepsilon/2,\delta} \circ \mathcal{E}_\varepsilon \, (\gamma_{\varepsilon/2} \, (\tilde{L}_\varepsilon(w_\Phi + v) - \mathbb{L}_{M_k(\varepsilon/2)} \, w_\Phi + Q_\varepsilon(w_\Phi + v)))$$

where $G_{\varepsilon/2,\delta}$ is the right inverse provided in Proposition 2.3 and $\mathcal{E}_\varepsilon$ is an extension (linear) operator

$$\mathcal{E}_\varepsilon : \mathcal{C}^{0,\alpha}_\delta\big(M_k^T(\varepsilon/2)\big) \longrightarrow \mathcal{C}^{0,\alpha}_\delta(M_k(\varepsilon/2)),$$

defined by $\mathcal{E}_\varepsilon \, v = v$ in $M_k^T(\varepsilon/2)$, $\mathcal{E}_\varepsilon \, v = 0$ on the image of $[s_\varepsilon + 1, +\infty) \times S^1$ by $X_{t,\varepsilon/2}$ and the image of $(-\infty, -s_\varepsilon - 1] \times S^1$ by $X_{b,\varepsilon/2}$ and $\mathcal{E}_\varepsilon \, v$ interpolate between these so that, for example,

$$(\mathcal{E}_\varepsilon v) \circ X_{t,\varepsilon/2}(t, \theta) = ((1 + s_\varepsilon - t) \, v) \circ X_{t,\varepsilon/2}(s_\varepsilon, \theta)$$

for $(t, \theta) \in [s_\varepsilon, s_\varepsilon + 1] \times S^1$. Here $\mathcal{C}^{0,\alpha}_\delta(M_k^T(\varepsilon/2))$ is the space of functions of $\mathcal{C}^{0,\alpha}_{loc}(M_k^T(\varepsilon/2))$ endowed with the weighted norm induced by the weighted norm in $\mathcal{C}^{0,\alpha}_\delta(M_k(\varepsilon/2))$. In particular, that

$$\|u\|_{\mathcal{C}^{0,\alpha}_\delta(M_k(\varepsilon/2))} = \|u|_{M_k^T(\varepsilon/2)}\|_{\mathcal{C}^{0,\alpha}_\delta(M_k^T(\varepsilon/2))}$$

for all $u \in \mathcal{C}^{0,\alpha}_\delta(M_k(\varepsilon/2))$.

As in the previous section, the existence of a fixed point $v \in \mathcal{C}^{2,\alpha}_\delta(M_k(\varepsilon/2))$ for (60) follows at once from the technical:

Lemma 6.1. *There exist constants $c_\kappa > 0$ and $\varepsilon_\kappa > 0$, such that*

$$\|T(\Phi; 0)\|_{\mathcal{C}^{2,\alpha}_\delta} \leq c_\kappa\, \varepsilon^2 \tag{61}$$

and, for all $\varepsilon \in (0, \varepsilon_\kappa)$

$$\|T(\Phi; v_2) - T(\Phi; v_1)\|_{\mathcal{C}^{2,\alpha}_\delta} \leq \frac{1}{2}\, \|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}_\delta}$$

and

$$\|T(\Phi_2; v_2) - T(\Phi_1; v_1)\|_{\mathcal{C}^{2,\alpha}_\delta} \leq c_\kappa\, \varepsilon\, \|\Phi_2 - \Phi_1\|_{\mathcal{C}^{2,\alpha}}$$

for all $v, v_1, v_2 \in \mathcal{C}^{2,\alpha}_\delta(M_k(\varepsilon/2))$ whose norm is bounded by $2\, c_\kappa\, \varepsilon^2$ and for all boundary data Φ, Φ_1, Φ_2 that are even, orthogonal to 1 *and* $\cos\theta$ *and satisfy* (59).

Proof. The proof is similar to the one in the proof of Lemma 5.1. Again, we use the result of Lemma 4.3 to obtain the estimate

$$\|\mathcal{E}_\varepsilon(\gamma_{\varepsilon/2}\, \mathbb{L}_{M_k(\varepsilon/2)}\, w_\Phi)\|_{\mathcal{C}^{0,\alpha}_\delta} \leq c_\kappa\, \varepsilon^2$$

and, using the properties of $\tilde{L}_\varepsilon$, we obtain

$$\|\mathcal{E}_\varepsilon(\gamma_{\varepsilon/2}\, \tilde{L}_\varepsilon\, w_\Phi)\|_{\mathcal{C}^{0,\alpha}_\delta} \leq c_\kappa\, \varepsilon^{\frac{4+\delta}{2}}.$$

Finally, we have

$$\|\mathcal{E}_\varepsilon(\gamma_{\varepsilon/2}\;\, Q_\varepsilon(w_\Phi))\|_{\mathcal{C}^{0,\alpha}_\delta} \leq c_\kappa\, \varepsilon^{3+\delta/2}.$$

We leave the details to the reader. □

The previous Lemma shows that, provided ε is chosen small enough, the nonlinear mapping $T(\Phi;\, \cdot)$ is a contraction mapping from the ball of radius $2\, c_\kappa\, \varepsilon^2$ in $\mathcal{C}^{2,\alpha}_\delta(M_k(\varepsilon/2))$ into itself. Consequently T has a unique fixed point v_Φ in this ball. Taking the graph over $M_k^T(\varepsilon/2)$ for the function $w_\Phi + v_\Phi$, we obtain a minimal surface $M_k(\varepsilon, \varphi_t, \varphi_b)$ that is close to $M_k^T(\varepsilon/2)$, has one horizontal end and two boundaries. Using the expansion of Lemma 2.2, we see that, close to its upper boundary, this surface is by construction a vertical graph over the annulus $\bar{B}_{r_\varepsilon} - B_{r_\varepsilon/2}$ for some function $\bar{U}_t$ that can be expanded as

$$\bar{U}_t(r, \theta) = \sigma_{t,\varepsilon/2} + \log(2r) - \frac{\varepsilon}{2}\, r\, \cos\theta + H_{\varphi_t}(s_\varepsilon - \log(2r), \theta) + \bar{V}_t(r, \theta)$$

and this surface is, close to its lower boundary, a vertical graph over the annulus $\bar{B}_{r_\varepsilon} - B_{r_\varepsilon/2}$ for some function $\bar{U}_b$ that can be expanded as

$$\begin{aligned}\bar{U}_b(r, \theta) =\; & -\sigma_{b,\varepsilon/2} - \log(2r) - \frac{\varepsilon}{2}\, r\, \cos\theta \\ & + H_{\varphi_b}(s_\varepsilon - \log(2r), \theta) + \bar{V}_b(r, \theta)\end{aligned}$$

where $\bar{V}_t = \bar{V}_t(\varepsilon, \Phi)$ and $\bar{V}_b = \bar{V}_b(\varepsilon, \Phi)$ depend nonlinearly on ε and Φ. The boundaries of the surface

$$M_k(\varepsilon, \varphi_t, \varphi_b)$$

correspond to $r = r_\varepsilon$.

Furthermore for $\bar{V} = \bar{V}_t$ or $\bar{V} = \bar{V}_b$, we have the estimates

$$\|\bar{V}(\varepsilon, \Phi)(r_\varepsilon \cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1 - B_{1/2})} \leq c\,\varepsilon$$

and

$$\|\bar{V}(\varepsilon, \Phi)(r_\varepsilon \cdot) - \bar{V}(\varepsilon, \Phi')(r_\varepsilon \cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1 - B_{1/2})} \leq c\,\varepsilon^{1-\delta/2}\,\|\Phi - \Phi'\|_{\mathcal{C}^{2,\alpha}} \quad (62)$$

where the constant $c > 0$ does not depend on ε or κ and c_κ only depends on κ but not on ε.

7 The matching of Cauchy data and the proof of the main result

We collect the results we have obtained in the last two sections and complete the proof of the main result. Using the result of Sect. 5, we obtain two minimal surfaces that are perturbations of the upper (rep. the lower end) of a Riemann's surface. The first surface

$$\Sigma^t_{\varepsilon+\eta_t}(\lambda_t, \sigma_{t,\varepsilon/2} + \sigma_t, \varsigma_t, \varphi_t)$$

depends on the parameters $\eta_t, \lambda_t, \sigma_t, \varsigma_t$ and the function φ_t and can be parameterized, in a collar neighborhood of its boundary as the vertical graph of

$$\begin{aligned} U_t(r, \theta) := (1 + \lambda_t)\,\log(2r) + \sigma_{t,\varepsilon/2} + \sigma_t - \frac{\varepsilon + \eta_t}{2}\, r\,\cos\theta \\ - \frac{\varsigma_t}{r}\,\cos\theta + H_{\varphi_t}(\log(2r) - s_\varepsilon, \theta) + \mathcal{O}_{\mathcal{C}^{2,\alpha}_b}(\varepsilon) \end{aligned}$$

over $\bar{B}_{2\,r_\varepsilon} - B_{r_\varepsilon}$. The second surface

$$\Sigma^b_{\varepsilon+\eta_b}(\lambda_b, \sigma_{b,\varepsilon/2} + \sigma_b, \varsigma_b, \varphi_b)$$

depends on the parameters $\eta_b, \lambda_b, \sigma_b, \varsigma_b$ and the function φ_b and can be parameterized, in a collar neighborhood of its boundary as the vertical graph of

$$\begin{aligned} U_b(r, \theta) := -\,(1 + \lambda_b)\,\log(2r) - \sigma_{b,\varepsilon/2} - \sigma_b - \frac{\varepsilon + \eta_b}{2}\, r\,\cos\theta \\ + \frac{\varsigma_b}{r}\,\cos\theta + H_{\varphi_b}(\log(2r) - s_\varepsilon, \theta) + \mathcal{O}_{\mathcal{C}^{2,\alpha}_b}(\varepsilon) \end{aligned}$$

over $\bar{B}_{2\,r_\varepsilon} - B_{r_\varepsilon}$.

Now, using the result of the last section, we obtain a minimal surface

$$M_k(\varepsilon, \tilde{\varphi}_b, \tilde{\varphi}_t)$$

that is a perturbation of a truncated genus k Costa–Hoffman–Meeks surface, has two boundaries and one end asymptotic to a horizontal plane. Moreover, this surface can be parameterized, in a collar neighborhood of its upper boundary as the vertical graph of

$$\begin{aligned}\bar{U}_t(r,\theta) := \; & \log(2r) + \sigma_{t,\varepsilon/2} - \frac{\varepsilon}{2} r \, \cos\theta \\ & + H_{\tilde{\varphi}_t}(s_\varepsilon - \log(2r), \theta) + \mathcal{O}_{\mathcal{C}^{2,\alpha}_b}(\varepsilon)\end{aligned}$$

over $\bar{B}_{r_\varepsilon} - B_{r_\varepsilon/2}$, while it can be parameterized in a collar neighborhood of its lower boundary as the vertical graph of

$$\begin{aligned}\bar{U}_b(r,\theta) := \; & -\log(2r) - \sigma_{b,\varepsilon/2} - \frac{\varepsilon}{2} r \, \cos\theta \\ & + H_{\tilde{\varphi}_b}(s_\varepsilon - \log(2r), \theta) + \mathcal{O}_{\mathcal{C}^{2,\alpha}_b}(\varepsilon)\end{aligned}$$

over $\bar{B}_{r_\varepsilon} - B_{r_\varepsilon/2}$.

We assume that the parameters and the boundary functions are chosen so that

$$\begin{aligned}\varepsilon^{-1/2}\,(|\eta_t| + |\eta_b|) + \varepsilon^{1/2}\,(|\varsigma_t| + |\varsigma_b|) + |\sigma_t + \log(2r_\varepsilon)\,\lambda_t| \\ + |\sigma_b + \log(2r_\varepsilon)\,\lambda_b| + |\lambda_t| + |\lambda_b| + \|\varphi_t\|_{\mathcal{C}^{2,\alpha}} + \|\varphi_b\|_{\mathcal{C}^{2,\alpha}} \\ + \|\tilde{\varphi}_t\|_{\mathcal{C}^{2,\alpha}} + \|\tilde{\varphi}_b\|_{\mathcal{C}^{2,\alpha}} \leq \kappa\,\varepsilon\end{aligned}$$

where the constant $\kappa > 0$ is fixed large enough. Recall that the functions φ_t, φ_b, $\tilde{\varphi}_t$ and $\tilde{\varphi}_b$ are assumed to be even and L^2-orthogonal to the functions 1 and $\theta \longmapsto \cos\theta$.

Here, the functions $\mathcal{O}_{\mathcal{C}^{2,\alpha}_b}(\varepsilon)$ in the above expansions are precisely the functions V_t, V_b, $\bar{V}_t$ and $\bar{V}_b$ that appear at the end of Sects. 5 and 6. They depend nonlinearly on the different parameters and boundary data functions but they are bounded by a constant (independent of κ and ε) times ε in $\mathcal{C}^{2,\alpha}_b$ topology, where partial derivatives are taken with respect to the vector fields $r\,\partial_r$ and ∂_θ.

It remains to show that, for all ε small enough, it is possible to choose the parameters and boundary functions in such a way that the surface

$$\begin{aligned}& \Sigma^t_{\varepsilon+\eta_t}(\lambda_t, \sigma_{t,\varepsilon/2} + \sigma_t, \varsigma_t, \varphi_t) \cup M_k(\varepsilon, \tilde{\varphi}_b, \tilde{\varphi}_t) \\ & \cup \Sigma^b_{\varepsilon+\eta_b}(\lambda_b, \sigma_{b,\varepsilon/2} + \sigma_b, \varsigma_b, \varphi_b)\end{aligned}$$

is a $\mathcal{C}^1$ surface across the boundaries of the different summands. Regularity theory will then ensure that this surface is in fact smooth and by construction is has the desired properties. This will therefore complete the proof of the main theorem.

Granted the description of the surfaces close to their respective boundaries it is enough to fulfill the following system of equations

$$U_t(r_\varepsilon\,\cdot) = \bar U_t(r_\varepsilon\,\cdot) \qquad U_b(r_\varepsilon\,\cdot) = \bar U_b(r_\varepsilon\,\cdot)$$
$$\partial_r U_t(r_\varepsilon\,\cdot) = \partial_r \bar U_t(r_\varepsilon\,\cdot) \qquad \partial_r U_b(r_\varepsilon\,\cdot) = \partial_r \bar U_b(r_\varepsilon\,\cdot)$$

on S^1.

The first two equations lead to the system

$$\begin{cases} \log(2r_\varepsilon)\,\lambda_t + \sigma_t - \left(\frac{1}{2}\,r_\varepsilon\,\eta_t + \frac{1}{r_\varepsilon}\,\varsigma_t\right)\,\cos\theta + \varphi_t - \tilde\varphi_t = \mathcal{O}_{\mathcal{C}^{2,\alpha}}(\varepsilon) \\ -\log(2\,r_\varepsilon)\,\lambda_b - \sigma_b - \left(\frac{1}{2}\,r_\varepsilon\,\eta_b - \frac{1}{r_\varepsilon}\,\varsigma_b\right)\,\cos\theta + \varphi_b - \tilde\varphi_b = \mathcal{O}_{\mathcal{C}^{2,\alpha}}(\varepsilon) \end{cases} \tag{63}$$

while the last two equations give the system

$$\begin{cases} \lambda_t - \left(\frac{1}{2}\,r_\varepsilon\,\eta_t - \frac{1}{r_\varepsilon}\,\varsigma_t\right)\,\cos\theta + \partial_t(H_{\varphi_t+\tilde\varphi_t})(0,\cdot) = \mathcal{O}_{\mathcal{C}^{1,\alpha}}(\varepsilon) \\ -\lambda_b - \left(\frac{1}{2}\,r_\varepsilon\,\eta_b + \frac{1}{r_\varepsilon}\,\varsigma_b\right)\,\cos\theta + \partial_t(H_{\varphi_b+\tilde\varphi_b})(0,\cdot) = \mathcal{O}_{\mathcal{C}^{1,\alpha}}(\varepsilon). \end{cases} \tag{64}$$

Here, the functions $\mathcal{O}_{\mathcal{C}^{\ell,\alpha}}(\varepsilon)$ in the above expansions are depend nonlinearly on the different parameters and boundary data functions but they are bounded by a constant (independent of κ and ε) times ε in $\mathcal{C}^{\ell,\alpha}(S^1)$ topology.

Projecting every equation of this system over the L^2-orthogonal complement of Span$\{1, \cos\theta\}$, we obtain the system

$$\varphi_t - \tilde\varphi_t = \mathcal{O}_{\mathcal{C}^{2,\alpha}}(\varepsilon) \qquad \varphi_b - \tilde\varphi_b = \mathcal{O}_{\mathcal{C}^{2,\alpha}}(\varepsilon)$$
$$\partial_t H_{\varphi_t+\tilde\varphi_t}(0,\cdot) = \mathcal{O}_{\mathcal{C}^{1,\alpha}}(\varepsilon) \qquad \partial_t H_{\varphi_b+\tilde\varphi_b}(0,\cdot) = \mathcal{O}_{\mathcal{C}^{1,\alpha}}(\varepsilon). \tag{65}$$

The following result is well known and we refer to [4] for a proof.

Lemma 7.1. *The operator*

$$\begin{aligned} \mathcal{C}^{2,\alpha}(S^1) &\longrightarrow \mathcal{C}^{1,\alpha}(S^1) \\ \varphi &\longmapsto \partial_t H_\varphi(0,\cdot) \end{aligned}$$

acting on functions that are orthogonal to 1 *and* $\theta \longmapsto \cos\theta$ *in the* L^2*-sense and are even, is invertible.*

Proof. Observe that if we decompose

$$\varphi = \sum_{j\geq 2} \varphi_j\,\cos(j\theta),$$

then

$$\partial H_\varphi(0,\theta) = -\sum_{j\geq 2} j\,\varphi_j\,\cos(j\theta),$$

that is clearly invertible from $H^1(S^1)$ into $L^2(S^1)$. Now elliptic regularity theory implies that this is also the case when this operator is defined between Hölder spaces. □

Using this result, the last system (65) can be rewritten as

$$(\varphi_t, \tilde{\varphi}_t, \varphi_b, \tilde{\varphi}_b) = \mathcal{O}_{\mathcal{C}^{2,\alpha}}(\varepsilon). \tag{66}$$

Recall that the right hand side depends nonlinearly on φ_t, $\tilde{\varphi}_t$, φ_b, $\tilde{\varphi}_b$ (and also on the parameters λ_t, λ_b, σ_t, σ_b, ς_t, ς_b, η_t and η_b). We look at this equation as a fixed point problem and fix κ large enough. Thanks to (58) and (62) we can use a fixed point theorem for contraction mapping in the ball of radius $\kappa\varepsilon$ in $(\mathcal{C}^{2,\alpha}(S^1))^4$ to obtain, for all ε small enough, a solution $(\varphi_t, \tilde{\varphi}_t, \varphi_b, \tilde{\varphi}_b)$ of (66). This solution being obtained a fixed point for contraction mapping and the right hand side of (66) being continuous with respect to all data, we see that this fixed point $(\varphi_t, \tilde{\varphi}_t, \varphi_b, \tilde{\varphi}_b)$ depends at continuously (and in fact smoothly) on the parameters λ_t, λ_b, σ_t, σ_b, ς_t, ς_b, η_t and η_b.

Inserting this solution $(\varphi_t, \tilde{\varphi}_t, \varphi_b, \tilde{\varphi}_b)$ into (63) and (64), we see that it remains to solve a system of the form

$$\begin{cases} \log(2\, r_\varepsilon)\, \lambda_t + \sigma_t - \left(\frac{1}{2}\, r_\varepsilon\, \eta_t + \frac{1}{r_\varepsilon}\, \varsigma_t\right) \cos\theta = \mathcal{O}(\varepsilon) \\ -\log(2\, r_\varepsilon)\, \lambda_b - \sigma_b - \left(\frac{1}{2}\, r_\varepsilon\, \eta_b - \frac{1}{r_\varepsilon}\, \varsigma_b\right) \cos\theta = \mathcal{O}(\varepsilon) \\ \lambda_t - \left(\frac{1}{2}\, r_\varepsilon\, \eta_t - \frac{1}{r_\varepsilon}\, \varsigma_t\right) \cos\theta = \mathcal{O}(\varepsilon) \\ -\lambda_b - \left(\frac{1}{2}\, r_\varepsilon\, \eta_b + \frac{1}{r_\varepsilon}\, \varsigma_b\right) \cos\theta = \mathcal{O}(\varepsilon) \end{cases} \tag{67}$$

where this time, the right hand sides only depend nonlinearly on λ_t, λ_b, σ_t, σ_b, ς_t, ς_b, η_t and η_b.

There are in fact 8 equations that are obtained by projecting this system over the constant function and the function $\theta \longmapsto \cos\theta$. This system can be rewritten as

$$(\lambda_t, \lambda_b, \bar{\sigma}_t, \bar{\sigma}_b, \bar{\varsigma}_t, \bar{\varsigma}_b, \bar{\eta}_t, \bar{\eta}_b) = \mathcal{O}(\varepsilon) \tag{68}$$

where we have set

$$(\bar{\varsigma}_t, \bar{\varsigma}_b) := \varepsilon^{1/2}\, (\varsigma_t, \varsigma_b),\ (\bar{\sigma}_t, \bar{\sigma}_b) := (\sigma_t + \log(2\, r_\varepsilon)\, \lambda_t, \sigma_b + \log(2\, r_\varepsilon)\, \lambda_b)$$

and

$$(\bar{\eta}_t, \bar{\eta}_b) := \varepsilon^{-1/2}\, (\eta_t, \eta_b).$$

This time, provided κ has been fixed large enough, we can use Schauder's Fixed point Theorem in the ball of radius $\kappa\varepsilon$ in $\mathbb{R}^8$ to solve (68), for all ε small enough. This provides a set of parameters and a set of boundary data such that (63) and (64) hold and hence this completes the proof of a solution of (63)–(64) and hence the proof of the main theorem.

Remark 7.1. Alternatively, with more work, one could have used a fixed point argument for contraction mapping to solve (68). We have not chosen to follow this route since this would have considerably increased the technicalities of the proof of the result.

8 Appendix A

We consider the surface parameterized by

$$X = X_c + w\, n_c.$$

The coefficients of g_w, the first fundamental form of this surface, are given by

$$\begin{aligned}|\partial_s X|^2 &= \cosh^2 s - 2\,w + \frac{1}{\cosh^2 s}\, w^2 + (\partial_s w)^2\\ |\partial_\theta X|^2 &= \cosh^2 s + 2\,w + \frac{1}{\cosh^2 s}\, w^2 + (\partial_\theta w)^2\end{aligned}$$

and

$$\partial_s X \cdot \partial_\theta X = \partial_s w\, \partial_\theta w.$$

It follows from these that the determinant of the metric g_w can be expanded as

$$\begin{aligned}|g_w| =& \cosh^4 s \left(1 + \frac{1}{\cosh^2 s}\left((\partial_s w)^2 + (\partial_\theta w)^2 - \frac{2}{\cosh^2 s}\, w^2\right)\right.\\ &\left. + \left(\frac{1}{\cosh s}\, P_3\left(\frac{w}{\cosh s}, \frac{\nabla w}{\cosh s}\right) + P_4\left(\frac{w}{\cosh s}, \frac{\nabla w}{\cosh s}\right)\right)\right)\end{aligned}$$

where the P_i are homogeneous polynomials of degree i, whose coefficients are bounded smooth functions of s and θ.

We consider the area energy

$$A(w) := \int \sqrt{|g_w|}\, ds\, d\theta.$$

The surface parameterized by X_w is minimal if and only if the first variation of A at w is 0. This can be written as

$$2\, D_w A(v) = \int \frac{1}{\sqrt{|g_w|}}\, D_w |g_w|\,(v)\, ds\, d\theta.$$

Observe that

$$\begin{aligned}\frac{1}{\sqrt{|g_w|}}\, D_w|g_w|\,(v) =& \partial_s w\, \partial_s v + \partial_\theta w\, \partial_\theta v - \frac{2}{\cosh^2 s}\, w\, v\\ &+ \left(\tilde{Q}_2\left(\frac{w}{\cosh s}, \frac{\nabla w}{\cosh s}\right) + \cosh s\, \tilde{Q}_3\left(\frac{w}{\cosh s}, \frac{\nabla w}{\cosh s}\right)\right) v\\ &+ \left(\tilde{Q}'_2\left(\frac{w}{\cosh s}, \frac{\nabla w}{\cosh s}\right) + \cosh s\, \tilde{Q}'_3\left(\frac{w}{\cosh s}, \frac{\nabla w}{\cosh s}\right)\right) \partial_s v\\ &+ \left(\tilde{Q}''_2\left(\frac{w}{\cosh s}, \frac{\nabla w}{\cosh s}\right) + \cosh s\, \tilde{Q}''_3\left(\frac{w}{\cosh s}, \frac{\nabla w}{\cosh s}\right)\right) \partial_\theta v\end{aligned} \tag{69}$$

where the operators $Q_2, \dots$ and the operators $\tilde{Q}_3, \dots$ enjoy properties similar to the one enjoyed by Q_2 and Q_3 in the statement of the result. The result then follows at once.

9 Appendix B

This appendix is essentially a generalization of the corresponding analysis in [13]. Let be Σ be a smooth oriented surface embedded in a Riemannian manifold (M, g). We denote by N the unit normal vector field compatible with the orientation of Σ. Suppose that $\tilde{N}$ is another unit vector field transverse to Σ, the implicit function theorem implies that, given $p_0 \in \Sigma$, there exist neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of $(p_0, 0) \in \Sigma \times \mathbb{R}$ and a diffeomorphism $(p, s) \longmapsto (\varphi(p, s), \psi(p, s))$ from $\mathcal{U}$ to $\mathcal{V}$ such that

$$\mathrm{Exp}_p^M(s\,\tilde{N}(p)) = \mathrm{Exp}^M_{\varphi(p,s)}(\psi(p, s)\, N(\varphi(p, s))) \tag{70}$$

where Exp^M denotes the exponential map in (M, g). In addition $\varphi(p, 0) = p$ and $\psi(p, 0) = 0$.

Differentiation of (70) with respect to s at $s = 0$ yields

$$\tilde{N}(p) = \partial_s\varphi(p, 0) + \partial_s\psi(p, 0)\, N(p). \tag{71}$$

Taking the scalar product with $N(p)$ we conclude that

$$g(\tilde{N}(p), N(p)) = \partial_s\psi(p, 0). \tag{72}$$

This immediately implies that $\psi(p, s) = g(\tilde{N}(p), N(p))\, s + \mathcal{O}(s^2)$. On the other hand, projection of (71) over $T_p\Sigma$ yields

$$\tilde{N}^t(p) = \partial_s\varphi(p, 0) \tag{73}$$

where $\tilde{N}^t(p)$ is the tangential component of $\tilde{N}$.

Next any surface $\tilde{\Sigma}$ sufficiently close to Σ can be either parameterized as a graph of the function w over Σ using the vector field $\tilde{N}$ or the graph of the function $\bar{w}$ for the normal vector field N. Thanks to the above analysis we can write

$$\bar{w}(\varphi(p, w(p))) = \psi(p, w(p)).$$

Now, the mean curvature of the surface Σ at the point $\mathrm{Exp}_p^M(w(p)\,\tilde{N}(p))$ and at the point $\mathrm{Exp}_{\bar{p}}^M(\bar{w}(\bar{p})\, N(\bar{p}))$ are the same if $\bar{p} = \varphi(p, w(p))$. We phrase this property as

$$H_{\tilde{N},w}(p) = H_{N,\bar{w}}(\bar{p}).$$

Differentiation with respect to w at $w = 0$ yields

$$D_w H_{\tilde{N},0}(\cdot) = D_{\bar{w}}\, H_{N,0}(\partial_s\psi\,\cdot) + D_{\bar{p}}\, H_{N,0}(\partial_s\varphi\,\cdot).$$

Taking into account the partial derivatives of φ and ψ, which are given in (73) and (72), we conclude that

$$D_w H_{\tilde{N},0}(\cdot) = D_{\bar{w}} H_{N,0}(g(\tilde{N}(p), N(p)) \cdot) + (\nabla_{\tilde{N}^t(p)} H_{N,0}) \cdot$$

for any smooth function u defined on Σ. In the special case where Σ has constant mean curvature, we simply get

$$D_w H_{\tilde{N},0}(u) = D_{\bar{w}} H_{N,0}(g(\tilde{N}(p), N(p))\, u)$$

which gives the relation between the Jacobi operator about Σ and $D_w H_{\tilde{N},0}$ the linearized mean curvature operator when the normal vector field N is changed into a transverse vector field $\tilde{N}$.

References

1. Brezis, H.: Analyse Fonctionnelle. Dunod (1983)
2. Costa, C.J.: Imersões minimas en $\mathbb{R}^3$ de gênero un e curvatura total finita. PhD thesis, IMPA, Rio de Janeiro, Brasil (1982)
3. Costa, C.J.: Example of a complete minimal immersion in $\mathbb{R}^3$ of genus one and three embedded ends. Bol. Soc. Bras. Mat. **15**(1–2), 47–54 (1984)
4. Fakhi, S., Pacard, F.: Existence of complete minimal hypersurfaces with finite total curvature. Manuscr. Math. **103**, 465–512 (2000)
5. Hauswirth, L.: Minimal surfaces of Riemann type in three-dimensional product manifolds. Pac. J. Math. **224**(1), 91–117 (2006)
6. Hoffman, D., Karcher, H.: Complete embedded minimal surfaces of finite total curvature. In: Geometry, V, Encyclopaedia Math. Sci., vol. 90, pp. 5–93, 267–272. Springer, Berlin (1997)
7. Hoffman, D., Meeks, W.H., III: The asymptotic behavior of properly embedded minimal surfaces of finite topology. J. Am. Math. Soc. **4**(2), 667–681 (1989)
8. Hoffman, D., Meeks, W.H., III: Embedded minimal surfaces of finite topology. Ann. Math. **131**, 1–34 (1990)
9. Jleli, M.: Constant mean curvature hypersurfaces. PhD Thesis, University Paris 12 (2004)
10. Korevaar, N., Kusner, R., Solomon, B.: The structure of complete embedded surfaces with constant mean curvature. J. Differ. Geom. **30**, 465–503 (1989)
11. Kusner, R., Mazzeo, R., Pollack, D.: The moduli space of complete embedded constant mean curvature surfaces. Geom. Funct. Anal. **6**, 120–137 (1996)
12. Mazzeo, R.: Elliptic theory of edge operators I. Commun. Partial Differ. Equations **16**(10), 1616–1664 (1991)
13. Mazzeo, R., Pacard, F.: Constant mean curvature surfaces with Delaunay ends. Commun. Anal. Geom. **9**(1), 169–237 (2001)
14. Melrose, R.: The Atiyah–Patodi–Singer index theorem. Res. Notes Math. (1993)
15. Meeks, W.H., III, Perez, J., Ros, A.: Uniqueness of the Riemann minimal examples. Invent. Math. **131**, 107–132 (1998)
16. Meeks, W.H., III, Perez, J., Ros, A.: The geometry of minimal surfaces of finite genus I; curvature estimates and quasiperiodicity. J. Differ. Geom. **66**(1), 1–45 (2004)
17. Meeks, W.H., III, Perez, J., Ros, A.: The geometry of minimal surfaces of finite genus II; non existence of one limit end examples. Invent. Math. **158**(2), 323–341 (2004)
18. Nayatani, S.: Morse index of complete minimal surfaces. In: Rassias, T.M. (ed.) The Problem of Plateau, pp. 181–189 (1992)
19. Nayatani, S.: Morse index and Gauss maps of complete minimal surfaces in Euclidean 3-space. Comment. Math. Helv. **68**(4), 511–537 (1993)

20. Perez, J., Ros, A.: The space of properly embedded minimal surfaces with finite total curvature. Indiana Univ. Math. J. **45**(1), 177–204 (1996)
21. Riemann, B.: Oeuvres Mathématiques de Riemann. Gauthiers-Villards, Paris (1898)
22. Shiffman, M.: On surfaces of stationary area bounded by two circles, or convex curves, in parallel planes. Ann. Math. (2) **63**, 77–90 (1956)