Proc. London Math. Soc. (3) 98 (2009) 445–470
doi:10.1112/plms/pdn038

Half-space theorem, embedded minimal annuli and minimal graphs in the Heisenberg group

Benoît Daniel and Laurent Hauswirth

Abstract

We construct a one-parameter family of properly embedded minimal annuli in the Heisenberg group Nil_3 endowed with a left-invariant Riemannian metric. These annuli are not rotationally invariant. This family of annuli is used to prove a vertical half-space theorem which is then applied to prove that each complete minimal graph in Nil_3 is entire. Also, it is shown that the sister surface of an entire minimal graph in Nil_3 is an entire constant mean curvature (CMC) $\frac{1}{2}$ graph in $\mathbb{H}^2 \times \mathbb{R}$, and vice versa. This gives a classification of all entire CMC $\frac{1}{2}$ graphs in $\mathbb{H}^2 \times \mathbb{R}$. Finally we construct properly embedded CMC $\frac{1}{2}$ annuli in $\mathbb{H}^2 \times \mathbb{R}$.

1. *Introduction*

This paper deals with global properties of minimal and constant mean curvature (CMC) surfaces in Riemannian homogeneous manifolds. Some interesting properties are the existence of a Hopf-type holomorphic quadratic differential (see [**1**, **2**]) and Lawson-type local isometric correspondences, in particular, between minimal surfaces in the Heisenberg group Nil_3 endowed with a left-invariant Riemannian metric and CMC $\frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$ (see [**8**]). Two surfaces related by this correspondence are called sister surfaces.

In this paper we first construct a one-parameter family of properly embedded minimal annuli that are somewhat analogous to catenoids of $\mathbb{R}^3$ with a 'horizontal axis'.

Theorem (Theorem 5.6). *There exists a one-parameter family $(\mathcal{C}_\alpha)_{\alpha>0}$ of properly embedded minimal annuli in* Nil_3*, called 'horizontal catenoids', having the following properties:*
- *the annulus $\mathcal{C}_\alpha$ is not invariant by a one-parameter group of isometries;*
- *the intersection of $\mathcal{C}_\alpha$ and any vertical plane of equation $x_2 = c (c \in \mathbb{R})$ is a non-empty closed embedded convex curve;*
- *the annulus $\mathcal{C}_\alpha$ is invariant by rotations of angle π around the x_1, x_2 and x_3 axes and the x_2-axis is contained in the 'interior' of $\mathcal{C}_\alpha$;*
- *the annulus $\mathcal{C}_\alpha$ is conformally equivalent to $\mathbb{C} \setminus \{0\}$.*

(*The model we use for* Nil_3 *is described in Section* 2.)

Up to now, the only known examples of complete minimal surfaces in Nil_3 were surfaces invariant by a one-parameter group of isometries [**11**], periodic surfaces [**2**] and entire graphs [**7**]. The annuli we construct are the first non-trivial examples of annuli; they are very different from the rotationally invariant catenoids, which are of hyperbolic conformal type. The existence of annuli of parabolic and hyperbolic conformal type suggests that there might be a rich theory of properly embedded minimal annuli in Nil_3.

These 'horizontal catenoids' are obtained using the Weierstrass-type representation for minimal surfaces in Nil_3 (see [**7**]). We start with a suitable harmonic map into the hyperbolic disk; this harmonic map is expressed explicitly in terms of a solution of an ordinary differential

Received 14 December 2007; revised 3 June 2008; published online 7 October 2008.

2000 *Mathematics Subject Classification* 53A10, 53C42 (primary), 53A35, 53C43 (secondary).

equation (ODE) and it will be the Gauss map of the surface. We integrate the equations and then solve a period problem. Consequently, we obtain an explicit expression for these horizontal catenoids in terms of a solution of an ODE (see Proposition 5.1).

The second main point in this paper is to prove a half-space theorem. Discovered by Hoffman and Meeks [**16**], the half-space theorem for minimal surfaces of $\mathbb{R}^3$ is used to understand the global geometry of proper examples. In Nil_3, we prove that our family of 'horizontal catenoids' converges to a punctured vertical plane, and then we obtain a 'vertical half-space theorem' (vertical planes are defined in Section 2).

Theorem (Theorem 6.3). *Let Σ be a properly immersed minimal surface in* Nil_3. *Assume that Σ is contained on one side of a vertical plane P. Then Σ is a vertical plane parallel to P.*

We next deal with complete graphs. There is a natural notion of graph in Nil_3. Indeed, Nil_3 admits a Riemannian fibration $\pi : \mathrm{Nil}_3 \to \mathbb{R}^2$ over the Euclidean plane. We will denote by ξ a unit vector field tangent to the fibers and we will call it a vertical vector field; it is a Killing field. Then a surface Σ in Nil_3 is a multigraph if it is transverse to ξ, it is a graph if it is transverse to ξ and $\pi_{|\Sigma} : \Sigma \to \mathbb{R}^2$ is injective, and it is an entire graph if it is transverse to ξ and $\pi_{|\Sigma} : \Sigma \to \mathbb{R}^2$ is bijective.

A natural problem is to determine if a complete multigraph is necessarily entire. We solve this problem using our half-space theorem and applying the arguments of P. Collin and H. Rosenberg in their forthcoming paper "The Jenkins–Serrin theorem for minimal graphs in homogeneous 3-manifolds" and of Hauswirth, Rosenberg and Spruck in [**14**]. This is the following theorem.

Theorem (Theorem 3.1). *Let Σ be a complete minimal immersed surface in* Nil_3. *If Σ is transverse to the vertical Killing field ξ, then Σ is an entire graph.*

Also, recently, Fernandez and Mira solved the Bernstein problem in Nil_3. We denote by $\mathbb{C}$ the complex plane and by $\mathbb{D}$ the unit disk $\{z \in \mathbb{C}; |z| < 1\}$.

Theorem [**10**]. *Let Q be a holomorphic quadratic differential on $\mathbb{D}$ or a non-identically zero holomorphic quadratic differential on $\mathbb{C}$. Then there exists a two-parameter family of generically non-congruent entire minimal graphs in* Nil_3 *whose Abresch–Rosenberg differential is Q.*

Conversely, all entire minimal graphs belong to these families.

Together with our Theorem 3.1, this gives a classification of all complete minimal graphs in Nil_3.

In this paper we will also deal with CMC $\frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$. For these surfaces, Hauswirth, Rosenberg and Spruck [**14**] proved a half-space-type theorem and used it to show that complete multigraphs are entire, that is, graphs over the whole hyperbolic plane $\mathbb{H}^2$.

Our proof of Theorem 6.3 is different from the proof of the half-space-type theorem in [**14**]: the main point in their proof is the construction of a continuous family of compact annuli with boundaries, contained between two horocylinders of $\mathbb{H}^2 \times \mathbb{R}$, and converging to one of the horocylinders; they use Schauder's fixed point theorem in a quasi-linear equation; they have to control the mean curvature vector in the maximum principle. Our proof uses the family of complete annuli and the classical geometrical argument of Hoffman and Meeks [**16**].

Also, by our Theorem 3.1 and [**10**] we show that entire minimal graphs in Nil_3 correspond exactly to entire CMC $\frac{1}{2}$ graphs in $\mathbb{H}^2 \times \mathbb{R}$ by the sister surface correspondence (Corollary 3.3). Hence we obtain a classification of all entire CMC $\frac{1}{2}$ graphs in $\mathbb{H}^2 \times \mathbb{R}$ in the following theorem, which solves the Bernstein problem for CMC $\frac{1}{2}$ graphs in $\mathbb{H}^2 \times \mathbb{R}$.

Theorem. *Let Q be a holomorphic quadratic differential on $\mathbb{D}$ or a non-identically zero holomorphic quadratic differential on $\mathbb{C}$. Then there exists a two-parameter family of generically non-congruent entire CMC $\frac{1}{2}$ graphs in $\mathbb{H}^2 \times \mathbb{R}$ whose Abresch–Rosenberg differential is Q.*

Conversely, all entire CMC $\frac{1}{2}$ graphs belong to these families.

Observe that this theorem could not be obtained using the method of [**10**]; indeed their solution of the Bernstein problem for minimal surfaces in Nil_3 is based on the relations between minimal immersions in Nil_3 and space-like CMC immersions in Minkowski space $\mathbb{L}^3$, and their arguments do not apply in our case.

We also construct a one-parameter family of properly embedded CMC $\frac{1}{2}$ annuli in $\mathbb{H}^2 \times \mathbb{R}$ which are analogous to our minimal horizontal catenoids in Nil_3.

Theorem (see Section 8). *There exists a one-parameter family $(\mathcal{C}_\alpha)_{\alpha>0}$ of properly embedded CMC $\frac{1}{2}$ annuli in $\mathbb{H}^2 \times \mathbb{R}$, called 'horizontal catenoids', having the following properties:*

- *the annulus $\mathcal{C}_\alpha$ is not invariant by a one-parameter group of isometries;*
- *the annulus $\mathcal{C}_\alpha$ is invariant by the reflections with respect to a horizontal plane and two orthogonal vertical planes, and it is a bigraph over some domain in a horizontal plane;*
- *the annulus $\mathcal{C}_\alpha$ is conformally equivalent to $\mathbb{C} \setminus \{0\}$.*

The curve of intersection of $\mathcal{C}_\alpha$ with its horizontal symmetry plane is similar to the profile curve of a rotational CMC 1 catenoid cousin in hyperbolic space $\mathbb{H}^3$ (see [**3**, **18**]). Moreover, this family converges to two punctured horocylinders tangent to each other. Hence it can be used to give an alternative proof of the half-space-type theorem of [**14**].

These annuli are the sister surfaces of helicoidal-type minimal surfaces in Nil_3 (see Section 7). They are obtained in a way similar to that of the minimal horizontal catenoids in Nil_3: we start from a suitable harmonic map into the hyperbolic disk and integrate the equations of [**10**]; the period problem is solved automatically using the symmetries. Hence we obtain an explicit expression in terms of a solution of an ODE.

The paper is organized as follows. In Section 2, we introduce material about harmonic maps, minimal surfaces in Nil_3 and CMC $\frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$. In Section 3, we give the proof and consequences of Theorem 3.1 assuming the vertical half-space theorem. In Section 4, we present the family of harmonic maps that will be used in the later sections. Section 5 is devoted to the construction of properly embedded minimal annuli in Nil_3. In Section 6, we prove our half-space theorem. In Section 7, we construct periodic helicoidal surfaces with horizontal 'axis'. In Section 8, we construct properly embedded CMC $\frac{1}{2}$ annuli in $\mathbb{H}^2 \times \mathbb{R}$. Finally, in the appendix, we give the proofs of technical lemmas.

2. *Preliminaries*

2.1. *Harmonic maps and holomorphic quadratic differentials*

In the following, we will use the unit disk model for $\mathbb{H}^2$. We will let $\mathbb{H}^2 = (\mathbb{D}, \sigma^2(u)|du|^2)$ denote the disk with the hyperbolic metric $\sigma^2(u)|du|^2 = (4/(1-|u|^2)^2)|du|^2$. The harmonic map equation is

$$g_{z\bar{z}} + \frac{2\bar{g}}{(1-|g|^2)} g_z g_{\bar{z}} = 0. \tag{1}$$

In the theory of harmonic maps there is a global object to consider: the holomorphic quadratic Hopf differential associated to g,

$$Q(g) = \phi(z)dz^2 = (\sigma \circ g)^2 g_z \bar{g}_z dz^2. \tag{2}$$

The function ϕ depends on the choice of the complex coordinate z, whereas $Q(g)$ does not. If $Q(g)$ is holomorphic then g is harmonic. We define the function $\omega = \frac{1}{2}\log|g_z|/|g_{\bar{z}}|$.

For a given holomorphic quadratic differential $Q = \phi(z)dz^2$, Wan [19] on $\mathbb{D}$, Wan and Au [20] on $\mathbb{C}$, constructed a unique (up to isometries) harmonic map $g : \Sigma \to \mathbb{H}^2$ with non-negative Jacobian and such that the metric

$$\tau|dz|^2 = 4(\sigma \circ g)^2|g_z|^2|dz|^2 = 4e^{2\omega}|\phi||dz|^2$$

is complete. To do that, they construct a space-like CMC $\frac{1}{2}$ in Minkowski space $\mathbb{L}^3$ with Gauss map g and metric $\tau|dz|^2$. First they solve the Gauss equation for the local theory of these surfaces:

$$\Delta_0\omega = 2\sinh(2\omega)|\phi|, \tag{3}$$

where $\Delta_0\omega = 4\omega_{z\bar{z}}$. The Codazzi equation is a consequence of the fact that ϕ is holomorphic. Then a maximum principle of Cheng and Yau [4] implies that there is a unique solution of (3) with complete metric $\tau|dz|^2$. Then by integration of the Gauss and Codazzi equations there is a unique (up to isometries) space-like CMC $\frac{1}{2}$ immersion $\tilde{X} = (\tilde{F}, \tilde{h})$ in the Minkowski space $\mathbb{L}^3$. The Gauss map of $\tilde{X}$ is the map $g = \psi \circ \tilde{N} : \Sigma \to \mathbb{D}$, where ψ is the stereographic projection with respect to the southern pole of the quadric $\{|v|^2 = -1\}$. The data (Q, τ) determine g uniquely (up to isometries). When $\tau|dz|^2$ is complete we say that g is *τ-complete*.

In Section 4, we will construct a family of harmonic maps with $Q = c\,dz^2$ ($c \in \mathbb{C}$) and not necessarily τ-complete. We will use these examples to construct our horizontal catenoids.

We describe a notion of conjugate harmonic map. It is known that a harmonic map g with Q having even zeroes induces a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$. The immersion is given by $X = (g, \operatorname{Re}\int -2i\sqrt{Q})$ and the induced metric is $ds^2 = 4\cosh^2\omega|Q|$ (see [13]). Conversely, if $X = (g, t)$ is a conformal minimal immersion then g is harmonic and $Q(t) = -(t_z)^2\,dz^2$ is a holomorphic quadratic differential with $Q(t) = Q(g)$.

DEFINITION 2.1. Two conformal minimal immersions $Y, Y^* : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ are *conjugate* if they induce the same metric on Σ and if we have $Q(g^*) = -Q(g)$.

In [6, 15], it is proved that the conjugate immersion exists. If $Y^* = (g^*, h^*)$, then we say that g^* is the *conjugate harmonic map* of g. In particular, we will use $Q(g^*) = -Q(g)$ and $\cosh\omega^* = \cosh\omega$ (and $\tau = \tau^*$).

2.2. *Minimal surfaces in the Heisenberg group*

In what follows, we use the exponential coordinates to identify the Heisenberg group Nil_3 with $(\mathbb{R}^3, d\sigma^2)$, where $d\sigma^2$ is given by

$$d\sigma^2 = dx_1^2 + dx_2^2 + \left(dx_3 + \tfrac{1}{2}(x_2\,dx_1 - x_1 dx_2)\right)^2.$$

The projection $\pi : \mathrm{Nil}_3 \to \mathbb{R}^2, (x_1, x_2, x_3) \mapsto (x_1, x_2)$ is a Riemannian fibration. We consider the left-invariant orthonormal frame (E_1, E_2, E_3) defined by

$$E_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3}, \quad E_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3}, \quad E_3 = \frac{\partial}{\partial x_3} = \xi.$$

A vector is said to be *vertical* if it is proportional to ξ and *horizontal* if it is orthogonal to ξ. A surface is a *multigraph* if ξ is nowhere tangent to it, that is, if the restriction of π to the surface is a local diffeomorphism. The isometry group of Nil_3 is 4-dimensional and has two connected components: isometries preserving the orientation of the fibers and the base of the fibration, and those reversing both of them. Vertical translations are isometries. The Heisenberg group Nil_3 is a homogeneous manifold.

Lemma 2.2. *Let $X : \Sigma \to \mathrm{Nil}_3$ be an immersion. Let N be the unit normal vector to X, and let $\tilde{N}$ be the Euclidean unit normal vector to X considered as an immersion into $\mathbb{R}^3$. Then N points up if and only if $\tilde{N}$ points up.*

Proof. We consider a conformal coordinate $z = u + iv$. In the frame (E_1, E_2, E_3) we have

$$X_u = \begin{bmatrix} x_{1u} \\ x_{2u} \\ x_{3u} + \frac{1}{2}(x_2 x_{1u} - x_1 x_{2u}) \end{bmatrix}, \quad X_v = \begin{bmatrix} x_{1v} \\ x_{2v} \\ x_{3v} + \frac{1}{2}(x_2 x_{1v} - x_1 x_{2v}) \end{bmatrix}.$$

Thus the third coordinate of $X_u \times X_v$ is $x_{1u}x_{2v} - x_{1v}x_{2u}$, which is also the third coordinate in the frame $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ of $X_u \wedge X_v$, where $\wedge$ is the Euclidean vector product. $\square$

We will call *vertical planes* surfaces of equation $a_1x_1 + a_2x_2 = b$ for some constants a_1, a_2 and b with $(a_1, a_2) \neq (0, 0)$. Such surfaces are minimal and flat, but not totally geodesic. Two vertical planes will be said to be *parallel* if their images by the projection π are two parallel straight lines in $\mathbb{R}^2$.

A graph $\{x_3 = f(x_1, x_2)\}$ is minimal if f satisfies the quasi-linear equation

$$(1 + q^2)r - 2pqs + (1 + p^2)t = 0$$

with

$$p = f_{x_1} + \frac{x_2}{2}, \quad q = f_{x_2} - \frac{x_1}{2},$$
$$r = f_{x_1x_1}, \quad s = f_{x_1x_2}, \quad t = f_{x_2x_2}.$$

The Bernstein problem deals with the existence and the unicity of entire solutions of this quasi-linear equation. We use conformal parametrization of surfaces. Let $X : \Sigma \to \mathrm{Nil}_3$ be a conformal immersion. We denote by $F = \pi \circ X$ the horizontal projection of X and by $h : \Sigma \to \mathbb{R}$ the third coordinate of X. We regard F as a complex-valued function, identifying $\mathbb{C}$ and $\mathbb{R}^2$. We denote the metric by $ds^2 = \lambda|dz|^2$ and by $N : \Sigma \to \mathbb{S}^2$ the unit normal vector to X, where $\mathbb{S}^2$ is the unit sphere in the Lie-algebra of Nil_3.

The Gauss map of X is the map $g = \psi \circ N : \Sigma \to \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, where ψ is the stereographic projection with respect to the southern pole; that is, g is defined by

$$N = \frac{1}{1 + |g|^2} \begin{bmatrix} 2\,\mathrm{Re}\, g \\ 2\,\mathrm{Im}\, g \\ 1 - |g|^2 \end{bmatrix}$$

in (E_1, E_2, E_3). The first author proved in [7] that the Gauss map g satisfies

$$(1 - |g|^2)g_{z\bar{z}} + 2\bar{g}g_z g_{\bar{z}} = 0. \tag{4}$$

It is important to keep in mind that $|g| = 1$ exactly at points where the surface is not transverse to ξ.

If Σ is a multigraph then, up to a change of orientations, g takes values in the unit disk $\mathbb{D}$. When $\mathbb{D}$ is endowed with the hyperbolic metric $(4/(1 - |z|^2)^2)|dz|^2$, g is a harmonic map from Σ to $\mathbb{H}^2$. Conversely, we can recover a minimal immersion from a given harmonic map using the following theorem.

Theorem 2.3 [7]. *Let Σ be a simply connected Riemann surface. Let $g : \Sigma \to \mathbb{H}^2$ be a harmonic map that is nowhere antiholomorphic. Let $z_0 \in \Sigma$, $F_0 \in \mathbb{C}$ and $h_0 \in \mathbb{R}$.*

Then there exists a unique conformal minimal immersion $X : \Sigma \to \mathrm{Nil}_3$ such that g is the Gauss map of X and $X(z_0) = (F_0, h_0)$.

Moreover, the immersion $X = (F, h)$ satisfies

$$F_z = -4i\frac{g_z}{(1-|g|^2)^2}, \quad F_{\bar z} = -4i\frac{g^2\bar g_{\bar z}}{(1-|g|^2)^2},$$
$$h_z = 4i\frac{\bar g g_z}{(1-|g|^2)^2} - \frac{i}{4}(\bar F F_z - F\bar F_z).$$

The metric of the immersion is given by

$$ds^2 = 16\frac{(1+|g|^2)^2}{(1-|g|^2)^4}|g_z|^2|dz|^2.$$

The hypothesis 'nowhere antiholomorphic' forces $\lambda|dz|^2$ to be a metric without branch points. The metrics $\lambda|dz|^2$ and $\tau|dz|^2$ are related by

$$\lambda = \frac{\tau}{\nu^2},$$

where

$$\nu = \frac{1-|g|^2}{1+|g|^2},$$

is the third coordinate of N. In the case of a multigraph we have $0 < |\nu| \leqslant 1$, and so, by the above relation between λ and τ, it is clear that the metric $\lambda|dz|^2$ is complete if $\tau|dz|^2$ is complete.

In Section 4, we will use a family of harmonic maps to construct explicitly minimal annuli which will be the unions of two non-complete graphs.

It is worth mentioning some recent results of Fernandez and Mira.

Theorem 2.4 [**10**]. *Every τ-complete nowhere antiholomorphic harmonic map induces an entire minimal graph in* Nil_3*. Conversely, every entire minimal graph in* Nil_3 *admits a τ-complete harmonic Gauss map g.*

This theorem proves that, starting from a holomorphic quadratic differential Q, there is a one-to-one canonical way to associate a two-parameter family of entire minimal graphs in Nil_3 [**10**].

This is not enough to prove that complete multigraphs of Nil_3 are entire graphs and then coming from a τ-complete harmonic Gauss map. This fact will be the object of Section 3. In other words, we will prove that, for a multigraph, if $\lambda|dz|^2$ is complete, then $\tau|dz|^2$ is also complete.

2.3. *Constant mean curvature $\frac{1}{2}$ surfaces in $\mathbb{H}^2\times\mathbb{R}$*

Abresch and Rosenberg [**1**] constructed a holomorphic quadratic differential Q_0 associated to CMC $\frac{1}{2}$ surfaces in $\mathbb{H}^2\times\mathbb{R}$; it generalizes the Hopf differential associated to CMC surfaces of $\mathbb{R}^3$. When the surface is a graph, Fernandez and Mira [**9**] constructed a harmonic 'hyperbolic Gauss map' from the surface to $\mathbb{H}^2$ whose associated Hopf differential is $Q = -Q_0$. In addition, given a harmonic map g from a surface to $\mathbb{H}^2$ plus some additional data (described below), they construct CMC $\frac{1}{2}$ graphs on $\mathbb{H}^2\times\mathbb{R}$ with this harmonic map as Gauss map.

Fernandez and Mira constructed CMC $\frac{1}{2}$ multigraph immersions $X^* = (F^*, h^*) : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ depending on the data (Q, τ).

Theorem 2.5 [9]. *Let Σ be a simply connected Riemann surface and let $g^* : \Sigma \to \mathbb{H}^2$ be a harmonic map admitting data $(-Q, \tau)$. Then for any $\theta_0 \in \mathbb{C}$ there exists a unique CMC $\frac{1}{2}$ immersion $X^* = (F^*, h^*) : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ satisfying*

- *$\tau = \lambda \nu^2$, where λ is the conformal factor of the metric of X^* and ν is the vertical coordinate of the unit normal Gauss map,*
- *$h_z^*(z_0) = \theta_0$.*

Moreover, with $G = (2g^*/(1-|g^*|^2), (1+|g^*|^2)/(1-|g^*|^2))$ we have

$$F^* = \frac{8\mathrm{Re}\left(G_z(4\bar{Q}h_z^* + \tau h_{\bar{z}}^*)\right)}{\tau^2 - 16|Q|^2} + G\sqrt{\frac{\tau + 4|h_z^*|^2}{\tau}},$$

and $h^* : \Sigma \to \mathbb{R}$ is the unique (up to an additive constant) solution to the differential system below with $h_z^*(z_0) = \theta_0$:

$$h_{zz}^* = (\log \tau)_z h_z^* + Q\sqrt{\frac{\tau + 4|h_z^*|^2}{\tau}},$$
$$h_{z\bar{z}}^* = \frac{1}{4}\sqrt{\tau(\tau + 4|h_z^*|^2)}.$$

The metric can be expressed as

$$\lambda = \frac{\tau}{\nu^2} = \tau + 4|h_z^*|^2, \quad \nu = \sqrt{\frac{\tau}{\tau + 4|h_z^*|^2}}.$$

By the above relation between λ and τ, it is clear that the metric $ds^2 = \lambda|dz|^2$ is complete if $\tau|dz|^2$ is complete. Thus, associated to a holomorphic quadratic differential Q, one obtains a complete CMC $\frac{1}{2}$ multigraph in $\mathbb{H}^2 \times \mathbb{R}$.

It is known from [8] that a CMC $\frac{1}{2}$ immersion $X^* = (F^*, h^*)$ is locally isometric to a minimal immersion $X = (F, h)$ in Nil_3. These two immersions are called *sister immersions*. The third coordinate ν of the unit normal vector of X and X^* remains unchanged by this correspondence. In particular, the sister surface of a multigraph is a multigraph. The harmonic Gauss maps are conjugate ($Q(g) = -Q(g^*)$ and $\tau = \tau^*$).

We mention the following result of Fernandez and Mira.

Proposition 2.6 [10]. *If $X^* = (F^*, h^*)$ is a CMC $\frac{1}{2}$ minimal graph in $\mathbb{H}^2 \times \mathbb{R}$ with a τ-complete harmonic Gauss map g^*, then X^* is an entire graph.*

3. *Complete graphs*

In this section we use the half-space theorem, Theorem 6.3, to obtain results on complete graphs in Nil_3 and $\mathbb{H}^2 \times \mathbb{R}$.

Theorem 3.1. *Let Σ be a complete minimal surface in Nil_3. If Σ is transverse to the vertical Killing field ξ, then Σ is an entire graph.*

Corollary 3.2. *Let Σ be a complete minimal surface in Nil_3. If Σ is transverse to the vertical Killing field ξ, then its Gauss map is τ-complete.*

Proof. From [10] we know that an entire graph has a τ-complete Gauss map. $\square$

Corollary 3.3. *A minimal surface in* Nil_3 *is an entire graph if and only if its CMC* $\frac{1}{2}$ *sister surface in* $\mathbb{H}^2 \times \mathbb{R}$ *is an entire graph.*

Proof. By [**10**], an entire graph of Nil_3 has a τ-complete Gauss map. Then again by [**10**] the sister CMC $\frac{1}{2}$ surface is entire in $\mathbb{H}^2 \times \mathbb{R}$ (this fact comes from the completeness of $\tau|dz|^2$).

Conversely, the sister of an entire CMC $\frac{1}{2}$ graph in $\mathbb{H}^2 \times \mathbb{R}$ is a complete multigraph and then entire in Nil_3 by our theorem, Theorem 3.1. □

Corollary 3.4. *Let* Σ *be a complete CMC* $\frac{1}{2}$ *surface in* $\mathbb{H}^2 \times \mathbb{R}$. *If* Σ *is a multigraph, then its Gauss map is* τ*-complete.*

Proof. By the theorem of Hauswirth, Rosenberg and Spruck [**14**], we have that Σ is an entire graph and hence its sister too (Corollary 3.3). From [**10**] we know that an entire graph is τ-complete. □

We now prove Theorem 3.1. The proof is an adaptation to our case of the proof of [**14**, Theorem 1.2]. However, we give a detailed proof for the reader's convenience and since we need to take care about the meaning of horizontal vectors in Nil_3. Lemma 3.5 below is inspired by the work 'The Jenkins–Serrin theorem for minimal graphs in homogeneous 3-manifolds' of Collin and Rosenberg, (forthcoming).

Let Σ be a complete minimal surface in Nil_3 such that Σ is transverse to the vertical Killing field ξ.

We assume that Σ is not entire. We denote by N the unit normal vector field to Σ. Since Σ is a multigraph, it is orientable. The function $\nu = \langle N, \xi \rangle$ is a non-vanishing Jacobi function on Σ, so Σ is strongly stable and thus has bounded curvature. Hence there is a $\delta > 0$ such that, for each $p \in \Sigma$, there is a piece $G(p)$ of Σ around p that is a graph (in exponential coordinates) over the disk $D_{2\delta}(p) \subset T_p\Sigma$ of radius 2δ centered at the origin of $T_p\Sigma$. This graph $G(p)$ has bounded geometry. The δ is independent of p and the bound on the geometry of $G(p)$ is uniform as well.

We denote by $F(p)$ the image of $G(p)$ by the vertical translation mapping p at height $x_3 = 0$, and by $F_O(p)$ the image of $G(p)$ by the translation mapping p to $O = (0, 0, 0) \in \mathrm{Nil}_3$.

In what follows, we will call x_3*-graphs* graphs with respect to the Riemannian fibration $\pi : \mathrm{Nil}_3 \to \mathbb{R}^2$ (as explained in Section 2).

We will identify vectors at different points of Nil_3 by left multiplication, and horizontal vectors with vectors in $\mathbb{R}^2$.

Lemma 3.5. *Let* (p_n) *be a sequence of points of* Σ *such that* $N(p_n)$ *has a horizontal limit* N_∞ *when* $n \to +\infty$. *Then there is a subsequence of* $(F_O(p_n))_{n\in\mathbb{N}}$ *that converges to a* δ*-piece* P_δ *around* O *of the vertical plane* P *passing through* O *and having* N_∞ *as unit normal vector at* O.

The convergence is in the $\mathcal{C}^2$*-topology. By* δ*-piece around* O *we mean a piece of* P *containing all the points* $p \in P$ *such that* $x_3(p) \in [-\delta, \delta]$ *and* $\pi(p)$ *belongs to the closed segment of* $\pi(P) \subset \mathbb{R}^2$ *centered at* $\pi(O)$ *and of length* 2δ.

Proof. Let P be the vertical plane passing through O and having N_∞ as unit normal vector at O. We endow P with the orientation induced by N_∞.

Since the $F_O(p_n)$ have bounded geometry and are graphs over $D_{2\delta}(p_n) \subset \mathrm{T}_O(F_O(p_n))$, the $F_O(p_n)$ are bounded exponential graphs over a δ-piece $P_\delta \subset P$ around O. Thus a subsequence of these graphs converges to a piece of a minimal surface F_∞, which is tangent to P_δ at O and which is an exponential graph over P_δ. It suffices to show that F_∞ is a piece of P.

If this is not the case, then by [**5**, Theorem 5.3], in the neighborhood of O, the intersection of F_∞ and the vertical plane consists of m ($m \geqslant 2$) curves meeting at O. These curves separate F_∞ into $2m$ connected components and adjacent components lie on opposite sides of the vertical plane. Hence in a neighborhood of O, the Euclidean unit normal vector to F_∞ alternates from pointing up to pointing down as one goes from one component to the other. This is also the case for $F_O(p_n)$, for n large, since $F_O(p_n)$ converges to F_∞ in $\mathcal{C}^2$-topology. Then by Lemma 2.2, the unit normal vector to $F_O(p_n)$ for the metric of Nil_3 also alternates from pointing up to down. This contradicts the fact that $F_O(p_n)$ is transverse to ξ. $\square$

We now consider a piece of Σ that is the x_3-graph of a function f defined on the open disk B_R of radius R centered at some point A of $\mathbb{R}^2$. Since Σ is not an entire graph, we choose the largest R such that f exists.

In the sequel, for any point $q \in B_R$ we will write $F(q)$, $N(q)$, etc. instead of $F(q, f(q))$, $N(q, f(q))$, etc.

Let $q \in \partial B_R$ be such that f does not extend to any neighborhood of q (to a function satisfying the minimal graph equation).

Lemma 3.6. *There exists a unit horizontal vector $N_\infty(q)$ such that, for any sequence $q_n \in B_R$ converging to q, $N(q_n) \to N_\infty(q)$ when $n \to +\infty$. Moreover, $N_\infty(q)$ is normal to ∂B_R at q.*

Proof. We first observe that $\nu(q_n) \to 0$ when $n \to +\infty$ (that is, tangent planes become vertical); otherwise, the exponential graph of bounded geometry $G(q_n)$ would extend to an x_3-graph beyond q for q_n close enough to q, and thus the map f would extend, which is a contradiction.

Let $N_\infty(q)$ be the horizontal unit vector at q normal to ∂B_R and pointing inside B_R. We will now prove that $N(q_n) \to N_\infty(q)$.

Assume that there exists a subsequence such that $N(q_n)$ converges to a horizontal vector $v \neq N_\infty(q)$. By Lemma 3.5 there exists a subsequence such that the pieces $F(q_n)$ converge to a δ-piece of a vertical plane Q having v as unit normal vector at q. Since $N_\infty(q)$ is normal to ∂B_R at q, there are points of $\pi(Q_\delta)$ in B_R. Consequently, there are a point $\hat{q} \in \pi(Q_\delta) \cap B_R$ and a sequence $(\hat{q}_n)$ of points of B_R converging to $\hat{q}$ such that $(\hat{q}_n, f(\hat{q}_n)) \in G(q_n)$. Since the $F(q_n)$ converge to a δ-piece of a vertical plane in the $\mathcal{C}^2$-topology, Σ has a horizontal normal at $(\hat{q}, f(\hat{q}))$, which contradicts the fact that Σ is transverse to ξ. $\square$

We denote by P the vertical plane passing through q and having $N_\infty(q)$ as unit normal vector at q. Without loss of generality we can assume that $q = (R, 0) \in \mathbb{R}^2$ and that P is the vertical plane of equation $x_1 = R$ in Nil_3. We will say that a point in Nil_3 is on the left-hand side (or on the right-hand side) of P if $x_1 < R$ (or $x_1 > R$).

Lemma 3.7. *We have $f(t, 0) \to \pm\infty$ when $t \to R$.*

Proof. Let $\varphi(t) = f(t, 0)$ and $\gamma(t) = (t, 0, \varphi(t))$.

We first claim that for t close enough to R we have $\varphi'(t) \neq 0$. Indeed, assume that there exists a t_0 such that $\varphi'(t_0) = 0$. We have $\gamma'(t) = E_1 + \varphi'(t)\xi$ so $\gamma'(t_0)$ is horizontal. Also, $G(\gamma(t_0))$ is an exponential graph over $D_{2\delta}(\gamma(t_0)) \subset T_{\gamma(t_0)}\Sigma$, and $T_{\gamma(t_0)}\Sigma$ contains the horizontal vector $\gamma'(t_0)$. Consequently, the projection of $G(\gamma(t_0))$ on $\mathbb{R}^2$ contains an open neighborhood of $\{(t, 0); t_0 - \delta < t < t_0 + \delta\}$. Hence, if t_0 is close enough to R, this implies that f extends beyond $q = (R, 0)$, which is a contradiction. This proves the claim.

Thus we can assume that $\varphi(t)$ is increasing as t converges to R. If $\varphi(t)$ were bounded from above, then it would have a finite limit l and the curve $t \mapsto (t, 0, f(t, 0))$ in Σ would have finite length up to (q, l). Since Σ is complete we would have $(q, l) \in \Sigma$, but then Σ would have a vertical tangent plane at (q, l) (otherwise f would extend to some neighborhood of q), which gives a contradiction. $\square$

From now on we assume that $f(t, 0) \to +\infty$ when $t \to R$ (the case where $f(t, 0) \to -\infty$ is similar). We set

$$\Gamma = \pi(P) = \{(R, s); s \in \mathbb{R}\}$$

and, for $\varepsilon > 0$ and $s \in \mathbb{R}$,

$$U_\varepsilon =]R - \varepsilon, R[\times\mathbb{R},$$

$$\gamma_{s,\varepsilon} = \{(x_1, s); R - \varepsilon < x_1 < R + \varepsilon\},$$

$$\gamma^+_{s,\varepsilon} = \{(x_1, s); R - \varepsilon < x_1 < R\}.$$

We fix $\varepsilon_0 > 0$ and consider a sequence (t_n) of real numbers such that $q_n = (t_n, 0)$ is in B_R, $q_n \to q$ when $n \to +\infty$ and such that

$$G = \bigcup_{n\in\mathbb{N}} G(q_n)$$

is connected. By Lemmas 3.5 and 3.7 and the fact that the $G(q_n)$ are pieces of bounded geometry, G is asymptotic to a part of P as one goes up. Moreover, we can choose the q_n close enough to R and to each other such that, for all $s \in [-\delta, \delta]$, the curve

$$C_s = \pi^{-1}(\gamma_{s,\varepsilon_0}) \cap G$$

is connected and has no horizontal or vertical tangents. This is possible since the $F(q_n)$ are $\mathcal{C}^2$-close to P_δ and since Σ is transverse to ξ.

Lemma 3.8. *Each $G(q_n)$ is disjoint from P and, for $s \in [-\delta, \delta]$, C_s is an x_3-graph over $\gamma^+_{s,\varepsilon(s)}$ for some $\varepsilon(s) \in (0, \varepsilon_0]$. Moreover, $\varepsilon(s)$ can be chosen to be continuous.*

Proof. The curve C_s is an x_3-graph over an interval in γ_{s,ε_0}. We show that this interval is in $\gamma^+_{s,\varepsilon_0}$.

Suppose this is not the case for some $s_0 \in [-\delta, \delta]$. Then C_{s_0} has some points on the right side of P. However, the curve C_0 stays on the left-hand side of P (otherwise f would extend beyond q). Therefore, for some $s_1 \in]0, s_0]$, the curve C_{s_1} has points on both sides of P.

However, G is asymptotic to a part of P as one goes up, so the curve C_{s_1} is asymptotic to $\pi^{-1}(R, s_1)$ as the height goes to $+\infty$. This obliges C_{s_1} to have a vertical tangent on the right-hand side of P, which is a contradiction since Σ is transverse to ξ. $\square$

Consequently $\bigcup_{s\in[-\delta,\delta]} C_s$ is the x_3-graph of a function g on $\bigcup_{s\in[-\delta,\delta]} \gamma^+_{s,\varepsilon(s)}$. The functions f and g coincide on the intersection of their domains of definition. The graph of g on each $\gamma^+_{s,\varepsilon(s)}$ is the curve C_s and the graph of g is asymptotic to P as the height goes to $+\infty$.

We can apply this process again replacing C_0 by the curve C_δ, then $C_{-\delta}$, and so on. Analytic continuation yields an extension h of g to a domain Ω contained on the left-hand side of Γ. The domain Ω is an open neighborhood of Γ in its left-hand side. We have $h \to +\infty$ as one approaches Γ in Ω; the graph of h is asymptotic to P as the height goes to $+\infty$.

Lemma 3.9. *There exists an $\varepsilon > 0$ such that Ω contains U_ε.*

Proof. The surface Σ contains a graph over Ω composed of curves C_s such that each C_s is a graph over $\gamma^+_{s,\varepsilon(s)}$ for some $\varepsilon(s) > 0$. Also, for each $s \in \mathbb{R}$, $h(t,s)$ is strictly increasing in t when $t \to R$ and $h(t,s) \to +\infty$ when $t \to R$.

Let $\varepsilon_1 \in (0,\delta)$ such that $\varepsilon_1 \leqslant \varepsilon(s)$ for all $s \in [-\delta,\delta]$.

Suppose that for some $s_0 \in \mathbb{R}$ we have $\varepsilon(s_0) < \varepsilon_1$. We set $\varphi(t) = h(t,s_0)$ and consider the curve

$$c : (R - \varepsilon(s_0), R) \ni t \mapsto (t, s_0, \varphi(t)) \in \Sigma.$$

We first claim that this curve has no horizontal tangent; indeed, if there is some t_0 such that $c'(t_0)$ is horizontal, then since $G(c(t_0))$ is an exponential graph over $D_{2\delta}(c(t_0)) \subset T_{c(t_0)}\Sigma$, and since $t_0 + \delta > R$, the curve c would go on the right side of P, which is a contradiction.

We have $c'(t) = E_1 + (\varphi'(t) + s_0/2)\xi$, so the fact that c has no horizontal tangent implies that $\varphi'(t) > -s_0/2$. This implies that $\varphi(t)$ cannot tend to $+\infty$ when $t \to R - \varepsilon(s_0)$. Therefore $\varphi(t) \to -\infty$ when $t \to R - \varepsilon(s_0)$ (otherwise the curve c would extend or have a vertical tangent at $R - \varepsilon(s_0)$).

The previous discussion where we showed that the graph over Ω exists and is asymptotic to P now applies to show that there is a vertical plane $\tilde{P}$ passing through $\tilde{p} = (R - \varepsilon(s_0), s_0, 0) \in \mathrm{Nil}_3$ such that a δ-neighborhood of c in Σ is asymptotic to a δ-vertical strip in $\tilde{P}$ as one goes down to $-\infty$. We know that this δ-neighborhood of c in Σ is asymptotic to a δ-vertical strip in P, as one goes up to $+\infty$.

For each $s \in [s_0 - \delta, s_0 + \delta]$, the curve $c_s : t \mapsto (t, s, h(t,s))$ is asymptotic to some vertical line in $\tilde{P}$ as one goes down to $-\infty$. By analytic continuation of the δ-neighborhoods, one continues this process along Γ.

If P and $\tilde{P}$ are parallel, then the process continues along all of Γ, and Ω is the region bounded by $\pi(P)$ and $\pi(\tilde{P})$. Then all the $\varepsilon(s)$ are equal, and this concludes the proof.

Therefore, we can assume that P and $\tilde{P}$ intersect along some vertical line $\pi^{-1}(\hat{p})$. Let us write $\hat{p} = (s_1, R)$. Consider the curves c_s as s goes from s_0 to s_1; they are graphs that become vertical both when the height goes to $+\infty$ and $-\infty$. Let $p(s)$ be the point of C_s at height 0; then, when $s \to s_1$, the path $p(s)$ has finite length (since the geometry of Σ is bounded), and so, since Σ is complete, $p(s)$ converges to a point of Σ, and the tangent plane at this point is vertical. This contradicts the fact that Σ is transverse to ξ. $\square$

We can now complete the proof of Theorem 3.1.

Proof of Theorem 3.1. We showed that Σ contains a graph G over some U_ε which is asymptotic to P as one approaches Γ in U_ε. We apply the proof of the vertical half-space Theorem 6.3 (this theorem is stated for complete surfaces without boundary, but the proof still works in our case since G is proper in some tubular neighborhood of P despite the fact that it has a non-compact boundary). This shows that such a graph G cannot exist.

Consequently, Σ is entire, and so it is an entire graph. $\square$

4. *The family of harmonic maps*

In this section we construct a family of harmonic maps that we will use to construct annuli. This family is derived from the two-parameter family of minimal surfaces of $\mathbb{H}^2 \times \mathbb{R}$ constructed in [**12**].

For $\alpha > 0$ and $\theta \in \mathbb{R}$ we define $g : \mathbb{C} \to \bar{\mathbb{C}}$ by

$$g(u+iv) = \frac{\sin\varphi(u) + i\sinh(\alpha v + \beta(u))}{\cos\varphi(u) + \cosh(\alpha v + \beta(u))} = \frac{\cosh(\alpha v + \beta(u)) - \cos\varphi(u)}{\sin\varphi(u) - i\sinh(\alpha v + \beta(u))},$$

where φ satisfies the following ODE:

$$\varphi'^2 = \alpha^2 + \cos(2\theta)\cos^2\varphi - \frac{\sin^2(2\theta)}{4\alpha^2}\cos^4\varphi, \tag{5}$$

and where β is defined by

$$\beta' = \frac{\sin(2\theta)}{2\alpha}\cos^2\varphi, \quad \beta(0) = 0.$$

The function φ is defined on the whole $\mathbb{R}$. We will study this function φ in Lemma 4.2. We also set

$$A = \alpha v + \beta(u), \quad D = \cos\varphi + \cosh A.$$

We notice that

$$1 - |g|^2 = \frac{2\cos\varphi}{D}. \tag{6}$$

PROPOSITION 4.1. *The function g satisfies*

$$(1 - |g|^2)g_{z\bar{z}} + 2\bar{g}g_z g_{\bar{z}} = 0,$$

and its Hopf differential is

$$Q = \tfrac{1}{4}e^{-2i\theta}\,dz^2.$$

Proof. To see that g satisfies the equation, it suffices to see that

$$Q = \frac{4}{(1-|g|^2)^2}g_z g_{\bar{z}}\,dz^2$$

is holomorphic.

We compute

$$g_u = \frac{\varphi' + i\beta'}{D^2}(1 + \cos\varphi\cosh A + i\sin\varphi\sinh A),$$
$$g_v = \frac{i\alpha}{D^2}(1 + \cos\varphi\cosh A + i\sin\varphi\sinh A).$$

From this and (6) we get

$$Q = \frac{\varphi'^2 - (\alpha + i\beta')^2}{4\cos^2\varphi}dz^2.$$

Using (5) and the definition of β we get $Q = \frac{1}{4}e^{-2i\theta}\,dz^2$. □

For $\alpha > 0$ and $\theta \in \mathbb{R}$, we set

$$C = C_{\alpha,\theta} = \frac{\sin(2\theta)}{2\alpha}, \quad P_{\alpha,\theta}(x) = \alpha^2 + \cos(2\theta)x^2 - C_{\alpha,\theta}^2 x^4,$$

so that (5) is equivalent to

$$\varphi'^2 = P_{\alpha,\theta}(\cos\varphi).$$

We set $\theta_\alpha^+ = \pi/2$ if $\alpha > 1$ and $\theta_\alpha^+ = (1/2)\arccos(1 - 2\alpha^2) \in (0, \pi/2]$ if $\alpha \leqslant 1$. Let $\Omega = \{(\alpha,\theta) \in \mathbb{R}^2; \alpha > 0, \theta \in (-\theta_\alpha^+, \theta_\alpha^+)\}$.

If $2\theta \notin \pi\mathbb{Z}$, we have

$$P_{\alpha,\theta}(x) = C_{\alpha,\theta}^2(\rho_{\alpha,\theta}^- - x^2)(\rho_{\alpha,\theta}^+ + x^2)$$

with

$$\rho_{\alpha,\theta}^- = \frac{2\alpha^2}{1 - \cos(2\theta)}, \quad \rho_{\alpha,\theta}^+ = \frac{2\alpha^2}{1 + \cos(2\theta)}.$$

Thus, if $2\theta \notin \pi\mathbb{Z}$ and $(\alpha,\theta) \in \Omega$, then $\rho^-_{\alpha,\theta} > 1$. Also, we have $P_{\alpha,0}(x) = \alpha^2 + x^2$. From this we deduce that

$$\forall(\alpha,\theta) \in \Omega, \quad \forall x \in [-1,1], \quad P_{\alpha,\theta}(x) > 0.$$

Thus, if $(\alpha,\theta) \in \Omega$, then the term on the right-hand side of (5) does not vanish.

Lemma 4.2. *Let $(\alpha,\theta) \in \Omega$. Let φ be the solution of (5) such that $\varphi(0) = 0$ and $\varphi'(0) \leqslant 0$. Then:*

(1) *$\forall u, \varphi'(u) < 0$;*
(2) *the function φ is a decreasing bijection from $\mathbb{R}$ onto $\mathbb{R}$;*
(3) *there exists a real number $U > 0$ such that*

$$\forall u \in \mathbb{R}, \quad \varphi(u+U) = \varphi(u) - \pi;$$

(4) *the function φ is odd.*

Proof. (1) Since the term on the right-hand side of (5) does not vanish, φ' does not vanish.
(2) The term on the right-hand side of (5) is bounded by two positive constants c_1 and c_2; hence $-\sqrt{c_2} \leqslant \varphi' \leqslant -\sqrt{c_1}$, which proves that φ is defined on the entire $\mathbb{R}$ and that $\varphi(u) \to -\infty$ when $u \to +\infty$ and $\varphi(u) \to +\infty$ when $u \to -\infty$.
(3) There exists a $U > 0$ such that $\varphi(U) = -\pi$. Then the function $\tilde{\varphi} : u \mapsto \varphi(u+U) + \pi$ satisfies (5) with $\tilde{\varphi}(0) = 0$ and $\tilde{\varphi}' < 0$; hence $\tilde{\varphi} = \varphi$.
(4) The function $\hat{\varphi} : u \mapsto -\varphi(-u)$ satisfies (5) with $\hat{\varphi}(0) = 0$ and $\hat{\varphi}' < 0$; hence $\hat{\varphi} = \varphi$. □

In what follows, we will use the function $G : \mathbb{R} \to \mathbb{R}$ defined by

$$G' = \frac{C^2\cos^2\varphi - \cos(2\theta)}{\alpha - \varphi'}, \quad G(0) = 0.$$

(We recall that $\alpha - \varphi' > 0$.) The functions β and G are odd and satisfy

$$\forall u \in \mathbb{R}, \quad \beta(u+U) = \beta(u) + \beta(U), \qquad \forall u \in \mathbb{R}, \quad G(u+U) = G(u) + G(U). \tag{7}$$

Lemma 4.3. *We have*

$$\varphi\left(\frac{U}{2}\right) = -\frac{\pi}{2}, \quad \beta\left(\frac{U}{2}\right) = \frac{\beta(U)}{2}, \quad G\left(\frac{U}{2}\right) = \frac{G(U)}{2}. \tag{8}$$

Proof. We have $\varphi(U/2) = \varphi(-U/2) - \pi = -\varphi(U/2) - \pi$, which gives the first formula. We prove the other formulas in the same way. □

Lemma 4.4. *The following identities hold.*

$$\varphi' + \alpha = G'\cos^2\varphi, \tag{9}$$

$$\varphi'' = -(\cos(2\theta) - 2C^2\cos^2\varphi)\sin\varphi\cos\varphi, \tag{10}$$

$$G''\cos\varphi = (2\varphi' G' - \cos(2\theta) + 2C^2\cos^2\varphi)\sin\varphi, \tag{11}$$

$$G'' = \frac{2C^2\alpha - \cos(2\theta)G'}{\alpha - \varphi'}\sin\varphi\cos\varphi, \tag{12}$$

$$G'' = (C^2 + G'^2)\sin\varphi\cos\varphi. \tag{13}$$

Proof. Formulas (9–12) are straightforward to obtain. Using (12) and the definition of G we get

$$(\varphi' - \alpha)^2 G'' = (\cos^2(2\theta) - C^2 \cos(2\theta)\cos^2\varphi - 2\alpha C^2(\varphi' - \alpha))\sin\varphi\cos\varphi.$$

On the other hand, we have

$$(\varphi' - \alpha)^2 (C^2 + {G'}^2) = \cos^2(2\theta) - C^2 \cos(2\theta)\cos^2\varphi - 2\alpha C^2(\varphi' - \alpha).$$

This proves (13). $\square$

For $(\alpha, \theta) \in \Omega$ we set

$$L(\alpha,\theta) = \int_{-1}^{1} \frac{2\alpha C_{\alpha,\theta}^2 x^2 - \alpha\cos(2\theta) + C_{\alpha,\theta}^2 x^2 \sqrt{P_{\alpha,\theta}(x)}}{\sqrt{(1-x^2)P_{\alpha,\theta}(x)}(\alpha + \sqrt{P_{\alpha,\theta}(x)})} dx.$$

We will prove in the appendix the following technical lemmas.

Lemma 4.5. *Let $\alpha > 0$. Then there exists a unique $\tilde{\theta}_\alpha \in (0, \theta_\alpha^+) \cap (0, \pi/4)$ such that*

$$L(\alpha, \tilde{\theta}_\alpha) = 0.$$

Lemma 4.6. *We have*

$$\lim_{\alpha \to +\infty} \tilde{\theta}_\alpha = \frac{\pi}{4}.$$

5. *Horizontal catenoids in* Nil_3

In this section we construct a one-parameter family of properly embedded minimal annuli in Nil_3. We use the notations of Section 4.

We will start from the map g which satisfies (1) (by Proposition 4.1) outside points where $|g| = 1$ but which does not take values in $\mathbb{D}$. However, in this case we can still recover a minimal immersion (but not a multigraph) by Theorem 2.3 provided that the map we obtain is well defined when $|g| = 1$ and provided that the metric we obtain has no singularity. In fact these two kinds of problems do not appear in our case, as shown by the following proposition.

Proposition 5.1. *The conformal minimal immersion $X = (F, h) : \mathbb{C} \to \mathrm{Nil}_3$ whose Gauss map is g is given (up to a translation) by*

$$F(u + iv) = \frac{G'}{\alpha}\cos\varphi\sinh A - \frac{C}{\alpha}\sin\varphi\cosh A + i(Cv - G),$$

$$\begin{aligned} h(u+iv) = &-\frac{1}{\alpha}\left(G'\sin\varphi + \frac{C^2}{\alpha}\sin\varphi + \frac{(Cv-G)G'}{2}\cos\varphi\right)\sinh A \\ &+ \frac{1}{\alpha}\left(-C\cos\varphi + \frac{CG'}{\alpha}\cos\varphi + \frac{C(Cv-G)}{2}\sin\varphi\right)\cosh A. \end{aligned}$$

The metric of X is given by

$$ds^2 = ({G'}^2 + C^2)\cosh^2 A |dz|^2.$$

Proof. We first recover F using Theorem 2.3 and the above computations. We get

$$F_z = -\frac{i}{2\cos^2\varphi}(\varphi' + i\beta' + \alpha)(1 + \cos\varphi\cosh A + i\sin\varphi\sinh A),$$
$$F_{\bar z} = -\frac{i}{2\cos^2\varphi}(\varphi' - i\beta' + \alpha)(1 - \cos\varphi\cosh A + i\sin\varphi\sinh A).$$

Hence

$$\begin{aligned} F_u &= \frac{\beta'\cos\varphi\cosh A - i(\varphi'+\alpha)(1 + i\sin\varphi\sinh A)}{\cos^2\varphi} \\ &= C\cos\varphi\cosh A - iG'(1 + i\sin\varphi\sinh A), \\ F_v &= \frac{(\varphi'+\alpha)\cos\varphi\cosh A + i\beta'(1 + i\sin\varphi\sinh A)}{\cos^2\varphi} \\ &= G'\cos\varphi\cosh A + iC(1 + i\sin\varphi\sinh A). \end{aligned}$$

This gives F.

Then we get

$$\begin{aligned} h_z = \frac{G'+iC}{4}\Big(&2\cos\varphi\sinh A + 2i\sin\varphi\cosh A - \frac{G'}{\alpha}\cos\varphi\sinh A + \frac{C}{\alpha}\sin\varphi\cosh A \\ &+ i(Cv - G)\cos\varphi\cosh A - (Cv - G)\sin\varphi\sinh A\Big). \end{aligned}$$

This gives h.

Using (6) and computations done in the proof of Proposition 4.1, we get

$$1 + |g|^2 = \frac{2\cosh A}{D}, \quad |g_z|^2 = \frac{(\varphi'+\alpha)^2 + \beta'^2}{4D^2},$$

and so by Theorem 2.3 we obtain the formula. □

PROPOSITION 5.2. *Let $\alpha > 0$ and $\theta = \tilde\theta_\alpha$. Then the corresponding immersion X is simply periodic, that is, there exists a $Z \in \mathbb{C}\setminus\{0\}$ such that*

$$\forall z \in \mathbb{C}, \quad X(z + Z) = X(z).$$

Proof. Let $C_\alpha = C_{\alpha,\tilde\theta_\alpha}$ and $P_\alpha(x) = P_{\alpha,\tilde\theta_\alpha}(x)$. We set

$$V = -\frac{\beta(U)}{\alpha}.$$

Then, by (7), for all $(u,v) \in \mathbb{R}^2$, we have $A(u + U + i(v + V)) = A(u + iv)$.

We claim that, for all $(u,v) \in \mathbb{R}^2$, we have $\operatorname{Im} F(u + U + i(v + V)) = \operatorname{Im} F(u + iv)$, that is, that

$$\alpha G(U) + C\beta(U) = 0. \tag{14}$$

We have

$$G(U) = \int_0^U G'(u)\,du, \quad \beta(U) = \int_0^U \beta'(u)\,du.$$

We do the change of variables $x = \cos\varphi(u)$, and hence $dx = -\varphi'\sin\varphi\,du = \varphi'\sqrt{1-x^2}\,du$ since $\varphi \in [-\pi, 0]$. We get

$$\begin{aligned} G(U) &= \int_{-1}^1 \frac{C_\alpha^2 x^2 - \cos(2\tilde\theta_\alpha)}{\sqrt{(1-x^2)P_\alpha(x)}(\alpha + \sqrt{P_\alpha(x)})}dx, \\ \beta(U) &= \int_{-1}^1 \frac{C_\alpha x^2}{\sqrt{(1-x^2)P_\alpha(x)}}dx, \end{aligned}$$

and so $\alpha G(U) + C\beta(U) = L(\alpha, \tilde{\theta}_\alpha) = 0$ by Lemma 4.5. This proves the claim.

Hence $A(u+iv)$ and $\operatorname{Im} F(u+iv) = Cv - G(u)$ are $(U+iV)$-periodic. We set $Z = 2(U + iV)$ (we have $Z \neq 0$ since $U > 0$). Then it follows from the expressions of F and h that they are Z-periodic. $\square$

Definition 5.3. Let $\alpha > 0$. The surface given by X when $\theta = \tilde{\theta}_\alpha$ is called a *horizontal catenoid* of parameter α with respect to the x_2-axis. It will be denoted by $\mathcal{C}_\alpha$.

The coordinates (x_1, x_2, x_3) of $\mathcal{C}_\alpha$ are

$$\begin{aligned}
x_1 &= \frac{G'(u)}{\alpha} \cos\varphi(u) \sinh A - \frac{C}{\alpha} \sin\varphi(u) \cosh A, \\
x_2 &= \frac{C}{\alpha} A - \frac{C}{\alpha} \beta(u) - G(u), \\
x_3 &= -\frac{x_1 x_2}{2} + \frac{C}{\alpha} \left(\frac{G'(u)}{\alpha} - 1 \right) \cos\varphi(u) \cosh A - \frac{1}{\alpha} \left(\frac{C^2}{\alpha} + G'(u) \right) \sin\varphi(u) \sinh A.
\end{aligned}$$

We now study the geometry of $\mathcal{C}_\alpha$. We first notice that

$$\begin{aligned}
x_1(u+U, v+V) &= -x_1(u,v), \\
x_2(u+U, v+V) &= x_2(u,v), \\
x_3(u+U, v+V) &= -x_3(u,v),
\end{aligned}$$

so $\mathcal{C}_\alpha$ is invariant by the rotation of angle π around the x_2-axis. We also have

$$\begin{aligned}
x_1(-u, -v) &= -x_1(u,v), \\
x_2(-u, -v) &= -x_2(u,v), \\
x_3(-u, -v) &= x_3(u,v),
\end{aligned}$$

so $\mathcal{C}_\alpha$ is invariant by the rotation of angle π around the x_3-axis. Since the composition of the rotations of angle π around the x_2 and x_3 axes is the rotation of angle π around the x_1-axis, $\mathcal{C}_\alpha$ is also invariant by this rotation.

It will be convenient to use the following coordinates in Nil_3:

$$y_1 = x_1, \quad y_2 = x_2, \quad y_3 = x_3 + \frac{x_1 x_2}{2}. \tag{15}$$

In these coordinates the metric of Nil_3 is given by

$$dy_1^2 + dy_2^2 + (dy_3 - y_1\, dy_2)^2.$$

In particular, in a vertical plane of equation $y_2 = c$ $(c \in \mathbb{R})$, the pair (y_1, y_3) is a pair of Euclidean coordinates.

We now study the intersection of $\mathcal{C}_\alpha$ with a vertical plane of equation $y_2 = c$ $(c \in \mathbb{R})$. On $\mathcal{C}_\alpha$, this intersection is given by

$$A = \frac{\alpha}{C} c + \beta(u) + \frac{\alpha}{C} G(u). \tag{16}$$

Hence, reporting this equality in the expressions of (x_1, x_2, x_3), we obtain a parametrization $u \mapsto \gamma(u)$ of this intersection.

LEMMA 5.4. *On a curve where y_2 is constant we have*

$$y_1'(u) = \frac{C^2 + G'^2}{C} \cos\varphi \cosh A,$$
$$y_3'(u) = -\frac{C^2 + G'^2}{C} \sin\varphi \cosh A.$$

Proof. Differentiating (16) we obtain $A' = C\cos^2\varphi + (\alpha/C)G'$. Hence we get

$$\begin{aligned} y_1'(u) = {} & \frac{1}{\alpha}\left(G''\cos\varphi - G'\varphi'\sin\varphi - C^2\sin\varphi\cos^2\varphi - \alpha G'\sin\varphi\right)\sinh A \\ & + \frac{1}{\alpha}\left(CG'\cos^3\varphi + \frac{\alpha}{C}G'^2\cos\varphi - C\varphi'\cos\varphi\right)\cosh A \end{aligned}$$

and

$$\begin{aligned} y_3'(u) = {} & \frac{1}{\alpha}\left(\frac{C}{\alpha}G''\cos\varphi - C\left(\frac{G'}{\alpha} - 1\right)\varphi'\sin\varphi - \left(\frac{C^2}{\alpha} + G'\right)\left(C\cos^2\varphi + \frac{\alpha}{C}G'\right)\sin\varphi\right)\cosh A \\ & + \frac{1}{\alpha}\left(C\left(\frac{G'}{\alpha} - 1\right)\left(C\cos^2\varphi + \frac{\alpha}{C}G'\right)\cos\varphi - G''\sin\varphi - \left(\frac{C^2}{\alpha} + G'\right)\varphi'\cos\varphi\right)\sinh A. \end{aligned}$$

We conclude using (9) and (13). □

PROPOSITION 5.5. *Let $c \in \mathbb{R}$. The intersection of $\mathcal{C}_\alpha$ and the vertical plane $\{y_2 = c\}$ is a non-empty closed embedded convex curve.*

Proof. This intersection is non-empty since setting $u = 0$ and $A = c$ gives $y_2 = c$. Also, by Lemma 5.4 we have ${y_1'}^2 + {y_3'}^2 > 0$, so the intersection of $\mathcal{C}_\alpha$ and the vertical plane $\{y_2 = c\}$ is a smooth curve γ. Also, we have $\gamma(u + 2U) = \gamma(u)$, so the curve is closed.

We now prove that γ is embedded and convex. We consider the half of γ corresponding to $u \in (-U/2, U/2)$. We have $\cos\varphi(u) > 0$. Then, by Lemma 5.4, $u \mapsto y_1(u)$ is injective and increasing. We get

$$\frac{dy_3}{dy_1} = -\tan\varphi(u),$$

so $\frac{dy_3}{dy_1}$ is an increasing function of u, and also of y_1. Consequently, the half of γ corresponding to $u \in (-U/2, U/2)$ is an embedded convex arc and is situated below the segment linking its endpoints.

Finally, since $\gamma(u + U) = -\gamma(u)$, the whole curve is embedded and is convex. □

THEOREM 5.6. *The horizontal catenoid $\mathcal{C}_\alpha$ has the following properties.*

(1) *The intersection of $\mathcal{C}_\alpha$ and any vertical plane of equation $x_2 = c$ ($c \in \mathbb{R}$) is a non-empty closed embedded convex curve.*

(2) *The surface $\mathcal{C}_\alpha$ is properly embedded.*

(3) *The horizontal catenoid $\mathcal{C}_\alpha$ is invariant by rotations of angle π around the x_1, x_2 and x_3 axes. The x_2-axis is contained in the 'interior' of $\mathcal{C}_\alpha$.*

(4) *It is conformally equivalent to $\mathbb{C} \setminus \{0\}$.*

Proof. (1) This is Proposition 5.5.

(2) The fact that $\mathcal{C}_\alpha$ is embedded is a consequence of Proposition 5.5. On a diverging path on $\mathcal{C}_\alpha$, the intersection A must be diverging and so x_2 is diverging. Consequently, $\mathcal{C}_\alpha$ is proper.

(3) The symmetries of $\mathcal{C}_\alpha$ have already been proved. The x_2-axis is contained in the 'interior' of $\mathcal{C}_\alpha$ since each curve $x_2 = c$ ($c \in \mathbb{R}$) is convex and symmetric with respect to the x_2-axis.

(4) The immersion $X = (F, h)$ induces a conformal bijective parametrization of $\mathcal{C}_\alpha$ by $\mathbb{C}/(Z\mathbb{Z})$, where Z is defined in Proposition 5.2. □

We now describe a few remarkable curves on $\mathcal{C}_\alpha$.

The curve corresponding to $u = 0$ is the set of the lowest points of the curves $y_2 = c$ ($c \in \mathbb{R}$). This curve is given by

$$y_1 = \frac{\alpha - \sqrt{\alpha^2 + \cos(2\theta) - C^2}}{\alpha} \sinh\left(\frac{\alpha}{C} y_2\right),$$
$$y_3 = -\frac{C\sqrt{\alpha^2 + \cos(2\theta) - C^2}}{\alpha} \cosh\left(\frac{\alpha}{C} y_2\right).$$

The curves along which $\mathcal{C}_\alpha$ is vertical correspond to $u = \pm U/2$ (because of formula (6)). They are symmetric one to the other with respect to the x_2-axis. By (14) and (8), the curve corresponding to $u = U/2$ is given by

$$y_1 = \frac{C}{\alpha} \cosh\left(\frac{\alpha}{C} y_2\right),$$
$$y_3 = \frac{2C^2 - \cos(2\theta)}{2\alpha^2} \sinh\left(\frac{\alpha}{C} y_2\right).$$

Consequently, the horizontal projection of $\mathcal{C}_\alpha$ is

$$\pi(\mathcal{C}_\alpha) = \left\{(y_1, y_2) \in \mathbb{R}^2; |y_1| \leqslant \frac{C}{\alpha} \cosh\left(\frac{\alpha}{C} y_2\right)\right\}$$

It is a remarkable fact that this projection coincides with the projection of a minimal catenoid of $\mathbb{R}^3$ of parameter C/α.

The curve given by $x_2 = 0$ is the analog of the 'waist circle' of minimal catenoids in $\mathbb{R}^3$.

Proposition 5.7. *On $\mathcal{C}_\alpha$, there exists some points with negative curvature and some points with positive curvature. Moreover, $\mathcal{C}_\alpha$ has infinite total absolute curvature.*

Proof. Setting $\lambda = ({G'}^2 + C^2)\cosh^2 A$, the curvature of ds^2 is given by

$$K = -\frac{1}{2\lambda}\Delta_0(\ln \lambda),$$

where Δ_0 is the Laplacian with respect to $|dz|^2$. Thus we have

$$\begin{aligned} K\lambda &= -\frac{\partial}{\partial u}(\beta' \tanh A) - \frac{\partial}{\partial v}(\alpha \tanh A) - \frac{\partial}{\partial u}\left(\frac{G'G''}{{G'}^2 + C^2}\right) \\ &= 2C\varphi' \sin\varphi \cos\varphi \tanh A - \frac{C^2\cos^4\varphi + \alpha^2}{\cosh^2 A} - (C^2 + {G'}^2)\sin^2\varphi\cos^2\varphi - G'\varphi'(2\cos^2\varphi - 1) \end{aligned}$$

by (13).

Hence, when $u = \pm U/2$ we have $K\lambda = -\alpha^2/\cosh^2 A + \cos(2\theta)/2$, which is positive for $|A|$ large enough. On the other hand, when $u = A = 0$ we get $K\lambda = -2\alpha^2 - \cos(2\theta) + \alpha\sqrt{\alpha^2 + \cos(2\theta) - C^2} < 0$.

Finally, the total absolute curvature of $\mathcal{C}_\alpha$ is

$$\int_{-U}^{U}\int_{-\infty}^{+\infty} |K|\lambda \, du \, dv = +\infty$$

since, in general, $K\lambda$ does not tend to 0 when $v \to +\infty$ and u fixed. □

6. *Limit of horizontal catenoids and vertical half-space theorem in* Nil_3

In this section we study the limit of $\mathcal{C}_\alpha$ when $\alpha \to +\infty$. As a corollary we obtain a vertical half-space theorem.

Since the parameter α will vary, the quantities and functions appearing in the construction of $\mathcal{C}_\alpha$ will be denoted by X_α, φ_α, U_α, C_α, β_α, etc. instead of X, φ, U, C, β, etc. They depend smoothly on α.

Proposition 6.1. *Let $(\hat{u}, \hat{v}) \in \mathbb{R}^2$. For $\alpha > 0$, let $u_\alpha = \hat{u}/\alpha$ and $v_\alpha = (4\ln\alpha + \hat{v})/\alpha$. Then, when $\alpha \to +\infty$,*

$$\begin{aligned}(y_1)_\alpha(u_\alpha, v_\alpha) &\longrightarrow \frac{\sin\hat{u}}{4} e^{\hat{v}/2},\\ (y_2)_\alpha(u_\alpha, v_\alpha) &\longrightarrow 0,\\ (y_3)_\alpha(u_\alpha, v_\alpha) &\longrightarrow -\frac{\cos\hat{u}}{4} e^{\hat{v}/2}.\end{aligned}$$

Proof. We have $\tilde{\theta}_\alpha \to \pi/4$ and so $C_\alpha \sim 1/2\alpha$. We have $|(\beta_\alpha)'| \leqslant 1/2\alpha$, thus $|\beta_\alpha(u)| \leqslant |u|/2\alpha$ and so $\beta_\alpha(u_\alpha) = \mathrm{O}(1/\alpha^2)$. Also, for $\alpha \geqslant 1/2$, we have

$$|(G_\alpha)'| = \left| \frac{C_\alpha^2 \cos^2\varphi_\alpha - \cos(2\tilde{\theta}_\alpha)}{\alpha - (\varphi_\alpha)'} \right| \leqslant \frac{1}{4\alpha^3} + \frac{\cos(2\tilde{\theta}_\alpha)}{\alpha},$$

thus $G_\alpha(u_\alpha) = \mathrm{O}(1/\alpha^2)$. From this we obtain that

$$(y_2)_\alpha(u_\alpha, v_\alpha) = C_\alpha v_\alpha - G_\alpha(u_\alpha) \to 0.$$

We also have

$$A = 4\ln\alpha + \hat{v} + \mathrm{O}\left(\frac{1}{\alpha^2}\right), \quad \cosh A \sim \frac{1}{2} e^{\hat{v}/2}\alpha^2, \quad \sinh A \sim \frac{1}{2} e^{\hat{v}/2}\alpha^2.$$

And since, for $\alpha \geqslant 1$, $-\sqrt{\alpha^2+1} \leqslant (\varphi_\alpha)' \leqslant -\sqrt{\alpha^2-1}$, we have $\varphi_\alpha(u_\alpha) \to -\hat{u}$. This concludes the proof. $\square$

This proposition means that, when $\alpha \to +\infty$, the half of $\mathcal{C}_\alpha$ corresponding to $A > 0$ converges to the punctured vertical plane $\{x_2 = 0\} \setminus \{(0,0,0)\}$. In the same way one can prove that the other half of $\mathcal{C}_\alpha$ converges to this punctured vertical plane.

Lemma 6.2. *The curve of equation $y_2 = 0$ in $\mathcal{C}_\alpha$ converges uniformly to 0 when $\alpha \to +\infty$.*

Proof. On the curve of equation $y_2 = 0$ in $\mathcal{C}_\alpha$ we have

$$\begin{aligned}|(y_1)_\alpha(u)| &\leqslant \frac{|G_\alpha'(u)|}{\alpha} \sinh|A_\alpha(u)| + \frac{C_\alpha}{\alpha}\cosh A_\alpha(u),\\ |(y_3)_\alpha(u)| &\leqslant \frac{C_\alpha}{\alpha}\left(\frac{|G_\alpha'(u)|}{\alpha} + 1\right)\cosh A_\alpha(u) + \frac{1}{\alpha}\left(\frac{C_\alpha^2}{\alpha} + |G_\alpha'(u)|\right)\sinh|A_\alpha(u)|\end{aligned}$$

with

$$A_\alpha(u) = \beta_\alpha(u) + \frac{\alpha}{C_\alpha} G_\alpha(u).$$

By the computations done in the proof of Proposition 6.1 we have, for $\alpha \geqslant 1$, $|G_\alpha'(u)| \leqslant 2/\alpha$ and $C_\alpha \leqslant 1/2\alpha$. Hence it suffices to prove that A_α is uniformly bounded when $\alpha \to +\infty$.

By (14), the function $\beta_\alpha + (\alpha/C_\alpha)G_\alpha$ is $2U_\alpha$-periodic with

$$U_\alpha = \int_0^{U_\alpha} du = \int_{-1}^{1} \frac{dx}{\sqrt{(1-x^2)P_\alpha(x)}}.$$

We n3ow assume that $\alpha \leqslant 1$. For $x \in [-1,1]$ we have $P_\alpha(x) \geqslant \alpha^2 - 1$, and so $U_\alpha \leqslant \pi/\sqrt{\alpha^2-1}$. Using the bounds on β'_α and G'_α and the periodicity, we get

$$|A_\alpha(u)| \leqslant \frac{\pi}{2\alpha\sqrt{\alpha^2-1}} + \frac{\alpha}{C_\alpha}\frac{2\pi}{\alpha\sqrt{\alpha^2-1}}.$$

Next, since $C_\alpha \sim 1/2\alpha$, we conclude that A_α is uniformly bounded when $\alpha \to +\infty$, which ends the proof. □

THEOREM 6.3 (Vertical half-space theorem). *Let Σ be a properly immersed minimal surface in* Nil_3. *Assume that Σ is contained on one side of a vertical plane P. Then Σ is a vertical plane parallel to P.*

Proof. We assume that Σ is not a vertical plane.

We proceed as in [**16**]. Up to an isometry of Nil_3 we can assume that P is the plane $\{y_2 = 0\}$, that $\Sigma \subset \{y_2 \leqslant 0\}$ and that Σ is not contained in any half-space $\{y_2 \leqslant -\varepsilon\}$ for $\varepsilon > 0$. By the maximum principle, we necessarily have $\Sigma \cap P = \emptyset$.

We use the coordinates (y_1, y_2, y_3) defined by (15). For $\varepsilon \in \mathbb{R}$, let $T_\varepsilon : (y_1, y_2, y_3) \mapsto (y_1, y_2 + \varepsilon, y_3)$ (this is a translation in the y_2 direction, an isometry of Nil_3). Then, for $\varepsilon > 0$ sufficiently small, we have $T_\varepsilon(\Sigma) \cap P \neq \emptyset$.

For $\alpha \geqslant 1$ we consider the half-horizontal catenoid $\mathcal{C}'_\alpha = \mathcal{C}_\alpha \cap \{y_2 \geqslant 0\}$. By Lemma 6.2, there exists a compact subset $\mathcal{D}$ of P containing 0 and $\mathcal{C}_\alpha \cap P$ for all $\alpha \geqslant 1$.

We claim that there exists an $\varepsilon > 0$ such that

$$T_\varepsilon(\Sigma) \cap P \neq \emptyset, \quad T_\varepsilon(\Sigma) \cap \mathcal{C}'_1 = \emptyset, \quad T_\varepsilon(\Sigma) \cap \mathcal{D} = \emptyset.$$

Assume that the claim is false. Since $T_\eta(\Sigma) \cap P \neq \emptyset$ for η small enough, this means that there exist a sequence (ε_n) of positive numbers converging to 0 and a sequence (q_n) of points such that $q_n \in T_{\varepsilon_n}(\Sigma)$ and $q_n \in \mathcal{C}'_1 \cup \mathcal{D}$ for all n. In particular, for n large enough, q_n belongs to the union of $\mathcal{D}$ and the part of $\mathcal{C}_1$ between the planes $\{y_2 = 0\}$ and $\{y_2 = 1\}$, which is compact. Hence, up to extraction of a subsequence, we can assume that q_n converges to a point q. We necessarily have $q \in P$ and, since Σ is proper, $q \in \Sigma$. This contradicts the fact that $\Sigma \cap P = \emptyset$, which proves the claim.

By Proposition 6.1, $\mathcal{C}'_\alpha$ converges smoothly, away from 0, to $P \setminus \{0\}$ when $\alpha \to +\infty$. Hence, for α large enough, $\mathcal{C}'_\alpha \cap T_\varepsilon(\Sigma) \neq \emptyset$. Also, by continuity of the family $(\mathcal{C}_\alpha)$, we have $\mathcal{C}'_\alpha \cap T_\varepsilon(\Sigma) = \emptyset$ for α close enough to 1.

Let $\Gamma = \{\alpha \geqslant 1; \mathcal{C}'_\alpha \cap T_\varepsilon(\Sigma) \neq \emptyset\}$ and $\gamma = \inf \Gamma$. We have $\gamma > 1$. We claim that $\gamma \in \Gamma$.

If γ is an isolated point, then it is clear. We now assume that γ is not isolated. Then there exists a decreasing sequence (α_n) converging to Γ and a sequence of points (p_n) such that $p_n \in \mathcal{C}'_{\alpha_n} \cap T_\varepsilon(\Sigma)$. We can write $p_n = X_{\alpha_n}(u_n, A_n)$ with $u_n \in [-U_{\alpha_n}, U_{\alpha_n}]$ and $A_n \in \mathbb{R}$. We have $0 \leqslant (y_2)_{\alpha_n}(p_n) \leqslant \varepsilon$, that is,

$$0 \leqslant \frac{C_{\alpha_n}}{\alpha_n}A_n - \frac{C_{\alpha_n}}{\alpha_n}\beta_{\alpha_n}(u_n) - G_{\alpha_n}(u_n) \leqslant \varepsilon.$$

Since for all n we have $\alpha_n \in [\gamma, \alpha_0]$, u_n is bounded and so $|(C_{\alpha_n}/\alpha_n)\beta_{\alpha_n}(u_n) + G_{\alpha_n}(u_n)|$ is also bounded; moreover C_{α_n} is bounded from below by a positive constant. From this we deduce that A_n is bounded. Consequently, up to extraction of a subsequence, we can assume that (u_n, A_n) converges to some $(u, A) \in \mathbb{R}$. Then, by continuity, p_n converges to a point lying in $T_\varepsilon(\Sigma)$ and in $\mathcal{C}'_\gamma$. This finishes proving the claim.

Thus there exists a point $p \in \mathcal{C}'_\gamma \cap T_\varepsilon(\Sigma)$. Since $\partial \mathcal{C}'_\gamma \subset \mathcal{D}$ (by construction of $\mathcal{D}$) and $T_\varepsilon(\Sigma) \cap \mathcal{D} = \emptyset$, it follows that p is an interior point of $\mathcal{C}'_\gamma$. Moreover, since $\mathcal{C}'_\alpha \cap T_\varepsilon(\Sigma) = \emptyset$ for all $\alpha < \gamma$, it follows that $\mathcal{C}'_\gamma$ lies on one side of $T_\varepsilon(\Sigma)$ in a neighborhood of p. Then, by the maximum principle we get $T_\varepsilon(\Sigma) = \mathcal{C}_\gamma$, which gives a contradiction since a horizontal catenoid is not contained in a half-space. □

Remark 6.4. Apart from the fact that horizontal catenoids converge to a punctured vertical plane, the key fact in this proof is that horizontal catenoids meet all vertical planes $\{y_2 = c\}$ for $c \in \mathbb{R}$ (Proposition 5.5). This ensures that the sequence (p_n) is bounded.

For example, for minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$, rotational catenoids have finite height (see [**17**]), and so there is no half-space theorem with respect to horizontal planes.

Remark 6.5. Abresch and Rosenberg [**2**] proved a half-space theorem with respect to surfaces of equation $x_3 = c$ ($c \in \mathbb{R}$). It relies on the fact that rotational catenoids converge to such a surface.

7. *Helicoidal minimal surfaces*

In this section we investigate the minimal surface $\mathcal{H}_\alpha$ in Nil_3 whose Gauss map is the function g defined in Section 4, with $\theta = 0$. The calculus of Proposition 5.1 still holds with $C = 0$, $\beta = 0$ and $A = \alpha v$.

The coordinates (y_1, y_2, y_3) of $\mathcal{H}_\alpha$ are

$$\begin{aligned} y_1 &= \frac{G'(u)}{\alpha} \cos\varphi(u) \sinh(\alpha v), \\ y_2 &= -G(u), \\ y_3 &= -\frac{G'(u)}{\alpha} \sin\varphi(u) \sinh(\alpha v). \end{aligned}$$

In particular, we have $y_3/y_1 = -\tan\varphi(u)$ and y_2 only depends on u. This means that the intersection of $\mathcal{H}_\alpha$ and any vertical plane $\{y_2 = c\}$ is a straight line. Moreover the surface is singly periodic since φ is periodic.

8. *A family of horizontal CMC $\frac{1}{2}$ annuli*

In this section we integrate the equations of Fernandez and Mira to construct a one-parameter family of horizontal annuli. These surfaces are the sister surfaces of the helicoidal surfaces constructed in section 7.

We consider the harmonic map $g : \Sigma \to \mathbb{H}^2$ given in Section 4, with $\theta = 0$ and g^* the conjugate harmonic map (see Section 4).

Lemma 8.1. *The following harmonic maps are conjugate:*

$$\begin{aligned} g &= \frac{\sin\varphi(u) + i\sinh(\alpha v)}{\cos\varphi(u) + \cosh(\alpha v)}, \\ g^* &= \frac{\sin\varphi_*(u) + i\sinh(\alpha_* v)}{\cos\varphi_*(u) + \cosh(\alpha_* v)} \end{aligned}$$

with $\varphi'^2 - \alpha^2 = \cos^2\varphi, \varphi(0) = 0$ *and* $\varphi_*'^2 - \alpha_*^2 = -\cos^2\varphi_*, \varphi_*(0) = 0$ *and* $\alpha_*^2 = \alpha^2 + 1$.

Proof. We remark that g^* is harmonic as g in Section 4. Moreover $Q(g^*) = -(1/4)dz^2$. The conformal minimal immersions in $\mathbb{H}^2 \times \mathbb{R}$ are given by $Y(u,v) = (g(u,v), v)$ and $Y^* = (g^*(u,v), u)$. To be isometric, it suffices to check that $\cosh\omega = \cosh\omega^*$, that is,

$$\cosh^2\omega = \frac{4|g^u|^2}{(1-|g|^2)^2} = 1 + \frac{4|g^v|^2}{(1-|g|^2)^2} = \frac{4|g^{*v}|^2}{(1-|g^*|^2)^2} = 1 + \frac{4|g^{*u}|^2}{(1-|g^*|^2)^2}.$$

These relations are equivalent to

$$\frac{\varphi'^2}{\cos^2\varphi} = 1 + \frac{\alpha^2}{\cos^2\varphi} = 1 + \frac{\varphi_*'^2}{\cos^2\varphi_*} = \frac{\alpha_*^2}{\cos^2\varphi_*}.$$

A straighforward computation shows that the functions $\varphi'/\cos\varphi$ and $\alpha_*/\cos\varphi_*$ are both solutions of

$$A'^2 = (A^2-1)(A^2-\alpha^2-1).$$

Moreover we have $\varphi(0) = \varphi_*(0) = 0$ because $\alpha_*^2 = \alpha^2 + 1$. This conclude the proof. □

In summary we have $Q(g) = (1/4)dz^2 = -Q(g^*)$ and $\tau = \tau^* = e^{2\omega}$. The map g induces locally a minimal graph in Nil_3 by Theorem 2.3, with metric

$$\lambda = \frac{\tau}{\nu^2}|dz|^2 = 16\frac{(1+|g|^2)^2}{(1-|g|^2)^4}|g_z|^2|dz|^2.$$

Then

$$\nu^2 = \frac{1-|g|^2}{1+|g|^2} = \frac{\cos^2\varphi}{\cosh^2(\alpha v)}, \qquad \tau = \frac{(\varphi'+\alpha)^2}{\cos^2\varphi} = \frac{(\varphi_*'+\alpha_*)^2}{\cos^2\varphi_*}. \tag{17}$$

This minimal multigraph is isometric to an immersed CMC $\frac{1}{2}$ surface in $\mathbb{H}^2 \times \mathbb{R}$ with harmonic Gauss map $g^* : \Sigma \to \mathbb{H}^2$ admitting data $(-Q, \tau)$. For $a_0 \in \mathbb{C}$, there is a unique solution h^* of the following system

$$\begin{aligned} h^*_{zz} &= (\log\tau)_z h^*_z + Q\sqrt{\frac{\tau + 4|h^*_z|^2}{\tau}}, \\ h^*_{z\bar z} &= \frac{1}{4}\sqrt{\tau(\tau + 4|h^*_z|^2)}, \\ h^*_z(z_0) &= a_0 \end{aligned}$$

with $\tau + 4|h^*_z|^2 = \lambda$, and using (17), $\varphi'' + \sin\varphi\cos\varphi = 0$, $Q = \frac{1}{4}$ we obtain

$$\begin{aligned} h^*_{zz} &= \alpha\tan\varphi h^*_z + \frac{\cosh(\alpha v)}{4\cos\varphi}, \\ h^*_{z\bar z} &= \frac{(\varphi'+\alpha)^2\cosh(\alpha v)}{4\cos^3\varphi}. \end{aligned}$$

Now set $H = h^*_z$, then

$$\begin{aligned} H_z &= \alpha\tan\varphi H + \frac{\cosh(\alpha v)}{4\cos\varphi}, \\ H_{\bar z} &= \frac{(\varphi'+\alpha)^2\cosh(\alpha v)}{4\cos^3\varphi}, \\ H(z_0) &= a_0. \end{aligned}$$

Then

$$H(u,v) = \frac{\cos\varphi}{2(\alpha-\varphi')}\left(i\sinh(\alpha v) - \tan\varphi\cosh(\alpha v)\right) + K_1(u)e^{i(\alpha\tan\varphi)v} + K_2(u),$$

where $K_i' = a\tan\varphi K_i$ and $K_1(u_0), K_2(u_0)$ are chosen to have $H(z_0) = a_0$. It is a two-parameter family and we are interested in a_0 such that, $K_1 = K_2 = 0$, that is, the solution with

$\tau + 4|H|^2 = \lambda$. The solution is periodic in u and by (17) we have

$$h^* = \frac{\cos\varphi\cosh(\alpha v)}{\alpha(\varphi' - \alpha)} = \frac{\cos\varphi_*\cosh(\alpha v)}{\alpha(\varphi'_* - \alpha_*)}.$$

Now we consider

$$(G_1, G_2, G_3) = \left(\frac{2g^*}{1-|g^*|^2}, \frac{1+|g^*|^2}{1-|g^*|^2}\right) = \left(\tan\varphi_*, \frac{\sinh(\alpha_* v)}{\cos\varphi_*}, \frac{\cosh(\alpha_* v)}{\cos\varphi_*}\right),$$

and we compute the horizontal component $F^* = (X_1 + iX_2)/(1+X_3)$ given by

$$X_j = \frac{8\mathrm{Re}\left(G_{j,z}(4\bar{Q}h^*_z + \tau h^*_{\bar{z}})\right)}{\tau^2 - 16|Q|^2} + G_j\sqrt{\frac{\tau + 4|h^*_z|^2}{\tau}},$$

that is

$$X_j = (\alpha_* - \varphi'_*)\left(\frac{G_{j,u}h^*_u}{\varphi'_*} + \frac{G_{j,v}h^*_v}{\alpha_*}\right) + G_j\frac{\cosh(\alpha v)}{\cos\varphi}.$$

Straightforward computations give

$$\begin{aligned}
h^*_u &= \frac{\alpha_*\sin\varphi_*\cosh(\alpha v)}{\alpha(\varphi'_* - \alpha_*)}, \quad h^*_v = \frac{\cos\varphi_*\sinh(\alpha v)}{(\varphi'_* - \alpha_*)},\\
G_{1,u} &= \frac{\varphi'_*}{\cos^2\varphi_*}, \quad G_{1,v} = 0,\\
G_{2,u} &= \frac{\varphi'_*\sin\varphi_*\sinh(\alpha_* v)}{\cos^2\varphi_*}, \quad G_{2,v} = \frac{\alpha_*\cosh(\alpha_* v)}{\cos\varphi_*},\\
G_{3,u} &= \frac{\varphi'_*\sin\varphi_*\cosh(\alpha_* v)}{\cos^2\varphi_*}, \quad G_{3,v} = \frac{\alpha_*\sinh(\alpha_* v)}{\cos\varphi_*}.
\end{aligned}$$

Inserting the explicit value above, setting $f(u) = (\alpha\cos\varphi_* - \alpha_*\cos\varphi)/(\alpha\cos\varphi\cos^2\varphi_*)$, we obtain

$$\begin{aligned}
X_1 &= \cosh(\alpha v)\sin\varphi_*(u)f(u),\\
X_2 &= \cosh(\alpha v)\sinh(\alpha_* v)\left(f(u) + \frac{\alpha_*}{\alpha}\right) - \cosh(\alpha_* v)\sinh(\alpha v),\\
X_3 &= \cosh(\alpha v)\cosh(\alpha_* v)\left(f(u) + \frac{\alpha_*}{\alpha}\right) - \sinh(\alpha_* v)\sinh(\alpha v).
\end{aligned}$$

Now we are interested in level curves at height zero. Then by (8), we have $\varphi(-U/2) = \varphi_*(-U/2) = +\pi/2$ and $\varphi(U/2) = \varphi_*(U/2) = -\pi/2$. We have $h^*(-U/2, v) = h^*(U/2, v) = 0$. Using the ODE of φ and φ_* we obtain $f(u) \to \gamma = -1/2\alpha\alpha_*$ when $u \to \pm U/2$. Then the horizontal curve $h^* = 0$ has two connected components given by $F^*(\pm U/2, v) = (X_1 + iX_2)/(1+X_3)$ with

$$\begin{aligned}
X_1 &= \cosh(\alpha v)\sin\varphi_*(\pm U/2)\gamma,\\
X_2 &= \cosh(\alpha v)\sinh(\alpha_* v)\left(\gamma + \frac{\alpha_*}{\alpha}\right) - \cosh(\alpha_* v)\sinh(\alpha v),\\
X_3 &= \cosh(\alpha v)\cosh(\alpha_* v)\left(\gamma + \frac{\alpha_*}{\alpha}\right) - \sinh(\alpha_* v)\sinh(\alpha v).
\end{aligned}$$

We remark that $\gamma + \alpha_*/\alpha = (2\alpha^2 + 1)/2\alpha\alpha_* \geqslant 1$. In the disk model $F^*(-U/2, v)$ and $F^*(U/2, v)$ are symmetric with respect to the y-axis of the disk ($X_1(-U/2, v) = -X_1(U/2, v)$). Then we will study $F^*(-U/2, v)$. It is a curve linking the point $(0, -1)$ to $(0, 1)$ in the unit disk and staying in $\mathrm{Re}\, F^* > 0$. We prove that this curve is embedded and behaves like a generatrix of a Bryant catenoid in hyperbolic three-space. In the half-plane model of $\mathbb{H}^2$, this curve is given by

$$\tilde{F}(-U/2,v)=\left(\frac{X_1}{X_3-X_2},\frac{1}{X_3-X_2}\right)$$
$$=\left(\frac{\gamma e^{\alpha_* v}}{(\gamma+\alpha_*/\alpha+\tanh(\alpha v))},\frac{e^{\alpha_* v}}{\cosh(\alpha v)(\gamma+\alpha_*/\alpha+\tanh(\alpha v))}\right).$$

By a straighforward computation we can see that the map

$$v\longmapsto\frac{\gamma e^{\alpha_* v}}{(\gamma+\alpha_*/\alpha+\tanh(\alpha v))}$$

is strictly increasing for $\alpha\leqslant 1$ and has exactly one point where the derivative is zero when $\alpha=1$. When $\alpha\geqslant 1$ the function has two local extrema at points v where $\tanh(\alpha v_\pm)=(-\sqrt{\alpha^2+1}\pm\sqrt{\alpha^2-1})/2\alpha<0$. For $v>0$, $v(x_1)$ is a well-defined function and $1/(X_3-X_2)=x_1^{1-\alpha/\alpha_*}q(x_1)$, where $q(x_1)$ is a bounded function having a positive limit at infinity. When $\alpha\to\infty$, the curve converges to a two-tangent horocycle.

The immersion for $u\in[-U/2,U/2]$ is a graph over a simply connected domain of $\mathbb{H}^2$. We complete it by reflection about the horizontal plane of height zero in $\mathbb{H}^2\times\mathbb{R}$ to obtain a properly embedded annulus.

Appendix. *Proofs of lemmas* 4.5 *and* 4.6

We use the notations of Section 4. For $(\alpha,\theta)\in\Omega$, we notice that $(-\alpha\sqrt{P_{\alpha,\theta}(x)}+\alpha^2)/x^2$ can be extended smoothly at $x=0$ and that

$$L(\alpha,\theta)=\int_{-1}^{1}\frac{-\alpha\sqrt{P_{\alpha,\theta}(x)}+\alpha^2+C_{\alpha,\theta}^2x^4}{x^2\sqrt{(1-x^2)}\sqrt{P_{\alpha,\theta}(x)}}dx.$$

Lemma 1 (Lemma 4.5). *Let $\alpha>0$. Then there exists a unique $\tilde{\theta}_\alpha\in(0,\theta_\alpha^+)\cap(0,\pi/4)$ such that*

$$L(\alpha,\tilde{\theta}_\alpha)=0.$$

Proof. We have

$$L(\alpha,\theta)=\int_{-1}^{1}\frac{l(\alpha,\theta,x)}{\sqrt{1-x^2}}dx=\int_{-\pi/2}^{\pi/2}l(\alpha,\theta,\sin t)\,dt,$$

where l is a smooth function on $\Omega\times[-1,1]$. Hence L is smooth on Ω.

We have $L(\alpha,0)<0$, since $C_{\alpha,0}=0$.

We first deal with the case where $\alpha>1/\sqrt{2}$, that is, $\theta_\alpha^+>\pi/4$. Since the integrand in $L(\alpha,\pi/4)$ is positive for all $x\in(-1,1)$, we have $L(\alpha,\pi/4)>0$. Hence, by continuity, there exists a $\tilde{\theta}_\alpha\in(0,\pi/4)$ such that $L(\alpha,\tilde{\theta}_\alpha)=0$.

We now deal with the case where $\alpha\leqslant 1/\sqrt{2}$, that is, $\theta_\alpha^+\leqslant\pi/4$. We have $C_{\alpha,\theta}^2\to 1-\alpha^2$, $\rho_{\alpha,\theta}^-\to 1$ and $\rho_{\alpha,\theta}^+\to\alpha^2/(1-\alpha^2)>0$ when $\theta\to\theta_\alpha^+$. We have

$$L(\alpha,\theta)=L_1(\alpha,\theta)+L_2(\alpha,\theta)$$

with

$$L_1(\alpha,\theta)=\int_{-1}^{1}-\frac{2\alpha C_{\alpha,\theta}^2}{\alpha+\sqrt{P_{\alpha,\theta}(x)}}\sqrt{\frac{1-x^2}{P_{\alpha,\theta}(x)}}dx,$$

$$L_2(\alpha,\theta)=\int_{-1}^{1}\frac{2\alpha C_{\alpha,\theta}^2-\alpha\cos(2\theta)+C_{\alpha,\theta}^2x^2\sqrt{P_{\alpha,\theta}(x)}}{\sqrt{(1-x^2)P_{\alpha,\theta}(x)}(\alpha+\sqrt{P_{\alpha,\theta}(x)})}dx.$$

We claim that $(1-x^2)/P_{\alpha,\theta}(x)$ is uniformly bounded (in x) when $\theta \to \theta_\alpha^+$. Indeed we have

$$\begin{aligned}\left|\frac{1-x^2}{P_{\alpha,\theta}(x)} - \frac{1}{\alpha^2+(1-\alpha^2)x^2}\right| &= \left|\frac{(1-2\alpha^2-\cos(2\theta))x^2-(1-\alpha^2-C_{\alpha,\theta}^2)x^4}{C_{\alpha,\theta}^2(\rho_{\alpha,\theta}^- - x^2)(\rho_{\alpha,\theta}^+ + x^2)(\alpha^2+(1-\alpha^2)x^2)}\right| \\ &\leqslant \frac{|1-2\alpha^2-\cos(2\theta)|+|1-\alpha^2-C_{\alpha,\theta}^2|}{C_{\alpha,\theta}^2(\rho_{\alpha,\theta}^- - 1)\rho_{\alpha,\theta}^+\alpha^2} \\ &\leqslant \frac{(\cos(2\theta)+1+2\alpha^2)(1-\cos(2\theta))}{4\alpha^2 C_{\alpha,\theta}^2\rho_{\alpha,\theta}^+}.\end{aligned}$$

This upper bound has a finite limit when $\theta \to \theta_\alpha^+$. This proves the claim. Consequently, $L_1(\alpha,\theta)$ is bounded when $\theta \to \theta_\alpha^+$. Moreover, we have $2\alpha C_{\alpha,\theta}^2 - \alpha\cos(2\theta) \to \alpha$ when $\theta \to \theta_\alpha^+$, so there exists a positive constant c_α such that, for θ close enough to θ_α^+,

$$L_2(\alpha,\theta) \geqslant \int_{-1}^{1} \frac{c_\alpha}{\sqrt{(1-x^2)(\rho_{\alpha,\theta}^- - x^2)}}dx.$$

Since $\rho_{\alpha,\theta}^- \to 1$ when $\theta \to \theta_\alpha^+$, we obtain that $L_2(\alpha,\theta) \to +\infty$ when $\theta \to \theta_\alpha^+$. We conclude that $L(\alpha,\theta) \to +\infty$ when $\theta \to \theta_\alpha^+$. Hence, by continuity, there exists a $\tilde\theta_\alpha \in (0,\theta_0)$ such that $L(\alpha,\tilde\theta_\alpha)=0$.

To prove the uniqueness of $\tilde\theta_\alpha$, it suffices to prove that $\theta \mapsto L(\alpha,\theta)$ is increasing on $(0,\theta_\alpha^+)\cap(0,\pi/4)$. A straightforward computation gives

$$\frac{\partial}{\partial\theta}\left(\frac{-\alpha\sqrt{P_{\alpha,\theta}(x)}+\alpha^2+C_{\alpha,\theta}^2x^4}{x^2\sqrt{P_{\alpha,\theta}(x)}}\right) = \frac{C_{\alpha,\theta}}{\alpha}\frac{B_1}{P_{\alpha,\theta}(x)^{3/2}}$$

with

$$B_1 = 2\cos(2\theta)x^2P_{\alpha,\theta}(x) + (2\alpha^2+\cos(2\theta)x^2)(\alpha^2+C_{\alpha,\theta}^2x^4) > 0.$$

This shows that $L(\alpha,\theta)$ is increasing in θ, which proves the uniqueness of $\tilde\theta_\alpha$. □

LEMMA 2 (Lemma 4.6). *We have*

$$\lim_{\alpha\to+\infty}\tilde\theta_\alpha = \frac{\pi}{4}.$$

Proof. We set $C_\alpha = C_{\alpha,\tilde\theta_\alpha}$ and $P_\alpha(x) = P_{\alpha,\tilde\theta_\alpha}(x)$. We first notice that $C_\alpha \leqslant 1/2\alpha$ and, for $\alpha \geqslant 1/2$ and $x\in[-1,1]$, $\alpha^2-1 \leqslant P_\alpha(x) \leqslant \alpha^2+1$.

We have

$$0 = \alpha L(\alpha,\tilde\theta_\alpha) = I_1(\alpha) - \cos(2\tilde\theta_\alpha)I_2(\alpha) + I_3(\alpha)$$

with

$$\begin{aligned}I_1(\alpha) &= \int_{-1}^{1}\frac{2\alpha^2C_\alpha^2}{\sqrt{P_\alpha(x)}(\alpha+\sqrt{P_\alpha(x)})}\frac{x^2dx}{\sqrt{1-x^2}} = \mathrm{O}\left(\frac{1}{\alpha^2}\right),\\ I_2(\alpha) &= \int_{-1}^{1}\frac{\alpha^2}{\sqrt{P_\alpha(x)}(\alpha+\sqrt{P_\alpha(x)})}\frac{dx}{\sqrt{1-x^2}} \geqslant \frac{\pi\alpha^2}{\sqrt{\alpha^2+1}(\alpha+\sqrt{\alpha^2+1})},\\ I_3(\alpha) &= \int_{-1}^{1}\frac{\alpha C_\alpha^2}{\alpha+\sqrt{P_\alpha(x)}}\frac{x^2dx}{\sqrt{1-x^2}} = \mathrm{O}\left(\frac{1}{\alpha^2}\right).\end{aligned}$$

Hence $\cos(2\tilde\theta_\alpha) \to 0$, which proves the lemma. □

References

1. U. Abresch and H. Rosenberg, 'A Hopf differential for constant mean curvature surfaces in $\mathbf{S}^2 \times \mathbf{R}$ and $\mathbf{H}^2 \times \mathbf{R}$', *Acta Math.*, 193 (2004) 141–174.
2. U. Abresch and H. Rosenberg, 'Generalized Hopf differentials', *Mat. Contemp.*, 28 (2005) 1–28.
3. R. Bryant, 'Surfaces of mean curvature one in hyperbolic space', *Astérisque* 154–155 (1988) 321–347, 353 [1987. Théorie des variétés minimales et applications (Palaiseau, 1983–1984)].
4. S. Y. Cheng and S. T. Yau, 'Differential equations on Riemannian manifolds and their geometric applications', *Comm. Pure Appl. Math.*, 28 (1975) 333–354.
5. T. H. Colding and W. P. Minicozzi II, *Minimal surfaces*, Courant Lecture Notes in Mathematics 4 (New York University Courant Institute of Mathematical Sciences, New York, 1999).
6. B. Daniel, 'Isometric immersions into $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ and applications to minimal surfaces,' Preprint, 2004, arXiv:math.DG/0406426, *Trans. Amer. Math. Soc.*, to appear.
7. B. Daniel, 'The Gauss map of minimal surfaces in the Heisenberg group,' Preprint, 2006, arXiv:math.DG/0606299.
8. B. Daniel, 'Isometric immersions into 3-dimensional homogeneous manifolds', *Comment. Math. Helv.* 82 (2007) 87–131.
9. I. Fernández and P. Mira, 'Harmonic maps and constant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$', *Amer. J. Math.* 129 (2007) 1145–1181.
10. I. Fernández and P. Mira, 'Holomorphic quadratic differentials and the Bernstein problem in Heisenberg space', Preprint, 2007, arXiv:0705.1436, *Trans. Amer. Math. Soc.*, to appear.
11. C. Figueroa, F. Mercuri and R. Pedrosa, 'Invariant surfaces of the Heisenberg groups', *Ann. Mat. Pura Appl.* (4), 177 (1999) 173–194.
12. L. Hauswirth, 'Minimal surfaces of Riemann type in three-dimensional product manifolds,' *Pacific J. Math.* 224 (2006) 91–117.
13. L. Hauswirth and H. Rosenberg, 'Minimal surfaces of finite total curvature in $\mathbb{H} \times \mathbb{R}$', *Mat. Contemp.*, 31 (2006) 65–80.
14. L. Hauswirth, H. Rosenberg, and J. Spruck, 'On complete mean curvature $\frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$', Preprint, 2005, http://perso-math.univ-mlv.fr/users/hauswirth.laurent, *Comm. Anal. Geom.*, to appear.
15. L. Hauswirth, R. Sá Earp and E. Toubiana, 'Associate and conjugate minimal immersion in $\mathbb{M} \times \mathbb{R}$', *Tohoku Math. J.* 60 (2008) 267–286.
16. D. Hoffman and W. H. Meeks III. 'The strong halfspace theorem for minimal surfaces', *Invent. Math.*, 101 (1990) 373–377.
17. B. Nelli and H. Rosenberg, 'Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$', *Bull. Braz. Math. Soc. (N.S.)* 33 (2002) 263–292.
18. M. Umehara and K. Yamada, 'Complete surfaces of constant mean curvature 1 in the hyperbolic 3-space', *Ann. of Math.* (2), 137 (1993) 611–638.
19. T. Y.-H. Wan, 'Constant mean curvature surface, harmonic maps, and universal Teichmüller space', *J. Differential Geom.* 35 (1992) 643–657.
20. T. Y.-H. Wan and T. K.-K. Au, 'Parabolic constant mean curvature spacelike surfaces', *Proc. Amer. Math. Soc.* 120 (1994) 559–564.

Benoît Daniel
Université Paris-Est
Laboratoire d'Analyse et de Mathématiques Appliquées
UMR 8050
UFR des Sciences et Technologies
61 avenue du Général de Gaulle
94010 Créteil cedex
France

daniel@univ-paris12.fr

Laurent Hauswirth
Université Paris-Est
Laboratoire d'Analyse et de Mathématiques Appliquées
UMR 8050
Cité Descartes, 5 bd Descartes
Champs-sur-Marne
77454 Marne-la-Vallée cedex 2
France

laurent.hauswirth@univ-mlv.fr