Surfaces of constant curvature in $\mathbb{R}^3$ with isolated singularities

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Abstract. We prove that finite area isolated singularities of surfaces with constant positive curvature $K > 0$ in $\mathbb{R}^3$ are removable singularities, branch points or immersed conical singularities. We describe the space of immersed conical singularities of such surfaces in terms of the class of real analytic closed locally convex curves in $\mathbb{S}^2$ with admissible cusp singularities, characterizing when the singularity is actually embedded. In the global setting, we describe the space of *peaked spheres* in $\mathbb{R}^3$, i.e. compact convex surfaces of constant curvature $K > 0$ with a finite number of singularities, and give applications to harmonic maps and constant mean curvature surfaces.

Résumé. Nous démontrons que les singularités isolées d'aire totale finie des surfaces de courbure de Gauss constante positive $K > 0$ de $\mathbb{R}^3$ sont des points de branchements, des singularités coniques immergées ou bien sont apparentes. Nous décrivons l'espace des singularités coniques immergées de ces surfaces en fonction des courbes analytiques réelles fermées et localement convexes de la sphère $\mathbb{S}^2$ pouvant admettre des singularités cuspidales. Nous charactérisons les courbes associées aux singularités plongées. Nous décrivons l'espace des sphères à *pics* de $\mathbb{R}^3$; celles qui sont des surfaces compactes convexes de courbure de Gauss constante $K > 0$ possédant un nombre fini de singularités. Nous donnons des applications de ces résultats aux surfaces de courbure moyenne constante et aux applications harmoniques à valeur dans $\mathbb{S}^2$.

1 Introduction

It is a classical result that any complete surface of constant curvature $K > 0$ in $\mathbb{R}^3$ is a round sphere. Thus, if one extends by analytic continuation a local piece of such a *K-surface* in $\mathbb{R}^3$ other than a sphere, singularities will eventually appear. It is hence natural to consider K-surfaces in $\mathbb{R}^3$ in the presence of singularities, and to investigate how the nature of these singularities determines the global geometry of the surface.

Surfaces of positive constant curvature with singularities are related to other natural geometric theories. For instance, they are parallel surfaces to constant mean curvature surfaces, their Gauss map is harmonic into $\mathbb{S}^2$ for the conformal structure of the second fundamental form, and this Gauss map can often be realized as the vertical projection of a minimal surface in the product space $\mathbb{S}^2 \times \mathbb{R}$. Also, when viewed as graphs, these surfaces are solutions to one of the most widely studied elliptic Monge-Ampère equations:

$$u_{xx}u_{yy} - u_{xy}^2 = K(1 + u_x^2 + u_y^2)^2, \qquad K > 0. \tag{1.1}$$

Thus, the regularity theory for K-surfaces in $\mathbb{R}^3$ is tightly linked to the regularity theory of Monge-Ampère equations.

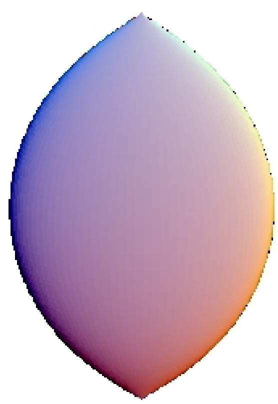

Figure 1: A rotational peaked sphere ($K = 1$) in $\mathbb{R}^3$

It is remarkable that surfaces of positive constant curvature in $\mathbb{R}^3$ can be regularly embedded around an isolated singularity, as shown by the rotational example in Figure 1. This type of singularities does not appear in many other geometric theories, and even for the case of K-surfaces in $\mathbb{R}^3$ they only appear in special circumstances. Still, K-surfaces in $\mathbb{R}^3$ having only isolated singularities are of central interest to the theory from several points of view. We explain this next, together with our contributions to the topic.

1. Singularity theory. The singularities of K-surfaces in $\mathbb{R}^3$ generically form curves in $\mathbb{R}^3$. Thus, K-surfaces only with isolated singularities constitute a rare phenomenon that happens when the surface with singularities has the biggest possible regularity (it is everywhere analytic except for some isolated points).

A natural problem in the theory of singularities is to understand when an isolated singularity is actually *removable* in an adequate sense. In this paper, we will show (Theorem 5) that an isolated singularity of a K-surface in $\mathbb{R}^3$ is *extendable* (i.e. the surface extends smoothly to the puncture with a well-defined tangent plane at it) if and only if the mean curvature is bounded around the singularity. Also, we prove that an isolated singularity of a K-surface in $\mathbb{R}^3$ has finite area if and only if it has finite total mean curvature (see Proposition 4).

2. Ends of K-surfaces in $\mathbb{R}^3$. Another global restriction of K-surfaces in $\mathbb{R}^3$ is that one cannot have complete ends in the theory (in particular, the exterior Dirichlet problem for (1.1) does not have a solution). This contrasts with the situation in other related theories, such as flat surfaces in hyperbolic 3-space [GMMi, GaMi, KUY, KRSUY] or solutions to the Hessian one equation $\det(D^2u) = 1$, see [FMM, GMM, ACG, Mar].

In any case, Figure 1 also suggests that these isolated singularities can be thought of as the most natural notion of an *end* for a K-surface in $\mathbb{R}^3$ (i.e. it is the most regular way in which a non-spherical K-surface can end). The study of such an *end*, although it is placed at a finite point in $\mathbb{R}^3$, is non-trivial. For instance, the unit normal does not extend continuously across that point, and the extrinsic conformal structure is that of an annulus (and not of a punctured disk).

In this paper we classify such *ends* under more general assumptions. We will show in Theorem 7 that any non-extendable isolated singularity of a K-surface in $\mathbb{R}^3$ with finite area has a *limit* unit normal at the singularity, which is a real analytic, closed strictly locally convex curve in $\mathbb{S}^2$, possibly having isolated singular points of cusp type of a certain shape. Conversely, we show that any such curve arises as the limit normal of a unique non-extendable isolated singularity of a K-surface in $\mathbb{R}^3$.

In the above classification, we also characterize when the surface is regularly embedded around the singularity. In this way, we prove (Theorem 12) that the space of embedded isolated singularities of K-surfaces in $\mathbb{R}^3$ is in one-to-one correspondence with the class of real analytic strictly convex Jordan curves on the sphere $\mathbb{S}^2$, and that K-surfaces can be analytically extended across any embedded isolated singularity by Schwarzian reflection for the conformal structure of the second fundamental form.

3. Isometric embedding of abstracts metrics in $\mathbb{R}^3$ and peaked spheres. A classical problem in differential geometry is to determine when a Riemannian surface (M^2, g) can be isometrically immersed (or embedded) into $\mathbb{R}^3$, and if such an isometric realization is unique up to rigid motions in $\mathbb{R}^3$. On the other hand, the class of conformal metrics of constant curvature 1 with conical singularities on a Riemann surface has been widely studied, both from a local and global point of view. It is hence natural to study the isometric embedding problem for such metrics of constant curvature with conical singularities.

In this paper we will show that any non-extendable isolated singularity of finite area of a K-surface in $\mathbb{R}^3$ (all of which are classified in Theorem 7) is, from an intrinsic point of view, a conical singularity, see Proposition 10. Hence, our results describe the space of possible local isometric realizations into $\mathbb{R}^3$ of an abstract conical metric of curvature 1.

In the global case, let us define a *peaked sphere* in $\mathbb{R}^3$ as a closed convex surface (i.e. the boundary of a convex region of $\mathbb{R}^3$) that is everywhere regular except for a finite set of points, and has constant curvature $K > 0$ at regular points. These are, from many different points of view, the most regular K-surfaces in $\mathbb{R}^3$ that one can consider globally. Note that a peaked sphere with no singularities is a round sphere, that there are no peaked spheres with exactly one singularity, and that a peaked sphere with exactly two singularities is a rotational example like in Figure 1.

Combining results by Alexandrov, Pogorelov, Troyanov and Luo-Tian, one can show that peaked spheres with an arbitrary number $n > 2$ of singularities exist, and that they constitute a $3n - 6$ dimensional family. These parameters come from the freedom in prescribing the intrinsic conformal structure of the peaked sphere (which is that of a finitely punctured sphere) and the cone angles at the singularities, all of which are in $(0, 2\pi)$ and must obey two inequalities (see Section 4). Thus, peaked spheres can be well described from an intrinsic point of view, but they are far from being understood in what regards their actual shape in $\mathbb{R}^3$. For instance, we do not know necessary and sufficient conditions under which a finite set of points $p_1, \dots, p_n \in \mathbb{R}^3$ can be realized as the set of singularities of a peaked sphere in $\mathbb{R}^3$.

By using the intrinsic classification of peaked spheres (see Theorem 15) and our analysis on embedded isolated singularities of K-surfaces in $\mathbb{R}^3$ in terms of their extrinsic conformal structure, we can provide some applications to harmonic maps into $\mathbb{S}^2$ and CMC surfaces in $\mathbb{R}^3$. In particular, we describe the space of solutions to the Neumann problem for harmonic diffeomorphisms into $\mathbb{S}^2$ (Theorem 16), we provide a new *reflection principle* for CMC surfaces in $\mathbb{R}^3$ (Corollary 18), and we obtain a family of complete branched CMC surfaces in $\mathbb{R}^3$ with an infinite number of ends, obtained by applying this reflection principle to a compact fundamental CMC piece whose boundary meets a finite set of spheres tangentially.

The paper goes as follows. In Section 2 we deal with generalities on surfaces of constant curvature $K > 0$ in $\mathbb{R}^3$ with wave-front singularities in terms of their *extrinsic conformal structure*, i.e. the conformal structure given by the second fundamental form of the surface. In Section 3 we prove the results explained above regarding the classification of isolated singularities of K-surfaces in $\mathbb{R}^3$. In Section 4 we expose the intrinsic classification of peaked spheres with n singularities. This result follows from the classification results by Troyanov and Luo-Tian on spherical cone metrics on $\mathbb{S}^2$ with cone angles in $(0, 2\pi)$, from Alexandrov's solution to the generalized Weyl's problem, and from Pogorelov's theorems of local regularity and rigidity for convex surfaces in $\mathbb{R}^3$. In Section 5 we transfer the intrinsic classification of peaked spheres in $\mathbb{R}^3$ to other geometric theories (harmonic maps into $\mathbb{S}^2$ and CMC surfaces in $\mathbb{R}^3$), by means of their analytic properties with respect to their extrinsic conformal structure and our results of Section 3. We end up the paper with some relevant open problems about the extrinsic geometry of peaked spheres in $\mathbb{R}^3$, and their relation to harmonic maps and constant mean curvature surfaces.

2 Surfaces of positive constant curvature in $\mathbb{R}^3$

Let $\psi : M^2 \to \mathbb{R}^3$ denote an immersed surface with constant curvature $K > 0$ in $\mathbb{R}^3$. Up to a dilation, we shall assume that $K = 1$. Such surfaces will be called from now on *K-surfaces* in $\mathbb{R}^3$.

By changing orientation if necessary, the second fundamental form II of the immersion ψ is positive definite, and thus it induces a conformal structure on M^2. This structure will be called the *extrinsic conformal structure* of the surface ψ.

Specifically, we may view the surface as an immersion $X : \Sigma \to \mathbb{R}^3$ from a Riemann surface Σ such that, if $z = u + iv$ is a complex coordinate on Σ and $N : \Sigma \to \mathbb{S}^2$ denotes the unit normal of the surface, then $\langle X_u, N_u\rangle = \langle X_v, N_v\rangle < 0$ and $\langle X_u, N_v\rangle = 0$. It is easy to check (see also [GaMa]) that the condition $K = 1$ implies

$$X_u = N \times N_v, \qquad X_v = -N \times N_u. \tag{2.1}$$

In particular, the unit normal $N : \Sigma \to \mathbb{S}^2$ is a harmonic map into $\mathbb{S}^2$, and the immersion X satisfies the equation

$$X_{uu} + X_{vv} = 2X_u \times X_v. \tag{2.2}$$

Observe that N is always a local diffeomorphism and that, by (2.1), the surface X is uniquely determined by N up to translations.

In order to deal with the possible appearance of singularities for the surface X, it is useful to introduce the following notions (see [KRSUY, SUY], for instance).

Definition 1 *A C^∞ map $f : M^2 \to \mathbb{R}^3$ from a surface M^2 is called a* frontal *if there exists a C^∞ map $N : M^2 \to \mathbb{S}^2$ such that $\langle df, N\rangle = 0$. Such a map is called the* unit normal *of the frontal. If in addition $\langle df, df\rangle + \langle dN, dN\rangle$ is positive definite, we say that f is a* front *in $\mathbb{R}^3$.*

Clearly, any regular surface in $\mathbb{R}^3$ is a front, and any front is a frontal. We will use this terminology from now on.

Conversely to the previous discussion on K-surfaces in $\mathbb{R}^3$ in terms of their Gauss maps, if $N : \Sigma \to \mathbb{S}^2$ is a harmonic map such that the smooth map X given by (2.1) is single valued (e.g. if Σ is simply connected), then $X : \Sigma \to \mathbb{R}^3$ is a frontal in $\mathbb{R}^3$, having $K = 1$ at its regular points. Moreover, the singular points of X are precisely the points at which N fails to be a local diffeomorphism. The frontal X will be a front except at the points where dN vanishes.

In terms of a conformal parameter $z = u + iv$ for the second fundamental form, the fundamental forms of a K-surface X are given by

$$\left\{\begin{array}{rcl} \langle dX, dX\rangle & = & Q\,dz^2 + 2\mu|dz|^2 + \bar{Q}d\bar{z}^2, \\ -\langle dX, dN\rangle & = & 2\rho|dz|^2, \\ \langle dN, dN\rangle & = & -Q\,dz^2 + 2\mu|dz|^2 - \bar{Q}d\bar{z}^2, \end{array}\right. \tag{2.3}$$

where:

1. $Qdz^2 := \langle X_z, X_z\rangle dz^2 = -\langle N_z, N_z\rangle dz^2$ is a holomorphic quadratic differential on Σ, which vanishes exactly at the umbilical points of the surface. We will call it the *extrinsic Hopf differential* of the surface.

2. $\mu, \rho : \Sigma \to (0,\infty)$ are smooth positive real functions, which by the condition $K = 1$ satisfy the relation
$$\rho^2 = \mu^2 - |Q|^2. \tag{2.4}$$

In order to define yet another geometric function of interest on the surface, let us assume preliminarily that $Q(z_0) \neq 0$ for some $z_0 \in \Sigma$. Then, around z_0, the real analytic function ω given by

$$\rho = |Q| \sinh \omega \tag{2.5}$$

is well defined, and positive. Moreover, as Q vanishes only at isolated points, ω can be defined globally on Σ, if we allow the possibility that $\omega = +\infty$ at the umbilical points of the surface. Besides, by the Gauss equation of the surface,

$$\omega_{z\bar{z}} + |Q| \sinh \omega = 0 \qquad \text{(or, equivalently, } \omega_{z\bar{z}} + \rho = 0\text{)}. \tag{2.6}$$

Observe that the mean curvature of X is given by any of these formulas:

$$H = \frac{\rho\,\mu}{\mu^2 - |Q|^2} = \frac{\mu}{\rho} = \coth(\omega). \tag{2.7}$$

Formulas (2.6) and (2.7) are well known, see for instance [Bob], pp. 118-119.

It is important for our purposes to understand in terms of the Gauss map how singularities can occur on a surface with $K = 1$. In order to do so, assume first that $N : \Sigma \to \mathbb{S}^2$ is a harmonic map such that $\langle dN, dN\rangle$ is given as in (2.3). Let $X : \Sigma \to \mathbb{R}^3$ be the $K = 1$ frontal in $\mathbb{R}^3$ described by (2.1), where we assume that Σ is simply connected. It is clear from (2.3) that N fails to be a local diffeomorphism exactly at the points where $\mu = |Q|$. Moreover, by (2.3), these points agree with the points at which $X : \Sigma \to \mathbb{R}^3$ is not an immersion.

Thus, if N is a local diffeomorphism, X will be an immersion. Let us analyze what happens on a *singular point* of N, i.e. a point $z_0 \in \Sigma$ such that $dN(z_0)$ has rank ≤ 1. We will rule out the case that N is singular on an open set of Σ, since this would imply that X is everywhere singular.

Let $z_0 \in \Sigma$ be a singular point of N. Then $\mu(z_0) = |Q|(z_0)$. If $\mu(z_0) = |Q|(z_0) \neq 0$, we have a point at which $dN(z_0)$ has rank one. Consequently, it is well known (see [Woo] for instance) that there exists a regular curve $\Gamma \subset \Sigma$ passing through z_0 such that dN has rank one at every point of Γ. It is also direct that the function ω vanishes identically along Γ. Contrastingly, if $\mu(z_0) = Q(z_0) = 0$, we have a point where $dN(z_0) = 0$. These points are necessarily isolated, since Q is holomorphic. There are two possible behaviors for the Gauss map around such points (see [Woo]):

1. z_0 is a meeting point of regular curves in Σ at which dN has rank one, or

2. The Gauss map has a *branch point* at z_0. That is, there are local coordinates (x, y) and (u, v) on Σ and $\mathbb{S}^2$ centered at z_0 and $N(z_0)$, respectively, such that N has the form $u + iv = (x + iy)^k$, for some integer $k > 1$.

In the first case, z_0 is not an isolated singularity for the $K = 1$ frontal X given by (2.1), since as we explained before, X fails to be an immersion whenever dN has rank ≤ 1 . In the second case the $K = 1$ frontal has an isolated *branch point*, at which its first and second fundamental forms are zero. In that case, the surface X is regularly immersed but not regularly embedded around the branch point. Indeed, if it were embedded, by Lemma 11 the surface would be a graph around the branch point, and hence the Gauss map would be injective, a contradiction.

Another important fact about surfaces with $K = 1$ in $\mathbb{R}^3$ is that they appear as parallel surfaces of constant mean curvature (CMC) surfaces in $\mathbb{R}^3$. Specifically, if $X : \Sigma \to \mathbb{R}^3$ is a $K = 1$ surface with Gauss map $N : \Sigma \to \mathbb{S}^2$, then $f = X + N : \Sigma \to \mathbb{R}^3$ and $f^\sharp = X - N : \Sigma \to \mathbb{R}^3$ are CMC surfaces with $H = -1/2$ and $H = 1/2$, respectively, possibly with singular points. The singular points $z_0 \in \Sigma$ for f (resp. $f^\sharp$) appear when $\mu(z_0) = \rho(z_0)$ (resp. $\mu(z_0) = -\rho(z_0)$). In both cases, by (2.4), it holds $Q(z_0) = 0$ for the singular points of f and $f^\sharp$. In fact, as the Hopf differential vanishes exactly at the umbilical points of the $K = 1$ immersion X, we see that the umbilical points of $X : \Sigma \to \mathbb{R}^3$ are exactly the points of Σ at which one of the two parallel CMC surfaces has a singular point (the other one will have an umbilical point).

Let us also remark that if $X : \Sigma \to \mathbb{R}^3$ is a frontal of $K = 1$ with singularities, constructed as explained above from a harmonic map $N : \Sigma \to \mathbb{S}^2$, it still makes sense to define the parallel CMC surfaces, and same criteria for the appearance of singular points of these CMC surfaces hold.

Rotational peaked spheres in $\mathbb{R}^3$

Let us consider the rotational $K = 1$ surfaces $X_A(u, v)$ given by

$$X_A(u, v) = (f(u) \cos v, f(u) \sin v, h(u)), \qquad (u, v) \in \Omega := (-\pi/2, \pi/2) \times \mathbb{R},$$

where $f(u) = A \cos u$ and $h(u) = \int_0^u \sqrt{1 - A^2 \sin^2 x}\, dx$, being $A \in (0, 1)$ a real parameter (see Figure 1). Clearly, X_A extends continuously to its closure $\bar{\Omega}$, so that $X_A(\pm\pi/2, \mathbb{R})$ are two points in the axis of rotation.

In this way, we have for each $A \in (0, 1)$ a rotational embedded surface with two isolated singularities, that will be called a *rotational peaked sphere*. The distance between these singularities agrees with the extrinsic diameter of the rotational peaked sphere, and varies continuously between 2 and π in terms of A. The family X_A varies between a sphere ($A = 1$) and a vertical segment of length π ($A = 0$). The unit normal N_A of X_A extends continuously to the boundary of Ω, and $\alpha(v) = N_A(\pm\pi/2, v)$ is a circle in $\mathbb{S}^2$ of constant geodesic curvature $k_g = \pm\sqrt{(2 - A^2)/(1 - A^2)}$.

Consider now the change of coordinates

$$s = s(u) = \int_0^u \frac{1}{\sqrt{1 - A^2 \sin^2 r}}\, dr, \qquad t = v.$$

It follows that (s,t) are global conformal parameters for X_A with respect to the second fundamental form. Using the conformal change $w = e^z$ (where $z = s + it$) we see that X_A has the extrinsic conformal structure of an annulus $\mathbb{A} = \{w : |w| \in (r,R)\}$, with $r = e^{-a}$ and $R = e^a$, being $a = \int_0^{\pi/2} 1/\sqrt{1 - A^2 \sin^2 u}\, du$. Thus, the modulus $R/r = e^{2a}$ of this annulus varies with A between e^π and ∞. In particular, conformal annuli with conformal modulus in $(1, e^\pi)$ cannot be realized as the extrinsic conformal structure of a rotational peaked sphere.

The surfaces X_A are parallel to the *unduloids*, i.e. the rotationally invariant embedded CMC surfaces in $\mathbb{R}^3$ ($H = 1/2$ in our situation).

The geometric Cauchy problem

Let $\beta(u) : I \subset \mathbb{R} \to \mathbb{R}^3$ and $V(u) : I \subset \mathbb{R} \to \mathbb{S}^2$ denote a real analytic regular curve and a real analytic map into $\mathbb{S}^2$, with $\langle \beta', V \rangle = 0$. By the Cauchy-Kovalevsky theorem, there is a unique solution $X(u,v)$ to the Cauchy problem for the equation (2.2) and the initial conditions $X(u,0) = \beta(u)$ and $X_v(u,0) = -V(u) \times V'(u)$. Using this fact and equations (2.1), (2.3) one can easily show that a necessary and sufficient condition for the existence of a regular $K = 1$ surface in $\mathbb{R}^3$ passing through β and whose unit normal along β is given by V is that $\langle \beta', V' \rangle \neq 0$ for all u. In these conditions, the solution is necessarily unique. (See [GaMi, GaMi2, ACG] for the solution to this *geometric Cauchy problem* in other geometric contexts).

Moreover, if we remove one of the regularity conditions $\beta' \neq 0$ or $\langle \beta', V' \rangle \neq 0$, a similar result holds for $K = 1$ frontals in $\mathbb{R}^3$. The resulting surface will have a singular point along β wherever $\langle \beta'(u_0), V'(u_0) \rangle = 0$. We can extract from here two consequences:

- Any regular analytic curve in $\mathbb{R}^3$ with non-vanishing curvature is realized as a curve of singularities of a unique $K = 1$ frontal in $\mathbb{R}^3$. This is easily seen taking into account that the metric relations $\langle \beta', V \rangle = \langle \beta', V' \rangle = 0$ actually determine the field V from the curve β.
- If $\beta(u) = a \in \mathbb{R}^3$ and $V(u) : I \to \mathbb{S}^2$ is real analytic, then there exists a unique $K = 1$ frontal $X : \Omega \subset \mathbb{C} \to \mathbb{R}^3$, Ω an open complex domain containing I, such that $X|_I = a \in \mathbb{R}^3$ and its unit normal along the real axis is given by $V(u)$.

This last consequence suggests a method for constructing isolated singularities of $K = 1$ surfaces in $\mathbb{R}^3$ as solutions to singular geometric Cauchy problems. This idea was first used in the authors' previous work [GaMi] on flat surfaces in hyperbolic 3-space, and it will be fully exploited for the present case in Section 3.

It must be remarked that the unique solution to the geometric Cauchy problem for $K = 1$ surfaces in $\mathbb{R}^3$ explained above can be described using loop groups. This is a consequence of the corresponding result for CMC surfaces by Brander and Dorfmeister [BrDo], and the relation between CMC surfaces and K-surfaces as parallel surfaces.

A boundary regularity result

In the next section we will use several times the following boundary regularity result for solutions to the Dirichlet problem of (2.2). The result comes from Jacobowsky's paper [Jac, Theorem 4.1] (see also the paper by Brezis and Coron [BrCo]).

We must point out that the boundary regularity result of [Jac] is only formulated in the continuous case; however, as explained in the introduction of [Jac], the result actually provides higher regularity at the boundary of the solution provided the boundary condition also has this higher regularity.

Lemma 2 ([Jac]) *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain whose boundary $\partial\Omega$ is C^∞. Let $X = (X_1, X_2, X_3) : \Omega \to \mathbb{R}^3$ be a solution to the Dirichlet problem*

$$X_{uu} + X_{vv} = 2\, X_u \times X_v \qquad \text{in } \Omega, \qquad\qquad X = \varphi \qquad \text{on } \partial\Omega,$$

where $\varphi \in C^\infty(\overline{\Omega}, \mathbb{R}^3)$. Assume that $X_k \in C^\infty(\Omega) \cap C(\overline{\Omega})$ and also that each X_k lies in the Sobolev space $H^1(\Omega) \equiv W^{1,2}(\Omega)$, for $k = 1, 2, 3$. Then $X_k \in C^\infty(\overline{\Omega})$.

3 The classification of isolated singularities

In what follows, let $D \subset \mathbb{R}^2$ denote a disc of center q, and $D^* := D \setminus \{q\}$ denote its associated punctured disc.

Definition 3 *Let $\psi : D^* \to \mathbb{R}^3$ denote an immersion of a punctured disc D^* into $\mathbb{R}^3$, and assume that ψ extends continuously to D. Then, the surface ψ is said to have an* isolated singularity *at $p = \psi(q) \in \mathbb{R}^3$.*

If ψ is an embedding around q, p will be called an embedded isolated singularity*. The singularity is called* extendable *if ψ and its unit normal N extend smoothly to D, and* removable *if it is extendable and $\psi : D \to \mathbb{R}^3$ is an immersion.*

An example of a non-extendable embedded isolated singularity is given by the graph of an arbitrary smooth function on a punctured planar topological disc $\mathcal{U}^*$ that extends continuously but not C^1 across the puncture. For example, the two isolated singularities of the surface in Figure 1 are non-extendable embedded isolated singularities.

Our aim is to classify locally the isolated singularities of K-surfaces in $\mathbb{R}^3$. For that, we will identify two K-surfaces having some point $p \in \mathbb{R}^3$ as an isolated singularity if they overlap on some common neighborhood of p. This is a natural identification since K-surfaces in $\mathbb{R}^3$ are real analytic, and our description in this section will be local around the isolated singularity p.

Proposition 4 *Let $\psi : D^* \to \mathbb{R}^3$ denote an immersed K-surface in $\mathbb{R}^3$ with an isolated singularity at $p = \psi(q)$. The following two conditions are equivalent:*

(i) ψ has finite area around p.

(ii) ψ has finite total mean curvature (i.e. $\int |H| dA < \infty$) around p.

In that case, p is called a finite isolated singularity.

Proof: From $H^2 - K \geq 0$ and $K = 1$, we clearly see that (ii) implies (i).

Conversely, assume that ψ has finite area around p. As we already said, we can reverse orientation if necessary to assume that $H \geq 1$. From (2.2) and (2.3) we see that (see also [GaMa])

$$\Delta^{II} \psi = 2N,$$

where Δ^{II} denotes the Laplacian for the second fundamental form II, which is a Riemannian metric. Therefore, if we consider the function $f = \frac{1}{2}\langle \psi - p, \psi - p\rangle : D^* \to \mathbb{R}$, a simple computation from (2.3) using that $\Delta^{II} = (2/\rho)\partial_{z\bar{z}}$ and $H = \mu/\rho$ gives

$$\Delta^{II} f = 2(H + \langle N, \psi - p\rangle). \tag{3.1}$$

Now, by Sard theorem, we know that for almost all values of $r > 0$ (sufficiently small), the sphere of $\mathbb{R}^3$ centered at p and of radius $r > 0$ meets $\psi(D^*)$ along a regular, not necessarily connected, curve. In that way, we may take values $r_1 > r_2 > 0$ for which the intersection of $\psi(D^*)$ with $\mathbb{S}^2(p, r_1)$ and $\mathbb{S}^2(p, r_2)$ is regular in the above sense. Also, by making r_1 small enough, we may assume that $\psi(\partial D)$ lies outside $\mathbb{S}^2(p, r_1)$.

Consider next the domain

$$\Omega_{r_1,r_2} = \{q \in D^* : r_1 \geq ||\psi(q) - p|| \geq r_2\}$$

and $C_{r_i} = \{q \in D^* : ||\psi(q) - p|| = r_i\}$. Using then (3.1) and the fact that, by (2.3), the area elements dA and dA_{II} for I and II agree, we have

$$\int_{\Omega_{r_1,r_2}} \Delta^{II} f \; dA_{II} = 2\int_{\Omega_{r_1,r_2}} (H + \langle N, \psi - p\rangle)\, dA_{II} = 2\int_{\Omega_{r_1,r_2}} (H + \langle N, \psi - p\rangle)\, dA. \tag{3.2}$$

On the other hand, by the divergence theorem,

$$\int_{\Omega_{r_1,r_2}} \Delta^{II} f \, dA_{II} = \int_{C_{r_1}} II(\nabla^{II} f, \nu) ds_{II} + \int_{C_{r_2}} II(\nabla^{II} f, \nu) ds_{II}. \tag{3.3}$$

Here ∇^{II} (resp. ds_{II}) is the gradient (resp. the boundary arc-length) with respect to the Riemannian metric II, and ν denotes the exterior unit normal for the domain Ω_{r_1,r_2}, again with respect to II. The first term in the right hand side of (3.3) is not interesting for what follows: call it $A_0(r_1)$. The second term is actually $\int_{C_{r_2}} \nu(f)\, ds_{II}$.

We claim now that $\nu(f) \leq 0$ along C_{r_2}. For that, we only need to observe that, as ν is the *exterior* unit normal, for each $x \in C_{r_2}$ there is a curve $\alpha(t)$ on $\Omega_{r_1,r_2} \subset D^*$ such that $\alpha(0) = x$ and $\alpha'(0) = \nu(x)$. Hence, $\nu(f)(x) = (f \circ \alpha)'(0)$ and as $(f \circ \alpha)(t) \geq r_1 = (f \circ \alpha)(0)$, we obtain $(f \circ \alpha)'(0) \leq 0$, as wished.

In addition,

$$\left| \int_{\Omega_{r_1,r_2}} \langle N, \psi - p\rangle \, dA \right| \leq \int_{\Omega_{r_1,r_2}} ||\psi - p|| \, dA \leq r_1 \, \mathcal{A}(\Omega_{r_1,r_2}) \leq r_1 \, \mathcal{A}(D^*),$$

where $\mathcal{A}$ stands for the area. Putting together this inequality with (3.2) and (3.3) we see that

$$\int_{\Omega_{r_1,r_2}} H\,dA \leq \frac{1}{2}A_0(r_1) + r_1\,\mathcal{A}(D^*).$$

As the right hand side of this inequality does not depend on r_2, by making $r_2 \to 0$ (through an adequate subsequence, so that the above explained regularity in the intersection with a ball holds) we obtain that $\int_{D^*} H\,dA$ is finite, as wished.

□

It is clear that there exist isolated singularities in $\mathbb{R}^3$ of infinite area and infinite total mean curvature around the singularity. However, we do not know if, in the case of K-surfaces in $\mathbb{R}^3$, all isolated singularities are actually finite.

Characterization of extendable isolated singularities

Let us recall that an isolated singularity of a surface $\psi : D^* \to \mathbb{R}^3$ is called *extendable* if both ψ and its unit normal N extend smoothly across the singularity. This means that the surface has a well defined tangent plane at the singularity, although it could be non-regular at it. The following result characterizes extendable singularities of K-surfaces. It generalizes theorems by Beyerstedt [Bey] and by Heinz and Beyerstedt [HeBe] for the case of graphs satisfying (1.1) on a punctured disk.

Theorem 5 *Let $\psi : D^* \to \mathbb{R}^3$ be an immersed K-surface with an isolated singularity at $p = \psi(q)$. The following conditions are equivalent.*

(i) The isolated singularity p is extendable.

(ii) The mean curvature of ψ is bounded around the singularity.

(iii) ψ has around the singularity the extrinsic conformal structure of a punctured disk.

(iv) The singularity p is removable, or it is a branch point.

Proof: We show first that (i) implies (iii), arguing by contradiction. For that, assume that the singularity is extendable, but that its extrinsic conformal structure is that of an annulus. We denote this conformal parametrization of the surface by X, and the conformal coordinate of the annulus $\mathbb{A}$ by $z = u + iv$.

As $\langle d\psi, d\psi\rangle$ is a smooth quadratic form on D, it is clear that ψ has finite area. So, by Proposition 4 and equation (2.3),

$$\int_{D^*} H\,dA = \int_{\mathbb{A}} \frac{\mu}{\rho}\,\rho\,dudv = \int_{\mathbb{A}} (\langle X_u, X_u\rangle + \langle X_v, X_v\rangle)dudv < \infty.$$

Now, since X satisfies (2.2), Lemma 2 shows that $X(u,v)$ can be extended smoothly to the boundary of $\mathbb{A}$. But by hypothesis, N can also be smoothly extended to the boundary, and it is constant on this boundary. This implies from (2.1) and the fact that

X is constant that dX vanishes identically along the boundary of the annulus. But that would imply from the uniqueness in the solution to the Cauchy problem for (2.2) that X is constant, a contradiction.

We prove next that (*iii*) implies (*ii*). Assume that the extrinsic conformal structure is that of the punctured disk $\mathbb{D}^*$. We may assume without loss of generality that the surface is smooth on $\partial\mathbb{D} \equiv \mathbb{S}^1$. Let $\omega : \mathbb{D}^* \to (0, \infty]$ be given by (2.5). We are going to prove that for every $z \in \mathbb{D}^*$ it holds

$$\omega(z) \geq \min\{\omega(\zeta) : |\zeta| = 1\} > 0.$$

This implies from (2.7) that H is bounded on $\mathbb{D}^*$, as wished.

In order to prove the above inequality, denote $\omega_0 = \min\{\omega(\zeta) : |\zeta| = 1\}$. Let $\{r_n\}$ be a strictly decreasing sequence of real numbers $r_n \in (0, 1)$, tending to 0. Let h_n denote the unique harmonic function on the annulus

$$\mathbb{A}_n = \{z \in \mathbb{C} : r_n \leq |z| \leq 1\},$$

with the Dirichlet conditions $h_n = \omega_0$ on $\mathbb{S}^1$ and $h_n = 0$ on $\{\zeta : |\zeta| = r_n\}$. By (2.6), we see that $\omega_{z\bar{z}} \leq 0$. Thus, by the maximum principle applied to ω and h_n we get that

$$0 \leq h_n(z) \leq h_{n+1}(z) \leq \omega(z), \qquad \text{for every } n \in \mathbb{N} \text{ and } z \in \mathbb{C} \text{ with } r_n \leq |z| \leq 1. \quad (3.4)$$

Thereby, we see that $\{h_n\}$ is an increasing sequence of harmonic functions, bounded from above by $\omega_0 < \infty$. So, they converge to some harmonic function h on $\mathbb{D}^* \cup \mathbb{S}^1$ which is constantly equal to ω_0 on $\mathbb{S}^1$. But as h is bounded, we deduce that $h(z) \equiv \omega_0$ on $\overline{\mathbb{D}}$. So, from (3.4) we get $\omega(z) \geq \omega_0$ for every $z \in \overline{\mathbb{D}}$, as desired. This shows that (*iii*) implies (*ii*).

In order to prove that (*ii*) implies (*iv*), we first remark that by [HaLa], if the surface has bounded mean curvature, then it has finite area. Thus, arguing as in the proof of (*i*) $\Rightarrow$ (*iii*) we have two possibilities:

1. If the surface has the extrinsic conformal structure of the punctured disk $\mathbb{D}^*$, then the finite area condition shows that

   $$\int_{\mathbb{D}^*} (\langle N_u, N_u\rangle + \langle N_v, N_v\rangle)\, dudv < \infty,$$

   i.e. $N \in \mathbb{H}^1(\mathbb{D}^*, \mathbb{S}^2) \equiv W^{1,2}(\mathbb{D}^*, \mathbb{S}^2)$. So, by Helein's regularity theorem [Hel] for harmonic maps into $\mathbb{S}^2$, N can be harmonically extended to $\mathbb{D}$. The surface X is then extended accordingly by means of (2.1). If dN is non-singular at 0, then X is immersed at 0, and so the singularity is removable. In contrast, if dN is singular at 0, a result by Wood [Woo] gives that, as dN is non-singular in $\mathbb{D}^*$, N has a branch point at z_0. Thus, (*iv*) holds.

2. If the surface has the extrinsic conformal structure of an annulus $\mathbb{A}$, as we explained previously, we may use Lemma 2 to extend $X(u, v)$ to its inner boundary, so that

X is constant there. But by (2.3) we get then that $\mu^2 - |Q|^2 = 0$ on this boundary. And as

$$\frac{|Q|^2}{\mu^2 - |Q|^2} = H^2 - 1 < \infty$$

on $\mathbb{A}$, we deduce that Q vanishes on this boundary curve. Thus $Q = 0$ everywhere, i.e. the surface is a piece of a round sphere with the extrinsic conformal structure of an annulus and that is constant on a boundary curve of the annulus. This is impossible, and rules out this second case.

Hence, $(ii) \Rightarrow (iv)$ holds. Finally, that (iv) implies (i) is immediate.

□

The classification of immersed conical singularities in $\mathbb{R}^3$

We study next the space of non-extendable finite isolated singularities of K-surfaces in $\mathbb{R}^3$. Let us point out that, by Theorem 5, the extrinsic conformal structure around such a singularity is that of an annulus.

We shall use the notation $\mathbb{A}_r = \{z \in \mathbb{C} : 1 < |z| < r\}$, where $r > 1$. We also need to introduce the following class of curves with singularities in $\mathbb{S}^2$.

Definition 6 *A smooth map $\alpha : I \subset \mathbb{R} \to \mathbb{S}^2 \subset \mathbb{R}^3$ is called a* locally convex curve with admissible cusps *if, for every $s \in I$, the quantity $||\alpha'(s)||k_\alpha(s)$ is a non-zero real number. Here $k_\alpha(s)$ is the geodesic curvature of α in $\mathbb{S}^2$, i.e.*

$$k_\alpha(s) = \frac{\langle \alpha''(s), J\alpha'(s)\rangle}{||\alpha'(s)||^3}$$

where J denotes the complex structure of $\mathbb{S}^2$.

It is clear that any regular locally convex curve in $\mathbb{S}^2$ satisfies this property. Indeed, for regular points of the curve, the condition of the definition is just that $k_\alpha \neq 0$, i.e. that α is locally convex around the point. Let us explain what happens locally around a singular point, i.e. a point s_0 with $\alpha'(s_0) = 0$. By definition, we must have

$$\lim_{s \to s_0} ||\alpha'(s)||k_\alpha(s) = c_0 \in \mathbb{R} \setminus \{0\}, \tag{3.5}$$

so in particular $k_\alpha \to \pm\infty$ when $s \to s_0$.

Assume without loss of generality that $\alpha(s_0) \in \mathbb{S}^2_+$, and let $\pi : \mathbb{S}^2_+ \to \mathbb{R}^2$ denote the totally geodesic embedding of $\mathbb{S}^2_+$ into $\mathbb{R}^2$, given by

$$\pi(x_1, x_2, x_3) = \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right).$$

If we let

$$\beta := \pi \circ \alpha, \tag{3.6}$$

then $\beta'(s_0) = 0$ and $||\beta'(s)||\kappa_\beta(s) \to c \neq 0$ as $s \to s_0$. Here κ_β stands for the curvature of β in the plane. Taking the Taylor series of β around s_0 and up to a rotation of $\mathbb{R}^2$ around the origin, it is easy to see that for the above limit to exist, we must have

$$\beta(s) = \big(a(s-s_0)^k, b(s-s_0)^{k+1}\big) + \text{ higher order terms }, \qquad (3.7)$$

where $a, b \in \mathbb{R}$ are non-zero, and $k \geq 2$. So, equations (3.6),(3.7) give the shape that the curve α must have around an admissible cusp point s_0.

Theorem 7 *Let $\alpha : \mathbb{S}^1 \to \mathbb{S}^2$ denote a closed, real analytic, locally convex curve with admissible cusps in $\mathbb{S}^2$. Then:*

i) There exists a unique harmonic map $N : \Omega \to \mathbb{S}^2$ from an open set $\Omega \subset \mathbb{C}$ containing $\mathbb{S}^1$ into $\mathbb{S}^2$, satisfying the initial conditions

$$N|_{\mathbb{S}^1} = \alpha, \qquad \left.\frac{\partial N}{\partial \mathbf{n}}\right|_{\mathbb{S}^1} = 0, \qquad (3.8)$$

where $\partial/\partial\mathbf{n}$ stands for the normal derivative along $\mathbb{S}^1$.

ii) If $X : \Omega \to \mathbb{R}^3$ is the map given in terms of N by the representation formula (2.1), then X is single valued, $X(\mathbb{S}^1) = p$ for some $p \in \mathbb{R}^3$, and for $r > 1$ sufficiently close to 1 the restriction of X to $\mathbb{A}_r$ is an immersion.

In particular, $X : \mathbb{A}_r \to \mathbb{R}^3$ is an immersed K-surface having $p \in \mathbb{R}^3$ as a non-extendable finite isolated singularity.

iii) Conversely, let $\mathcal{S}$ denote an immersed K-surface with a non-extendable finite isolated singularity at $p \in \mathbb{R}^3$. Then, $\mathcal{S}$ is one of the surfaces constructed above.

As a consequence, there exists a correspondence between the space of immersed surfaces with $K = 1$ in $\mathbb{R}^3$ having $p \in \mathbb{R}^3$ as a non-extendable finite isolated singularity and the class of closed, real analytic, locally convex curves with admissible cusps in $\mathbb{S}^2$.

Proof: We shall identify in the usual way $\mathbb{S}^1$ with $\mathbb{R}/(2\pi\mathbb{Z})$ and $\mathbb{C}\setminus\{0\}$ with $\mathbb{C}/(2\pi\mathbb{Z})$. It follows then from the Cauchy-Kowalevsky theorem applied to the equation for harmonic maps into $\mathbb{S}^2$ that there exists a harmonic map $N(u, v)$ defined on an open set $\mathcal{U} \subset \mathbb{R}^2 \equiv \mathbb{C}$ containing $\mathbb{R}$, such that

$$N(u, 0) = \alpha(u), \qquad N_v(u, 0) = 0, \qquad (3.9)$$

for all $u \in \mathbb{R}$. Moreover, as α is 2π-periodic, so is N (by uniqueness). Assertion *i)* follows then immediately.

Let us now prove *iii)*. Consider an immersed $K = 1$ surface $\mathcal{S} \subset \mathbb{R}^3$ with $p \in \mathbb{R}^3$ a non-extendable finite isolated singularity. Without loss of generality we may assume that $p = (0, 0, 0)$.

By Theorem 5, $\mathcal{S}$ has the extrinsic conformal structure of an annulus. This implies that, changing $\mathcal{S}$ by a proper subset of it if necessary, $\mathcal{S}$ can be conformally parametrized with respect to the second fundamental form by a map $X : \mathcal{U}^+ \subset \mathbb{C} \to \mathbb{R}^3$, so that $\mathcal{S} = X(\mathcal{U}^+)$, where $\mathcal{U}^+ := \{z \in \mathbb{C} : 0 < \text{Im} z < \delta\}$ for some $\delta > 0$, and X is 2π-periodic. Moreover, X is real analytic and extends continuously to the boundary $\mathbb{R} \subset \partial\mathcal{U}^+$ with $X(u, 0) = 0$ for all $u \in \mathbb{R}$. Also, observe that the other boundary component $\mathbb{R} + i\,\delta$ of $\partial\mathcal{U}^+$ is not relevant to our study: we can assume that X extends analytically across $\mathbb{R} + i\,\delta$ to a larger open set.

Write now $X(u, v) = (X_1(u, v), X_2(u, v), X_3(u, v))$, where $z = u + iv$. As the singularity has finite area we have

$$\int_{\mathcal{U}^+} (\langle X_u, X_u\rangle + \langle X_v, X_v\rangle) du\, dv < \infty,$$

and so we see that X_1, X_2, X_3 belong to the Sobolev space $H^1(\mathcal{U}^+) \equiv W^{1,2}(\mathcal{U}^+)$.

Now, recall that X is a solution to (2.2) that vanishes on $\mathbb{R} \cap \partial\mathcal{U}^+ \equiv \mathbb{R}$. In these conditions Lemma 2 ensures that X extends smoothly up to the boundary. It follows then that the extension of X to

$$\mathcal{U} := \{z \in \mathbb{C} : -\delta < \text{Im}\, z < \delta\} \tag{3.10}$$

given by $X(u, -v) = -X(u, v)$ is a real analytic map with $X(u, 0) = 0$. Hence,

$$X_u \times X_v : \mathcal{U} \to \mathbb{R}^3$$

satisfies $(X_u \times X_v)(u, -v) = -(X_u \times X_v)(u, v)$ and vanishes along $\mathbb{R}$. By analyticity, we can then write near $\mathbb{R}$

$$(X_u \times X_v)(u, v) = v^{2k+1} G(u, v)$$

for some $k \in \mathbb{N}$ and some real analytic $\mathbb{R}^3$-valued map $G(u, v)$ with $G(u, 0)$ not identically zero. Now, the 2π-periodic real analytic function $f(u) = G(u, 0)$ can be trivially written as $f(u) = f_1(u)f_2(u)$, where both $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}^3$ are real analytic and $f_2 \neq 0$.

All of this shows that we can write near $\mathbb{R}$

$$X_u \times X_v(u, v) = \xi(u, v)\, F(u, v),$$

where $\xi : \mathcal{U} \to \mathbb{R}$ is real analytic with $\xi(u, -v) = -\xi(u, v)$ and $\xi(u, 0) \equiv 0$, and $F : \mathcal{U} \to \mathbb{R}^3$ is also real analytic with $F(u, v) = F(u, -v)$ and $F \neq (0, 0, 0)$ along $\mathbb{R}$. Therefore, the unit normal N of X can be analytically extended across $\mathbb{R}$ by $N = F/||F||$. That is, N extends analytically to a map $N : \mathcal{U} \to \mathbb{S}^2$ as $N(u, v) = N(u, -v)$, and the surface X is recovered in terms of N by (2.1). Thus, denoting $\alpha(u) := N(u, 0)$, we see that α is real analytic, 2π periodic, and N is the unique solution to the harmonic map equation into $\mathbb{S}^2$ for the initial conditions (3.9).

Let us prove now that α is a locally convex curve with admissible cusps, following Definition 6. First, observe that the zeros of Q are isolated on $\mathcal{U}$, and that on $\mathbb{R}$ we have $||\alpha'(u)||^2 = |Q(u, 0)|$. Also, observe that the function ρ in (2.3) vanishes on Ω exactly at the points in the real axis (since in the general case, it vanishes at the singular points of X). This tells that the function ω given by (2.5) also satisfies $\omega(u,0)=0$. So, from (2.5), we have then

$$\rho_v(u,0) = ||\alpha'(u)||^2\omega_v(u,0). \tag{3.11}$$

In addition, if J denotes the complex structure of $\mathbb{S}^2$, from (2.1) we have at $(u,0)$,

$$\begin{aligned} \langle \alpha'', J\alpha' \rangle &= \langle N_{uu}, N\times N_u\rangle = \langle N\times N_u, N\times X_{uv}\rangle = \langle N_u, X_{uv}\rangle \\ &= \frac{\partial}{\partial v}\left(\langle N_u, X_u\rangle\right) - \langle N_{uv}, X_u\rangle = \rho_v, \end{aligned}$$

where we have used that $X_u(u,0)=0$. If we compare this with (3.11), we get the relation

$$\omega_v(u,0) = ||\alpha'(u)||k_\alpha(u), \tag{3.12}$$

where here $k_\alpha(u)$ stands for the geodesic curvature in $\mathbb{S}^2$ of $\alpha(u)$. This equation implies that

$$\lim_{u\to u_0} ||\alpha'(u)||k_\alpha(u) = c_0 \in \mathbb{R},$$

for every $u_0\in\mathbb{R}$, and we want to ensure that $c_0\neq 0$. For that, observe that if $c_0=0$ for some u_0, then $\nabla\omega(u_0)=0$. Now, as ω satisfies the elliptic PDE (2.6), this implies that there are at least two nodal curves of ω passing through u_0 (one of which is the real line). But as the zeros of ω are singular points of the surface, this contradicts the fact that X is regular on $\mathcal{U}\setminus\mathbb{R}$, i.e. the fact that the singularity is isolated.

So, we have proved that α is a locally convex curve with admissible cusps singularities, as desired. This completes the proof of *iii*).

In order to prove assertion *ii*), let $X:\mathcal{U}\to\mathbb{R}^3$ be the map given by the representation formula (2.1) in terms of the harmonic map $N:\mathcal{U}/2\pi\mathbb{Z}\to\mathbb{S}^2$ with initial conditions (3.9), where $\mathcal{U}\subset\mathbb{C}$ is given by (3.10) for some $\delta>0$. It is then immediate that $X(u,0)=p$ for some $p\in\mathbb{R}^3$, so it follows from (2.1) and the periodicity of N that X is also 2π-periodic. Since $\alpha(u)$ is locally convex with admissible cusps (see Definition 6, the map $\omega:\Omega\to\mathbb{R}$ given by (2.5) satisfies (3.12). Hence, $\omega_v(u,0)\neq 0$ for all $u\in\mathbb{R}$, and this implies that $\omega\neq 0$ on $\mathcal{U}\setminus\mathbb{R}$ (by taking a smaller $\delta>0$ in the definition of $\mathcal{U}$, if necessary). As a consequence, if $\mathcal{U}^+ := \mathcal{U}\cap\{\operatorname{Im} z>0\}$, then $X:\mathcal{U}^+/2\pi\mathbb{Z}\to\mathbb{R}^3$ is regular, and therefore it is a $K=1$ surface in $\mathbb{R}^3$ having p as an isolated singularity. By construction, the singularity is non-extendable (since it has the conformal type of an annulus) and finite (since it has finite area around the singularity). This concludes the proof of *ii*), and of the theorem.

□

Remark 8 Theorem 7 shows that any non-extendable isolated singularity with finite area of a K-surface in $\mathbb{R}^3$ has a well defined limit unit normal at the singularity, which is a real analytic closed strictly convex curve with admissible cusps in $\mathbb{S}^2$. In this sense, the following definition is natural.

Definition 9 *A non-extendable finite isolated singularity of a K-surface in $\mathbb{R}^3$ will be called an* immersed conical singularity*.*

Let us recall that there exists an *intrinsic* notion of conical singularity for a Riemannian metric on a punctured disk. Specifically, a conformal Riemannian metric $\lambda|dz|^2$ on $\mathbb{D}^*$ has a *conical singularity* of angle $2\pi\theta$ at 0 if

$$\lambda = |z|^{2\beta} f \, |dz|^2,$$

where f is a continuous positive function on $\mathbb{D}$ and $\beta = \theta - 1 > -1$.

The next result shows that immersed conical singularities of K-surfaces in $\mathbb{R}^3$ are indeed conical from an intrinsic point of view.

Proposition 10 *Let $\psi : D^* \to \mathbb{R}^3$ denote a K-surface with an immersed conical singularity at the puncture $q \in D$. Then its intrinsic metric has a conical singularity at q.*

Proof: Since the area of the surface around an immersed conical singularity is finite, in order to show that the intrinsic metric ds^2 of ψ has a conical singularity at q if suffices to show, by [Bry, Proposition 4], that the conformal type of ds^2 around q is that of a punctured disk.

If this is not the case, then we can parameterize conformally (for the intrinsic metric ds^2) a punctured neighborhood $\mathcal{U}^*$ of q as a quotient $\Omega/\mathbb{Z}$, where $\Omega := \{z \in \mathbb{C} : 0 < \text{Imz} < r\}$ for some $r > 0$ (here the real axis corresponds to the singularity). Thus, there is a meromorphic map $g : \Omega \to \bar{\mathbb{C}}$ such that

$$ds^2 = \frac{4|g'|^2}{(1+|g|^2)^2} \, |dz|^2,$$

and this metric is well defined on the conformal annulus $\Omega/\mathbb{Z}$.

Define now the curve

$$\gamma(t) = ir(1-t) : (0,1) \to \Omega \subset \mathbb{C}.$$

As by Theorem 7 we can deduce that the metric of the surface extends smoothly (as a tensor, not as a regular metric) to the singularity, i.e. to the real axis, it is clear that the length $L(\gamma)$ of γ is finite as $t \to 0$. In particular, any sequence $\{is_n\} \to 0$, where $0 < s_n < r$, has the property that it is a Cauchy sequence for the $K = 1$ metric ds^2.

So, as g provides a local isometry between (Ω, ds^2) and $\bar{\mathbb{C}}$ endowed with its canonical spherical metric, there exists $z_0 \in \bar{\mathbb{C}}$ such that $\{g(is_n)\} \to z_0$ for any sequence $\{is_n\}$ in the above conditions.

Let us show that g can be extended to $(0,1) \subset \mathbb{R}$ so that $g(s) = z_0$ for all $s \in (0,1)$. This would imply that the meromorphic function g is constant, a contradiction.

For that, take $r_0 \in (0,1)$ and $\{a_n + ib_n\}$ a sequence of points in Ω that converge to r_0 (for the flat metric in $\mathbb{C}$). We also assume that $a_n \in (0,1)$ and that the real sequence $\{b_n\}$ strictly decreases to 0.

Observe now that the curves β_b whose image is $[0,1] \times \{ib\}$ give rise to a foliation of $\Omega/\mathbb{Z}$ by closed curves, so that the lengths of β_b tend to zero when $b \to 0$ (again since ds^2 extends smoothly up to the real axis).

Therefore,

$$\begin{aligned} d(g(a_n+ib_n),z_0) &\leq d(g(a_n+ib_n),g(ib_n))+d(g(ib_n),z_0) \\ &\leq d(a_n+ib_n,ib_n)+d(g(ib_n),z_0) \\ &\leq L(\beta_{b_n})+d(g(ib_n),z_0), \end{aligned}$$

where d denotes indistinctly the spherical distance on $\bar{\mathbb{C}}$, or the distance for the ds^2 metric on $\Omega/\mathbb{Z}$ (recall that they are isometric via g). As the terms on the right part of the inequality tend to zero when $n\to\infty$, we deduce that $g(a_n+ib_n)\to z_0$, as wished.

This is a contradiction, as explained before. So, the conformal structure must be that of a punctured disk, and ds^2 has a conical singularity.

□

The classification of embedded isolated singularities

We now focus our study on classifying the embedded isolated singularities of K-surfaces in $\mathbb{R}^3$. First, we have:

Lemma 11 *Let $\psi: D^*\to\mathbb{R}^3$ be a strictly locally convex surface in $\mathbb{R}^3$ having $p=\psi(q)$ as an embedded isolated singularity. Then there is a neighborhood of q such that ψ is a convex graph (possibly singular at the puncture) over some plane of $\mathbb{R}^3$.*

In particular, every embedded isolated singularity of a K-surface in $\mathbb{R}^3$ has finite area. So, it is a removable singularity or one of the conical singularities constructed in Theorem 7.

Proof: The first assertion was proved by the first and third author in [GaMi]. By convexity, it is well known then that ψ has finite area around the singularity. The rest follows from Theorems 5 and 7, bearing in mind that a K-surface in $\mathbb{R}^3$ cannot be embedded around a branch point.

□

Once here, the next result characterizes which of the curves α in $\mathbb{S}^2$ described by Theorem 7 correspond to embedded isolated singularities.

Theorem 12 *Let $\psi: D^*\to\mathbb{R}^3$ denote a K-surface with an immersed conical singularity at $p=\psi(q)$. Let $\alpha:\mathbb{S}^1\to\mathbb{S}^2$ denote its limit unit normal at the singularity. Then p is an embedded isolated singularity if and only if α is a regular convex Jordan curve in $\mathbb{S}^2$.*

In particular, the conical angle at the singularity is given by $2\pi-\mathcal{A}(\alpha)$, where $\mathcal{A}(\alpha)$ stands for the area of the smallest region of $\mathbb{S}^2$ enclosed by the convex Jordan curve α.

Proof: Let $X:\mathcal{U}\to\mathbb{R}^3$ be a $K=1$ surface with a conical singularity, as constructed in Theorem 7, and assume that it is embedded. By Lemma 11, we can assume that $X(\mathcal{U}^+)$ is a convex graph in the x_3-axis direction. Hence, making $\mathcal{U}^+$ smaller if necessary, it is clear that there exists a compact convex body $K \subset \mathbb{R}^3$ such that $X(\mathcal{U}^+)$ is a piece of its boundary.

Besides, it is well known that the set of interior unit support vectors at an arbitrary boundary point p_0 of a convex body K is a convex set of the unit sphere $\mathbb{S}^2$. In particular, if the boundary of K is C^1 at p_0, this convex set is just a point in $\mathbb{S}^2$, which agrees with the Gauss map of K at p_0.

In this way, as $\alpha(\mathbb{S}^1)$ is the limit set of unit normals of $X(\mathcal{U}^+)$, we easily see that $\alpha(\mathbb{S}^1) \subset \mathbb{S}^2$ is the boundary of a convex set of $\mathbb{S}^2$. This ensures that the local behavior (3.7) is impossible for the curve $\beta = \pi \circ \alpha$. So, from Theorem 7 we see that β (and thus α) is regular and strictly locally convex, and a simple topological argument ensures that it is actually a convex Jordan curve (since it is a limit of locally convex Jordan curves in $\mathbb{S}^2$).

Therefore, we conclude that the curve α is a regular convex Jordan curve in $\mathbb{S}^2$, as wished.

For the converse, we will use the *Legendre transform* (see [LSZ])

$$\mathcal{L}_X = \left(\frac{-N_1}{N_3}, \frac{-N_2}{N_3}, -X_1 \frac{N_1}{N_3} - X_2 \frac{N_2}{N_3} - X_3 \right) : \mathcal{U}^+ \to \mathbb{R}^3, \tag{3.13}$$

where $X = (X_1, X_2, X_3)$ and $N = (N_1, N_2, N_3)$. It is classically known that $\mathcal{L}_X$ can be defined for convex multigraphs in the x_3-axis direction, so that $\mathcal{L}_X$ is also a convex multigraph in the x_3-axis direction. The interior unit normal of $\mathcal{L}_X$ is

$$\mathcal{N}_{\mathcal{L}} = \frac{1}{\sqrt{1 + X_1^2 + X_2^2}} \left(-X_1, -X_2, 1 \right). \tag{3.14}$$

So, assume that the K-surface $X : \mathcal{U} \to \mathbb{R}^3$ with a conical singularity is generated following Theorem 7 from a regular convex Jordan curve α in $\mathbb{S}^2$. We prove below that $X(\mathcal{U}^+)$ is a graph, taking $\delta > 0$ in (3.10) smaller if necessary, what concludes the proof.

Since α is a convex Jordan curve, it lies on a hemisphere of $\mathbb{S}^2$, say, $\mathbb{S}^2_+ = \{(x_1, x_2, x_3) \in \mathbb{S}^2 : x_3 > 0\}$, and so $X|_{\mathcal{U}^+}$ is a local graph in the x_3-axis direction.

Let now $\mathcal{L}_X : \mathcal{U}^+/2\pi\mathbb{Z} \to \mathbb{R}^3$ denote the Legendre transform (3.13) of X. It turns out that $\mathcal{L}_X(\mathbb{R}/2\pi\mathbb{Z})$ is a regular convex Jordan curve in the x_1, x_2-plane, and that the unit normal of $\mathcal{L}_X$ along $\mathbb{R}$ is $(0, 0, 1)$, constant.

Therefore, $\mathcal{L}_X$ lies in the upper half-space $\mathbb{R}^3_+$, and there is some $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ the intersection $\Upsilon_\varepsilon = \mathcal{L}_X(\mathcal{U}/2\pi\mathbb{Z}) \cap \{x_3 = \varepsilon\}$ is a regular convex Jordan curve. Consider now $S_{\varepsilon_1, \varepsilon_2}$ the portion of $\mathcal{L}_X$ that lies in the slab between the planes $\{x_3 = \varepsilon_1\}$ and $\{x_3 = \varepsilon_2\}$, where $0 < \varepsilon_2 < \varepsilon_1 < \varepsilon_0$. Then, as $S_{\varepsilon_1, \varepsilon_2}$ is convex and the curves Υ_ε are convex Jordan curves, we get that the unit normal $\mathcal{N}_{\mathcal{L}}$ of $\mathcal{L}_X$ in this slab is a global diffeomorphism onto its image in $\mathbb{S}^2$. Letting $\varepsilon_1 \to 0$ and choosing $\delta > 0$ sufficiently small, we get that $\mathcal{N}_{\mathcal{L}}$ is a global diffeomorphism from $\mathcal{U}^+/2\pi\mathbb{Z}$ onto its spherical image in $\mathbb{S}^2$.

Consequently, by (3.14), $X(\mathcal{U}^+)$ is a graph over a region in the x_1, x_2-plane. This concludes the proof of the first statement of the theorem.

In order to compute from an extrinsic point of view the angle of this conical singularity, we consider the limit tangent cone $C := t\,\nu(s)$ at the singularity, where $t > 0$ and $\nu(s)$ is a convex curve on $\mathbb{S}^2$. The unit normal of this tangent cone at the origin agrees with the limit unit normal $\alpha(s) : \mathbb{R}/(2\pi\mathbb{Z}) \to \mathbb{S}^2$ of the surface at the singularity, which is also a convex Jordan curve. Let us recall that, by Theorem 7, the surface is uniquely determined by $\alpha(s)$, which can also be chosen arbitrarily. Parameterizing α by arc-length, we get working on C, that

$$\nu(s) = \alpha(s) \times \alpha'(s).$$

The cone angle θ is given by the length of the convex curve $\nu(s)$, and this implies by the Gauss-Bonnet theorem that

$$\theta = \int ||\nu'(s)||ds = \int k_g(\alpha(s))ds = 2\pi - \mathcal{A}(\alpha) \in (0, 2\pi),$$

where $k_g(\alpha(s))$ and $\mathcal{A}(\alpha)$ denote, respectively, the geodesic curvature of $\alpha(s)$ in $\mathbb{S}^2$ and the area of the smallest spherical region enclosed by α.

□

Putting together Lemma 11 and Theorem 12, we conclude:

Corollary 13 *The space of non-removable embedded isolated singularities of $K = 1$ surfaces in $\mathbb{R}^3$ is in one-to-one correspondence with the class of real analytic regular convex Jordan curves in $\mathbb{S}^2$.*

This correspondence assigns to each embedded isolated singularity its associated limit unit normal at the singularity.

4 The intrinsic classification of peaked spheres

Definition 14 *A* peaked sphere *in $\mathbb{R}^3$ is a closed convex surface $S \subset \mathbb{R}^3$ (i.e. the boundary of a bounded convex set of $\mathbb{R}^3$) that is a regular surface everywhere except for a finite set of points $p_1, \dots, p_n \in S$, and such that $S \setminus \{p_1, \dots, p_n\}$ has constant curvature 1.*

The points $p_1, \dots, p_n$ are called the singularities *of the peaked sphere S.*

Equivalently, a peaked sphere can also be defined as an embedding

$$\phi : \mathbb{S}^2 \setminus \{q_1, \dots, q_n\} \to \mathbb{R}^3$$

of constant curvature 1, such that ϕ extends continuously to $\mathbb{S}^2$. If ϕ does not C^1-extend across q_j, then $p_j := \phi(q_j) \in \mathbb{R}^3$ is a singularity of $S := \phi(\mathbb{S}^2) \subset \mathbb{R}^3$. That these two definitions agree follows from a simple topological argument and the local convexity of $K = 1$ surfaces in $\mathbb{R}^3$.

It is clear from our analysis in Section 3 that the singularities of a peaked sphere in $\mathbb{R}^3$ are conical, with conic angles in $(0, 2\pi)$. Thus, from an intrinsic point of view, peaked spheres in $\mathbb{R}^3$ are well known objects.

There are no peaked spheres with exactly one singularity. For the case of two singularities, there exist rotational peaked spheres, and a simple application of Alexandrov reflection principle shows that any peaked sphere with exactly two singularities is one of these rotational examples. So, there is exactly a 1-parameter family of non-congruent peaked spheres with $n = 2$ singularities, all of them rotational (see Section 2 and Figure 1).

For $n > 2$, peaked spheres in $\mathbb{R}^3$ with n singularities exist, and can be classified from an intrinsic point of view. This follows from some classical results by Alexandrov and Pogorelov (see also [BuSh]) on the isometric realization and regularity in $\mathbb{R}^3$ of singular metrics of non-positive curvature, together with the intrinsic classification of cone metrics of constant positive curvature on $\mathbb{S}^2$ whose cone angles lie in $(0, 2\pi)$ by Troyanov [Tro] and Luo-Tian [LuTi]. Specifically, the classification is:

Theorem 15 *Let Λ denote a conformal structure of $\mathbb{S}^2$ minus n points, $n > 2$, and let $\theta_1, \dots, \theta_n \in (0, 1)$. Then, a necessary and sufficient condition for the existence of a peaked sphere $S \subset \mathbb{R}^3$ with n singularities $p_1, \dots, p_n$ of given conic angles $2\pi\theta_1, \dots, 2\pi\theta_n$, and such that Λ is the conformal structure of $S \setminus \{p_1, \dots, p_n\}$ for its intrinsic metric, is that*

$$n - 2 < \sum_{j=1}^{n} \theta_j < n - 2 + \min_j\{\theta_j\}. \tag{4.1}$$

Moreover, any peaked sphere in $\mathbb{R}^3$ is uniquely determined up to rigid motions by the conformal structure of $S \setminus \{p_1, \dots, p_n\}$ and by the cone angles $2\pi\theta_1, \dots, 2\pi\theta_n$.

In particular, the space of peaked spheres in $\mathbb{R}^3$ with $n > 2$ singularities is a $3n - 6$ parameter family, modulo rigid motions.

Proof: In [Tro], Troyanov proved that (4.1) is a sufficient condition for the existence of a metric of constant curvature 1 on $\mathbb{S}^2 \setminus \{q_1, \dots, q_n\}$, and such that this metric has at each q_j a conical singularity of angle $2\pi\theta_j$. Then, Luo and Tian proved in [LuTi] that if $\theta_j \in (0, 1)$ for all j, then (4.1) is also a necessary condition for the existence of such a metric. Moreover, this metric is unique under the above hypotheses.

So, in order to prove Theorem 15 it suffices to show that all these metric are isometrically embeddable into $\mathbb{R}^3$ as peaked spheres, and that the intrinsic metric of any peaked sphere is isometric to one of the Troyanov-Luo-Tian cone metrics on the sphere, with all angles in $(0, 2\pi)$. The second property is clear, since we showed in Section 3 that any non-removable embedded isolated singularity of a $K = 1$ surface in $\mathbb{R}^3$ is a conical singularity of angle in $(0, 2\pi)$.

As regards the isometric realization in $\mathbb{R}^3$ of these abstract cone metrics, Alexandrov proved in [Ale] the following solution to the generalized Weyl's embedding problem: *any 2-manifold with a singular metric of non-negative curvature homeomorphic to a sphere is isometric to a closed convex surface in $\mathbb{R}^3$; conversely, any closed convex surface in $\mathbb{R}^3$ is, intrinsically, a 2-manifold of non-negative curvature.*

A precise definition of the concept of a 2-dimensional manifold of non-negative curvature in the Alexandrov sense, together with an explanation of this deep theorem, can be consulted in page 24 of [BuSh].

It turns out that $\mathbb{S}^2$ endowed with any of the cone metrics on $\mathbb{S}^2$ with angles in $(0, 2\pi)$, classified by Troyanov and Luo-Tian, is a manifold of non-negative curvature in the Alexandrov sense. This is just a consequence of the convexity condition $\theta_j \in (0,1)$ at the singularities. An alternative proof can be given as follows. First, it follows from the work of Bryant [Bry, Proposition 4] that any 2-dimensional Riemannian metric of constant positive curvature around a conical singularity is isometric to a radial Riemannian metric. Moreover, if the conic angle belongs to $(0, 2\pi)$, such a radial Riemannian metric can be realized in $\mathbb{R}^3$ as the first fundamental form of some rotationally invariant $K = 1$ surface. But this implies by convexity and embeddedness of this rotational surface that the original abstract Riemannian metric has non-negative curvature in the Alexandrov sense, see Theorem 2.2.1 in page 24 of [BuSh].

As a consequence, if g is a cone metric of constant curvature 1 on $\mathbb{S}^2$ whose cone angles are all in $(0, 2\pi)$, then there exists a (singular) closed convex surface $S \subset \mathbb{R}^3$ that is isometric to $(\mathbb{S}^2, g)$. We claim that S is a peaked sphere. Indeed, this follows from Pogorelov's regularity theorem in [Pog2] (see Theorem 3.1.1. in page 27 of [BuSh]), as we explain next. As the metric g is everywhere regular, except for a finite number of points $q_1, \dots, q_n \in \mathbb{S}^2$, the regularity theorem of Pogorelov ensures that S is a regular, smooth surface everywhere except on a finite set of points $p_1, \dots, p_n \in S$. Obviously, in the regular part of S, the Gaussian curvature is $K = 1$. Thus, S is a peaked sphere, as claimed. This completes the proof.

□

5 Peaked spheres and harmonic maps

The Neumann problem for harmonic diffeomorphisms into $\mathbb{S}^2$

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain whose boundary $\partial\Omega$ consists of a finite number of regular Jordan curves. The Neumann problem for harmonic diffeomorphisms asks about the existence (and uniqueness) of a harmonic map $g : \Omega \cup \partial\Omega \to \mathbb{S}^2$ that is a diffeomorphism onto its image, and such that

$$\left.\frac{\partial g}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0 \qquad (\mathbf{n} \text{ is the exterior normal derivative along } \partial\Omega). \tag{5.1}$$

We remark that this problem is conformally invariant, i.e. only the conformal equivalence class of the complex domain Ω matters for the problem. So, this domain can always be assumed to be a *bounded circular domain*, i.e. a disk $D_1 \subset \mathbb{C}$ with a finite collection of interior disjoint disks removed.

Theorem 16 *A harmonic map $g : \Omega \to \mathbb{S}^2$ is a solution to the Neumann problem for harmonic diffeomorphisms if and only if it is the Gauss map of a peaked sphere in $\mathbb{R}^3$, with respect to its extrinsic conformal structure.*

As a consequence, the spaces of harmonic maps into $\mathbb{S}^2$ that solve the above Neumann problem for some planar domain with $n > 2$ boundary components is a $3n-6$ dimensional family (here the planar domain Ω is not *fixed; only the number n is).*

Proof: Let $S \subset \mathbb{R}^3$ denote a peaked sphere in $\mathbb{R}^3$. As explained in Theorems 7 and 12, S has around any of its singularities the conformal structure of an annulus with respect to the second fundamental form. By uniformization, the conformal type of $S \subset \mathbb{R}^3$ for the second fundamental form is that of a bounded circular domain Ω. Also, the Gauss map $N : \Omega \to \mathbb{S}^2$ is a harmonic map, which is a diffeomorphism onto its image. Moreover, when parameterized conformally for its extrinsic conformal structure, the peaked sphere $X : \Omega \to \mathbb{R}^3$ satisfies that X is constant along each boundary component. So, by (2.1), its Gauss map satisfies the Neumann condition (5.1).

Conversely, let $N : \Omega \to \mathbb{S}^2$ denote a solution to the Neumann problem for harmonic diffeomorphisms into $\mathbb{S}^2$, where Ω is a bounded circular domain. Let $X : \Omega \to \mathbb{R}^3$ denote the surface with $K = 1$, possibly with singularities, determined from the Gauss map by the representation formula (2.1). Clearly, X is regular and constant along each boundary circle C_j of Ω, so we only need to ensure that X is single valued. In order to do this, as the fundamental group of Ω is generated by the boundary circles C_j, it suffices to show that X is single valued around each of these circles. But this property was already proved in the local classification theorem of conical singularities, see the existence part in Theorem 7. So, this concludes the proof.

□

As a consequence of this and the fact that peaked spheres in $\mathbb{R}^3$ with two singularities are rotational and their conformal structure is controlled (see Section 2), we have

Corollary 17 *Let $\mathbb{A}(r, R)$ be the annulus $\{z : r < |z| < R\}$. Then, the Neumann problem for harmonic diffeomorphisms $g : \mathbb{A}(r, R) \to \mathbb{S}^2$ has a solution if and only if $R/r > e^{\pi}$.*

In that case, the solution is unique and radially symmetric.

CMC surfaces with free boundary

A well studied problem in the theory of CMC surfaces is the free boundary problem (or *capillarity* problem) of finding all compact CMC surfaces that meet a certain support surface $S \subset \mathbb{R}^3$ at a constant angle along their boundary. Our analysis on $K = 1$ surfaces in $\mathbb{R}^3$ with isolated singularities provides some interesting consequences in this context.

First, one has the following reflection principle in the spirit of the usual Schwarz's reflection principle (i.e. that CMC surfaces meeting a plane orthogonally along its boundary can be analytically extended by reflection across this plane).

Let $\mathcal{U}$ be a bounded symmetric domain in $\mathbb{C}$, i.e. $\mathcal{U} = \mathcal{U}^* := \{\bar{z} : z \in \mathcal{U}\}$. Assume that $\mathcal{U} \cap \mathbb{R} \neq \emptyset$ and call $\mathcal{U}^+ := \mathcal{U} \cap \mathbb{C}_+$.

Corollary 18 *Let $f : \mathcal{U}^+ \to \mathbb{R}^3$ be a conformally immersed $H = 1/2$ surface in $\mathbb{R}^3$. Assume that f extends C^1 to $\Gamma := \overline{\mathcal{U}^+} \cap \mathbb{R}$, so that $f|_\Gamma$ is contained in the sphere $\mathbb{S}^2(1)$ of radius one, and meets this sphere tangentially.*

Then, f can be analytically extended to $\mathcal{U}$ by the formula

$$f(\bar{z}) = -f(z) - 2N(z), \tag{5.2}$$

where here $N : \mathcal{U}^+ \to \mathbb{S}^2$ is the unit normal of f.

Proof: Consider the parallel $K = 1$ frontal $X = f + N : \mathcal{U}^+ \to \mathbb{R}^3$. Then X extends continuously to Γ, with $X|_\Gamma = 0 \in \mathbb{R}^3$. Observe that from the condition $K = 1$, the area of X agrees with the $\mathbb{S}^2$-area of the spherical image $N(\overline{\mathcal{U}^+})$, which is finite around any point $(u_0, 0) \in \Gamma$ since N is continuous. Thus, we are in the conditions of Lemma 2 around any such point, and so X can be extended across Γ by $X(u, -v) = -X(u, v)$ (see the proof of Theorem 7). The Gauss map of X (and f) is extended by $N(u, -v) = N(u, v)$. Once here, formula (5.2) follows directly.

□

Interestingly, the reflected part described by (5.2) is the parallel $H = -1/2$ surface of the original $H = 1/2$ surface $f : \mathcal{U}^+ \to \mathbb{R}^3$, composed with the central isometry of $\mathbb{R}^3$ about the center of the sphere. Indeed, if $f^\sharp$ is the parallel $H = -1/2$ surface of f, we see from (5.2) that $f(\bar{z}) = -f^\sharp(z)$.

So, the extended surface $f : \mathcal{U} \to \mathbb{R}^3$ is *self-parallel* (a CMC surface $f : \Sigma \to \mathbb{R}^3$ is *self-parallel* if there exists an antiholomorphic diffeomorphism $J : \Sigma \to \Sigma$ and an orientation preserving rigid motion Φ of $\mathbb{R}^3$ such that $\Phi \circ f^\sharp = f \circ J$, where $f^\sharp : \Sigma \to \mathbb{R}^3$ is the parallel CMC surface of f). Thus, we obtain a very general procedure for constructing self-parallel CMC surfaces in $\mathbb{R}^3$, just by solving an adequate geometric Cauchy problem.

In addition, the $H = 1/2$ surfaces that are parallel to peaked spheres with $K = 1$ in $\mathbb{R}^3$ also have interesting global properties. Indeed, by the above discussion, they are locally convex $H = 1/2$ immersions

$$f : \Omega \subset \mathbb{C} \to \mathbb{R}^3,$$

where $\Omega \subset \mathbb{C}$ is a bounded circular domain with $n \geq 2$ boundary components, satisfying:

1. Each boundary curve $\Gamma_j := f(C_j)$ (where $\partial\Omega = C_1 \cup \cdots \cup C_n$) is a regular convex Jordan curve contained in a sphere $\mathbb{S}^2(p_j; 1)$ of radius one centered at some $p_j \in \mathbb{R}^3$, $j = 1, \ldots, n$.

2. The surface meets $\mathbb{S}^2(p_j; 1)$ tangentially along Γ_j for each $j = 1, \ldots, n$.

So, these CMC surfaces are solutions to a free boundary (or capillarity) problem where the support surface S is a collection of $n \geq 2$ spheres of radius one in $\mathbb{R}^3$, and the surface meets this configuration tangentially along all its boundary components.

When $n = 2$, the peaked sphere is rotational, and so the parallel CMC surface S_H is a compact convex piece of an unduloid, bounded by two parallel circles (actually, the largest piece of the unduloid with these conditions).

When $n > 2$, by the results of Section 4, the space of such $H = 1/2$ surfaces S_H with spherical boundaries is a $3n - 6$ dimensional family.

To all these $H = 1/2$ surfaces S_H that are parallel to peaked spheres in $\mathbb{R}^3$, the reflection principle of Corollary 18 applies. When $n = 2$ we obtain the complete unduloid, by repeatedly applying this reflection principle. In contrast, when $n > 2$ we obtain complete branched CMC surfaces with an infinite number of ends, again by reflection across all boundary components and iterating this process.

It must be emphasized that, although the starting compact surface with boundary S_H is regular (i.e. free of branch points), its reflected part will always encounter branch points. This is a consequence of the following facts.

1. The reflected piece S_H^* of S_H across a boundary curve in the sense of Corollary 18 is its parallel surface, up to a rigid motion. So S_H^* will be an immersion (i.e. free of branch points) if and only if S_H is free of umbilic points.

2. Consider the *double Riemann surface* $\bar{M}_g$ of Ω, which is a compact Riemann surface of genus $g = n - 1$. Let $Q\,dz^2$ denote the Hopf differential of S_H, which is defined on Ω. As $\text{Im}Q\,dz^2$ vanishes along $\partial\Omega$, we can extend it to be a holomorphic quadratic differential on the compact Riemann surface $\bar{M}_g$, by Schwarz reflection principle. But as $n > 2$, the genus of the surface is greater than one. This means that $Q\,dz^2$ must vanish somewhere on $\bar{M}_g$, and thus by symmetry we conclude that Q vanishes somewhere in Ω, i.e. that S_H is not free of umbilic points.

As a consequence, we get the existence of a $3n - 6$ dimensional family (for $n > 2$) of complete, branched self-parallel CMC surfaces in $\mathbb{R}^3$ with genus zero and an infinite number of ends.

Open problems

Let us recall that the extrinsic conformal structure (i.e. the conformal structure induced by the second fundamental form) of a peaked sphere is that of a bounded circular domain in $\mathbb{C}$.

Problem 1. Which bounded circular domains in $\mathbb{C}$ are realizable as the extrinsic conformal structure of a peaked sphere in $\mathbb{R}^3$? Is a peaked sphere uniquely determined by its extrinsic conformal structure?

Problem 1 has a strong connection with the Neumann problem for harmonic diffeomorphisms. Indeed, a classification of peaked spheres in terms of their extrinsic conformal structure would solve completely the Neumann problem for harmonic diffeomorphisms into $\mathbb{S}^2$. Problem 1 follows the spirit of some previous classification theorems of entire solutions to elliptic PDEs with a finite number of singularities, in terms of some underlying conformal structure. See [GMM, FLS, Fer, CMM].

Problem 2. Find necessary and sufficient conditions for a set of points $p_1, \dots, p_n \in \mathbb{R}^3$ to be realized as the set of singularities of a peaked sphere in $\mathbb{R}^3$. Are two peaked spheres with the same singularities $p_j \in \mathbb{R}^3$ necessarily the same?

Problem 2 is connected with the free boundary capillarity problem for CMC surfaces, in the case that one wishes to prescribe the centers of the spheres (and not just the number of spheres and their common radius). In any case, it is clear that an arbitrary configuration of n points will not be in general the singular set of a peaked sphere in $\mathbb{R}^3$.

Problem 3. Can one realize any conformal metric of constant curvature 1 on $\mathbb{S}^2$ with a finite number of conical singularities as the intrinsic metric of an immersed $K = 1$ surface in $\mathbb{R}^3$?

The results by Alexandrov and Pogorelov show that these metrics are realized as the intrinsic metric of peaked spheres, *provided* all conical angles are in $(0, 2\pi)$, but there are many other abstract cone metrics. Such an isometric realization in $\mathbb{R}^3$ must necessarily be non-embedded. It must be emphasized that, by our local study, any conical singularity of arbitrary angle can be realized as an immersed $K = 1$ surface in $\mathbb{R}^3$ in many different ways.

Let us also point out that a complete classification for conformal metrics of positive constant curvature on $\mathbb{S}^2$ with n conical singularities remains open if $n > 3$ (see [UmYa, Ere] for the case of three conical singularities).

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