Deformations of constant mean curvature 1/2 surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with vertical ends at infinity

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Abstract

We study constant mean curvature $1/2$ *surfaces in* $\mathbb{H}^2 \times \mathbb{R}$ *that admit a compactification of the mean curvature operator. We show that a particular family of complete entire graphs over* $\mathbb{H}^2$ *admits a structure of infinite dimensional manifold with local control on the behaviors at infinity. These graphs also appear to have a half-space property and we deduce a uniqueness result at infinity. Deforming non degenerate constant mean curvature* $1/2$ *annuli, we provide a large class of (non rotational) examples and construct (possibly embedded) annuli without axis, i.e. with two vertical, asymptotically rotational, non aligned ends.*

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1 Introduction

This paper concerns the theory of constant mean curvature (*CMC* for short) surfaces $H = 1/2$ in $\mathbb{H}^2 \times \mathbb{R}$. The value $H = 1/2$ is critical in the sense that there is no compact CMC sphere for $H \leq 1/2$ while for $H > 1/2$ there are rotational compact examples. A half-space theorem in $\mathbb{H}^2 \times \mathbb{R}$ (see [8]) proves that for CMC $H = 1/2$, complete multigraphs are entire graphs over $\mathbb{H}^2$. Entire graphs are classified by I. Fernández and P. Mira [5] and their moduli space is modeled on the set of quadratic holomorphic differential Q defined on the complex plane $\mathbb{C}$ or the unit disk $\mathbb{D}$. The link between Q and the geometry of the graph is not very well understood.

We first deal with complete conformal immersions of the disk $\mathbb{D}$, properly immersed into the half-space $\mathbb{H}^2 \times \mathbb{R}_+$ ($x_3 \geq 0$), which are entire vertical graphs over $\mathbb{H}^2$. We assume that the third coordinate $x_3 \to +\infty$ on any diverging sequence of points in $\mathbb{D}$, which means the height function is proper.

Up to this date, the only simply connected example is a rotational example called the *hyperboloid* S_0. In the Poincaré disk model of $\mathbb{H}^2 \times \mathbb{R}$ — see (2) below — with polar coordinates (r, θ), a parametrization of S_0 as a graph over $\mathbb{H}^2$ is:

$$(r,\theta) \in [0,1) \times \mathbb{S}^1 \mapsto \left(re^{i\theta}, \frac{2}{\sqrt{1-r^2}} \right) \in \mathbb{H}^2 \times \mathbb{R}.$$

We describe a family of examples endowed with a structure of infinite dimensional smooth manifold. The manifold structure arises from a suitable *compactification* of the mean curvature operator at infinity (Theorem 2.5) and is diffeomorphic to a codimension one submanifold of $\mathcal{C}^{2,\alpha}(\mathbb{S}^1) \times \mathbb{R}$ (Theorem 3.10). This construction comes with a control of the asymptotic behavior in terms of the horizontal (hyperbolic) distance from the hyperboloid S_0, namely:

Theorem (Theorem 3.9)**.** *For any small $\gamma \in \mathcal{C}^{2,\alpha}(\mathbb{S}^1)$ such that $e^{-\gamma}$ has unit $L^2(\mathbb{S}^1)$-norm, there exists a CMC-1/2 complete entire graph at asymptotic horizontal signed distance 2γ from S_0.*

These graphs are interesting, since any connected complete embedded CMC-1/2 surface in $\mathbb{H}^2 \times \mathbb{R}$ which is contained in the half-space $\mathbb{H}^2 \times \mathbb{R}_+$ and has a proper height function is a vertical entire graph. Indeed, apply Alexandrov reflection principle to such an immersion. There will be no first point of tangent contact between the surface and the symmetry part of the constructed bigraph — i.e. the part of the bigraph which is not a part of the surface — since there is no compact CMC-1/2 sphere in $\mathbb{H}^2 \times \mathbb{R}$.

We also prove a half-space property for these entire graphs:

Theorem (Theorem 4.2)**.** *Let Σ be a CMC-1/2 surface which is properly immersed in $\mathbb{H}^2 \times \mathbb{R}$ and lies on one side of a CMC-1/2 entire graph S in the family. Then Σ coincides with S up to a vertical translation.*

We use this result to show an asymptotic rigidity in our family of CMC-1/2 entire graphs (Theorem 4.3). Namely, if two graphs in the family are at the same asymptotic horizontal signed distance from the hyperboloid S_0, they coincide up to a vertical translation.

In $\mathbb{H}^2 \times \mathbb{R}$, R. Sa Earp and E. Toubiana [3] construct a one-parameter family of CMC $H = 1/2$ annuli which are rotationally invariant around a vertical geodesic. Recently, L. Mazet has shown [9] that for $H > 1/2$, CMC annuli which are cylindrically bounded around a vertical geodesic are rotational examples.

Though annuli are not cylindrically bounded for $H = 1/2$, we prove that in a bounded tubular neighborhood of a rotational example, there are annuli, eventually embedded, which are asymptotic to different rotational examples with different axis:

Theorem (Theorem 5.9)**.** *There exist CMC-*$1/2$ *annuli in* $\mathbb{H}^2 \times \mathbb{R}$ *with vertical ends, that are asymptotic — regarding the horizontal hyperbolic distance — to rotational examples with different vertical axis.*

It means that contrary to the case of embedded minimal surfaces in $\mathbb{R}^3$ with finite total curvature and horizontal ends [11], the notion of *axis* is not relevant in general for CMC-1/2 annuli with vertical ends in $\mathbb{H}^2 \times \mathbb{R}$.

Notations

Let $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ be the open unit disk, $\overline{\mathbb{D}} = \{z \in \mathbb{C} | |z| \leq 1\}$ its closure and (r, θ) the polar coordinates on $\overline{\mathbb{D}}$. We use two standard models of $\mathbb{H}^2 \times \mathbb{R}$, which are the Minkowski model:

$$\mathbb{H}^2 \times \mathbb{R} = \Big(\Big\{ (x_0, \ldots, x_3) \in \mathbb{R}^4 \Big| x_1^2 + x_2^2 - x_0^2 = -1 \Big\}, \\ ds_L^2 = dx_1^2 + dx_2^2 + dx_3^2 - dx_0^2 \Big), \quad (1)$$

where $\mathbb{H}^2 \times \mathbb{R}$ is seen as a subspace of the 4-dimensional Minkowski space $\mathbb{L}^4$, and the Poincaré disk model:

$$\mathbb{H}^2 \times \mathbb{R} = \Big(\{(w, x_3) \in \mathbb{D} \times \mathbb{R}\}, \\ ds_P^2 + dx_3^2 = \frac{4}{(1 - |w|^2)^2} |dw|^2 + dx_3^2 \Big). \quad (2)$$

The vector field associated to the third coordinate is denoted e_3. In the Poincaré disk model (2), the hyperbolic radius $\rho_{\mathbb{H}}(w)$ of a point w is:

$$\rho_{\mathbb{H}}(w) = 2 \operatorname{argtanh} |w| = \log \left(\frac{1 + |w|}{1 - |w|} \right),$$

and we will need the following formula in the proof of Proposition 2.2:

$$\cosh \frac{\rho_{\mathbb{H}}(w)}{2} = \frac{1}{\sqrt{1 - |w|^2}}.$$

We call vertical graphs (resp. vertical annuli) in $\mathbb{H}^2 \times \mathbb{R}$, immersions which are complete graphs (resp. bi-graphs) over an open subset of the slice $\mathbb{H}^2 \equiv \mathbb{H}^2 \times \{0\}$.

Given surfaces S, S' in $\mathbb{H}^2 \times \mathbb{R}$ admitting parametrizations in the Poincaré disk model respectively:

$$(f(t,\theta)e^{i\theta}, t) \quad \text{and} \quad (f'(t,\theta)e^{i\theta}, t),$$

the hyperbolic horizontal signed distance $d_{\mathbb{H}}(S,S')(t,\theta)$ between S and S' at height t and in the direction θ is the difference of their hyperbolic radii in the slice $\mathbb{H}^2 \times \{t\}$ and direction θ:

$$\begin{aligned} d_{\mathbb{H}}(S,S')(t,\theta) &= \rho_{\mathbb{H}}(S')(t,\theta) - \rho_{\mathbb{H}}(S)(t,\theta) \\ &= 2\left(\operatorname{argtanh} f'(t,\theta) - \operatorname{argtanh} f(t,\theta)\right). \end{aligned}$$

When it exists, the *asymptotic hyperbolic horizontal signed distance* between S and S' in the direction θ is the limit $\lim_{t\to+\infty} d_{\mathbb{H}}(S,S')(t,\theta)$.

For any $R \in [0,1)$, let $\Omega_R \subset \mathbb{D}$ be the domain $\Omega_R = \{R \leq r < 1\}$. We consider the set of *admissible domains* $\mathcal{D} = \{\Omega_R | 0 \leq R < 1\}$. The boundary at infinity $\partial_\infty \mathbb{H}^2$ of $\mathbb{H}^2$ is identified with $\mathbb{S}^1$.

Given $\Omega \in \mathcal{D}$, the spaces $\mathcal{C}^{k,\alpha}(\overline{\Omega})$ and $\mathcal{C}_0^{k,\alpha}(\overline{\Omega})$, with $k \geq 0$ and $0 < \alpha < 1$, are respectively the usual Hölder space and the subspace of functions that are zero on the exterior boundary $\{r = 1\}$. Finally, we consider the spaces $L^2(\cdot)$ endowed with the natural scalar product denoted $\langle \cdot, \cdot \rangle_{L^2(\cdot)}$ and Hilbert norm $|\cdot|_{L^2(\cdot)}$.

2 The mean curvature operator

Consider a surface S parametrized by an immersion $X : \mathbb{D} \to \mathbb{H}^2 \times \mathbb{R}$ with complete induced metric g. By *compactification* of S, we mean a conformal change $\overline{g}$ of metric such that $\overline{g}$ extends to a metric on $\overline{\mathbb{D}}$.

The process is sensible to the parametrization. For instance, consider the hyperboloid S_0. It is a vertical graph over $\mathbb{H}^2$ parametrized by:

$$(r,\theta) \in \mathbb{D} \mapsto \left(re^{i\theta}, \frac{2}{\sqrt{1-r^2}}\right) \in \mathbb{H}^2 \times \mathbb{R},$$

in the Poincaré disk model (2), with induced metric:

$$g = \frac{4}{(1-r^2)^3}\begin{pmatrix} 2-r^2 & 0 \\ 0 & 1-r^2 \end{pmatrix}.$$

But g cannot be conformally extended to the boundary $\{r = 1\}$ of $\mathbb{D}$, since the terms of g have different rates of explosion when $r \to 1$. The resulting metric would degenerate for $r = 1$.

To ensure the extension of the induced metric, we use a conformal parametrization S_0, namely the immersion $X^0 : \mathbb{D} \to \mathbb{H}^2 \times \mathbb{R}$ defined by:

$$X^0(r,\theta) = \left(F(r,\theta), \frac{2}{\sqrt{1 - |F(r,\theta)|^2}} \right) = \left(F(r,\theta), 2\frac{1+r^2}{1-r^2} \right),$$

where $F : \mathbb{D} \to \mathbb{H}^2$ is the $\mathcal{C}^1$-diffeomorphism defined in the Poincaré disk model (2) by:

$$F(r,\theta) = \frac{2r}{1+r^2} e^{i\theta}$$

and in the Minkowski model (1) by:

$$F(r,\theta) = \big(\cosh\chi(r,\theta), \sinh\chi(r,\theta)\cos\theta, \sinh\chi(r,\theta)\sin\theta \big)$$
$$\text{with} \quad \chi(r,\theta) = 2\log\left(\frac{1+r}{1-r}\right).$$

Definition 2.1. A surface in $\mathbb{H}^2 \times \mathbb{R}$ is said to *admit graph coordinates at infinity*, if there exist an admissible domain $\Omega \subset \mathcal{D}$ and a function $h : \Omega \to \mathbb{R}$ such that a part of the surface can be parametrized as the immersion $X : (r,\theta) \in \Omega \mapsto \big(F(r,\theta), h(r,\theta)\big) \in \mathbb{H}^2 \times \mathbb{R}$ on Ω.
When defined, we call such a parametrization *graph coordinates at infinity*.

In the sequel, we use graph coordinates at infinity to compactify surfaces and quantify their asymptotic behaviour. Surfaces are thus considered as *compact surfaces with boundary* and we can apply the method first developped by B. White in [14].

2.1 The family $\mathcal{E}$

Let $\mathcal{E}$ be the set of immersed surfaces in $\mathbb{H}^2 \times \mathbb{R}$, which admit — up to a symmetry with respect to the slice $\mathbb{H}^2 \times \{0\}$ — graph coordinates at infinity written as:

$$X^\eta : (r,\theta) \in \Omega \mapsto \left(F(r,\theta), 2e^{\eta(r,\theta)} \frac{1+r^2}{1-r^2} \right) \in \mathbb{H}^2 \times \mathbb{R}, \tag{3}$$

for some admissible domain $\Omega \in \mathcal{D}$ and $\eta \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$. Elements of $\mathcal{E}$ have *vertical ends* [4] i.e. topological annuli with no asymptotic point at finite height — i.e. topological annuli properly embedded in $(\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2) \times \mathbb{R}$.

The hyperboloid S_0 itself is in $\mathcal{E}$ with $\Omega = \mathbb{D}$ and $\eta \equiv 0$. And so are the rotational examples of E. Toubiana and R. Sa Earp studied in Section 5, owing the asymptotic development (11).

We highlight two properties of the family $\mathcal{E}$. The first is that it contains normal deformations of the hyperboloid S_0. Namely:

Proposition 2.2. *A normal graph $S = \exp_{S_0}(\zeta N)$ over S_0, where N is the upward pointing normal to S_0 and $\zeta \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}})$, is in $\mathcal{E}$. In other words, there exist $\Omega \in \mathcal{D}$ and $\eta \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$ such that the end of S admits graph coordinates at infinity as in* (3).
Furthermore, the asymptotic value of η is linked with the asymptotic horizontal (hyperbolic) distance between S and S_0:

$$\eta|_{\partial\mathbb{D}} = \frac{1}{2}\zeta|_{\partial\mathbb{D}},$$

Proof. We use the Minkowski model (1) of $\mathbb{H}^2 \times \mathbb{R}$, where the map F reads:

$$F(r,\theta) = (\cosh\chi(r,\theta), \sinh\chi(r,\theta)\cos\theta, \sinh\chi(r,\theta)\sin\theta)$$
$$\text{with} \quad \chi(r,\theta) = 2\log\left(\frac{1+r}{1-r}\right).$$

A computation shows the unit normal N to S_0 is:

$$N = -\frac{2r}{1+r^2}\left(\sinh\chi\frac{\partial}{\partial x_0} + \cosh\chi\cos\theta\frac{\partial}{\partial x_1} + \cosh\chi\sin\theta\frac{\partial}{\partial x_2}\right) + \frac{1-r^2}{1+r^2}\frac{\partial}{\partial x_3},$$

in the canonical basis of $\mathbb{L}^4$. Hence, S is parametrized by the immersion:

$$\left(\cosh\left(\chi - \frac{2r\zeta}{1+r^2}\right), \sinh\left(\chi - \frac{2r\zeta}{1+r^2}\right)\cos\theta,\right.$$
$$\left.\sinh\left(\chi - \frac{2r\zeta}{1+r^2}\right)\sin\theta, 2\frac{1+r^2}{1-r^2} + \frac{1-r^2}{1+r^2}\zeta\right).$$

We want to find new coordinates $(\widetilde{r},\widetilde{\theta})$ on an admissible domain verifying:

$$\chi(\widetilde{r},\widetilde{\theta}) = \chi(r,\theta) - \frac{2r}{1+r^2}\zeta(r,\theta), \quad \cos\widetilde{\theta} = \cos\theta \quad \text{and} \quad \sin\widetilde{\theta} = \sin\theta,$$

to have graph coordinates at infinity on S as in (3). Taking $\widetilde{\theta} = \theta$, compute:

$$\begin{aligned}\frac{\partial}{\partial r}\left(\chi(r,\theta) - \frac{2r}{1+r^2}\zeta(r,\theta)\right) &= \frac{4}{1-r^2} - \frac{2}{1+r^2}\left(\frac{1-r^2}{1+r^2}\zeta + r\zeta_r\right)\\ &= \frac{4}{1-r^2} + O(1).\end{aligned}$$

If r is sufficiently close to 1, the map $r \mapsto \chi - 2r\zeta/(1+r^2)$ is strictly increasing (uniformly in θ), which ensures existence and uniqueness of $\widetilde{r}$.

To compute the asymptotic horizontal distance, consider a horizontal slice $\mathbb{H}^2 \times \{t\}$ intersecting S and S_0. The hyperbolic radii of S and S_0 at height t and in the direction θ respectively denoted $\rho_{\mathbb{H}}(S)(t,\theta)$ and $\rho_{\mathbb{H}}(S_0)(t,\theta)$ verify:

$$\begin{aligned} t &= \frac{2e^\eta}{\sqrt{1-|F|^2}} = 2e^\eta \cosh\frac{\rho_{\mathbb{H}}(S)(t,\theta)}{2}\\ \text{and} \quad t &= \frac{2}{\sqrt{1-|F|^2}} = 2\cosh\frac{\rho_{\mathbb{H}}(S_0)(t,\theta)}{2},\end{aligned}$$

and we deduce:

$$\begin{aligned}\rho_{\mathbb{H}}(S)(t,\theta) = 2\operatorname{argcosh}\frac{te^{-\eta}}{2} = 2\log t - 2\eta + O\left(\frac{1}{t^2}\right)\\ \text{and} \quad \rho_{\mathbb{H}}(S_0)(t,\theta) = 2\operatorname{argcosh}\frac{t}{2} = 2\log t + O\left(\frac{1}{t^2}\right).\end{aligned}$$

Therefore, the hyperbolic horizontal signed distance $d_{\mathbb{H}}(S,S_0)(t,\theta)$ between S and S_0 at height t and in the direction θ is:

$$d_{\mathbb{H}}(S,S_0)(t,\theta) = \rho_{\mathbb{H}}(S_0)(t,\theta) - \rho_{\mathbb{H}}(S)(t,\theta) = 2\eta + O\left(\frac{1}{t^2}\right),$$

which establishes the equality $\zeta|_{\partial\mathbb{D}} = 2\eta|_{\partial\mathbb{D}}$ at infinity. Indeed, $\zeta|_{\partial\mathbb{D}}$ is the normal signed distance between S and S_0 at infinity, and also the horizontal distance at infinity, since N is asymptotically horizontal. □

Proposition 2.2 emphasizes the fact that the relevant information at infinity is the asymptotic horizontal distance from the hyperboloid. And as suggested by (11) in Section 5, the asymptotic horizontal distance is also relevant for deformed annuli, since the rotational examples are at finite constant asymptotic horizontal distance from each other.

Therefore a general principle in our purpose is to fix a convenient surface, the *model surface*, and to construct deformations of the model surface prescribing the asymptotic horizontal distance from the model surface. It is also the supporting idea of the compactification of the mean curvature operator (Theorem 2.5).

A second interesting property of $\mathcal{E}$ is the following:

Proposition 2.3. *The image of any element of $\mathcal{E}$ under the action of any isometry of $\mathbb{H}^2 \times \mathbb{R}$ is still an element of $\mathcal{E}$.*

Proof. Consider a surface $S \in \mathcal{E}$ with graph coordinates at infinity (F, h) defined on $\Omega \in \mathcal{D}$, and denote by (F, h') the graph coordinates at infinity of its image S' under an isometry ψ of $\mathbb{H}^2 \times \mathbb{R}$. Using parametrization (3), we know that in the Poincaré disk model (2):

$$h = \frac{2e^{\eta}}{\sqrt{1-|F|^2}} \quad \text{with} \quad \eta \in \mathcal{C}^{2,\alpha}(\overline{\Omega}).$$

It is sufficient to examine the cases when ψ is either an isometry of $\mathbb{H}^2$ fixing the coordinate x_3 or a vertical translation. If ψ is a vertical translation of $t_0 \in \mathbb{R}$, we have:

$$h' = \frac{2e^{\eta}}{\sqrt{1-|F|^2}} + t_0 = 2\exp\left(\eta + \log\left(1 + t_0 \frac{e^{-\eta}}{2}\frac{1-r^2}{1+r^2}\right)\right)\frac{1}{\sqrt{1-|F|^2}},$$

eventually after a restriction to a domain $\Omega' \in \mathcal{D}$ for which $h|_{\Omega'} > -t_0$.

If ψ reduces to an isometry of $\mathbb{H}^2$ preserving the orientation of $\mathbb{H}^2$, there exist $w_0 \in \mathbb{D}$ and $\delta_0 \in \mathbb{R}$ such that:

$$\psi(w) = \frac{w + w_0}{1 + \overline{w_0} w} e^{i\delta_0}.$$

If $\psi' = F^{-1} \circ \psi^{-1} \circ F$, then:

$$\begin{aligned} h' = h \circ \psi' = \frac{2e^{\eta\circ\psi'}}{\sqrt{1-|\psi^{-1}\circ F|^2}} &= \left(e^{\eta\circ\psi'}\frac{|1-\overline{w_0}F|}{\sqrt{1-|w_0|^2}}\right)\frac{2}{\sqrt{1-|F|^2}} \\ &= \exp\left(\eta\circ\psi' + \log\left(\frac{|1-\overline{w_0}F|}{\sqrt{1-|w_0|^2}}\right)\right)\frac{2}{\sqrt{1-|F|^2}}, \end{aligned}$$

and $S' \in \mathcal{E}$. Changing F in $\overline{F}$, gives the result when ψ reduces to an isometry of $\mathbb{H}^2$ reversing the orientation. $\square$

Remark 2.4. The value $\eta|_{\partial\mathbb{D}}$ is invariant under vertical translations.

2.2 Compactification of the mean curvature

From now on, to ease the notations, we denote with indices $1, 2$ quantities related to coordinates r, θ respectively. Consider an admissible domain $\Omega \in \mathcal{D}$ and a function $a \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$. The model surface is the immersion X^a, written as in (3).

Theorem 2.5. *For any deformation $X^{a+\xi}$ of the model surface X^a, with $\xi \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$, the respective mean curvatures $H(a+\xi)$ and $H(a)$ verify the following:*

$$\sqrt{|g(a)|}(H(a+\xi) - H(a)) = \sum_{i,j} A_{ij}(r, \theta, a, D\xi)\xi_{ij} + B(r, \theta, a, \xi, D\xi), \quad (4)$$

where $|g(a)|$ is the determinant of the metric induced by X^a, A_{ij} and B are $\mathcal{C}^{0,\alpha}$ functions on $\overline{\Omega}$ which are real-analytic in their variables, and $A = (A_{ij})$ is a coercive matrix on $\overline{\Omega}$.

Proof (See Appendix 6 for computation details). Denote σ the pullback metric $F^* ds_P^2$, i.e. in matrix terms:

$$\sigma = \frac{16}{(1-r^2)^4} \begin{pmatrix} (1-r^2)^2 & 0 \\ 0 & r^2(1+r^2) \end{pmatrix}.$$

Differential properties of a surface in $\mathbb{H}^2 \times \mathbb{R}$ with graph coordinates at infinity (F, h) are the ones of the actual graph of h in $\mathbb{D} \times \mathbb{R}$ endowed with the metric $\sigma + dx_3^2$. Following J. Spruck [12], the mean curvature $H(a+\xi)$ is:

$$H(a+\xi) = \frac{1}{2} \operatorname{div}_\sigma \left(\frac{\nabla_\sigma h(a+\xi)}{W(a+\xi)} \right) \quad \text{with} \quad W(a+\xi) = \sqrt{1 + |\nabla_\sigma h(a+\xi)|_\sigma^2},$$

with quantities computed with respect to σ. If (Γ_{ij}^k) denote the Christoffel symbols associated to σ, we have:

$$H(a+\xi) = \frac{1}{2W(a+\xi)} \sum_{i,j} g^{ij}(a+\xi) \left(\partial_{ij} h(a+\xi) - \sum_k \Gamma_{ij}^k \partial_k h(a+\xi) \right),$$

where the non zero Christoffel symbols are:

$$\Gamma_{11}^1 = \frac{2r}{1-r^2}, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1+6r^2+r^4}{r(1+r^2)(1-r^2)}$$
$$\text{and} \quad \Gamma_{22}^1 = -\frac{r(1+r^2)(1+6r^2+r^4)}{(1-r^2)^3}.$$

The induced metric $g(a)$ reads:

$$g_{11}(a) = \frac{16(1+r^2)^2 e^{2a}}{(1-r^2)^4}\left[1 + \frac{2ra_1}{1+r^2}(1-r^2) + \left(\frac{a_1^2}{4} + \frac{e^{-2a}-1}{(1+r^2)^2}\right)(1-r^2)^2\right],$$
$$g_{12}(a) = \frac{8(1+r^2)^2 a_2 e^{2a}}{(1-r^2)^3}\left[\frac{2r}{1+r^2} + \frac{a_1}{2}(1-r^2)\right]$$
$$\text{and} \quad g_{22}(a) = \frac{16r^2(1+r^2)^2}{(1-r^2)^4}\left[1 + \frac{a_2^2 e^{2a}}{4r^2}(1-r^2)^2\right],$$

and the expression of $W(a)$ is the following:

$$W(a) = \frac{(1+r^2)e^a}{2(1-r^2)}\left[1 + \frac{2ra_1}{1+r^2}(1-r^2) + \left(\frac{a_1^2}{4} + \frac{e^{-2a}-1}{(1+r^2)^2}\right)(1-r^2)^2 + \frac{a_2^2}{4r^2(1+r^2)^2}(1-r^2)^4\right]^{1/2}. \quad (5)$$

The computation detailed in Appendix 6 gives the expression (4) with the desired regularity and:

$$A_{11} = e^{-a} + O(1-r^2), \quad A_{12} = A_{21} = O(1-r^2) \quad \text{and} \quad A_{22} = e^a + O(1-r^2),$$

which shows that A is coercive on $\Omega \cup \partial\mathbb{D}$. □

The quantity $\sqrt{g(a)}(H(\eta) - H(a))$ with $\eta \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$ can be called a *compactification* of the mean curvature of X^η since it can be extended to the exterior boundary $\{r = 1\}$ of Ω. It is strongly linked with the compactification of the induced metric $g(a)$ by the following equality:

$$A^{-1} = \begin{pmatrix} e^a & 0 \\ 0 & e^{-a} \end{pmatrix} + O(1-r^2) = \frac{1}{\sqrt{|g(a)|}} g(a) + O(1-r^2).$$

3 Moduli space of CMC-1/2 entire graphs

In this section, we are interested in the subset $\mathcal{G} \subset \mathcal{E}$ of CMC-1/2 complete entire graphs contained in the half-space $\mathbb{H}^2 \times \mathbb{R}_+^*$. Since elements of $\mathcal{G}$ are simply connected, they can be *globally* parametrized in graph coordinates at infinity over the whole disk $\mathbb{D}$ using (3):

$$X^\eta = \left(F, 2e^\eta \frac{1+r^2}{1-r^2}\right) \quad \text{with} \quad F(r,\theta) = \frac{2r}{1+r^2}e^{i\theta} \quad \text{and} \quad \eta \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}}),$$

and the geometrically defined function $\eta|_{\partial\mathbb{D}} : \mathbb{S}^1 \to \mathbb{R}$ is the *value at infinity* of the surface.

Consider a CMC-1/2 entire graph $S \in \mathcal{G}$, with graph coordinates at infinity X^a, where $a \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}})$, and denote $\gamma^a = a|_{\partial\mathbb{D}}$ the value at infinity. A simple computation shows that the vertical component $\varphi^a = \langle N^a, e_3\rangle$ of the upward pointing unit normal N^a to X^a can be expressed as:

$$\varphi^a = \frac{e^{-a}}{2c^a}\frac{1-r^2}{1+r^2} \quad \text{with} \quad c^a = \frac{e^{-a}}{2}\frac{1-r^2}{1+r^2}W(a), \tag{6}$$

where $W(a)$ is given by (5) and $\varphi^a = 1/W(a)$. Note that c^a is a positive function on $\overline{\mathbb{D}}$ such that $c^a|_{\partial\mathbb{D}} = 1/2$.

In the sequel, we make the following abuse of notation denoting H the operator:

$$H : \eta \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}}) \mapsto H(\eta) \in \mathcal{C}^{0,\alpha}(\overline{\mathbb{D}}),$$

where $H(\eta)$ is the mean curvature of X^η, and calling it the *mean curvature operator*.

Lemma 3.1. *The differential of the operator H at point a is:*

$$\forall \eta \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}}),\ DH(a)\cdot\eta = \frac{1}{2}L\left(\frac{\eta}{c^a}\right),$$

where L is the Jacobi operator of X^a.

Proof. If X^{η_t} is a differentiable family in the parameter t such that $\eta_0 = a$, it is a standard fact that:

$$\left.\frac{d}{dt}\right|_{t=0} H(\eta_t) = \frac{1}{2}L\left\langle \left.\frac{d}{dt}\right|_{t=0} X^{\eta_t}, N^a\right\rangle = \frac{1}{2}L\left(2e^a\varphi^a\frac{1+r^2}{1-r^2}\left.\frac{d\eta_t}{dt}\right|_{t=0}\right),$$

and the expression (6) of φ^a leads to the conclusion. □

Using Theorem 2.5, we define the *compactified mean curvature operator* to be:

$$\overline{H} : \xi \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}}) \mapsto \sqrt{|g(a)|}\left(H(a+2c^a\xi) - \frac{1}{2}\right) \in \mathcal{C}^{0,\alpha}(\overline{\mathbb{D}}). \tag{7}$$

The *compactified Jacobi operator* is $\overline{L} = D\overline{H}(0) : \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}}) \to \mathcal{C}^{0,\alpha}(\overline{\mathbb{D}})$ and using Lemma 3.1 we know that:

$$\overline{L} = \sqrt{|g(a)|}L.$$

Proposition 3.2 (Green identity)**.** *For any $u, v \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}})$, $\overline{L}$ satisfies the following identity:*

$$\int_{\overline{\mathbb{D}}} \left(u\overline{L}v - v\overline{L}u\right) d\overline{A} = \int_0^{2\pi} e^{-\gamma^a} \left(u\frac{\partial v}{\partial r} - v\frac{\partial u}{\partial r}\right)\bigg|_{r=1} d\theta,$$

with $d\overline{A}$ the Lebesgue measure on $\overline{\mathbb{D}}$.

Proof. Let $u, v \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}})$. For any $R \in (0,1)$, L satisfies a Green identity on $\{r \leq R\}$:

$$\int_{\{r\leq R\}} (uLv - vLu) dA = \int_{\{r=R\}} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right) ds,$$

where dA and ds are the measures corresponding to the metric induced by X^a on $\{r \leq R\}$ and $\{r = R\}$ respectively, and where $\partial \cdot / \partial \nu$ denotes the co-normal derivative. Notice that:

$$dA = \sqrt{|g(a)|}\, d\overline{A}, \quad ds = \sqrt{g_{22}(a)}\, d\theta$$
$$\text{and} \quad \nu = \frac{1}{\sqrt{g_{22}(a)|g(a)|}} (g_{22}(a)X_1^a - g_{12}(a)X_2^a),$$

with $d\overline{A}$ the Lebesgue measure on $\mathbb{R}^2$. Taking the limit when $R \to 1$, we obtain:

$$\lim_{R\to 1} \sqrt{g_{22}(a)}\frac{\partial}{\partial \nu} = e^{-\gamma^a} \left.\frac{\partial}{\partial r}\right|_{r=1},$$

and the identity follows. □

Corollary 3.3. *There is no solution $u \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}})$ to the equation:*

$$\begin{cases} \overline{L}u = 0 & \text{on } \overline{\mathbb{D}} \\ u|_{\partial\mathbb{D}} = 1 \end{cases}.$$

Proof. By contradiction, suppose such a u exist and apply Proposition 3.2 to φ^a and u:

$$\begin{aligned} 0 &= \int_{\overline{\mathbb{D}}} \left(\varphi^a\overline{L}u - u\overline{L}\varphi^a\right) d\overline{A} = \int_0^{2\pi} e^{-\gamma^a} \left(\varphi^a\frac{\partial u}{\partial r} - u\frac{\partial \varphi^a}{\partial r}\right)\bigg|_{r=1} d\theta \\ &= \int_0^{2\pi} e^{-2\gamma^a}\, d\theta, \end{aligned}$$

since:

$$\varphi^a|_{r=1} = 0 \quad \text{and} \quad \left.\frac{\partial \varphi^a}{\partial r}\right|_{r=1} = -e^{-\gamma^a}.$$

This is impossible. $\square$

Let $\overline{L}_0$ be the restriction of $\overline{L}$ to $\mathcal{C}_0^{2,\alpha}(\overline{\mathbb{D}})$ and $K = \ker \overline{L}_0$. Using the standard inclusions $\mathcal{C}_0^{2,\alpha}(\overline{\mathbb{D}}) \subset \mathcal{C}^{0,\alpha}(\overline{\mathbb{D}}) \subset L^2(\mathbb{D})$, we denote by $K^\perp$ the orthogonal to K in $\mathcal{C}^{0,\alpha}(\overline{\mathbb{D}})$ for the natural scalar product of $L^2(\mathbb{D})$ and $K_0^\perp = K^\perp \cap \mathcal{C}_0^{2,\alpha}(\overline{\mathbb{D}})$.

It is a standard fact that the restriction $\overline{L}_0$ is a Fredholm operator with index zero (see for instance [7]). Namely $K = \mathbb{R}\varphi^a$ and $\overline{L}_0(\mathcal{C}_0^{2,\alpha}(\overline{\mathbb{D}})) = K^\perp$.

3.1 General deformations

Let $\mu_a : \mathcal{C}^{2,\alpha}(\mathbb{S}^1) \to \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}})$ be the operator such that $\mu_a(\gamma)$ is the harmonic function on $\overline{\mathbb{D}}$ (for the flat laplacian) with value $\gamma - \gamma^a$ on the boundary $\partial\mathbb{D}$. Denote Π_K and $\Pi_{K^\perp}$ be the orthogonal projections on K and $K^\perp$ respectively. Following B. White [14], we show:

Lemma 3.4. *Consider the map* $\Phi : \mathcal{C}^{2,\alpha}(\mathbb{S}^1) \times \mathbb{R} \times K_0^\perp \to K^\perp$ *defined by:*

$$\Phi(\gamma, \lambda, \sigma) = \Pi_{K^\perp} \circ \overline{H}(\mu_a(\gamma) + \lambda\varphi^a + \sigma).$$

Then $D_3\Phi(\gamma^a, 0, 0) : K_0^\perp \to K^\perp$ *is an isomorphism.*

Proof. A direct computation gives $D_3\Phi(\gamma^a, 0, 0) = \Pi_{K^\perp} \circ \overline{L}_0|_{K_0^\perp}$ and we know $K^\perp$ is the range of $\overline{L}_0$, which means $D_3\Phi(\gamma^a, 0, 0) : K_0^\perp \to K^\perp$ is an isomorphism. $\square$

Therefore, we can apply the implicit function theorem to Φ, which states that there exist an open neighborhood U_a of $(\gamma^a, 0)$ in $\mathcal{C}^{2,\alpha}(\mathbb{S}^1) \times \mathbb{R}$ and a unique smooth map $\sigma : U_a \to K_0^\perp$ such that:

$$\forall (\gamma, \lambda) \in U_a,\ \Phi\big(\gamma, \lambda, \sigma(\gamma, \lambda)\big) = 0.$$

Then we define the smooth maps $\xi_a : U_a \to \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}})$, $\eta_a : U_a \to \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}})$ and $\kappa_a : U_a \to K$ by:

$$\xi_a(\gamma, \lambda) = \mu_a(\gamma) + \lambda\varphi^a + \sigma(\gamma, \lambda), \quad \eta_a(\gamma, \lambda) = a + 2c^a\xi_a(\gamma, \lambda)$$
$$\text{and} \quad \kappa_a(\gamma, \lambda) = \Pi_K \circ \overline{H}\big(\xi_a(\gamma, \lambda)\big).$$

If a surface in $\mathcal{E}$, defined on $\mathbb{D}$, admits $X^{\eta_a(\gamma,\lambda)}$ as graph coordinates at infinity, we say that $\{\gamma, \lambda\}$ are the *data* of the surface with respect to S or to a.

Lemma 3.5. *The maps* η_a *and* ξ_a *have the following properties:*

1. $\xi_a(\gamma^a, 0) = 0$ *and* $\eta_a(\gamma^a, 0) = a$.

2. $\forall(\gamma, \lambda) \in U_a, \ \eta_a(\gamma, \lambda)|_{\partial\mathbb{D}} = \gamma$.

3. $D_2\xi_a(\gamma^a, 0) : \lambda \in \mathbb{R} \mapsto \lambda\varphi^a \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}})$.

Proof. Point 1 comes from the definition of μ_a and from the uniqueness in the implicit function theorem. Point 2 is a direct computation:

$$\begin{aligned}\eta_a(\gamma, \lambda)|_{\partial\mathbb{D}} &= a|_{\partial\mathbb{D}} + 2c^a|_{\partial\mathbb{D}}\big(\mu_a(\gamma)|_{\partial\mathbb{D}} + \lambda\varphi^a|_{\partial\mathbb{D}} + \sigma(\gamma, \lambda)|_{\partial\mathbb{D}}\big)\\ &= \gamma^a + 2\frac{1}{2}\big((\gamma - \gamma^a)\big) = \gamma.\end{aligned}$$

For Point 3, it is sufficient to show $D_2\sigma(\gamma^a, 0) = 0$. To do so we compute:

$$\begin{aligned}0 &= \frac{d}{dt}\bigg|_{t=0} \Phi\big(\gamma^a, t, \sigma(\gamma^a, t)\big) = \Pi_{K^\perp} \circ \overline{L}_0\big(\varphi^a + D_2\sigma(\gamma^a, 0) \cdot 1\big)\\ &= \overline{L}_0\big(D_2\sigma(\gamma^a, 0) \cdot 1\big).\end{aligned}$$

Hence, $D_2\sigma(\gamma^a, 0) \cdot 1 \in K \cap K_0^\perp = \{0\}$, which means $D_2\sigma(\gamma^a, 0) = 0$. □

Remark 3.6. Consider $S, S' \in \mathcal{G}$ admitting respectively $X^a, X^{a'}$ as graph coordinates at infinity and suppose there exist a surface in $\mathcal{E}$ with data $\{\gamma, \lambda\}$ and $\{\gamma', \lambda'\}$ with respect to S and S' respectively. Therefore, this surface admits graph coordinates at infinity $X^{\eta_a(\gamma,\lambda)}$ and $X^{\eta_{a'}(\gamma',\lambda')}$ — i.e. $\eta_a(\gamma, \lambda) = \eta_{a'}(\gamma', \lambda')$ — and we get:

$$\gamma' = \gamma \quad \text{and} \quad \lambda' = \frac{1}{|\varphi^{a'}|^2_{L^2(\mathbb{D})}}\left\langle \frac{\eta_a(\gamma, \lambda) - a'}{2c^{a'}} - \mu_{a'}(\gamma), \varphi^{a'} \right\rangle_{L^2(\mathbb{D})}. \tag{8}$$

The identity on values at infinity comes from Lemma 3.5 Point 2, and the expression of λ' is just the projection along $\varphi^{a'}$.

Note that a converse to this decompostion is the subject of Theorem 4.3, namely if $X^{\eta_a(\gamma,\lambda)}$ admits data with respect to S', these data are $\{\gamma', \lambda'\}$ as defined in (8).

Lemma 3.5 Point 2 also shows that the value at infinity of a surface $X^{\eta_a(\gamma,\lambda)}$ does not depend on λ, which means there exists a 1-parameter family of surfaces admitting the same value at infinity. This family is nothing but the vertical translations of $X^{\eta_a(\gamma,\lambda)}$:

Proposition 3.7. *Let* $(\gamma, \lambda) \in U_a$. *The surface* $X^{\eta_a(\gamma,\lambda')}$ *exists for any* $\lambda' \in \mathbb{R}$ *and coincides with* $X^{\eta_a(\gamma,\lambda)}$ *up to a vertical translation.*

Proof. To ease the writing, denote $\widetilde{a} = \eta_a(\gamma, \lambda)$, $h(\widetilde{a})$ the height function of $X^{\widetilde{a}}$ and $m > 0$ the minimum of $h(\widetilde{a})$ on $\mathbb{D}$. We know from Proposition 2.3 that the graph coordinates at infinity of the image of $X^{\widetilde{a}}$ under the vertical translation by some $t \in \mathbb{R}$ can still be written $X^{a'(t)}$ defined on $\mathbb{D}$ if and only if $t > -m$ and in that case:

$$a'(t) = \widetilde{a} + \log\left(1 + t\frac{e^{-\widetilde{a}}}{2}\frac{1-r^2}{1+r^2}\right) = \widetilde{a} + \log\left(1 + \frac{t}{h(\widetilde{a})}\right).$$

We also know that $a'(t)|_{\partial\mathbb{D}} = \widetilde{a}|_{\partial\mathbb{D}}$, which implies $\mu_a(\gamma^{a'(t)}) = \mu_a(\gamma)$. Writing:

$$a'(t) = a + 2c^a\Big(\mu_a(\gamma) + \lambda'(t)\varphi^a + \sigma(\gamma, \lambda'(t))\Big),$$

we only have to show that $\lambda'(t)$ is a bijection in the variable t from the interval $(-m, +\infty)$ of possible translations onto $\mathbb{R}$. We have:

$$\frac{a'(t) - a}{2c^a} = \frac{a'(t) - \widetilde{a}}{2c^a} + \frac{\widetilde{a} - a}{2c^a} = \frac{1}{2c^a}\log\left(1 + \frac{t}{h(\widetilde{a})}\right) + \xi_a(\gamma, \lambda),$$

and using (8), the expression of $\lambda'(t)$ is:

$$\begin{aligned}\lambda'(t) &= \lambda + \frac{1}{2\pi|\varphi^a|^2_{L^2(\mathbb{D})}}\int_{\mathbb{D}}\frac{\varphi^a}{c^a}\log\left(1 + \frac{t}{h(\widetilde{a})}\right)\\ &= \lambda + \frac{1}{2\pi|\varphi^a|^2_{L^2(\mathbb{D})}}\int_{\mathbb{D}}(\varphi^a)^2 h(a)\log\left(1 + \frac{t}{h(\widetilde{a})}\right) \quad \text{since} \quad \frac{1}{c^a} = \varphi^a h(a).\end{aligned}$$

Compute:

$$\frac{d\lambda'(t)}{dt} = \frac{1}{2\pi|\varphi^a|^2_{L^2(\mathbb{D})}}\int_{\mathbb{D}}\frac{(\varphi^a)^2 h(a)}{t + h(\widetilde{a})} > 0$$

i.e. $\lambda'(t)$ is a strictly increasing bijection from $(-m, +\infty)$ into $\mathbb{R}$. Also:

$$\lambda'(t) \overset{(t\leq 0)}{\leq} \lambda + \left[\frac{1}{2\pi|\varphi^a|^2_{L^2(\mathbb{D})}}\int_{\mathbb{D}}(\varphi^a)^2 h(a)\right]\log\left(1 + \frac{t}{m}\right) \xrightarrow[t\to -m]{} -\infty.$$

If $M > 0$ is the maximum of $h(\widetilde{a})$ on the disk $\{0 \leq r \leq 1/2\}$, we get:

$$\begin{aligned}\lambda'(t) &\overset{(t\geq 0)}{\geq} \lambda + \frac{1}{2\pi|\varphi^a|^2_{L^2(\mathbb{D})}}\int_{\{0\leq r\leq 1/2\}}(\varphi^a)^2 h(a)\log\left(1 + \frac{t}{h(a)}\right)\\ &\geq \lambda + \left[\frac{1}{2\pi|\varphi^a|^2_{L^2(\mathbb{D})}}\int_{\{0\leq r\leq 1/2\}}(\varphi^a)^2 h(a)\right]\log\left(1 + \frac{t}{M}\right) \xrightarrow[t\to +\infty]{} +\infty,\end{aligned}$$

which ensures that $\lambda'(t)$ is bijective from $(-m, +\infty)$ *onto* $\mathbb{R}$. □

3.2 CMC-1/2 deformations

The values of the mean curvature of deformations $X^{\eta_a(\gamma,\lambda)}$ of S are determined by κ_a. Indeed, for $(\gamma, \lambda) \in U_a$ we have $\Phi(\gamma, \lambda, \sigma(\gamma, \lambda)) = 0$ and:

$$\overline{H}(\xi_a(\gamma, \lambda)) = \kappa_a(\gamma, \lambda) + \Phi(\gamma, \lambda, \sigma(\gamma, \lambda)) = \kappa_a(\gamma, \lambda). \tag{9}$$

In particular:

$$\forall (\gamma, \lambda) \in U_a,\ H\big(\eta_a(\gamma, \lambda)\big) = \frac{1}{2} \iff \kappa_a(\gamma, \lambda) = 0.$$

Consider $\mathcal{U}_a = \kappa_a^{-1}(\{0\}) \cap U_a$. Using Proposition 3.7, we can take $\mathcal{U}_a = \Gamma_a \times \mathbb{R}$ with Γ_a a subset of $\mathcal{C}^{2,\alpha}(\mathbb{S}^1)$. Furthermore, since the construction is local, we can suppose Γ_a connected.

Proposition 3.8. *Γ_a is a codimension 1 smooth submanifold of $\mathcal{C}^{2,\alpha}(\mathbb{S}^1)$. The tangent space to Γ_a at γ^a is the orthogonal space $\langle e^{-2\gamma^a} \rangle^\perp$ to $e^{-2\gamma^a}$ in $\mathcal{C}^{2,\alpha}(\mathbb{S}^1)$ for the scalar product of $L^2(\mathbb{S}^1)$ and Γ_a is a subset of:*

$$\left\{ \gamma \in \mathcal{C}^{2,\alpha}(\mathbb{S}^1) \,\middle|\, |e^{-\gamma}|_{L^2(\mathbb{S}^1)} = 1 \right\}.$$

Proof. We first show that κ_a is a submersion at $(\gamma^a, 0)$. Using (9), compute:

$$\begin{aligned} D_2\kappa_a(\gamma^a, 0) \cdot 1 &= \frac{d}{dt}\Big|_{t=0} \kappa_a(\gamma^a, t) = \frac{d}{dt}\Big|_{t=0} \overline{H}(\xi_a(\gamma^a, t)) \\ &= \overline{L}\big(D_2\xi_a(\gamma^a, 0) \cdot 1\big) = \overline{L}_0(\varphi^a) = 0, \end{aligned}$$

since $\varphi^a \in K$. Remains to find $\gamma \in \mathcal{C}^{2,\alpha}(\mathbb{S}^1)$ such that $D_1\kappa_a(\gamma^a, 0) \cdot \gamma$ is not identically zero. We can take $\gamma = 1$. Indeed, using (9):

$$\begin{aligned} D_1\kappa_a(\gamma^a, 0) \cdot 1 &= \frac{d}{dt}\Big|_{t=0} \kappa_a(\gamma^a + t, 0) = \frac{d}{dt}\Big|_{t=0} \overline{H}(\xi_a(\gamma^a + t, 0)) \\ &= \overline{L}\big(D_1\xi_a(\gamma^a, 0) \cdot 1\big) \neq 0, \end{aligned}$$

using Corollary 3.3 with $\big(D_1\xi_a(\gamma^a, 0) \cdot 1\big)|_{\partial\mathbb{D}} = 1$ deduced from Lemma 3.5 Point 2. Since $D\kappa_a$ is continuous and non zero at $(\gamma^a, 0)$, there exists an open neighborhood of $(\gamma^a, 0)$ in $\mathcal{C}^{2,\alpha}(\mathbb{S}^1) \times \mathbb{R}$ on which κ_a is a submersion. Therefore, up to a restriction on Γ_a, we can suppose κ_a is a submersion on $\Gamma_a \times \{0\}$, which implies Γ_a is a submanifold of $\mathcal{C}^{2,\alpha}(\mathbb{S}^1)$ of codimension 1.

Consider a smooth path γ_t in Γ_a with $\gamma_0 = \gamma^a$ and tangent vectors $\dot{\gamma}_t$. Note that similarly:

$$0 = D\kappa_a(\gamma^a, 0) \cdot (\dot{\gamma}_0, 0) = \frac{d}{dt}\Big|_{t=0} \kappa_a(\gamma_t, 0) = \overline{L}\big(D_1\xi_a(\gamma^a, 0) \cdot \dot{\gamma}_0\big).$$

Denote $v = D_1\xi_a(\gamma^a, 0)\cdot\dot\gamma_0 \in \ker \overline{L}$. Knowing that:

$$\varphi^a|_{r=1} = 0, \qquad \left.\frac{\partial \varphi^a}{\partial r}\right|_{r=1} = -e^{-\gamma^a} \quad \text{and} \quad v|_{r=1} = \dot\gamma_0,$$

apply Proposition 3.2 to φ^a and v:

$$\begin{aligned} 0 &= \int_{\overline{\mathbb{D}}} \left(\varphi^a \overline{L} v - v \overline{L}\varphi^a\right) d\overline{A} = \int_0^{2\pi} e^{-\gamma^a} \left.\left(\varphi^a \frac{\partial v}{\partial r} - v\frac{\partial \varphi^a}{\partial r}\right)\right|_{r=1} d\theta \\ &= \int_0^{2\pi} \dot\gamma_0 e^{-2\gamma^a} d\theta = 2\pi \langle \dot\gamma_0, e^{-2\gamma^a}\rangle_{L^2(\mathbb{S}^1)}. \end{aligned} \tag{10}$$

Thus $\langle e^{-2\gamma^a}\rangle^\perp$ is the tangent space to Γ_a at γ^a, since it is of codimension 1.

The stated inclusion for Γ_a expresses the nullity of the vertical flux of an entire graph. If $\gamma \in \Gamma_a$ is the value at infinity of a surface $S' \in \mathcal{G}$, consider the subset V_R, for $R \in (0,1)$, of $\mathbb{H}^2 \times \mathbb{R}$ inside the vertical cylinder C_R of (euclidean) radius $|F(R,\cdot)| = 2R/(1+R^2)$ in the Poincaré disk model (2), delimited below by the slice $\mathbb{H}^2 \equiv \mathbb{H}^2 \times \{0\}$ and above by the surface S'. Since e_3 is a Killing vector field, using Stokes theorem we have:

$$0 = \int_{V_R} \operatorname{div} e_3 = \int_{S'_R} \langle N^a, e_3\rangle + \int_{D_R} \langle -e_3, e_3\rangle,$$

where S'_R is the part of S' inside C_R and D_R the disk in $\mathbb{H}^2$ of (euclidean) radius $|F(R,\cdot)| = 2R/(1+R^2)$.

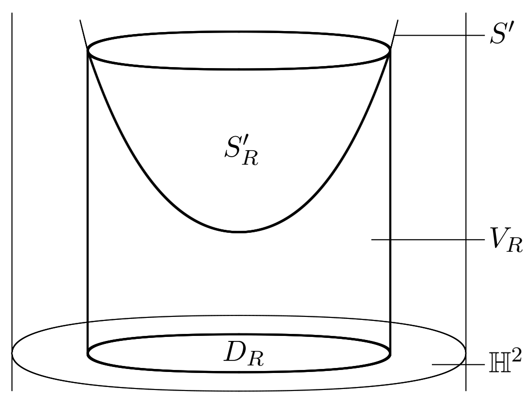

Figure 1: Decomposition of the boundary of V_R

We use notations of Appendix 6. If $X^{a'}$ are the graph coordinates at infinity of S', we have $\Delta_{g(a')} X^{a'} = 2H(a')N^{a'} = N^{a'}$, since $H(a') = 1/2$, and the first integral writes:

$$\int_{S'_R} \langle N^{a'}, e_3\rangle = \int_{\{r\le R\}} \Delta_{g(a')} h(a')\ dA = \int_{\{r=R\}} \frac{\partial h(a')}{\partial \nu} ds$$
$$= \frac{8R^2}{(1-R^2)^2} \int_0^{2\pi} \frac{1}{w(a')} \left[1 + \frac{(1+r^2)a'_r}{4r}(1-r^2) \right]\Bigg|_{r=R} d\theta$$
$$= \frac{16\pi R^2}{(1-R^2)^2} + 2\pi \left(1 - |e^{-a'(R,\cdot)}|_{L^2(\mathbb{S}^1)}\right) + O(1-R^2).$$

The second is the area of D_R:

$$\int_{D_R} 1 = 2\pi \int_0^R \sqrt{|\sigma|} dr = 16\pi \int_0^{R^2} \frac{1+r}{(1-r)^3} dr = \frac{16\pi R^2}{(1-R^2)^2}.$$

Making $R \to 1$, we get $|e^{-\gamma}|_{L^2(\mathbb{S}^1)} = 1$ and the inclusion for Γ_a. □

A. E. Treibergs showed (see [13]) that given a $\mathcal{C}^2$ curve — generalized to continuous curves by H. I. Choi and A. E. Treibergs in [1] — $\gamma : \mathbb{S}^1 \to \mathbb{R}$, there exists a CMC-1/2 complete entire vertical graph in the 3-dimensional Minkowski space which is asymptotically at signed distance γ from the light cone. Namely, it is the graph of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ such that:

$$f(x) = |x| + \gamma\left(\frac{x}{|x|}\right) + \varepsilon(x) \quad \text{with} \quad \lim_{|x|\to+\infty} \varepsilon(x) = 0.$$

Proposition 3.8 is indeed a $\mathcal{C}^{2,\alpha}$ local version of this result in $\mathbb{H}^2 \times \mathbb{R}$:

Theorem 3.9. *Let $\gamma \in \mathcal{C}^{2,\alpha}(\mathbb{S}^1)$, small in the $\mathcal{C}^{2,\alpha}$-norm, be such that $|e^{-\gamma}|_{L^2(\mathbb{S}^1)} = 1$. Then there exists a surface in $\mathcal{G}$ with γ as value at infinity. In other words, there exists a CMC-1/2 complete entire vertical graph at asymptotic horizontal signed distance γ from the hyperboloid S_0.*

Proof. If γ is sufficiently small in the $\mathcal{C}^{2,\alpha}$ norm, then $\gamma \in \Gamma_0$ and $X^{\eta_0(\gamma,0)}$ is a CMC-1/2 entire graph admitting γ as value at infinity. □

Another consequence of Proposition 3.8 is the global structure of $\mathcal{G}$:

Theorem 3.10. *The family $\mathcal{G}$ can be endowed with a structure of infinite dimensional smooth manifold.*

Proof. Consider a surface $S \in \mathcal{G}$ with graph coordinates at infinity X^a and $\mathcal{V}_a \subset \mathcal{G}$ the set of surfaces admitting data in $\mathcal{U}_a$. From uniqueness in the implicit function theorem we know that the map:

$$\tau_a : S' \in \mathcal{V}_a \mapsto (\gamma, \lambda) \in \mathcal{U}_a,$$

where $\{\gamma, \lambda\}$ are the data of S' with respect to a, is a bijection. To prove that the couple $(\mathcal{V}_a, \tau_a)$ form a smooth atlas, it only remains to show that the transition maps are smooth. But identities (8) are precisely the transition map from $(\mathcal{V}_a, \tau_a)$ to $(\mathcal{V}_{a'}, \tau_{a'})$, which concludes the proof. $\square$

4 A half-space theorem

In [10], B. Nelli and R. Sa Earp show a half-space theorem for the hyperboloid S_0. We extend this result to the family $\mathcal{G}$ of CMC-1/2 entire graphs with appropriate graph coordinates at infinity. The proof is based on the idea of B. Daniel, W. H. Meeks and H. Rosenberg [2] in Heisenberg space. A key-ingredient is to construct a family of surfaces with boundary. Our tool to do this is the following:

Lemma 4.1. *Let E be a CMC-1/2 surface with boundary admitting graph coordinates at infinity X^a defined on an admissible domain $\Omega_R \in \mathcal{D}$, with $R \in (0,1)$ and $a \in \mathcal{C}^{2,\alpha}(\overline{\Omega_R})$. Denote $\gamma^a_{int} = a|_{\{r=R\}}$ and $\gamma^a_{ext} = a|_{\partial\mathbb{D}}$. Then for any $(\gamma_{int}, \gamma_{ext})$ in a neighborhood of $(\gamma^a_{int}, \gamma^a_{ext})$ in $(\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2$, there exists a CMC-1/2 surface admitting graph coordinates at infinity $X^{a'}$ defined on Ω_R such that $a|_{\{r=R\}} = \gamma_{int}$ and $a|_{\partial\mathbb{D}} = \gamma^a_{ext}$.*

Proof. Consider the map $\Phi : (\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2 \times \mathcal{C}^{2,\alpha}_0(\overline{\Omega_R}) \to \mathcal{C}^{0,\alpha}(\overline{\Omega_R})$ defined by:

$$\Phi(\gamma_{int}, \gamma_{ext}, \sigma) = \overline{H}\big(\mu_a(\gamma_{int}, \gamma_{ext}) + \sigma\big),$$

where $\overline{H}$ is the compactified mean curvature operator as defined in (7) and $\mu_a : (\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2 \to \mathcal{C}^{2,\alpha}(\overline{\Omega_R})$ is the operator such that $\mu_a(\gamma_{int}, \gamma_{ext})$ is the harmonic function on Ω_R with value $(\gamma_{int} - \gamma^a_{int}, \gamma_{ext} - \gamma^a_{ext})$ on the boundary of Ω_R.

We know that E is strictly stable since the third coordinate φ^a of the upward pointing normal is a positive Jacobi function (see [6]). Hence, $D_3\Phi(\gamma^a_{int}, \gamma^a_{ext}, 0) : \mathcal{C}^{2,\alpha}_0(\overline{\Omega_R}) \to \mathcal{C}^{0,\alpha}(\overline{\Omega_R})$ is an isomorphism and we can apply the implicit function theorem as in Section 3.1. There exist a neighborhood U of $(0,0)$ in $(\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2$ and a smooth map $\sigma : U \to \mathcal{C}^{0,\alpha}(\overline{\Omega_R})$ such that:

$$\forall (\gamma_{int}, \gamma_{ext}) \in U,\ \Phi\big(\gamma_{int}, \gamma_{ext}, \sigma(\gamma_{int}, \gamma_{ext})\big) = 0.$$

We can take $a' = a + 2c^a\big(\mu_a(\gamma_{int}, \gamma_{ext}) + \sigma(\gamma_{int}, \gamma_{ext})\big)$. $\square$

We now can show the following half-space result:

Theorem 4.2. *Let Σ be a CMC-$1/2$ surface which is properly immersed in $\mathbb{H}^2 \times \mathbb{R}$ and lies on one side of a CMC-$1/2$ entire graph $S \in \mathcal{G}$ admitting graph coordinates at infinity X^a with $a \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}})$. Then Σ coincides with S up to a vertical translation.*

Proof. Without loss of generality, we suppose Σ is above S. We denote $T^c : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2 \times \mathbb{R}$ the vertical translation by $c \in \mathbb{R}$ and:

$$c_0 = \inf \left\{ c \in \mathbb{R} \middle| \Sigma \cap T^c(S) \neq \emptyset \right\}.$$

If $\Sigma \cap T^{c_0}(S) \neq \emptyset$ then by maximum principle, Σ coincides with $T^{c_0}(S)$.

From now on, suppose $\Sigma \cap T^{c_0}(S) = \emptyset$ and — up to a vertical translation — $c_0 = 0$. In other words:

$$\Sigma \cap S = \emptyset \quad \text{and} \quad \forall c > 0,\ \Sigma \cap T^c(S) \neq \emptyset.$$

We want to construct a CMC-1/2 surface with boundary intersecting Σ in an interior point. To do so, consider $R \in (0, 1/2)$ and admissible domains $\Omega_R, \Omega_{2R} \in \mathcal{D}$. There exists $\delta > 0$ such that Σ intersects $T^c(S)$ only inside the exterior domain $\Omega_{2R} \times \mathbb{R}$ for any $0 < c < 2\delta$:

$$\forall c < 2\delta,\ \left(T^c(S) \cap \Sigma\right) \subset \Omega_{2R} \times \mathbb{R}.$$

Denote $E = T^\delta(S) \cap (\Omega_R \times \mathbb{R})$. E is a CMC-1/2 surface with boundary. We can apply Lemma 4.1 to deform E and construct a family $\left(E(\varepsilon)\right)_{\varepsilon \geq 0}$ of CMC-1/2 surfaces such that:

- $E(\varepsilon)$ is at constant asymptotic horizontal signed distance $-\varepsilon$ from E;
- $E(\varepsilon)$ coincides with E on the interior boundary $\{|w| = R\} \times \mathbb{R}$ of $\Omega_R \times \mathbb{R}$;
- $E(0) = E$.

Since $E(\varepsilon)$ converges to E when $\varepsilon \to 0$ and $E \cap \Sigma \neq \emptyset$, there exists $\varepsilon_0 > 0$ such that $E(\varepsilon_0) \cap \Sigma \neq \emptyset$.

At infinity $E(\varepsilon_0)$ is outside S, thus $T^c\left(E(\varepsilon_0)\right) \cap \Sigma = \emptyset$ for large $c < 0$. Consider:

$$c_1 = \sup \left\{ c < 0 \middle| T^c\left(E(\varepsilon_0)\right) \cap \Sigma = \emptyset \right\} \leq 0.$$

We know that $T^{c_1}\left(E(\varepsilon_0)\right) \cap \Sigma \neq \emptyset$ since the first intersection point cannot be at infinity. And this intersection does not occur on the boundary of $T^{c_1}\left(E(\varepsilon_0)\right)$, since the boundary lies outside $\Omega_{2R} \times \mathbb{R}$. Therefore, the first intersection point is point interior to $T^{c_1}\big(E(\varepsilon_0)\big)$ and by maximum principle Σ coincides with $T^{c_1}\big(E(\varepsilon_0)\big)$ over Ω_R, which is impossible. □

We can deduce from Theorem 4.2 a uniqueness result at infinity for the family $\mathcal{G}$:

Theorem 4.3. *Let S, S' be CMC-1/2 entire graphs in $\mathcal{G}$ admitting graph coordinates at infinity $X^a, X^{a'}$ respectively, with $a, a' \in \mathcal{C}^{2,\alpha}(\overline{\mathbb{D}})$. Suppose there exist a surface Σ admitting data $(\gamma, \lambda) \in \Gamma_a \times \mathbb{R}$ with respect to S and, as in (8), denote:*

$$\lambda' = \frac{1}{|\varphi^{a'}|^2_{L^2(\mathbb{D})}} \left\langle \frac{\eta_a(\gamma,\lambda) - a'}{2c^{a'}} - \mu_{a'}(\gamma), \varphi^{a'} \right\rangle_{L^2(\mathbb{D})},$$

with $X^{\eta_a(\gamma,\lambda)}$ the graph coordinates at infinity of Σ. Suppose $\gamma \in \Gamma_{a'}$, then Σ admits data $\{\gamma, \lambda'\}$ with respect to S'; in other words, $\eta_a(\gamma,\lambda) = \eta_{a'}(\gamma,\lambda')$.

Proof. We first make two remarks:

- If Σ admits data with respect to S', then from (8) and the definition of λ' above, we know that the data are precisely $\{\gamma, \lambda'\}$.
- To show that Σ admits data with respect to S', we only have to show that a vertical translate $T^{c_0}(\Sigma)$ of Σ, with $c_0 \geq 0$, admits data with respect to S'.

Consider graph coordinates at infinity X^η for $T^c(\Sigma)$ with $c \geq 0$. Suppose there exist $r_0 \in (0,1)$ such that the height functions of $T^c(\Sigma)$ and $X^{\eta_{a'}(\gamma,\lambda')}$ verify $h(\eta) > h\big(\eta_{a'}(\gamma,\lambda')\big)$ for any $(r,\theta) \in [r_0, 1) \times \mathbb{S}^1$. We take:

$$c_0 = c + \max_{[0,r_0]\times\mathbb{S}^1} \Big| h\big(\eta_{a'}(\gamma,\lambda')\big) - h(\eta) \Big|,$$

so that $T^{c_0}(\Sigma)$ is above $X^{\eta_{a'}(\gamma,\lambda')}$. Applying Theorem 4.2, we deduce that $T^{c_0}(\Sigma)$ is a vertical translate of $X^{\eta_{a'}(\gamma,\lambda')}$.

Remains to show the existence of r_0. We have:

$$\begin{aligned} &\eta_{a'}(\gamma,\lambda') = a' + 2c^{a'}\big[\mu_{a'}(\gamma) + \lambda'\varphi^{a'} + \sigma(\gamma,\lambda')\big] \\ \text{and} \quad &\eta_a(\gamma,\lambda') = a' + 2c^{a'}\big[\mu_{a'}(\gamma) + \lambda'\varphi^{a'} + \sigma\big], \end{aligned}$$

by definition of λ', which gives:

$$\eta - \eta_{a'}(\gamma,\lambda') = \underbrace{2c^{a'}\big[\sigma - \sigma(\gamma,\lambda')\big]}_{\in \mathcal{C}^{2,\alpha}_0(\overline{\mathbb{D}})} + \log\left(1 + c\frac{e^{-\eta_a(\gamma,\lambda)}}{2}\frac{1-r^2}{1+r^2}\right).$$

We know $\eta - \eta_{a'}(\gamma, \lambda')$ is identically zero on $\partial\mathbb{D}$ and we remark that:

$$\frac{\partial}{\partial r}(\eta - \eta_{a'}(\gamma, \lambda'))\Big|_{\partial\mathbb{D}} = 2\,\frac{\partial}{\partial r}\Big(c^{a'}[\sigma - \sigma(\gamma, \lambda')]\Big)\Big|_{\partial\mathbb{D}} - c\frac{e^{-\gamma}}{2}.$$

Taking:

$$c = 2\max_{\partial\mathbb{D}} e^{\gamma}\left(1 + 2\frac{\partial}{\partial r}\Big(c^{a'}[\sigma - \sigma(\gamma, \lambda')]\Big)\right),$$

we get:

$$\frac{\partial}{\partial r}(\eta - \eta_{a'}(\gamma, \lambda'))\Big|_{\partial\mathbb{D}} \le -1.$$

Thus, there exists $r_0 \in (0,1)$ such that $\eta - \eta_{a'}(\gamma, \lambda')$ is strictly decreasing (in r) on $[r_0, 1) \times \mathbb{S}^1$, which means $\eta - \eta_{a'}(\gamma, \lambda') > 0$ on $[r_0, 1) \times \mathbb{S}^1$ and concludes the proof. □

5 Deformations of CMC-1/2 annuli

R. Sa Earp and E. Toubiana showed in [3] that — up to a not necessarily orientation preserving isometry of $\mathbb{H}^2 \times \mathbb{R}$ — a rotational CMC-1/2 vertical annulus is a bigraph, symmetrical with respect to the slice $\mathbb{H}^2 \times \{0\}$. The upper graph part of such an annulus admits graph coordinates at infinity (F, h_β), with β a positive real number, $\beta \neq 1$ and h_β defined by:

$$h_\beta(r) = \int_{|\log\beta|}^{2\log\left(\frac{1+r}{1-r}\right)} \frac{\cosh t - \beta}{\sqrt{2\beta\cosh t - 1 - \beta^2}}dt \quad \text{where} \quad r \ge \left|\frac{\sqrt{\beta}-1}{\sqrt{\beta}+1}\right| = R_\beta.$$

We denote by A_β this annulus, which is embedded if $0 < \beta < 1$ and only immersed when $\beta > 1$.
We have the following asymptotic development as $r \to 1$:

$$h_\beta(r) = \frac{1}{\sqrt{\beta}}\frac{1+r}{1-r} + O(1), \tag{11}$$

which means that the restriction of (F, h_β) to the exterior domain Ω_{R_β} is in $\mathcal{E}$ with constant value $-\log\beta$ at infinity. Therefore, the method developed in Section 3 should adapt to the study of deformations of these annuli.

For our purpose, we slightly change the notations. Fix $\beta > 0$ with $\beta \neq 1$; the annulus A_β is now the model surface. To deform rotational annuli, we need conformal coordinates to provide a compactification of the

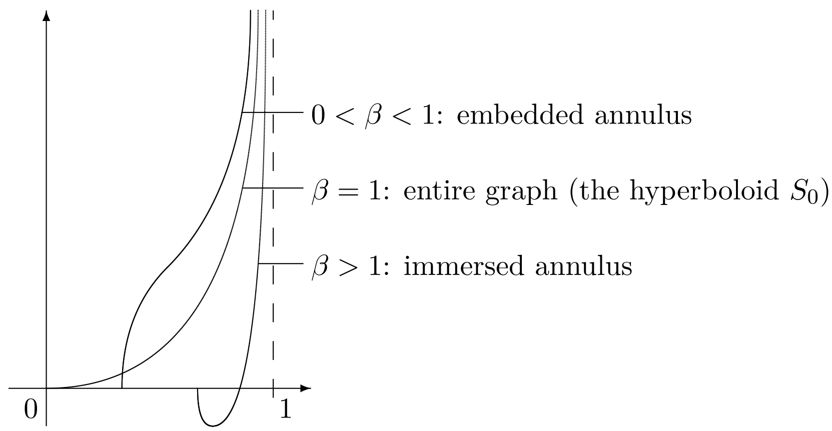

Figure 2: Profile curves of rotational CMC-1/2 examples in the Poincaré disk model (2)

mean curvature. A conformal parametrization of the annulus A_β, written in cylindrical coordinates, is the following:

$$X^0 : (s,\theta) \in \Omega^\beta \mapsto \Big(F\big(r(s),\theta\big), \varepsilon(s) h_\beta\big(r(s)\big)\Big) \quad \text{with} \quad \Omega^\beta = (-T,T)\times\mathbb{S}^1, \tag{12}$$

$$T = \frac{4}{|\beta-1|}\int_{R_\beta}^{1} \frac{dt}{\sqrt{(t^2-R_\beta^2)(R_\beta^{-2}-t^2)}}, \quad \varepsilon(s) = \operatorname{sign}(s),$$

$$\frac{dr}{ds} = \frac{|\beta-1|}{4}\sqrt{\big(r^2(s)-R_\beta^2\big)\big(R_\beta^{-2}-r^2(s)\big)} \quad \text{and} \quad r(0) = R_\beta.$$

We also identify functions over A_β with functions over Ω^β. The *cylindrical parametrization* of a deformed annulus is the following immersion:

$$X^\eta : (s,\theta) \in \Omega^\beta \mapsto \Big(F\big(r(s),\theta\big), \varepsilon(s) e^{\eta(s,\theta)} h_\beta\big(r(s)\big)\Big) \quad \text{with} \quad \eta \in \mathcal{C}^{2,\alpha}(\overline{\Omega^\beta}).$$

The determinant of the first fundamental form is $|g(\eta)|$, the mean curvature $H(\eta)$ and the values at infinity are the couple $\big(\eta(T,\cdot),\eta(-T,\cdot)\big) \in (\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2$.

5.1 Non degeneracy of rotational annuli

As in Section 3, we need to understand the Jacobi functions in order to control the deformations. Thus, we focus the study on annuli in $\mathcal{E}$ that are non degenerate in the following sense:

Definition 5.1. A surface in $\mathcal{E}$ is said to be *non degenerate* if the only Jacobi functions that are zero at infinity on each end of the surface (i.e. when $r = 1$ in the graph coordinates at infinity of the ends) come from isometries of $\mathbb{H}^2 \times \mathbb{R}$.

A direct consequence of the proof of Proposition 2.3 and the shape of the ends is that if an annulus in $\mathcal{E}$ is non degenerate, then the space of Jacobi functions which are zero on the boundary is 1-dimensional, generated by the vertical component of the unit normal.

Another fact is that, since the rank of the Jacobi operator is locally constant, small deformations of a non degenerate annulus are still non degenerate.

Therefore, the method used in Section 3 can be strictly transposed to the study of deformations in a small neighborhood of a non degenerate example.

Proposition 5.2. *The annulus A_β is non degenerate for any value of β ($\neq 1$).*

Proof. If L denotes the Jacobi operator of A_β, the compactified Jacobi operator $\overline{L} = \sqrt{|g(0)|}L$ of A_β can be written $\Delta + q(s)$ in the conformal parametrization (12), with Δ the flat laplacian and $q \in \mathcal{C}^0([-T, T])$. Moreover, A_β being symmetric with respect to $\mathbb{H}^2 \times \{0\}$, the function q is even.

Since a Jacobi function is 2π-periodic in θ, using the Fourier decomposition, we reduce the problem to solving a family (D_n) of Dirichlet problems on $\mathcal{C}^2([-T, T])$ for $n \in \mathbb{N}$:

$$\begin{cases} u'' + (q(s) - n^2)u = 0 \\ u(-T) = u(T) = 0 \end{cases} . \qquad (\mathrm{D}_n)$$

We make two immediate observations:

- Considering a solution of (D_n) for any $n \in \mathbb{N}$, its odd and even parts are also solutions of (D_n). Hence, we only have to consider odd and even solutions.
- The vertical component φ of the unit normal to A_β is an odd solution of (D_0) which does not vanish on $(0, T)$.

Let $n \in \mathbb{N}$. *An odd solution of* (D_n) *is proportional to* φ. Otherwise, using Sturm comparison theorem with $q - n^2 \leq q$, φ should vanish once in $(0, T)$. *There is no even solution to* (D_n). Suppose such a function exist. Using Sturm comparison theorem, this function vanishes nowhere in $(-T, T)$, which means n^2 is the first eigenvalue of the elliptic operator:

$$\frac{d^2}{ds^2} + q(s),$$

which contradicts the existence of φ. $\square$

5.2 Deformations of annuli

Consider a β-*deformable* CMC-1/2 annulus A i.e. a surface such that:

- A admits X^b, with $b \in \mathcal{C}^{2,\alpha}(\overline{\Omega^\beta})$, as a cylindrical parametrization;
- A is non degenerate;
- the values at infinity are the couple $\gamma^b = (\gamma^b_+, \gamma^b_-) = b|_{\partial\Omega^\beta}$ satisfying the condition:
$$|e^{-\gamma^b_+}|_{L^2(\mathbb{S}^1)} = |e^{-\gamma^b_-}|_{L^2(\mathbb{S}^1)}.$$

Again, the vertical component φ^b of the unit normal to A reads:

$$\varphi^b = \varepsilon \frac{e^{-b}}{h_\beta(r)} \frac{1}{c^b} \quad \text{with} \quad c^b|_{\partial\Omega^\beta} = \frac{1}{2},$$

and we use a similar definition to Section 3 for the compactified mean curvature operator:

$$\overline{H} : \xi \in \mathcal{C}^{2,\alpha}(\overline{\Omega^\beta}) \mapsto \sqrt{|g(b)|}\left(H(b + 2c^b\xi) - \frac{1}{2}\right) \in \mathcal{C}^{0,\alpha}(\overline{\Omega^\beta}).$$

The compactified Jacobi operator is still $\overline{L} = D\overline{H}(0)$, $\overline{L}_0$ is its restriction to $\mathcal{C}^{2,\alpha}_0(\overline{\Omega^\beta})$ and $K, K^\perp, K^\perp_0$ are defined as before. The non degeneracy hypothesis on A means $\ker \overline{L}_0 = \mathbb{R}\varphi^b$.

Again, define $\mu_b : (\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2 \to \mathcal{C}^{2,\alpha}(\overline{\Omega^\beta})$ to be the harmonic function on $\overline{\Omega^\beta}$ with values $\gamma - \gamma^b$ on $\partial\Omega^\beta$.

The compactified Jacobi operator satisfies a Green identity similar to Proposition 3.2 for entire graphs:

Proposition 5.3 (Green identity). *For any $u, v \in \mathcal{C}^{2,\alpha}(\overline{\Omega^\beta})$, the compactified Jacobi operator satisfies the following identity:*

$$\int_{\overline{\Omega^\beta}} \left(u\overline{L}v - v\overline{L}u\right) d\overline{A} = \sqrt{\beta}\int_0^{2\pi} e^{-\gamma^b_+} \left(u\frac{\partial v}{\partial s} - v\frac{\partial u}{\partial s}\right)\Big|_{s=T} d\theta - \sqrt{\beta}\int_0^{2\pi} e^{-\gamma^b_-} \left(u\frac{\partial v}{\partial s} - v\frac{\partial u}{\partial s}\right)\Big|_{s=-T} d\theta,$$

with $d\overline{A}$ the Lebesgue measure on $\overline{\Omega^\beta}$.

And we also have the equivalent of Corollary 3.3:

Corollary 5.4. *There is no solution* $u \in \mathcal{C}^{2,\alpha}(\overline{\Omega^\beta})$ *to the equation:*

$$\begin{cases} \overline{L}u = 0 & \text{on } \overline{\Omega^\beta} \\ u|_{\partial\Omega^\beta} = (1,-1) \end{cases}.$$

As in Section 3.1, let Π_K and $\Pi_{K^\perp}$ be the orthogonal projections on K and $K^\perp$. Lemma 3.4 still holds:

Lemma 5.5. *Consider the map* $\Phi : (\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2 \times \mathbb{R} \times K_0^\perp \to K^\perp$ *defined by:*

$$\Phi(\gamma, \lambda, \sigma) = \Pi_{K^\perp} \circ \overline{H}\big(\mu_b(\gamma) + \lambda\varphi^b + \sigma\big).$$

Then $D_3\Phi(\gamma^b, 0, 0) : K_0^\perp \to K^\perp$ *is an isomorphism.*

We can apply again the implicit function theorem to Φ, which states that there exist an open neighborhood U_b of $(\gamma^b, 0)$ in $(\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2 \times \mathbb{R}$ and a unique smooth map $\sigma : U_b \to K_0^\perp$ such that:

$$\forall(\gamma, \lambda) \in U_b,\ \Phi\big(\gamma, \lambda, \sigma(\gamma, \lambda)\big) = 0.$$

We define similarly the smooth maps $\xi_b : U_b \to \mathcal{C}^{2,\alpha}(\overline{\Omega^\beta})$, $\eta_b : U_b \to \mathcal{C}^{2,\alpha}(\overline{\Omega^\beta})$ and $\kappa_b : U_b \to K$ by:

$$\xi_b(\gamma, \lambda) = \mu_b(\gamma) + \lambda\varphi^b + \sigma(\gamma, \lambda), \quad \eta_b(\gamma, \lambda) = b + 2c^b\xi_b(\gamma, \lambda)$$
$$\text{and} \quad \kappa_b(\gamma, \lambda) = \Pi_K \circ \overline{H}\big(\xi_b(\gamma, \lambda)\big).$$

Also, if an annulus, defined on Ω^β, admits $X^{\eta_b(\gamma,\lambda)}$ as a parametrization, we say that $\{\gamma, \lambda\}$ are the *data* of the annulus with respect to A or to b.

Properties of ξ_b and η_b are similar to those of ξ_a and η_a in Section 3.1:

Lemma 5.6. *The maps* η_b *and* ξ_b *have the following properties:*

1. $\xi_b(\gamma^a, 0) = 0$ *and* $\eta_b(\gamma^b, 0) = b$.
2. $\forall(\gamma, \lambda) \in U_b,\ \eta_b(\gamma, \lambda)|_{\partial\Omega^\beta} = \gamma$.
3. $D_2\xi_b(\gamma^b, 0) : \lambda \in \mathbb{R} \mapsto \lambda\varphi^b \in \mathcal{C}^{2,\alpha}(\overline{\Omega^\beta})$.

Consider A, A' β-deformable annuli admitting respectively $X^b, X^{b'}$ as cylindrical parametrizations and suppose there exist an annulus with data

$\{\gamma, \lambda\}$ and $\{\gamma', \lambda'\}$ with respect to A and A' respectively. Therefore, this surface can be described as $X^{\eta_b(\gamma,\lambda)}$ and $X^{\eta_{b'}(\gamma',\lambda')}$ and we get:

$$\gamma' = \gamma \quad \text{and} \quad \lambda' = \frac{1}{|\varphi^{b'}|^2_{L^2(\Omega^\beta)}} \left\langle \frac{\eta_b(\gamma,\lambda) - b'}{2c^{b'}} - \mu_{b'}(\gamma), \varphi^{b'} \right\rangle_{L^2(\Omega^\beta)}.$$

Lemma 5.6 Point 2 shows that the values at infinity are still independent from the parameter λ, and the meaning of the parameter λ is the same as in the case of entire graphs:

Proposition 5.7. *Let $(\gamma, \lambda) \in U_b$. The surface $X^{\eta_b(\gamma,\lambda')}$ exists for any $\lambda' \in \mathbb{R}$ and coincides with $X^{\eta_b(\gamma,\lambda)}$ up to a vertical translation.*

We are now interested in deformations $X^{\eta_b(\gamma,\lambda)}$ of the annulus A that are CMC-1/2, which means deformations such that $\kappa_b(\gamma, \lambda) = 0$. Consider $\mathcal{U}_b = \kappa_b^{-1}(\{0\}) \cap U_b$. Again, using Proposition 5.7, we can take $\mathcal{U}_b = \Gamma_b \times \mathbb{R}$ with Γ_b a connected subset of $(\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2$.

Proposition 5.8. *The set Γ_b is a codimension 1 smooth submanifold of $(\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2$ which is a subset of:*

$$\left\{ (\gamma_+, \gamma_-) \in (\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2 \,\middle|\, |e^{-\gamma_+}|_{L^2(\mathbb{S}^1)} = |e^{-\gamma_-}|_{L^2(\mathbb{S}^1)} \right\}.$$

Proof. As in Proposition 3.8, if κ_b is a submersion is a submersion at $(\gamma^b, 0)$, then it is a submersion in a neighborhood of $(\gamma^b, 0)$ in $(\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2 \times \mathbb{R}$ and, up to a restriction, Γ_b is a smooth submanifold of $(\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2$ of codimension 1. Again $D_2\kappa_b(\gamma^b, 0) = 0$ since:

$$\begin{aligned} D_2\kappa_b(\gamma^b, 0) \cdot 1 &= \left.\frac{d}{dt}\right|_{t=0} \kappa_b(\gamma^b, t) = \left.\frac{d}{dt}\right|_{t=0} \overline{H}(\xi_b(\gamma^b, t)) \\ &= \overline{L}(D_2\xi_b(\gamma^b, 0) \cdot 1) = \overline{L}_0(\varphi^b) = 0, \end{aligned}$$

with $\varphi^b \in K$. Consider $\gamma = (1, -1) \in (\mathcal{C}^{2,\alpha}(\mathbb{S}^1))^2$ and compute:

$$\begin{aligned} D_1\kappa_b(\gamma^b, 0) \cdot \gamma &= \left.\frac{d}{dt}\right|_{t=0} \kappa_b(\gamma^b + t\gamma, 0) = \left.\frac{d}{dt}\right|_{t=0} \overline{H}(\xi_b(\gamma^b + t\gamma, 0)) \\ &= \overline{L}(D_1\xi_b(\gamma^b, 0) \cdot \gamma). \end{aligned}$$

Lemma 5.6 Point 2 implies $(D_1\xi_b(\gamma^b, 0) \cdot \gamma)|_{\partial\mathbb{D}} = (1, -1)$ and using Corollary 5.4, we know that $D_1\kappa_b(\gamma^b, 0) \cdot (1, -1)$ is not identically zero.

Consider a smooth path $\gamma_t = ((\gamma_+)_t, (\gamma_-)_t)$ in Γ_b with $\gamma_0 = \gamma^b$ and tangent vector at t $\dot\gamma_t = (\dot{(\gamma_+)}_t, \dot{(\gamma_-)}_t)$. Note that similarly:

$$0 = D\kappa_b(\gamma^b, 0) \cdot (\dot\gamma_0, 0) = \left.\frac{d}{dt}\right|_{t=0} \kappa_b(\gamma_t, 0) = \overline{L}(D_1\xi_b(\gamma^b, 0) \cdot (\dot\gamma_0, 0)).$$

Denote $v = D_1\xi_b(\gamma^b, 0) \cdot (\dot\gamma_0, 0) \in \ker \overline{L}$. Knowing that:

$$\varphi^b|_{s=T} = \varphi^b|_{s=-T} = 0, \quad \left.\frac{\partial\varphi^b}{\partial s}\right|_{s=T} = -e^{-\gamma_+^b}, \quad \left.\frac{\partial\varphi^b}{\partial s}\right|_{s=-T} = -e^{-\gamma_-^b},$$
$$v|_{s=T} = \dot{(\gamma_+)}_0 \quad \text{and} \quad v|_{s=-T} = \dot{(\gamma_-)}_0,$$

apply Proposition 5.3 to φ^b and v:

$$\begin{aligned} 0 = \int_{\Omega^\beta} (\varphi^b \overline{L} v - v \overline{L} \varphi^b) d\overline{A} &= \sqrt{\beta} \int_0^{2\pi} \dot{(\gamma_+)}_0 e^{-2\gamma_+^b} d\theta - \sqrt{\beta} \int_0^{2\pi} \dot{(\gamma_-)}_0 e^{-2\gamma_-^b} d\theta \\ &= 2\pi\sqrt{\beta} \left(\left\langle \dot{(\gamma_+)}_0, e^{-2\gamma_+^b} \right\rangle_{L^2(\mathbb{S}^1)} - \left\langle \dot{(\gamma_-)}_0, e^{-2\gamma_-^b} \right\rangle_{L^2(\mathbb{S}^1)} \right). \end{aligned} \tag{13}$$

For a fixed t, consider the reparametrized path $\gamma'_s = \gamma_{s+t}$ and denote $b' = \eta_b(\gamma_t, 0)$. There exists $\varepsilon > 0$ such that $\gamma'_s \in \Gamma_{b'}$ for any $|s| < \varepsilon$. Hence, the path of surfaces $X^{\eta_b(\gamma'_s, 0)}$ can be described by a path of data $\{\gamma'_s, \lambda'_s\}$ in $\mathcal{U}_{b'}$, $|s| < \varepsilon$, with $\lambda'_0 = 0$ and tangent vector $\dot{\gamma'}_0 = \dot\gamma_t$ at $s = 0$. The result (10) applies to (γ'_s, λ'_s) i.e.:

$$\begin{aligned} &\frac{d}{dt}\left(|e^{-(\gamma_+)_t}|^2_{L^2(\mathbb{S}^1)} - |e^{-(\gamma_-)_t}|^2_{L^2(\mathbb{S}^1)} \right) \\ &\qquad = \left\langle \dot{(\gamma_+)}_t, e^{-2(\gamma_+)_t} \right\rangle_{L^2(\mathbb{S}^1)} - \left\langle \dot{(\gamma_-)}_t, e^{-2(\gamma_-)_t} \right\rangle_{L^2(\mathbb{S}^1)} = 0, \end{aligned}$$

for any t, and thus:

$$|e^{-(\gamma_+)_t}|^2_{L^2(\mathbb{S}^1)} - |e^{-(\gamma_-)_t}|^2_{L^2(\mathbb{S}^1)} = |e^{-\gamma_+^b}|^2_{L^2(\mathbb{S}^1)} - |e^{-\gamma_-^b}|^2_{L^2(\mathbb{S}^1)} = 0,$$

since the annulus A is β-deformable. □

The condition on the values at infinity defining Γ_b is indeed the conservation of the vertical flux in the deformed annuli.

5.3 Annuli with non aligned ends

For minimal surfaces in $\mathbb{R}^3$, one can define two Nœther vector-invariants associated to isometries, namely the flux — associated to translations — and the torque — associated to rotations. In the case of a minimal catenoidal end with growth $\alpha > 0$ and vertical axis $\{x_1 = u, x_2 = v\}$, the flux and the torque are respectively $(0, 0, 2\pi\alpha)$ and $2\pi\alpha(v, -u, 0)$. In other words, the growth and the position of the axis of the end are determined by the vertical component of the flux and horizontal components of the torque.

In $\mathbb{H}^2 \times \mathbb{R}$, Nœther invariants are constructed similarly but the torque is not a vector anymore, since remain only rotations around vertical axis. In the case of a vertical rotational end with parameter $\beta > 0$, the flux is vertical with third component β and the torque is always zero, no matter where the rotation axis is situated. The fact the position of the axis is no longer caught by Nœther invariants, indicates that the construction of CMC-1/2 annuli with vertical ends should be more flexible regarding the relative positions of the axis of the ends.

Theorem 5.9. *There exist CMC-1/2 annuli in $\mathbb{H}^2 \times \mathbb{R}$ with vertical ends, that are asymptotic — regarding the horizontal hyperbolic distance — to rotational examples with different vertical axis.*

Proof. Fix $\beta > 0$, $\beta \neq 1$. From Proposition 2.3, we know that, in the Poincaré disk model (2), a horizontal translation of $w_\varepsilon = \varepsilon e^{i\theta_0} \in \mathbb{D}^*$ changes the top value at infinity of the rotational annulus A_β into:

$$\gamma_\varepsilon(\theta) = \log\left(\frac{|1 - \varepsilon e^{i(\theta-\theta_0)}|}{\sqrt{1-\varepsilon^2}}\right).$$

A direct computation shows:

$$|e^{-\gamma_\varepsilon}|_{L^2(\mathbb{S}^1)} = 1 \quad \text{and} \quad |\gamma_\varepsilon|_{\mathcal{C}^{2,\alpha}(\mathbb{S}^1)} \leq C\varepsilon \quad \text{with} \quad C \in \mathbb{R}_+^*.$$

Thus, for ε sufficiently small, we have $\big((\gamma_\varepsilon, 0), 0\big) \in \mathcal{U}_0$ and the CMC-1/2 annulus $X^{\eta_0\left((\gamma_\varepsilon,0),0\right)}$ exists.

Moreover, the top end of $X^{\eta_0\left((\gamma_\varepsilon,0),0\right)}$ is asymptotic to the top end of the image of S_0 under the horizontal translation by w_ε — since it has the same value at infinity — and is therefore asymptotically rotational. Similarly, the bottom end of $X^{\eta_0\left((\gamma_\varepsilon,0),0\right)}$ is asymptotically rotational, being asymptotic to the bottom end of S_0.

And finally, the ends of $X^{\eta_0\left((\gamma_\varepsilon,0),0\right)}$ are not aligned since the axis of the top end is $\{w_\varepsilon\} \times \mathbb{R}$ and the one of the bottom end is $\{0\} \times \mathbb{R}$. □

Remark 5.10. In the proof of Theorem 5.9, we see that the ends of the constructed annulus are asymptotic to *the same* rotational example, up to isometries. This is indeed a necessary condition since the ends have to preserve the vertical flux, which is determined by the parameter β of the rotational annulus — namely, the vertical flux of the annulus A_β is $2\pi(1-\beta)$.

6 Appendix: Compactification of the mean curvature

Consider the product metric $\sigma + dx_3^2$ on $\mathbb{D}\times\mathbb{R}$ where:

$$\sigma = F^* ds_P^2 \quad \text{and} \quad F : (r,\theta) \in \mathbb{D} \mapsto \frac{2r}{1+r^2} e^{i\theta} \in \mathbb{H}^2,$$

in the Poincaré disk model (2). To ease the notations, we use indices 1, 2 for quantities respectively related to coordinates r, θ on $\mathbb{D}$. In matrix terms, the metric is $\sigma = (\sigma_{ij})$ with:

$$\sigma_{11} = \frac{16}{(1-r^2)^2}, \quad \sigma_{12} = \sigma_{21} = 0, \quad \sigma_{22} = \frac{16r^2(1+r^2)^2}{(1-r^2)^4}$$
$$\text{and} \quad |\sigma| = \left(\frac{16r(1+r^2)}{(1-r^2)^3}\right)^2.$$

The Christoffel symbols (Γ_{ij}^k) associated to σ for the Levi-Civita connection verify:

$$\Gamma_{ij}^k = \frac{1}{2}\sum_m \sigma^{km}\left(\partial_i\sigma_{jm} + \partial_j\sigma_{im} - \partial_m\sigma_{ij}\right),$$

which means:

$$\Gamma_{11}^1 = \frac{2r}{1-r^2}, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1+6r^2+r^4}{r(1+r^2)(1-r^2)}$$
$$\text{and} \quad \Gamma_{22}^1 = -\frac{r(1+r^2)(1+6r^2+r^4)}{(1-r^2)^3},$$

the other terms being zero.

Fix $\Omega \in \mathcal{D}$. A surface in $S \in \mathcal{E}$ defined on Ω with graph coordinates at infinity:

$$(r,\theta) \in \Omega \mapsto \big(F(r,\theta), h(\eta)\big) \quad \text{with} \quad \eta \in \mathcal{C}^{2,\alpha}(\overline{\Omega}) \quad \text{and} \quad h(\eta) = 2e^\eta \frac{1+r^2}{1-r^2},$$

can be reparametrized as the actual graph of the function $h(\eta) : \Omega \to \mathbb{R}$ in $\mathbb{D} \times \mathbb{R}$ endowed with metric $\sigma + dx_3^2$. As shown by J. Spruck [12], the metric $g(\eta) = (g_{ij}(\eta))$ induced by $h(\eta)$ is given by:

$$g_{ij}(\eta) = \sigma_{ij} + \partial_i h(\eta) \partial_j h(\eta),$$

and denoting $\eta_i = \partial_i \eta$, for $i = 1, 2$, we obtain:

$$g_{11}(\eta) = \frac{16(1+r^2)^2 e^{2\eta}}{(1-r^2)^4} \left[1 + \frac{2r\eta_1}{1+r^2}(1-r^2) + \left(\frac{\eta_1^2}{4} + \frac{e^{-2\eta}-1}{(1+r^2)^2} \right)(1-r^2)^2 \right],$$

$$g_{12}(\eta) = \frac{8(1+r^2)\eta_2 e^{\eta}}{(1-r^2)^3} \left[\frac{2r}{1+r^2} + \frac{\eta_1}{2}(1-r^2) \right]$$

$$\text{and} \quad g_{22}(\eta) = \frac{16r^2(1+r^2)^2}{(1-r^2)^4} \left[1 + \frac{\eta_2^2 e^{2\eta}}{4r^2}(1-r^2)^2 \right].$$

The determinant $|g(\eta)|$ of the induced metric is:

$$|g(\eta)| = \left(\frac{16r(1+r^2)^2 e^{\eta}}{(1-r^2)^4} \right)^2 w^2(\eta)$$

with $w(\eta)$ denoting:

$$w(\eta) = \left[1 + \frac{2r\eta_1}{1+r^2}(1-r^2) + \left(\frac{\eta_1^2}{4} + \frac{e^{-2\eta}-1}{(1+r^2)^2} \right)(1-r^2)^2 + \frac{\eta_2^2}{4r^2(1+r^2)^2}(1-r^2)^4 \right]^{1/2}.$$

In the metric $\sigma + dx_3^2$, the mean curvature $H(\eta)$ of S can be expressed as:

$$H(\eta) = \frac{1}{2} \operatorname{div}_\sigma \left(\frac{\nabla_\sigma h(\eta)}{W(\eta)} \right) = \frac{1}{2W(\eta)} \sum_{i,j} g^{ij}(\eta) \left(\partial_{ij} h(\eta) - \sum_k \Gamma_{ij}^k \partial_k h(\eta) \right)$$

$$\text{with} \quad W(\eta) = \sqrt{1 + |\nabla_\sigma h(\eta)|_\sigma^2} = \frac{(1+r^2)e^{\eta}}{1-r^2} w(\eta),$$

where the quantities are computed with respect to the metric σ on $\mathbb{D}$, and $g^{-1}(\eta) = (g^{ij}(\eta))$.

In order to ease the notations, denote:

$$H_{ij}(\eta) = g^{ij}(\eta) \left(\partial_{ij} h(\eta) - \sum_k \Gamma_{ij}^k \partial_k h(\eta) \right).$$

For $H_{11}(\eta)$, compute:

$$\begin{aligned}H_{11}(\eta) &= g^{11}(\eta)\left(\partial_{11}h(\eta) - \Gamma^1_{11}\partial_1 h(\eta)\right)\\ &= \frac{W(\eta)}{w^3(\eta)}e^{-2\eta}(1-r^2)^2\left[\frac{1}{2(1+r^2)^2} + \frac{r\eta_1}{2(1+r^2)^3}(1-r^2)\right.\\ &\quad \left. + R_{11}(1-r^2)^2\right] + 2W(\eta)\frac{g^{11}(\eta)}{w(\eta)}\eta_{11},\end{aligned}$$

with $R_{11} = R_{11}(r, \eta, D\eta)$ defined on $\Omega \cup \partial\mathbb{D}$, identically zero if $\eta = 0$ and real-analytic in its variables. For $H_{12}(\eta)$:

$$\begin{aligned}H_{12}(\eta) &= g^{12}(\eta)\left(\partial_{12}h(\eta) - \Gamma^2_{12}\partial_2 h(\eta)\right)\\ &= \frac{W(\eta)}{w^3(\eta)}R_{12}(1-r^2)^4 + 2W(\eta)\frac{g^{12}(\eta)}{w(\eta)}\eta_{12},\end{aligned}$$

again with $R_{12} = R_{12}(r, \eta, D\eta)$ defined on $\Omega \cup \partial\mathbb{D}$, zero if $\eta = 0$ and real-analytic in its variables. And for $H_{22}(\eta)$:

$$\begin{aligned}H_{22}(\eta) &= g^{22}(\eta)\left(\partial_{22}h(\eta) - \Gamma^1_{22}\partial_1 h(\eta)\right)\\ &= \frac{W(\eta)}{w^3(\eta)}\left[\frac{1+6r^2+r^4}{2(1+r^2)^2} + \frac{(5-10r^2+29r^4)\eta_1}{2r(1+r^2)^3}(1-r^2)\right.\\ &\quad + \frac{4r^2}{(1+r^2)^2}\left(\frac{3\eta_1^2}{4} + \frac{e^{-2\eta}-1}{(1+r^2)^2}\right)(1-r^2)^2 + \frac{\eta_1}{r(1+r^2)}\left(\frac{\eta_1^2}{4}\right.\\ &\quad \left.\left. + \frac{e^{-2\eta}-1}{(1+r^2)^2}\right)(1-r^2)^3 + R_{22}(1-r^2)^4\right] + 2W(\eta)\frac{g^{22}(\eta)}{w(\eta)}\eta_{22},\end{aligned}$$

with $R_{22} = R_{22}(r, \eta, D\eta)$ defined on $\Omega \cup \partial\mathbb{D}$, zero if $\eta = 0$ and real-analytic in its variables. Hence, a Taylor expansion of the mean curvature $H(\eta)$ is:

$$\begin{aligned}H(\eta) &= \frac{1}{w(\eta)}\left(g^{11}(\eta)\eta_{11} + 2g^{12}(\eta)\eta_{12} + g^{22}(\eta)\eta_{22}\right)\\ &\quad + \frac{1}{2w^3(\eta)}\left[1 + \frac{3r\eta_1}{1+r^2}(1-r^2) + \frac{6r^2}{(1+r^2)^2}\left(\frac{\eta_1^2}{2}\right.\right.\\ &\quad \left. + \frac{e^{-2\eta}-1}{(1+r^2)^2}\right)(1-r^2)^2 + \frac{\eta_1}{2r(1+r^2)}\left(\frac{\eta_1^2}{2}\right.\\ &\quad \left.\left. + \frac{3(e^{-2\eta}-1)}{(1+r^2)^2}\right)(1-r^2)^3\right] + R_H(1-r^2)^4,\end{aligned}$$

with as before $R_H = R_H(r, \eta, D\eta)$ defined on $\Omega \cup \partial\mathbb{D}$, identically zero if $\eta = 0$ and real-analytic in its variables.

The Taylor expansion of $w^{-3}(\eta)$ is the following:

$$\frac{1}{w^3(\eta)} = 1 - \frac{3r\eta_1}{1+r^2}(1-r^2) - \frac{3}{2(1+r^2)}\left(4r^2\eta_1^2 + (e^{-2\eta}-1)\right)(1-r^2)^2 \\ - \frac{5r\eta_1}{(1+r^2)^3}\left(2r^2\eta_1^2 - \frac{3(e^{-2\eta}-1)}{2}\right)(1-r^2)^3 + R_w(1-r^2)^4,$$

with $R_w = R_w(r, \eta, D\eta)$ defined on $\Omega \cup \partial\mathbb{D}$, zero if $\eta = 0$ and real-analytic in its variables. Finally, we obtain:

$$H(\eta) = \frac{1}{2} + \frac{1}{w(\eta)}\left(g^{11}(\eta)\eta_{11} + 2g^{12}(\eta)\eta_{12} + g^{22}(\eta)\eta_{22}\right) + R(1-r^2)^4, \quad (14)$$

with $R = R(r, \eta, D\eta)$ defined on $\Omega \cup \partial\mathbb{D}$, identically zero if $\eta = 0$ and real-analytic in its variables.

Taking $\eta = a + \xi$ with $a, \xi \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$, the Taylor expansion (14) reads:

$$H(a+\xi) = H(a) + \frac{1}{\sqrt{|g(a)|}}\sum_{i,j} A_{ij}\xi_{ij} + \frac{1}{\sqrt{|g(a)|}}B,$$

$$\begin{aligned}
\text{with}\quad A_{11} &= \frac{1}{w(a+\xi)}\sqrt{|g(a)|}g^{11}(a+\xi) = \frac{1}{w(a+\xi)}\frac{g_{22}(a+\xi)}{\sqrt{|g(a)|}} \\
&= e^{-a} + O(1-r^2), \\
A_{12} &= \frac{1}{w(a+\xi)}\sqrt{|g(a)|}g^{12}(a+\xi) = -\frac{1}{w(a+\xi)}\frac{g_{12}(a+\xi)}{\sqrt{|g(a)|}} \\
&= O(1-r^2) \\
\text{and}\quad A_{22} &= \frac{1}{w(a+\xi)}\sqrt{|g(a)|}g^{22}(a+\xi) = \frac{1}{w(a+\xi)}\frac{g_{11}(a+\xi)}{\sqrt{|g(a)|}} \\
&= e^{a} + O(1-r^2).
\end{aligned}$$

Moreover $A_{ij} = A_{ij}(r, a, \xi, D\xi)$ and $B = B(r, a, \xi, D\xi)$ are defined on $\Omega \cup \partial\mathbb{D}$ and real-analytic in their variables, the matrix $A = (A_{ij})$ is coercive on $\Omega \cup \partial\mathbb{D}$, and B is identically zero if $\xi = 0$.

References

[1] H. I. Choi and A. E. Treibergs, *Gauss map of spacelike constant mean curvature hypersurfaces in Minkowski space*, J. Differential Geom. **32** (1990), no. 3, 775–817.

[2] B. Daniel, W. H. Meeks III, and H. Rosenberg, *Half-space theorems for minimal surfaces in Nil_3 and Sol_3*, J. Differential Geom. **88** (2011), no. 1, 41–59.

[3] R. Sa Earp and E. Toubiana, *Screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$*, Illinois J. Math. **49** (2005), no. 4, 1323–1362 (electronic).

[4] M. F. Elbert, B. Nelli, and R. Sa Earp, *Existence of vertical ends of mean curvature $1/2$ in $\mathbb{H}^2 \times \mathbb{R}$*, Trans. Amer. Math. Soc. **364** (2012), no. 3, 1179–1191.

[5] I. Fernández and P. Mira, *Harmonic maps and constant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$*, Amer. J. Math. **129** (2007), no. 4, 1145–1181.

[6] D. Fischer-Colbrie and R. Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math. **33** (1980), no. 2, 199–211.

[7] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, 2001, Reprint of the 1998 edition.

[8] L. Hauswirth, H. Rosenberg, and J. Spruck, *On complete mean curvature $1/2$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$*, Comm. Anal. Geom. **16** (2008), no. 5, 989–1005.

[9] L. Mazet, *Cylindrically bounded constant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$*, preprint arXiv:1203.2746v1, 2012.

[10] B. Nelli and R. Sa Earp, *A halfspace theorem for mean curvature $h = 1/2$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$*, J. Math. Anal. Appl. **365** (2010), no. 1, 167–170.

[11] J. Pérez and A. Ros, *The space of properly embedded minimal surfaces with finite total curvature*, Indiana Univ. Math. J. **45** (1996), 177–204.

[12] J. Spruck, *Interior gradient estimates and existence theorems for constant mean curvature graphs in $m^n \times \mathbb{R}$*, Pure Appl. Math. Q. **3** (2007), no. 3, Special Issue: In honor of Leon Simon. Part 2, 785–800.

[13] A. E. Treibergs, *Entire spacelike hypersurfaces of constant mean curvature in Minkowski space*, Invent. Math. **66** (1982), no. 1, 39–56.

[14] B. White, *The space of m-dimensional surfaces that are stationary for a parametric elliptic functional*, Indiana Univ. Math. J. **36** (1987), 567–603.

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