On doubly periodic minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with finite total curvature in the quotient space

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Abstract

In this paper we develop the theory of properly immersed minimal surfaces in the quotient space $\mathbb{H}^2 \times \mathbb{R}/G$, where G is a subgroup of isometries generated by a vertical translation and a horizontal isometry in $\mathbb{H}^2$ without fixed points. The horizontal isometry can be either a parabolic translation along horocycles in $\mathbb{H}^2$ or a hyperbolic translation along a geodesic in $\mathbb{H}^2$. In fact, we prove that if a properly immersed minimal surface in $\mathbb{H}^2 \times \mathbb{R}/G$ has finite total curvature then its total curvature is a multiple of 2π, and moreover, we understand the geometry of the ends. These theorems hold true more generally for properly immersed minimal surfaces in $M \times \mathbb{S}^1$, where M is a hyperbolic surface with finite topology whose ends are isometric to one of the ends of the above spaces $\mathbb{H}^2 \times \mathbb{R}/G$.

1 Introduction

Among all the minimal surfaces in $\mathbb{R}^3$, the ones of finite total curvature are the best known. In fact, if a minimal surface in $\mathbb{R}^3$ has finite total curvature then this minimal surface is either a plane or its total curvature is a non-zero multiple of 2π. Moreover, if the total curvature is -4π, then the minimal surface is either the Catenoid or the Enneper's surface [16].

In 2010, the first author jointly with Harold Rosenberg [10] developed the theory of complete embedded minimal surfaces of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$. In that work they proved that the total curvature of such surfaces must be a multiple of 2π, and they gave simply connected

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examples whose total curvature is $-2\pi m$, for each nonnegative integer m.

In the last few years, many people have worked on this subject and classified some minimal surfaces of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$ (see [8, 9, 15, 20]).

In [15] Morabito and Rodríguez constructed for $k \geq 2$ a $(2k-2)$-parameter family of properly embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ invariant by a vertical translation which have total curvature $4\pi(1-k)$, genus zero and $2k$ vertical Scherk-type ends in the quotient by the vertical translation. Moreover, independently, Morabito and Rodríguez [15] and Pyo [17] constructed for $k \geq 2$ examples of properly embedded minimal surfaces with total curvature $4\pi(1-k)$, genus zero and k ends, each one asymptotic to a vertical plane. In particular, we have examples of minimal annuli with total curvature -4π.

It was expected that each end of a complete embedded minimal surface of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$ was asymptotic to either a vertical plane or a Scherk graph over an ideal polygonal domain. However in [18], Pyo and Rodríguez constructed new simply-connected examples of minimal surfaces of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$, showing this is not the case.

In this work we consider $\mathbb{H}^2 \times \mathbb{R}$ quotiented by a subgroup of isometries $G \subset \mathrm{Isom}(\mathbb{H}^2 \times \mathbb{R})$ generated by a horizontal isometry in $\mathbb{H}^2$ without fixed points, ψ, and a vertical translation, $T(h)$, for some $h > 0$. The isometry ψ can be either a parabolic translation along horocycles in $\mathbb{H}^2$ or a hyperbolic translation along a geodesic in $\mathbb{H}^2$. We prove that if a properly immersed minimal surface in $\mathbb{H}^2 \times \mathbb{R}/G$ has finite total curvature then its total curvature is a multiple of 2π, and moreover, we understand the geometry of the ends. More precisely, we prove that each end of a properly immersed minimal surface of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}/G$ is asymptotic to either a horizontal slice, or a vertical geodesic plane or the quotient of a *Helicoidal plane*. Where by *Helicoidal plane* we mean a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ which is parametrized by $X(x, y) = (x, y, ax + b)$ when we consider the halfplane model for $\mathbb{H}^2$.

Let us mention that these results hold true for properly immersed minimal surfaces in $M \times \mathbb{S}^1$, where M is a hyperbolic surface $(K_M = -1)$ with finite topology whose ends are either isometric to $\mathcal{M}_+$ or $\mathcal{M}_-$, which we define in the next section.

2 Preliminaries

Unless otherwise stated, we use the Poincaré disk model for the hyperbolic plane, that is

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

with the hyperbolic metric $g_{-1} = \sigma g_0 = \frac{4}{(1-x^2-y^2)^2} g_0$, where g_0 is the Euclidean metric in $\mathbb{R}^2$. In this model, the asymptotic boundary $\partial_\infty \mathbb{H}^2$ of $\mathbb{H}^2$ is identified with the unit circle and we denote by p_o the point $(1, 0) \in \partial_\infty \mathbb{H}^2$.

We write $\overline{pq}$ to denote the geodesic arc between the two points p, q.

We consider the quotient spaces $\mathbb{H}^2 \times \mathbb{R}/G$, where G is a subgroup of $\mathrm{Isom}(\mathbb{H}^2 \times \mathbb{R})$ generated by a horizontal isometry on $\mathbb{H}^2$ without fixed points, ψ, and a vertical translation, $T(h)$, for some $h > 0$. The horizontal isometry ψ can be either a horizontal translation along horocycles in $\mathbb{H}^2$ or a horizontal translation along a geodesic in $\mathbb{H}^2$.

Let us analyse each one of these cases for ψ.

Consider any geodesic γ that limits to p_o at infinity parametrized by arc length. Let $c(s)$ be the horocycles in $\mathbb{H}^2$ tangent to p_o at infinity that intersects γ at $\gamma(s)$ and write $d(s)$ to denote the horocylinder $c(s) \times \mathbb{R}$ in $\mathbb{H}^2 \times \mathbb{R}$. Taking two points $p, q \in c(s)$, let $\psi : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2 \times \mathbb{R}$ be the parabolic translation along $d(s)$ such that $\psi(p) = q$. We have $\psi(d(s)) = d(s)$ for all s. If $G = [\psi, T(h)]$, then the manifold $\mathcal{M}$ which is the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by G is diffeomorphic to $\mathbb{T}^2 \times \mathbb{R}$, where $\mathbb{T}^2$ is the 2-torus. Moreover, $\mathcal{M}$ is foliated by the family of tori $\mathbb{T}(s) = d(s)/G$, which are intrinsically flat and have constant mean cuvature $\frac{1}{2}$. (See Figure 1).

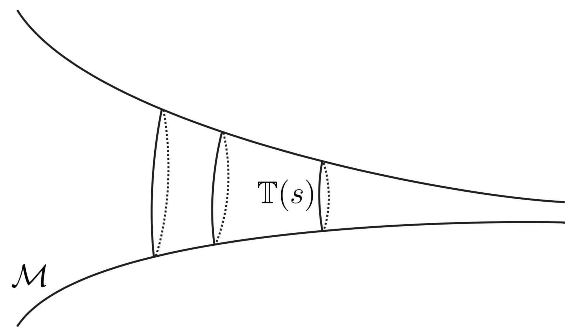

Figure 1: $\mathcal{M} = \mathbb{H}^2 \times \mathbb{R}/[\psi, T(h)]$, where ψ is a parabolic isometry.

Now take a geodesic γ in $\mathbb{H}^2$ and consider $c(s)$ the family of equidistant curves to γ, with $c(0) = \gamma$. Write $d(s)$ to denote the plane $c(s) \times \mathbb{R}$ in $\mathbb{H}^2 \times \mathbb{R}$. Given two points $p, q \in c(s)$, let $\psi : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2 \times \mathbb{R}$ be the hyperbolic translation along γ such that $\psi(p) = q$. We have $\psi(d(s)) =$

$d(s)$ for all s. If $G = [\psi, T(h)]$, then the manifold $\mathcal{M}$ which is the quotient of $\mathbb{H}^2 \times \mathbb{R}$ by G is also diffeomorphic to $\mathbb{T}^2 \times \mathbb{R}$ and $\mathcal{M}$ is foliated by the family of tori $\mathbb{T}(s) = d(s)/G$, which are intrinsically flat and have constant mean cuvature $\frac{1}{2}\tanh(s)$. (See Figure 2).

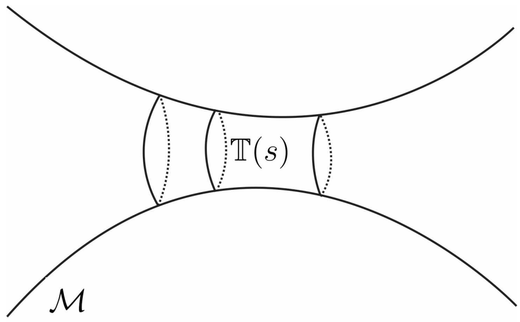

Figure 2: $\mathcal{M} = \mathbb{H}^2 \times \mathbb{R}\,/[\psi, T(h)]$, where ψ is a hyperbolic isometry.

In these quotient spaces we have two different types of ends. One where the injectivity radius goes to zero at infinity, which we denote by $\mathcal{M}_+$, and another one where the injectivity radius is strictly positive, which we denote by $\mathcal{M}_-$.

Hence $\mathcal{M}_+ = \bigcup_{s\geq 0} d(s)/[\psi, T(h)]$, where ψ is a parabolic translation along horocycles, and $\mathcal{M}_- = \bigcup_{s\geq 0} d(s)/[\psi, T(h)]$, for ψ hyperbolic translation along a geodesic in $\mathbb{H}^2$, or $\mathcal{M}_- = \bigcup_{s\leq 0} d(s)/[\psi, T(h)]$, where ψ can be either a parabolic translation along horocycles or a hyperbolic translation along a geodesic in $\mathbb{H}^2$. (See Figure 3).

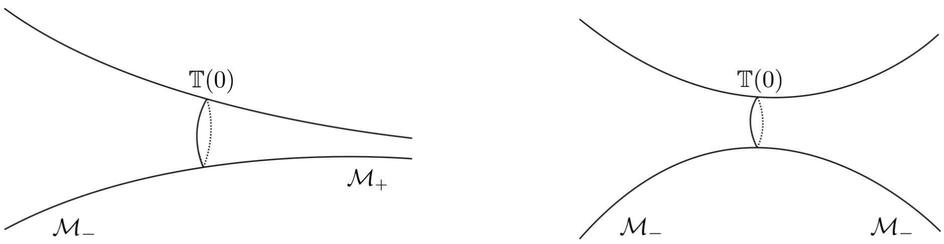

Figure 3: $\mathcal{M}_+$ and $\mathcal{M}_-$.

From now one we will not distinguish between the two quotient spaces above. We will denote both by $\mathcal{M}$.

Let Σ be a Riemannian surface and $X : \Sigma \to \mathcal{M}$ be a minimal immersion. As

$$\mathcal{M} = \mathbb{H}^2 \times \mathbb{R}\,/[\psi, T(h)] \cong \mathbb{H}^2/[\psi] \times \mathbb{S}^1,$$

we can write $X = (F, h) : \Sigma \to \mathbb{H}^2/[\psi] \times \mathbb{S}^1$, where $F : \Sigma \to \mathbb{H}^2/[\psi]$ and $h : \Sigma \to \mathbb{S}^1$ are harmonic maps. We consider local conformal parameters

$z = x + iy$ on Σ. Hence

$$\begin{aligned} |F_x|_\sigma^2 + (h_x)^2 &= |F_y|_\sigma^2 + (h_y)^2 \\ \langle F_x, F_y \rangle_\sigma + h_x . h_y &= 0 \end{aligned} \tag{2.1}$$

and the metric induced by the immersion is given by

$$ds^2 = \lambda^2(z)|dz|^2 = (|F_z|_\sigma + |F_{\bar z}|_\sigma)^2 |dz|^2. \tag{2.2}$$

Considering the universal covering $\pi : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2/[\psi] \times \mathbb{S}^1$ we can take $\widetilde{\Sigma}$, a connected component of the lift of Σ to $\mathbb{H}^2 \times \mathbb{R}$, and we have $\widetilde{X} = (\widetilde{F}, \widetilde{h}) : \widetilde{\Sigma} \to \mathbb{H}^2 \times \mathbb{R}$ such that $\pi(\widetilde{\Sigma}) = \Sigma$ and $\widetilde{F} : \widetilde{\Sigma} \to \mathbb{H}^2, \widetilde{h} : \widetilde{\Sigma} \to \mathbb{R}$ are harmonic maps. We denote by $\widetilde{\partial}_t, \partial_t$ the vertical vector fields in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{H}^2/[\psi] \times \mathbb{S}^1$, respectively. Observe that the functions $n_3 : \Sigma \to \mathbb{R}$, $\tilde n_3 : \widetilde{\Sigma} \to \mathbb{R}$, given by $n_3 = \langle \partial_t, N \rangle, \tilde n_3 = \left\langle \widetilde{\partial}_t, \widetilde{N} \right\rangle$, where $N, \widetilde{N}$ are the unit normal vectors of $\Sigma, \widetilde{\Sigma}$, respectively, satisfy $\tilde n_3 = n_3 \circ \pi$. Then if we define the functions $\omega : \Sigma \to \mathbb{R}, \tilde\omega : \widetilde{\Sigma} \to \mathbb{R}$ so that $\tanh(\omega) = n_3$ and $\tanh(\tilde\omega) = \tilde n_3$, we get $\tilde\omega = \omega \circ \pi$.

As we consider X a conformal minimal immersion, we have

$$n_3 = \frac{|F_z|^2 - |F_{\bar z}|^2}{|F_z|^2 + |F_{\bar z}|^2} \tag{2.3}$$

and

$$\omega = \frac{1}{2} \ln \frac{|F_z|}{|F_{\bar z}|}. \tag{2.4}$$

Note that the same formulae are true for $\tilde n_3$ and $\tilde\omega$.

We know that for local conformal parameters $\tilde z$ on $\widetilde{\Sigma}$, the holomorphic quadratic Hopf differential associated to $\widetilde{F}$, given by

$$\widetilde{Q}(\widetilde{F}) = (\sigma \circ \widetilde{F})^2 \widetilde{F}_{\tilde z} \overline{\widetilde{F}}_{\tilde z} (d\tilde z)^2,$$

can be written as $(\widetilde{h}_{\tilde z})^2 (d\tilde z)^2 = -\widetilde{Q}$. Then, since $\widetilde{h}$ and h differ by a constant in a neighborhood, $(h_z)^2(dz)^2 = -Q$ is also a holomorphic quadratic differential on Σ for local conformal parameters z on Σ. We note Q has two square roots globally defined on Σ. Writing $Q = \phi(dz)^2$, we denote by $\eta = \pm 2i\sqrt{\phi}dz$ a square root of Q, where we choose the sign so that

$$h = \operatorname{Re} \int \eta.$$

Using (2.2), (2.4) and the definition of Q, we have

$$ds^2 = 4(\cosh^2\omega)|Q|. \tag{2.5}$$

As the Jacobi operator of the minimal surface Σ is given by

$$J = \frac{1}{4\cosh^2\omega|\phi|}\left[\Delta_0 - 4|\phi| + \frac{2|\nabla\omega|^2}{\cosh^2\omega}\right]$$

and $Jn_3 = 0$, then

$$\Delta_0\omega = 2\sinh(2\omega)|\phi|, \tag{2.6}$$

where Δ_0 denotes the Laplacian in the Euclidean metric $|dz|^2$, that is, $\Delta_0 = 4\partial^2_{z\bar{z}}$.

The sectional curvature of the tangent plane to Σ at a point z is $-n_3^2$ and the second fundamental form is

$$II = \frac{\omega_x}{\cosh\omega}dx\otimes dx - \frac{\omega_x}{\cosh\omega}dy\otimes dy + 2\frac{\omega_y}{\cosh\omega}dx\otimes dy.$$

Hence, using the Gauss equation, the Gauss curvature of (Σ, ds^2) is given by

$$K_\Sigma = -\tanh^2\omega - \frac{|\nabla\omega|^2}{4(\cosh^4\omega)|\phi|}. \tag{2.7}$$

3 Main results

In this section, besides prove the main theorem of this paper, we will firstly demonstrate some properties of an end when it is properly immersed in $\mathcal{M}_+$ or in $\mathcal{M}_-$, which are interesting by theirselves.

We will write $[d(0), d(s)]$ to denote the slab $\cup_{0\le t\le s} d(t)$ in $\mathbb{H}^2\times\mathbb{R}$ whose boundary is $d(0)\cup d(s)$.

Lemma 1. *There is no proper minimal end E in $\mathcal{M}_+$ with $\partial\mathcal{M}_+\cap E = \partial E$ whose lift is an annulus in $\mathbb{H}^2\times\mathbb{R}$.*

Proof. Let us prove it by contradiction. Suppose we have a proper minimal end E in $\mathcal{M}_+$ with $\partial\mathcal{M}_+\cap E = \partial E$ whose lift $\widetilde{E}$ is a proper minimal annulus in $\mathbb{H}^2\times\mathbb{R}$. Hence $\partial\widetilde{E}\subset d(0)$, $\widetilde{E}\subset\bigcup_{s\ge 0} d(s)$ and $\widetilde{E}\cap d(s)\neq\emptyset$ for any s, where $d(s) = c(s)\times\mathbb{R}$, $c(s)$ horocycle tangent at infinity to p_o.

Choose $p\neq p_o\in\partial_\infty\mathbb{H}^2$ such that $(\overline{pp_o}\times\mathbb{R})\cap\partial\widetilde{E} = \emptyset$.

Now consider $q\in\partial_\infty\mathbb{H}^2$ contained in the halfspace determined by $\overline{pp_o}\times\mathbb{R}$ that does not contain $\partial\widetilde{E}$ such that $(\overline{pq}\times\mathbb{R})\cap d(0) = \emptyset$. Let q go to p_o. If there exists some point q_1 such that $(\overline{pq_1}\times\mathbb{R})\cap\widetilde{E}\neq\emptyset$, then, as $p, q_1\notin d(s)$ for any s, and E is proper, that intersection is a compact set in $\widetilde{E}$. Therefore, when we start with q close to p and let q go to q_1, there will be a first contact point between $\overline{pq_0}\times\mathbb{R}$ and $\widetilde{E}$, for some point q_0. By the maximum principle this yields a contradiction. Therefore, we conclude that $\overline{pp_0}\times\mathbb{R}$ does not intersect $\widetilde{E}$. Choosing another point $\bar{p}$ in the same halfspace determined by $\overline{pp_o} \times \mathbb{R}$ as $\widetilde{E}$ such that $(\overline{\bar{p}p_o} \times \mathbb{R}) \cap \partial\widetilde{E} = \emptyset$, we can use the same argument above and conclude that $\widetilde{E}$ is contained in the region between $\overline{pp_o} \times \mathbb{R}$ and $\overline{\bar{p}p_o} \times \mathbb{R}$. Call $\alpha = \overline{pp_o}$ and $\bar{\alpha} = \overline{\bar{p}p_o}$.

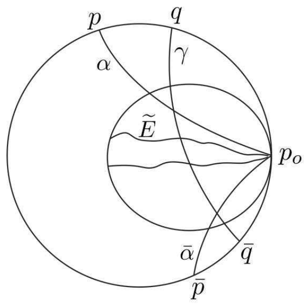

Figure 4: Curve γ.

Now consider a horizontal geodesic γ with endpoints $q, \bar{q}$ such that q is contained in the halfspace determined by $\alpha \times \mathbb{R}$ that does not contain $\widetilde{E}$, and $\bar{q}$ is contained in the halfspace determined by $\bar{\alpha} \times \mathbb{R}$ that does not contain $\widetilde{E}$ (see Figure 4). Up to translation, we can suppose $\widetilde{E} \cap (\gamma \times \mathbb{R}) \neq \emptyset$. As E is proper, the part of $\widetilde{E}$ between $\partial\widetilde{E}$ and $\widetilde{E} \cap (\gamma \times \mathbb{R})$ is compact, then there exists $M \in \mathbb{R}$ such that the function $\widetilde{h}$ restrict to this part satisfies $-M \leq \widetilde{h} \leq M$. Consider the function v that takes the value $+\infty$ on γ and take the value M on the asymptotic arc at infinity of $\mathbb{H}^2$ between q and $\bar{q}$ that does not contain p_o. The graph of v is a minimal surface that does not intersect $\widetilde{E}$. When we let $q, \bar{q}$ go to p_o we get, using the maximum principle, $\widetilde{E}$ is under the graph of v and then $\widetilde{h}|_{\widetilde{E}}$ is bounded above by M, since v converges to the constant function M uniformly on compact sets as $q, \bar{q}$ converge to p_o (see section B, [12]). Using a similar argument, we can show that $\widetilde{h}|_{\widetilde{E}}$ is also bounded below by $-M$. Therefore $\widetilde{E}$ is an annulus contained in the region bounded by $\alpha \times \mathbb{R}, \bar{\alpha} \times \mathbb{R}, \mathbb{H}^2 \times \{-M\}$ and $\mathbb{H}^2 \times \{M\}$.

Take four points $p_1, p_2, p_3, p_4 \in \partial_\infty \mathbb{H}^2$ such that p_1, p_2 is contained in the halfspace determined by $\alpha \times \mathbb{R}$ that does not contain $\widetilde{E}$, and p_3, p_4 is contained in the halfspace determined by $\bar{\alpha} \times \mathbb{R}$ that does not contain $\widetilde{E}$. Moreover, choose these points so that there exists a complete minimal surface $\mathcal{A}$ taking value 0 on $\overline{p_1p_2}$ and $\overline{p_3p_4}$, and taking value $+\infty$ on $\overline{p_2p_4}$ and $\overline{p_1p_3}$ (see Figure 5). This minimal surface exists by [2].

Up to a vertical translation, $\mathcal{A}$ does not intersect $\widetilde{E}$ and $\mathcal{A}$ is above $\widetilde{E}$. Pushing down $\mathcal{A}$ (under vertical translation) and using the maximum principle, we conclude that $\mathcal{A} = \widetilde{E}$, what is impossible. $\square$

Remark 1. *We do not use any assumption on the total curvature of the end to prove the previous lemma.*

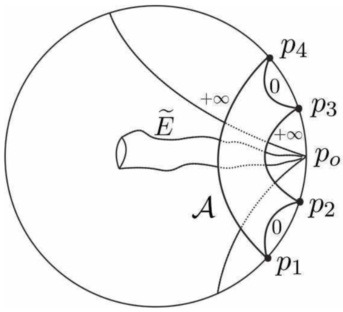

Figure 5: Minimal graph $\mathcal{A}$.

Lemma 2. *If a proper minimal end E with finite total curvature is contained in $\mathcal{M}_-$, then E has bounded curvature and infinite area.*

Proof. Suppose E does not have bounded curvature. Then there exists a divergent sequence $\{p_n\}$ in E such that $|A(p_n)| \geq n$, where A denotes the second fundamental form of E. As the injectivity radius of $\mathcal{M}_-$ is strictly positive, there exists $\delta > 0$ such that for all n, the exponential map $\exp_{\mathcal{M}} : D(0,\delta) \subset T_{p_n}\mathcal{M} \to B_{\mathcal{M}}(p_n,\delta)$ is a diffeomorphism, where $B_{\mathcal{M}}(p_n,\delta)$ is the extrinsic ball of radius δ centered at p_n in $\mathcal{M}$. Without loss of generality, we can suppose $B_{\mathcal{M}}(p_n,\delta) \cap B_{\mathcal{M}}(p_k,\delta) = \emptyset$.

The properness of the end implies the existence of a curve $c \subset E$ homotopic to ∂E such that every point in the connected component of $E \setminus c$ that does not contain ∂E is at a distance greater than δ from ∂E. Call E_1 this component. Hence each point of E_1 is the center of an extrinsic ball of radius δ disjoint from ∂E.

Denote by C_n the connected component of p_n in $B_{\mathcal{M}}(p_n,\delta) \cap E_1$ and consider the function $f_n : C_n \to \mathbb{R}$ given by

$$f_n(q) = d(q, \partial C_n)|A(q)|,$$

where d is the extrinsic distance.

The function f_n restricted to the boundary is identically zero and $f_n(p_n) = \delta|A(p_n)| > 0$. Then f_n attains a maximum in the interior. Let q_n be such maximum. Hence $\delta|A(q_n)| \geq d(q_n, \partial C_n)|A(q_n)| = f_n(q_n) \geq f_n(p_n) = \delta|A(p_n)| \geq \delta n$, what yields $|A(q_n)| \geq n$.

Now consider $r_n = \frac{d(q_n,\partial C_n)}{2}$ and denote by B_n the connected component of q_n in $B_{\mathcal{M}}(q_n, r_n) \cap E_1$. We have $B_n \subset C_n$. If $q \in B_n$, then $f_n(q) \leq f_n(q_n)$ and

$$\begin{aligned} d(q_n, \partial C_n) &\leq d(q_n, q) + d(q, \partial C_n) \\ &\leq \tfrac{d(q_n,\partial C_n)}{2} + d(q, \partial C_n) \\ \Rightarrow d(q_n, \partial C_n) &\leq 2d(q, \partial C_n), \end{aligned}$$

hence we conclude that $|A(q)| \leq 2|A(q_n)|$.

Call g the metric on E and take $\lambda_n = |A(q_n)|$. Consider Σ_n the homothety of B_n by λ_n, that is, Σ_n is the ball B_n with the metric $g_n = \lambda_n g$. We can use the exponential map at the point q_n to lift the surface Σ_n to the tangent plane $T_{q_n}\mathcal{M} \approx \mathbb{R}^3$, hence we obtain a surface $\widetilde{\Sigma}_n$ in $\mathbb{R}^3$ which is a minimal surface with respect to the lifted metric $\tilde{g}_n$, where $\tilde{g}_n$ is the metric such that the exponential map $\exp_{q_n}$ is an isometry from $(\widetilde{\Sigma}_n, \tilde{g}_n)$ to (Σ_n, g_n).

We have $\widetilde{\Sigma}_n \subset B_{\mathbb{R}^3}(0, \lambda_n r_n)$, $|A(0)| = 1$ and $|A(q)| \leq 2$ for all $q \in \widetilde{\Sigma}_n$.

Note that $2\lambda_n r_n = f_n(q_n) \geq f_n(p_n) \geq \delta n$, hence $\lambda_n r_n \to +\infty$ as $n \to \infty$.

Fix n. The sequence $\left\{\widetilde{\Sigma}_k \cap B_{\mathbb{R}^3}(0, \lambda_n r_n)\right\}_{k \geq n}$ is a sequence of compact surfaces in $\mathbb{R}^3$, with bounded curvature, passing through the origin and the metric g_k converges to the canonical metric g_0 in $\mathbb{R}^3$. Then a subsequence converges to a minimal surface in $(\mathbb{R}^3, g_0)$ passing through the origin with the norm of the second fundamental form at the origin equal to 1. We can apply this argument for each n and using the diagonal sequence argument, we obtain a complete minimal surface $\widetilde{\Sigma}$ in $\mathbb{R}^3$, with $0 \in \widetilde{\Sigma}$ and $|A(0)| = 1$. In particular, $\widetilde{\Sigma}$ is not the plane. Then by Osserman's theorem [16] we know $\int_{\widetilde{\Sigma}} |A|^2 \geq 4\pi$.

We know that the integral $\int_{\Sigma} |A|^2$ is invariant by homothety of Σ, hence

$$\int_{B_n} |A|^2 = \int_{\Sigma_n} |A|^2 = \int_{\widetilde{\Sigma}_n} |A|^2.$$

Consider a compact $K \subset \widetilde{\Sigma}$ sufficiently large so that $\int_K |A|^2 \geq 2\pi$. Fix n such that $K \subset B(0, \lambda_n r_n)$. As a subsequence of $\widetilde{\Sigma}_k \cap B_{\mathbb{R}^3}(0, \lambda_n r_n)$ converges to $\widetilde{\Sigma} \cap B_{\mathbb{R}^3}(0, \lambda_n r_n)$, we have for k sufficiently large that

$$\int_{\widetilde{\Sigma}_k \cap B(0, \lambda_n r_n)} |A|^2 \geq 2\pi - \epsilon,$$

for some small ϵ. It implies $\int_{B_k} |A|^2 \geq 2\pi - \epsilon$, for k sufficiently large. As $B_i \cap B_j = \emptyset$, we conclude that $\int_E |A|^2 = +\infty$. But this is not possible, since

$$\int_E |A|^2 = \int_E -2K_E + 2K_{\sec_{\mathcal{M}(E)}} \leq -2 \int_E K_E < +\infty.$$

Therefore, E has necessarily bounded curvature.

Since E is complete, there exist $\epsilon > 0$ and a sequence of points $\{p_n\}$ in E such that p_n diverges in $\mathcal{M}_-$ and $B_E(p_k, \epsilon) \cap B_E(p_j, \epsilon) = \emptyset$, where $B_E(p_k, \epsilon) \subset E$ is the intrinsic ball centered at p_k with radius ϵ. As E has bounded curvature, then there exists $\tau < \epsilon$ such that $B_E(p_k, \tau)$ is a graph with bounded geometry over a small disk $D(0,\tau)$ of radius τ in $T_{p_k}E$, and the area of $B_E(p_k,\tau)$ is greater or equal to the area of $D(0,\tau)$. Therefore,

$$\text{area}(E) \geq \sum_{n\geq 1} \text{area}\left(B_E(p_n,\tau)\right) = \infty.$$

□

Definition 1. We write *Helicoidal plane* to denote a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ which is parametrized by $X(x,y) = (x, y, ax+b)$ when we consider the halfplane model for $\mathbb{H}^2$.

Now we can state the main result of this paper.

Theorem 1. *Let $X : \Sigma \hookrightarrow \mathcal{M} = \mathbb{H}^2 \times \mathbb{R}\,/[\psi, T(h)]$ be a properly immersed minimal surface with finite total curvature. Then*

1. *Σ is conformally equivalent to a compact Riemann surface $\overline{M}$ with genus g minus a finite number of points, that is, $\Sigma = \overline{M}\backslash\{p_1,...,p_k\}$.*
2. *The total curvature satisfies*

$$\int_\Sigma K d\sigma = 2\pi(2-2g-k).$$

3. *If we parametrize each end by a punctured disk then either Q extends to zero at the origin (in the case where the end is asymptotic to a horizontal slice) or Q extends meromorphically to the puncture with a double pole and residue zero. In this last case, the third coordinate satisfies $h(z) = b\arg(z) + O(|z|)$ with $b \in \mathbb{R}$.*
4. *The ends contained in $\mathcal{M}_-$ are necessarily asymptotic to a vertical plane $\gamma\times\mathbb{S}^1$ and the ends contained in $\mathcal{M}_+$ are asymptotic to either*
   - *a horizontal slice $\mathbb{H}^2\,/[\psi] \times \{c\}$, or*
   - *a vertical plane $\gamma \times \mathbb{S}^1$, or*
   - *the quotient of a Helicoidal plane.*

Proof. The proof of this theorem uses arguments of harmonic diffeomorphisms theory as can be found in the work of Han, Tam, Treibergs and Wan [5, 6, 22] and Minsky [14].

From a result by Huber [11], we deduce that Σ is conformally a compact Riemann surface $\overline{M}$ minus a finite number of points $\{p_1,...,p_k\}$, and the ends are parabolic.

We consider $\overline{M}^* = \overline{M} - \cup_i B(p_i,r_i)$, the surface minus a finite number of disks removed around the punctures p_i. As the ends are parabolic, each punctured disk $B^*(p_i, r_i)$ can be parametrized conformally by the exterior of a disk in $\mathbb{C}$, say $U = \{z \in \mathbb{C}; |z| \geq R_0\}$.

Using the Gauss-Bonnet theorem for $\overline{M}^*$, we get

$$\int_{\overline{M}^*} K d\sigma + \sum_{i=1}^{k} \int_{\partial B(p_i,r_i)} k_g \; ds = 2\pi(2 - 2g - k). \tag{3.1}$$

Therefore, in order to prove the second item of the theorem is enough to show that for each i, we have

$$\int_{\partial B(p_i,r_i)} k_g \; ds = \int_{B(p_i,r_i)} K d\sigma.$$

In other words, we have to understand the geometry of the ends. Let us analyse each end.

Fix i, denote $E = B^*(p_i, r_i)$ and let $X = (F, h) : U = \{|z| \geq R_0\} \to \mathbb{H}^2/[\psi] \times \mathbb{S}^1$ be a conformal parametrization of the end E. In this parameter we express the metric as $ds^2 = \lambda^2 |dz|^2$ with $\lambda^2 = 4(\cosh^2 \omega)|\phi|$, where $\phi(dz)^2 = Q$ is the holomorphic quadratic differential on the end.

If $Q \equiv 0$ then $\phi \equiv 0$ and $h \equiv$ constant, what yields that the end E of Σ is contained in some slice $\mathbb{H}^2/[\psi] \times \{c_0\}$. Then, in fact, the minimal surface Σ is the slice $\mathbb{H}^2/[\psi] \times \{c_0\}$. Note that by our hypothesis on Σ this case is possible only when the horizontal slices of $\mathcal{M}$ have finite area. Therefore, we can assume $Q \not\equiv 0$.

Following the ideas of [6] and section 3 of [10], we can show that finite total curvature and non-zero Hopf differential Q implies that Q has a finite number of isolated zeroes on the surface Σ. Moreover, for $R_0 > 0$ large enough we can show that there is a constant α such that $(\cosh^2 \omega)|\phi| \leq |z|^\alpha |\phi|$ and then, as the metric ds^2 is complete, we use a result by Osserman [16] to conclude that Q extends meromorphically to the puncture $z = \infty$. Hence we can suppose that ϕ has the following form:

$$\phi(z) = \left(\sum_{j \geq 1} \frac{a_{-j}}{z^j} + P(z) \right)^2,$$

for $|z| > R_0$, where P is a polynomial function.

Since ϕ has a finite number of zeroes on U, we can suppose without loss of generality that ϕ has no zeroes on U, and then the minimal surface E is transverse to the horizontal sections $\mathbb{H}^2/[\psi] \times \{c\}$.

As in a conformal parameter z, we express the metric as $ds^2 = \lambda^2 |dz|^2$, where $\lambda^2 = 4(\cosh^2 \omega)|\phi|$, then on U

$$-K_\Sigma \lambda^2 = 4(\sinh^2 \omega)|\phi| + \frac{|\nabla \omega|^2}{\cosh^2 \omega} \geq 0. \tag{3.2}$$

Hence,

$$\begin{aligned} -\int_U KdA &= \int_U 4(\sinh^2\omega)|\phi||dz|^2 + \int_U \frac{|\nabla\omega|^2}{\cosh^2\omega}|dz|^2 \\ &= \int_U 4(\cosh^2\omega)|\phi||dz|^2 - \int_U 4|\phi||dz|^2 + \int_U \frac{|\nabla\omega|^2}{4(\cosh^4\omega)|\phi|}dA \\ &= \text{area}(E) - 4\int_U |\phi||dz|^2 + \int_U \frac{|\nabla\omega|^2}{4(\cosh^4\omega)|\phi|}dA, \end{aligned}$$

where the last term in the right hand side is finite by (3.2), once we have finite total curvature.

By the above equality, we conclude that area(E) is finite if, and only if, $\phi = \left(\sum_{j\geq 2}\frac{a_{-j}}{z^j}\right)^2$. Equivalently, area$(E)$ is infinite if, and only if, $\phi = \left(\sum_{j\geq 1}\frac{a_{-j}}{z^j} + P(z)\right)^2$, with $P \not\equiv 0$ or $a_{-1} \neq 0$.

Claim 1: If the area of the end is infinite, then the function ω goes to zero uniformly at infinity.

Proof. To prove this we use estimates on positive solutions of sinh-Gordon equations by Han [5], Minsky [14] and Wan [22] to our context.

Given V any simply connected domain of $U = \{|z| \geq R_0\}$, we have the conformal coordinate $w = \int\sqrt{\phi}dz = u + iv$ with the flat metric $|dw|^2 = |\phi||dz|^2$ on V. In the case where $P \not\equiv 0$, the disk $D(w(z), |z|/2)$ contains a ball of radius at least $c|z|$ in the metric $|dw|^2$ where c does not depend on z.

In the case where $a_{-1} \neq 0$, we consider the conformal universal covering $\tilde{U}$ of the annulus U given by the conformal change of coordinate $w = \ln(z) + f(z)$ where $f(z)$ extends holomorphically by zero at the puncture. Any point z in U lifts to the center $w(z)$ of a ball $D(w(z), \ln(|z|/2)) \subset \tilde{U}$ for $|z| > 2R_0$ large enough.

The function ω lifts to the function $\tilde{\omega}\circ w(z) := \omega(z)$ on the w-plane which satisfies the equation

$$\Delta_{|\phi|}\tilde{\omega} = 2\sinh 2\tilde{\omega}$$

where $\Delta_{|\phi|}$ is the Laplacian in the flat metric $|dw|^2$. On the disc $D(w(z), 1)$ we consider the hyperbolic metric given by

$$d\sigma^2 = \mu^2|dw|^2 = \frac{4}{(1-|w-w(z)|^2)^2}|dw|^2.$$

Then μ takes infinite values on $\partial D(w(z),1)$ and since the curvature of the metric $d\sigma^2$ is $K=-1$, the function $\omega_2=\ln\mu$ satisfies the equation

$$\Delta_{|\phi|}\omega_2 = e^{2\omega_2} \geq e^{2\omega_2} - e^{-2\omega_2} = 2\sinh\omega_2,$$

Then the function $\eta(w)=\tilde\omega(w)-\omega_2(w)$ satisfies

$$\Delta_{|\phi|}\eta = e^{2\tilde\omega} - e^{-2\tilde\omega} - e^{2\omega_2} = e^{2\omega_2}\left(e^{2\eta} - e^{-4\omega_2}e^{-2\eta} - 1\right),$$

which can be written in the metric $d\tilde\sigma^2 = e^{2\omega_2}|dw|^2$ as

$$\Delta_{\tilde\sigma}\eta = e^{2\eta} - e^{-4\omega_2}e^{-2\eta} - 1.$$

Since ω_2 goes to $+\infty$ on the boundary of the disk $D_{|\phi|}(w(z),1)$, the function η is bounded above and attains its maximum at an interior point q_0. At this point $\eta_0=\eta(q_0)$ we have

$$e^{2\eta_0} - e^{-4\omega_2}e^{-2\eta_0} - 1 \leq 0.$$

which implies

$$e^{2\eta_0} \leq \frac{1+\sqrt{1+4a^2}}{2},$$

where $a = e^{-2\omega_2(q_0)} \leq \ \sup\frac{1}{\mu^2} \leq \frac{1}{4}$. Thus at any point of the disk $D_{|\phi|}(z,1)$, $\tilde\omega$ satisfies

$$\tilde\omega \leq \omega_2 + \frac{1}{2}\ln(\frac{2+\sqrt{5}}{4}).$$

We observe that the same estimate above holds for $-\tilde\omega$. Then at the point z, we have

$$|\omega(z)| = |\tilde\omega(w(z))| \leq \ln 4 + \frac{1}{2}\ln(\frac{2+\sqrt{5}}{4}) := K_0$$

uniformly on $R \geq R_0$. Using this estimate we can apply a maximum principle as in Minsky [14]. We know that for $|z|$ large, we can find a disk $D_{|\phi|}(w(z),r)$ with r large too. Now, consider the function

$$F(u,v) = \frac{K_0}{\cosh r}\cosh\sqrt{2}u\cosh\sqrt{2}v.$$

Then $F \geq K_0 \geq \omega$ on $\partial D_{|\phi|}(w(z),r)$ and at q_0 we have $\Delta_{|\phi|}F = 4F$. Suppose the minimum of $F-\tilde\omega$ is a point q_0 where $\tilde\omega(q_0) \geq F(q_0)$. Then $0 \leq \tilde\omega(q_0) \leq \sinh\tilde\omega(q_0)$ and

$$\Delta_{|\phi|}(F-\tilde\omega) = 4F - 2\sinh 2\tilde\omega \leq 4(F(q_0) - \tilde\omega(q_0)) \leq 0.$$

Therefore we have necessarily $\tilde{\omega} \leq F$ on the disk. Considering the same argument to $F + \tilde{\omega}$ we can conclude $|\tilde{\omega}| \leq F$. Hence

$$|\tilde{\omega}(w(z))| \leq \frac{K_0}{\cosh r} \tag{3.3}$$

and then $|\tilde{\omega}| \to 0$ uniformly at the puncture, consequently $|\omega| \to 0$ uniformly at infinity. □

Claim 2: If $P \not\equiv 0$ then the end E is not proper in $\mathcal{M}$.

Proof. Suppose $P \not\equiv 0$. Up to a change of variable, we can assume that the coefficient of the leading term of P is one. Then, for suitable complex number $a_0, ..., a_{k-1}$, we have

$$P(z) = z^k + a_{k-1}z^{k-1} + \ + a_0 \text{ and } \sqrt{\phi} = z^k(1 + o(1)).$$

Let us define the function

$$w(z) = \int \sqrt{\phi(z)}dz = \int \left(\sum_{j \geq 1} \frac{a_{-j}}{z^j} + a_0 + ... + z^k \right).$$

If $a_{-1} = a + ib$ and we denote by $\theta \in \mathbb{R}$ a determination of the argument of $z \in U$, then locally

$$\text{Im}(w)(z) = b\log|z| + a\theta + \frac{|z|^{k+1}}{k+1}(\sin(k+1)\theta + o(1)) \tag{3.4}$$

and

$$\text{Re}(w)(z) = a\log|z| - b\theta + \frac{|z|^{k+1}}{k+1}(\cos(k+1)\theta + o(1)). \tag{3.5}$$

If $C_0 > \max\{|\text{Im}(w)(z)|; |z| = R_0\}$, then the set $U \cap \{\text{Im}(w)(z) = C_0\}$ is composed of $k+1$ proper and complete curves without boundary $L_0, ..., L_k$ (see Figure 6).

Take $\mathcal{R}$ a simply connected component of $U \cap \{\text{Im}(w)(z) \geq C_0\}$. The holomorphic map $w(z)$ gives conformal parameters $w = u + iv, v \geq C_0$, to $X(\mathcal{R}) \subset E$.

Then $\widetilde{X}(w) = (\widetilde{F}(w), v)$ is a conformal immersion of $\mathcal{R}$ in $\mathbb{H}^2 \times \mathbb{R}$ and we have

$$|\widetilde{F}_u|^2_\sigma = |\widetilde{F}_v|^2_\sigma + 1 \text{ and } \left\langle \widetilde{F}_u, \widetilde{F}_v \right\rangle_\sigma = 0.$$

Hence the holomorphic quadratic Hopf differential is

$$Q_{\widetilde{F}} = \phi(w)(dw)^2 = \frac{1}{4}\left(|\widetilde{F}_u|^2_\sigma - |\widetilde{F}_v|^2_\sigma + 2i\left\langle \widetilde{F}_u, \widetilde{F}_v \right\rangle_\sigma \right) = \frac{1}{4}(dw)^2$$

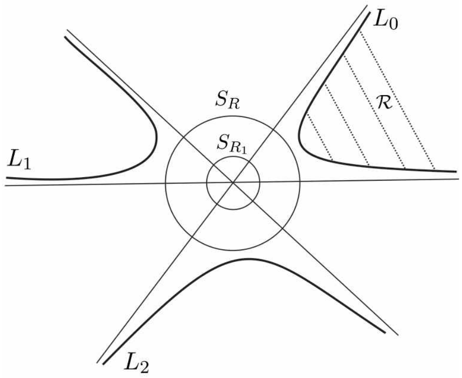

Figure 6: L_j for $k = 2$.

and the induced metric on these parameters is given by $ds^2 = \cosh^2\widetilde{\omega}|dw|^2$.

Consider the curve $\gamma(v) = \widetilde{X}(u_0 + iv) = (\widetilde{F}(u_0, v), v)$. We have

$$d_{\mathbb{H}^2}(\widetilde{F}(u_0, C_0), \widetilde{F}(u_0, v)) \leq \int_{C_0}^{v} |\widetilde{F}_v| dv = \int_{C_0}^{v} |\sinh \tilde{\omega}| dv < \infty,$$

once we know $|\tilde{\omega}| \to 0$ at infinity by Claim 1.

Thus, when we pass the curve γ to the quotient by the third coordinate, we obtain a curve in E which is not properly immersed in the quotient space $\mathcal{M}$. Therefore, the claim is proved and we have $P \equiv 0$ necessarily. □

Suppose $E \subset \mathcal{M}_+$. We have $E = X(U)$ homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$. Up to translation (along a geodesic not contained in $\mathbb{T}(0)$), we can suppose that E is transverse to $\mathbb{T}(0)$. Then $E \cap \mathbb{T}(0)$ is k jordan curves $d_1, ..., d_j, \alpha_1, ..., \alpha_l, j + l = k$, where each d_i is homotopically zero in E and each α_i generates the fundamental group of $E, \pi_1(E)$.

We will prove that $l = 1$ necessarily and the subannulus bounded by α_1 is contained in $\cup_{s \geq 0} \mathbb{T}(s)$.

Assume $l \neq 1$. Then there exist $\alpha_1, \alpha_2 \subset \mathbb{T}(0)$ generators of $\pi_1(E)$. As $E \cong \mathbb{S}^1 \times \mathbb{R}$, there exists $F \subset E$ such that $F \cong \mathbb{S}^1 \times [0, 1]$ and $\partial F = \alpha_1 \cup \alpha_2$. So F is compact and its boundary is on $\mathbb{T}(0)$. By the maximum principle, $F \cap (\cup_{s<0} \mathbb{T}(0)) = \emptyset$. Hence $F \subset \cup_{s \geq 0} \mathbb{T}(s)$ and then, since $E \subset \mathcal{M}_+$, there exist a third jordan curve α_3 that generates $\pi_1(E)$ and another cylinder G such that $G \cap (\cup_{s<0} \mathbb{T}(0)) \neq \emptyset$ and ∂G is either $\alpha_1 \cup \alpha_3$ or $\alpha_2 \cup \alpha_3$, but we have just seen that such G can not exist. Therefore $l = 1$, that is, $E \cap \mathbb{T}(0) = \alpha \cup d_1 \cup ... \cup d_j$, where α generates $\pi_1(E)$, the subannulus bounded by α is contained in $\cup_{s \geq 0} \mathbb{T}(s)$, and each $d_i \subset E$ bounds a disk on E contained in $\cup_{s \geq 0} \mathbb{T}(s)$.

Remark 2. *The same holds true for $E \subset \mathcal{M}_-$, that is, if $E \subset \mathcal{M}_-$ and E is transversal to $\mathbb{T}(s)$ then $E \cap \mathbb{T}(s)$ is $l_s + 1$ curves $\alpha, d_1, ..., d_{l_s}$, where d_i is homotopically zero in E and α generates $\pi_1(E)$.*

Take a point p in the horocycle $c(0) \subset \mathbb{H}^2$ and consider $e_1 = c(0)/[\psi]$, $e_2 = p \times \mathbb{R}/[T(h)]$. The curves e_1, e_2 are generators of $\pi_1(\mathbb{T}(0))$.

As $E \subset \mathcal{M}_+$ and $\pi_1(\mathcal{M}_+) = \pi_1(\mathbb{T}(0))$, we can consider the inclusion map $i_* : \pi_1(E) \to \pi_1(\mathbb{T}(0))$ and $i_*([\alpha]) = n[e_1] + m[e_2]$, where m, n are integers.

Case 1.1: $n = m = 0$. This case is impossible.

In fact, $n = m = 0$ implies that E lifts to an annulus in $\mathbb{H}^2 \times \mathbb{R}$ and we already know by Lemma 1 that this is not possible.

Case 1.2: $n \neq 0, m = 0$.

We can assume, without loss of generality, that $\partial E \subset \mathbb{T}(0)$. Call $\widetilde{E}$ a connected component of $\pi^{-1}(E \cap \mathcal{M}_+)$ such that $\pi(\widetilde{E}) = E$. We have that $\widetilde{E}$ is a proper minimal surface and its boundary $\partial\widetilde{E} = \pi^{-1}(\partial E)$ is a curve in $d(0)$ invariant by ψ^n. Moreover, the horizontal projection of $\widetilde{E}$ on $\cup_{s \geq 0} c(s) \subset \mathbb{H}^2$ is surjective.

By the Trapping Theorem in [4], $\widetilde{E}$ is contained in a horizontal slab. Hence $\widetilde{h}|_{\widetilde{E}}$ is a bounded harmonic function, and then $h|_E$ is a bounded harmonic function defined on a punctured disk. Therefore h has a limit at infinity, and then we can say that Q extends to a constant at the origin, say zero. In particular, $\widetilde{h}$ has a limit at infinity.

The end of $\widetilde{E}$ is contained in a slab of width $2\epsilon > 0$ and by a result of Collin, Hauswirth and Rosenberg [3], $\widetilde{E}$ is a graph outside a compact domain of $\mathbb{H}^2 \times \mathbb{R}$. This implies that $\widetilde{E}$ has bounded curvature. Then there exists $\delta > 0$ such that for any $p \in E$, $B_E(p, \delta)$ is a minimal graph with bounded geometry over the disk $D(0, \delta) \subset T_pE$.

Now fix s and consider a divergent sequence $\{p_n\}$ in E. Applying hyperbolic translations to $\{p_n\}$(horizontal translations along a geodesic of $\mathbb{H}^2$ that sends p_n to a point in $\mathbb{T}(s)$), we get a sequence of points in $\mathbb{T}(s)$ which we still call $\{p_n\}$. As $\mathbb{T}(s)$ is compact, the sequence $\{p_n\}$ converges to a point $p \in \mathbb{T}(s)$ and the sequence of graphs $B_E(p_n, \delta)$ converges to a minimal graph $B_E(p, \delta)$ with bounded geometry over $D(0, \delta) \subset T_pE$.

As h has a limit at infinity, this limit disk $B_E(p, \delta)$ is contained in a horizontal slice. Then we conclude $n_3 \to 1$ and $|\nabla h| \to 0$ uniformly at infinity, what yields a C^1-convergence of E to a horizontal slice. Now using elliptic regularity we get E converges in the C^2-topology to a horizontal slice. In particular, the geodesic curvature of α_s goes to 1 and its length goes to zero, where α_s is the curve in $E \cap \mathbb{T}(s)$ that generates $\pi_1(E)$.

Denote by E_s the part of the end E bounded by ∂E and α_s. Applying the Gauss-Bonnet theorem for E_s, we obtain

$$\int_{E_s} K + \int_{\alpha_s} k_g - \int_{\partial E} k_g = 0.$$

By our analysis in the previous paragraph, we have $\int_{\alpha_s} k_g \to 0$, when $s \to \infty$. Then when we let s go to infinity, we get

$$\int_E K = \int_{\partial E} k_g,$$

as we wanted to prove.

Claim 3: If $m \neq 0$ then the area of the end is infinite.

Proof. In fact, consider $g : \Sigma \to \mathbb{R}$ the extrinsic distance function to $\mathbb{T}(0)$, that is, $g = d_{\mathcal{M}}(\ .\ , \mathbb{T}(0))$. Hence $|\nabla^{\mathcal{M}} g| = 1$ and $g^{-1}(s) = \Sigma \cap \mathbb{T}(s)$. We know for almost every s, $\Sigma \cap \mathbb{T}(s) = \alpha_s \cup d_1 \cup ... \cup d_l$, where α_s generates $\pi_1(E)$ and d_i is homotopic to zero in E. Then, by the coarea formula,

$$\begin{aligned} \int_{\{g \le s\}} 1 dA &= \int_{-\infty}^{s} \left(\int_{\{g=\tau\}} \frac{ds_\tau}{|\nabla^\Sigma g|} \right) d\tau \;\geq\; \int_0^s |\alpha_\tau| d\tau \\ &\geq \int_0^s |e_2| d\tau \;=\; s|e_2|, \end{aligned}$$

where the last inequality follows from the fact we are supposing that $i_*[\alpha_s]$ has a component $[e_2]$, and in the last equality we use that the curve e_2 has constant length. Hence when we let s go to infinity, we conclude the area of E is infinite. □

So if $E \subset \mathcal{M}_+$ and $m \neq 0$, then the area of E is infinite. Also, we know by Lemma 2 that all the ends contained in $\mathcal{M}_-$ have infinite area. Thus we will analyse all these cases together using the commom fact of infinite area.

Suppose we have an end E with infinite area. We can assume without loss of generality that $\partial E \subset \mathbb{T}(0)$. We know that $\phi = \left(\sum_{j \geq 1} \frac{a_{-j}}{z^j} \right)^2$ with $a_{-1} \neq 0$ for $|z| \geq R_0$, and $|\omega| \to 0$ uniformly at infinity by Claim 1. In particular, we know that the tangent planes to the end become vertical at infinity.

Let $X : D^*(0,1) \subset \mathbb{C} \to \mathcal{M}$ be a conformal parametrization of the end from a punctured disk (we suppose, without loss of generality, that the punctured disk is the unit punctured disk). Now consider the covering of $D^*(0,1)$ by the halfplane $HP := \{w = u + iv, u < 0\}$ through the holomorphic exponential map $e^w : HP \to D^*(0,1)$. Hence, we can take $\hat{X} = X \circ e^w : HP \to \mathcal{M}$ a conformal parametrization of the end from a halfplane.

We denote by $h, \hat{h}$ the third coordinates of X and $\hat{X}$, respectively. We already know $h(z) = a\ln|z| + b\arg(z) + p(z)$ for $z \in D^*(0,1)$, where either a or b is not zero, and p is a polynomial function. Hence $|p(z)| \to 0$ when $|z| \to 0$ and $\hat{h}(w) = au + bv + \hat{p}(w)$, where $u = \mathrm{Re}\,(w), v = \mathrm{Im}\,(w)$ and $\hat{p}(w) = p(e^w)$.

As the halfplane is simply connected, consider $\widetilde{X} : HP \to \mathbb{H}^2 \times \mathbb{R}$ the lift of $\hat{X}$ into $\mathbb{H}^2 \times \mathbb{R}$. We have $\widetilde{X} = (\widetilde{F}, \widetilde{h})$, where $\widetilde{h}(w) = au + bv + \widetilde{p}(w)$, with $|\widetilde{p}(w)| \to 0$ when $|w| \to \infty$. Up to a conformal change of parameter, we can suppose that $\widetilde{h}(w) = au + bv$.

Observe $\partial\widetilde{E} = \widetilde{X}(\{u = 0\})$ and the curve $\{\widetilde{h} = c\}$ is the straight line $\{au + bv = c\}$. We have three cases to analyse.

Case 2.1: $a = 0, b \neq 0$, that is, the third coordinate satisfies $h(z) = b\arg(z) + O(|z|)$.

Without loss of generality we can suppose $b = 1$. Hence in this case, $\widetilde{h}(w) = v$ and $\partial\widetilde{E} = \widetilde{X}(\{u = 0\})$.

We have $\widetilde{X}(w) = (\widetilde{F}(w), v)$ a conformal immersion of $\widetilde{E}$, and

$$|\widetilde{F}_u|^2_\sigma = |\widetilde{F}_v|^2_\sigma + 1 \text{ and } \left\langle \widetilde{F}_u, \widetilde{F}_v \right\rangle_\sigma = 0.$$

Hence the holomorphic quadratic Hopf differential is

$$\widetilde{Q}_{\widetilde{F}} = \tilde{\phi}(w)(dw)^2 = \frac{1}{4}\left(|\widetilde{F}_u|^2_\sigma - |\widetilde{F}_v|^2_\sigma + 2i\left\langle \widetilde{F}_u, \widetilde{F}_v \right\rangle_\sigma\right) = \frac{1}{4}(dw)^2$$

and the induced metric on these parameters is given by $ds^2 = \cosh^2\widetilde{\omega}|dw|^2$.

Moreover, by (3.3) there exists a constant $K_0 > 0$ such that

$$|\widetilde{\omega}(w)| \leq \frac{K_0}{\cosh r}, \tag{3.6}$$

for $r = \sqrt{u^2 + v^2}$ sufficiently large.

Using Schauder's estimates and (3.6), we obtain

$$|\widetilde{\omega}|_{2,\alpha} \leq C\,(|\sinh\widetilde{\omega}|_{0,\alpha} + |\widetilde{\omega}|_0) \leq Ce^{-r}.$$

Then

$$|\nabla\widetilde{\omega}| \leq Ce^{-r}. \tag{3.7}$$

Now consider the curve $\gamma_c = \widetilde{E} \cap \mathbb{H}^2 \times \{v = c\}$, that is, $\gamma_c(u) = (\widetilde{F}(u,c), c)$. Let $(V, \sigma(\eta)|d\eta|^2)$ be a local parametrization of $\mathbb{H}^2$ and define the local function φ as the argument of $\widetilde{F}_u$, hence

$$\widetilde{F}_u = \frac{1}{\sqrt{\sigma}}\cosh\widetilde{\omega}e^{i\varphi} \text{ and } \widetilde{F}_v = \frac{i}{\sqrt{\sigma}}\sinh\widetilde{\omega}e^{i\varphi}.$$

If we denote by k_g the geosedic curvature of γ_c in $(V, \sigma(\eta)|d\eta|^2)$ and by k_e the Euclidean geodesic curvature of γ_c in $(V, |d\eta|^2)$, we have

$$k_g = \frac{k_e}{\sqrt{\sigma}} - \frac{\langle \nabla \sqrt{\sigma}, n\rangle}{\sigma},$$

where $n = (-\sin\varphi, \cos\varphi)$ is the Euclidean normal vector to γ_c. If t denotes the arclength of γ_c, we have

$$k_e = \varphi_t = \frac{\varphi_u \sqrt{\sigma}}{\cosh \widetilde{\omega}}$$

and

$$\frac{\langle \nabla \sqrt{\sigma}, n\rangle}{\sigma} = \frac{\langle \nabla \log \sqrt{\sigma}, n\rangle}{\sqrt{\sigma}} = \frac{1}{2\sqrt{\sigma}}\left(\cos\varphi (\log\sigma)_{\eta_2} - \sin\varphi(\log\sigma)_{\eta_1}\right).$$

Then,

$$k_g = \frac{\varphi_u}{\cosh\widetilde{\omega}} - \frac{1}{2\sqrt{\sigma}}\left(\cos\varphi (\log\sigma)_{\eta_2} - \sin\varphi(\log\sigma)_{\eta_1}\right). \tag{3.8}$$

In the complex coordinate w, we have

$$\widetilde{F}_w = \frac{e^{\widetilde{\omega}+i\varphi}}{2\sqrt{\sigma}} \text{ and } \widetilde{F}_{\bar{w}} = \frac{e^{-\widetilde{\omega}+i\varphi}}{2\sqrt{\sigma}}. \tag{3.9}$$

Moreover, the harmonic map equation in the complex coordinate $\eta = \eta_1 + i\eta_2$ of $\mathbb{H}^2$ (see [21], page 8) is

$$\widetilde{F}_{w\bar{w}} + (\log\sigma)_\eta \widetilde{F}_w \widetilde{F}_{\bar{w}} = 0. \tag{3.10}$$

Then using (3.9) and (3.10) we obtain

$$\begin{aligned}
(-\widetilde{\omega} + i\varphi)_w &= -\sqrt{\sigma}\left(\tfrac{1}{\sqrt{\sigma}}\right)_w - (\log\sigma)_\eta \widetilde{F}_w \\
&= \tfrac{1}{2}(\log\sigma)_w - (\log\sigma)_\eta \widetilde{F}_w \\
&= \tfrac{1}{2}\left((\log\sigma)_\eta \widetilde{F}_w + (\log\sigma)_{\bar{\eta}} \bar{\widetilde{F}}_w\right) - (\log\sigma)_\eta \widetilde{F}_w \\
&= \tfrac{1}{2}(\log\sigma)_{\bar{\eta}} \bar{\widetilde{F}}_w - \tfrac{1}{2}(\log\sigma)_\eta \widetilde{F}_w,
\end{aligned} \tag{3.11}$$

where $2(\log\sigma)_\eta = (\log\sigma)_{\eta_1} - i(\log\sigma)_{\eta_2}$ and $\bar{\widetilde{F}}_w = \frac{1}{2\sqrt{\sigma}} e^{-\widetilde{\omega}-i\varphi}$.

Taking the imaginary part of (3.11), we get

$$\varphi_u + \widetilde{\omega}_v = \frac{\cosh\widetilde{\omega}}{2\sqrt{\sigma}}\left(\cos\varphi(\log\sigma)_{\eta_2} - \sin\varphi(\log\sigma)_{\eta_1}\right). \tag{3.12}$$

By (3.8) and (3.12), we deduce

$$k_g = -\frac{\widetilde{\omega}_v}{\cosh \widetilde{\omega}}. \tag{3.13}$$

Therefore, by (3.6) and (3.7), when $c \to +\infty$, $k_g(\gamma_c)(u) \to 0$ and also when we fix c and let u go to infinity the geodesic curvature of the curve γ_c goes to zero. In particular, for $\widetilde{h}$ sufficiently large, the asymptotic boundary of γ_c consists in only one point (see [8], Proposition 4.1).

We will prove that the family of curves γ_c has the same boundary point at infinity independently on the value c. Fix u_0 and consider α_{u_0} the projection onto $\mathbb{H}^2$ of the curve $\widetilde{X}(u_0, v) = (\widetilde{F}(u_0, v), v)$, that is, $\alpha_{u_0}(v) = \widetilde{F}(u_0, v) \in \mathbb{H}^2$. We have $\alpha'_{u_0}(v) = \widetilde{F}_v$ and $|\alpha'_{u_0}(v)|_\sigma = |\sinh \widetilde{\omega}|$. Then

$$d(\alpha_{u_0}(v_1), \alpha_{u_0}(v_2)) \le l(\alpha_{u_0}|_{[v_1,v_2]}) = \int_{v_1}^{v_2} |\sinh \widetilde{\omega}| dv \le \int_{v_1}^{v_2} \sinh \mathrm{e}^{-r} dv,$$

where $r = \sqrt{u_0^2 + v^2}$. Thus, for any v_1, v_2, we have $d(\alpha_{u_0}(v_1), \alpha_{u_0}(v_2)) \to 0$ when $u_0 \to -\infty$.

Therefore, the asymptotic boundary of all horizontal curves γ_c in $\widetilde{E}$ coincide, and we can write $\partial_\infty \widetilde{E} = p_0 \times \mathbb{R}$.

Observe that as $\widetilde{h}|_{\partial \widetilde{E}}$ is unbounded, then we have two possibilities for $\partial \widetilde{E}$, either $\partial \widetilde{E}$ is invariant by a vertical translation or is invariant by a screw motion $\psi^n \circ T(h)^m, n, m \neq 0$.

Subcase 2.1.1: $\partial \widetilde{E}$ invariant by vertical translation and $E \subset \mathcal{M}_+$.

In this case, by the Trapping Theorem in [4], $\widetilde{E}$ is contained in a slab between two vertical planes that limit to the same vertical line at infinity, $p_0 \times \mathbb{R}$. Moreover, since $|\widetilde{\omega}| \to 0$, then we get bounded curvature by (2.7). The same holds true for E in $\mathcal{M}_+$.

Thus, using the same argument as in Case 1.2, we can show that in fact E converges in the C^2-topology to a vertical plane. Therefore, the geodesic curvature of α_s goes to zero and its length stays bounded, where α_s is the curve in $E \cap \mathbb{T}(s)$ that generates $\pi_1(E)$.

Applying the Gauss-Bonnet theorem for E_s, the part of the end E bounded by ∂E and α_s, we obtain

$$\int_{E_s} K + \int_{\alpha_s} k_g - \int_{\partial E} k_g = 0.$$

By our analysis in the previous paragraph, we have $\int_{\alpha_s} k_g \to 0$, when $s \to \infty$. Then, when we let s go to infinity, we get

$$\int_E K = \int_{\partial E} k_g,$$

as we wanted to prove.

Subcase 2.1.2: $\partial\widetilde{E}$ invariant by vertical translation and $E \subset \mathcal{M}_-$.

As $\partial\widetilde{E}$ invariant by vertical translation, then we can find a horizontal geodesic γ in $\mathbb{H}^2$ such that γ limits to p_0 at infinity and $\gamma\times\mathbb{R}$ does not intersect $\partial\widetilde{E}$. Call q_0 the other endpoint of γ. Take $q\in\partial_\infty\mathbb{H}^2$ contained in the halfspace determined by $\gamma\times\mathbb{R}$ that does not contain $\partial\widetilde{E}$. As the asymptotic boundary of $\widetilde{E}$ is just $p_0\times\mathbb{R}$, then $\overline{qq_0}\times\mathbb{R}$ does not intersect $\widetilde{E}$ for q sufficiently close to q_0. Also note that for any q, $\overline{qq_0}\times\mathbb{R}$ can not be tangent at infinity to $\widetilde{E}$, because E is proper in $\mathcal{M}$. Thus, if we start with q close to q_0 and let q go to p_0, we conclude that in fact $\gamma\times\mathbb{R}$ does not intersect $\widetilde{E}$, by the maximum principle. Now if we consider another point $\bar{q}_0\in\partial_\infty\mathbb{H}^2$ contained in the same halfspace determined by $\gamma\times\mathbb{R}$ as $\partial\widetilde{E}$ and such that $\bar{\gamma}\times\mathbb{R}=\overline{\bar{q}_0p_0}\times\mathbb{R}$ does not intersect $\partial\widetilde{E}$, we can prove using the same argument above that $\bar{\gamma}\times\mathbb{R}$ does not intersect $\widetilde{E}$. Thus we conclude that $\widetilde{E}$ is contained in the region between two vertical planes that limit to $p_0\times\mathbb{R}$.

As $|\widetilde{\omega}|\to 0$, we get bounded curvature by (2.7). So $E\subset\mathcal{M}_-$ is a minimal surface with bounded curvature contained in a slab bounded by two vertical planes that limit to the same point at infinity. Hence, using the same argument as in Case 1.2, we can show that E converges in the C^2-topology to a vertical plane. Therefore, as in Subcase 2.1.1 above, we get

$$\int_E K = \int_{\partial E} k_g.$$

Subcase 2.1.3: $\partial\widetilde{E}$ invariant by screw motion and $E\subset\mathcal{M}_+$.

In this case, by the Trapping Theorem in [4], $\widetilde{E}$ is contained in a slab between two parallel Helicoidal planes and, since $|\widetilde{\omega}|\to 0$, we get bounded curvature by (2.7). Then E is a minimal surface in $\mathcal{M}_+$ with bounded curvature contained in a slab between the quotient of two parallel Helicoidal planes.

Thus, using the same argument as in Case 1.2, we can show that in fact E converges in the C^2-topology to the quotient of a Helicoidal plane. In particular, the geodesic curvature of α_s goes to zero and its length stays bounded, where α_s is the curve in $E\cap\mathbb{T}(s)$ that generates $\pi_1(E)$.

Applying the Gauss-Bonnet theorem for E_s, the part of the end E bounded by ∂E and α_s, we obtain

$$\int_{E_s} K + \int_{\alpha_s} k_g - \int_{\partial E} k_g = 0.$$

By our previous analysis, we have $\int_{\alpha_s} k_g \to 0$, when $s\to\infty$. Then, when we let s go to infinity, we get

$$\int_E K = \int_{\partial E} k_g,$$

as we wanted to prove.

Subcase 2.1.4: $\partial\widetilde{E}$ invariant by screw motion and $E \subset \mathcal{M}_-$.

By Remark 2, we know that for almost every $s \leq 0$, $\widetilde{E} \cap d(s)$ contains a curve invariant by screw motion, so it is not possible to have $p_0 \times \mathbb{R}$ as the only asymptotic boundary. Thus this subcase is not possible.

Case 2.2: $a \neq 0$. We will show this is not possible.

Consider the change of coordinates by the rotation $e^{i\theta}w : HP \to \widetilde{HP}$, where $\tan\theta = \frac{a}{b}$ (notice that if $b = 0$, then $\theta = \pi/2$) and $\widetilde{HP} = e^{i\theta}(HP) \subset \{\tilde{w} = \tilde{u} + i\tilde{v}\}$. From now on, when we write one curve in the plane $\tilde{w} = \tilde{u} + i\tilde{v}$, we mean the part of this curve contained in $\widetilde{HP}$.

In this new parameter $\tilde{w}$, we have $\partial\widetilde{E} = \widetilde{X}(\{b\tilde{u} + a\tilde{v} = 0\})$, the curve $\{\widetilde{h} = c\}$ is the straight line $\{\tilde{v} = \frac{c}{\sqrt{a^2+b^2}}\}$. (See Figure 7).

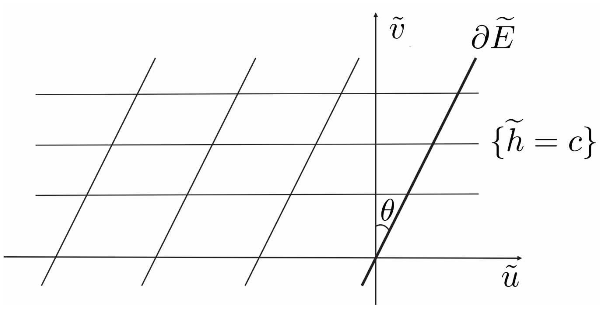

Figure 7: Parameter $\tilde{w} = \tilde{u} + i\tilde{v}$.

Now consider the curve $\beta(t) = (0, t), t \geq 0$. The angle between $\widetilde{X}(\beta)$ and $\partial\widetilde{E}$ is $\theta \neq 0$ and $\widetilde{X}(\beta)$ is a divergent curve in $\widetilde{E}$. However, the curve $\widetilde{F}(\beta) = \widetilde{F}(0, t)$ satisfies

$$l(\widetilde{F}(\beta)) = \frac{1}{|a|}\int_0^t |\widetilde{F}_{\tilde{v}}| d\tilde{v} = \frac{1}{|a|}\int_0^t |\sinh\widetilde{\omega}| d\tilde{v} \leq C,$$

for some constant C not depending on t, since we know by (3.3) that $|\tilde{\omega}| \to 0$ at infinity. This implies that when we pass the curve $\widetilde{X}(\beta)$ to the quotient space $\mathcal{M}$, we obtain a curve in E which is not proper in $\mathcal{M}$, what is impossible, once the end E is proper.

Therefore, analysing the geometry of all possible cases for the ends of a proper immersed minimal surface with finite total curvature Σ in $\mathcal{M}$, we have proved the theorem. $\square$

Remark 3. *The case of a Helicoidal end contained in $\mathcal{M}_+$ is in fact possible, as shows the example constructed by the second author in section 4.3 in [13]. The example is a minimal surface contained in $\mathcal{M}$ with two vertical ends and two Helicoidal ends.*

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