ON MEAN-CONVEX ALEXANDROV EMBEDDED SURFACES IN THE 3-SPHERE

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Abstract. We consider mean-convex Alexandrov embedded non compact surfaces in the 3-sphere, and show under which conditions it is possible to glue local mean-convex Alexandrov embedded patches together into a global mean-convex Alexandrov embedded surface. Applying this local result, we provide condition on space moduli of non compact surface with bounded geometry such that Alexandrov embedding is an open property. We apply this to deformation of surfaces via integrable system.

Introduction

Preserving the property of mean-convex Alexandrov embeddedness during continuous deformations is quite sensitive when the surface M is not compact. In general one gets continuity only with respect to the topology defined by uniform distances on compact subsets and this not necessarily preserves embeddness or Alexandrov embeddness of non compact surfaces.

The existence of an Alexandrov embedded tubular neighborhood T_c of M with uniform width $c > 0$ and uniform deformation M_t of M such that M_t is contained into T_c preserve certainly Alexandrov embeddness, but this look so restrictive for applications to minimal and constant mean curvature surface theory.

We consider mean-convex Alexandrov embeddings $\mathrm{f} : M \to \mathbb{S}^3$ of 2-manifolds M into the round 3-sphere, with particular emphasis on applications for the constant mean curvature case. In the literature we only found the notion of Alexandrov embeddings for compact domains on the one hand, and the concept of properly Alexandrov embedded immersions from open manifolds into open Riemannian manifolds on the other hand. Since we are interested in immersions of open manifolds into the compact Riemannian manifold $\mathbb{S}^3$, we propose the following

Definition 0.1. *A mean-convex Alexandrov embedding in* $\mathbb{S}^3$ *is a smooth complete immersion* $\mathrm{f} : M \to \mathbb{S}^3$ *from a connected surface* M *which extends as an immersion to a connected 3-manifold* N *with boundary* $M = \partial N$ *with the following properties:*

(i) *The mean curvature of* M *in* $\mathbb{S}^3$ *with respect to the inward normal is non-negative everywhere.*
(ii) *The manifold* N *is complete with respect to the metric induced by* f.

An immersion $\mathrm{f} : M \to \mathbb{S}^3$ *just obeying condition* (ii) *is called an Alexandrov embedding.*

We consider the space $\mathcal{M}(\kappa_{max}, C')$ of complete mean-convex immersion $\mathrm{f} : M \to \mathbb{S}^3$ with bounded geometry i.e. immersions having

(i) Principal curvatures of f are uniformly bounded by $max(|\kappa_1|, |\kappa_2|) \leq \kappa_{max}$
(ii) There is a constant C' which bound the covariant derivative of the second fundamental form on the immersion

$$|(\nabla_X \mathfrak{h})(Y, Z)| \leq C' \cdot |X| \cdot |Y| \cdot |Z| \qquad \text{for all } p \in M \text{ and } X, Y, Z \in T_pM, \tag{0.1}$$

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In order to preserve the mean-convex Alexandrov embeddedness during continuous deformations, we localize the concept and divide the surface into compact mean-convex Alexandrov embedded pieces. Then we show first that along continuous deformations the compact pieces stay mean-convex Alexandrov embedded.

For this purpose we need the collar of the 3-manifold with boundary. We assume that there is $c > 0$ a constant which bound the cut locus function from below ($c_{\mathrm{f}}(q) \geq c > 0, q \in M$, see section 1). In the case of bounded curvature constant mean curvature Alexandrov embedded parabolic surfaces, this bound come from Hopf's maximum principle at infinity.

Secondly, we have to prove that these local mean-convex Alexandrov embedded pieces can be glued to a global mean-convex Alexandrov embedding. This we show with the help of a chord arc bound, that inside large enough overlapping boundaries the chord distances of two such pieces are the same. This allows us to glue two overlapping mean-convex Aleandrov embeddings from the boundary to the interior and we obtain the following global criteria

Theorem 1: *Let $\tilde{\mathrm{f}} : M \to \mathbb{S}^3$ be an immersion of $\mathcal{M}(\kappa_{max}, C')$ such that for any $\epsilon > 0$, there is a constant $r > 0$ such that $\forall p \in M$, there exists a mean-convex Alexandrov embedding $\mathrm{f}_p \in \mathcal{M}(\kappa_{max}, C')$ (depending on the point p) with uniform bound of the cut locus function $c_{\mathrm{f}_p}(q) \geq c > 0, q \in M$ (c independent of f_p) such that $\tilde{\mathrm{f}}$ and f_p are locally close in $\mathcal{C}^3$-norm on $B_p = B(p, r)$*

$$||\tilde{\mathrm{f}} - \mathrm{f}_p||_{\mathcal{C}^3(B_p)} < \epsilon$$

Then $\tilde{f} : M \to \mathbb{S}^3$ extend to a mean-convex Alexandrov embedding.

Remark: Minimal and CMC surfaces are not isolated and comes into family of deformations. These deformations comes from the integration of some bounded Jacobi fields on the surface. We consider not only an immersed surface $\mathrm{f} : M \to \mathbb{S}^3$ but f with a whole family of deformation, called isospectral set $I(\mathrm{f})$. The fact that Jacobi fields are bounded is useful to prove that surfaces of $I(\mathrm{f})$ are all mean convex Alexandrov embedded, $I(\mathrm{f}) \subset \mathcal{M}(\kappa_{max}, C')$ and the cut locus bound is uniform with $c_{\mathrm{f}} \geq c > 0$ for all $\mathrm{f} \in I(\mathrm{f})$.

In the case where we deform a surface $\mathrm{f}_0 \in \mathcal{M}$ by integration of a non bounded Jacobi fields, we need to control the geometry at infinity of surfaces f_t which are closed on compact subset to f_0. The Theorem 1 say that if the geometry at infinity of f_t is uniformly close on some fixed ball to an element of $I(\mathrm{f}_0)$ (which is Alexandrov embedded), then f_t is Alexandrov embedded.

Application: In $\mathbb{S}^3$ a proof of the classification of minimal and CMC embedded tori is propose via integrable system theory (see [6]). In this setting the space moduli $\mathcal{M}$ focus on the study of CMC annuli Alexandrov embedded in $\mathbb{S}^3$ of finite type. Finite type mean that we are concerning with CMC annuli without umbilic point and with uniform bounded second fundamental form, conformally parabolic. CMC tori are infinitely covered by annuli of finite type. In the case where the torus is embedded, this annulus is mean convex Alexandrov embedded. We prove in [6] that finite type annuli of $\mathcal{M}$ satisfy the bound of the covariant derivative (0.1).

An application of the maximum principle at infinity in the following proposition 1.2. prove that constant mean curvature annuli of finite type satisfy the bound from below of the cut locus function $c_{\mathrm{f}}(p) \geq c > 0$ with c depending on κ_{max}. This comes from the fact that annuli of finite type have parabolic conformal type $\mathbb{C} \backslash \tau\mathbb{Z}$. This imply that we can apply Theorem 1 in the setting of finite type CMC immersed in $\mathbb{S}^3$.

We prove in [4] that immersed annuli $\mathrm{f} \in \mathcal{M}$ satisfy some algebraic condition called spectral data $(a, b, \lambda_1, \lambda_2) \in \mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda] \times \mathbb{S}^1 \times \mathbb{S}^1$ for $g \in \mathbb{N}$ depending on the immersion. In this setting we consider $\mathrm{f} : \mathbb{C} \to \mathbb{S}^3$ and we recover an annulus when there exists $\tau \in \mathbb{C}$ such that $\mathrm{f}(z + \tau) = \mathrm{f}(z)$ (The existence of τ is encoded in the polynomial b in our context).

On each annulus there is a family of $2g$ independent bounded Jacobi fields. This variational fields can be integrate. We prove in [4] that this deformation has long times existence and preserve the spectral data $(a, b, \lambda_1, \lambda_2)$. The set of this deformation is called isospectral set $I(a)$ and

$$\mathcal{M} = \cup\{I(a); \exists \tau \in \mathbb{C} \text{ with } \mathrm{f}(z+\tau) = \mathrm{f}(z)\}.$$

We consider an annulus $\tilde{\mathrm{f}}$ with spectral data $(\tilde{a}, \tilde{b}, \tilde{\lambda}_1, \tilde{\lambda}_2)$ and we construct a deformation in $\mathcal{M}$ as a section on the space of spectral data which end to spectral data with $g = 0$, which correspond to flat annuli covering the Clifford torus. The result depends on the fact that along the deformation we are able to preserve mean convex Alexandrov embeddness property to conclude. Along this deformation we use two type of deformations as in the remark above:

1)The first deformation concern continuous deformations of an annulus into the isospectral set $I(a)$ by integration of bounded algebraic Jacobi fields. We prove (see [6]) that a commuting property of algebraic variational field induce a commuting group action on $I(a)$:

$$\pi : \mathbb{C}^{2g} \times I(a) \to I(a)$$

Using this commuting property, we prove that for any $\epsilon > 0$, there is a $r > 0$ such that if the immersion $\tilde{\mathrm{f}}$ and f are in the same orbit of the action π and

$$||\tilde{\mathrm{f}} - \mathrm{f}||_{\mathcal{C}^3(B(0,r)} < \epsilon,$$

then for any $p \in \mathbb{C}$, there is an isometry I_p of $\mathbb{S}^3$ which depends only on p such that

$$||\tilde{\mathrm{f}} - I_p\mathrm{f}||_{\mathcal{C}^3(B(p,r)} < \epsilon$$

with the same $\epsilon > 0$. Moreover some compactness of $I(a)$ which is justify in [6], prove that the bound κ_{max} and C' are uniformly satisfy on $I(a)$. The Theorem 1, prove that if f is an annulus Alexandrov embedded, then $\tilde{\mathrm{f}}$ extend to a Alexandrov embedding in $\mathbb{S}^3$. In particular Alexandrov embedding is an open property of the isospectral set $I(a)$.

2) If the we consider $\tilde{\mathrm{f}} \in \mathcal{M}$ with corresponding spectral data $(\tilde{a}, \tilde{b}, \tilde{\lambda}_1, \tilde{\lambda}_2)$ and f with spectral data $(a, b, \lambda_1, \lambda_2)$ and assume that the immersions are $\mathcal{C}^1$ close on $B(0, r)$. Then using group action on the respective isospectral set $I(\tilde{a})$ and $I(a)$, we prove that for any $p \in \mathbb{C}$, the immersion $\tilde{\mathrm{f}}$ is close on $B(p, r)$ of an immersion $\mathrm{f}_p \in I(a)$. If we assume that $\mathrm{f} \in I(a)$, is mean convex Alexandrov embedded, then any element of the orbit f is mean convex Alexandrov embedded, in particular f_p is chosen as an element of the orbit of f. Moreover compactness of $I(a)$ prove that the constant κ_{max}, C' and c are uniformly satisfy on $I(a)$. Hence Theorem 1 apply and mean convex Alexandrov embedded is an open property in the space moduli of annuli parametrize by spectral data.

Now we describe the structure of the present paper. For our purposes we introduce the notion of a local mean-convex Alexandrov embedding, and provide a detailed analysis of mean-convex Alexandrov embedded surfaces in $\mathbb{S}^3$. We prove with a maximum principle at infinity that for CMC mean-convex Alexandrov embedded surfaces with bounded curvature the cut locus function is bounded from below (Proposition 1.1) and depends on κ_{max}. We show that mean-convex Alexandrov embedded surfaces obey a chord-arc bound, if the cut locus function has a positive lower bound, and the covariant derivative of the second fundamental form has a uniform upper bound (Proposition 2.1). Finally we explain how to proceed from local mean-convex Alexandrov embeddings to global mean-convex Alexandrov embeddings via local collar pertubations (Proposition 4.1) using chord arc bound.

1. Lower bound on the cut locus function

In the setting of Definition 0.1, a fixed orientation of $\mathbb{S}^3$ induces on N and $M = \partial N$ an orientation. Conversely, if M is endowed with an orientation, then there exists a unique normal, which points inward to the side of M in $\mathbb{S}^3$, which induces on the boundary M the given orientation of M. In this sense the orientation of M determines the inner normal of N.

For each point $p \in M$ of a hypersurface of a Riemannian manifold N there exists a unique arc-length parameterised geodesic $\gamma(p, \cdot)$ emanating from $p = \gamma(p, 0)$ and going in the direction of the inward normal at p. Such geodesics are called inward M-geodesics [7].

Let $\gamma(p, \cdot)$ be an inward M-geodesic. Points $q \in N$ in the ambient manifold that are 'close to one side' of M can thus be uniquely parameterised by $(p,\, t)$ where $p \in M$ and $q = \gamma(p, t)$ for some inward M-geodesic $\gamma(p, \cdot)$ and some $t \in \mathbb{R}^+_0$. The value of t is the geodesic distance of q to M. Extending the geodesic further into N it might eventually encounter a point past which $\gamma(p, t)$ has distance smaller than t to M. Such a point is called a cut point. The cut locus of M in N consists of the set of cut points along all inward M-geodesics. We define the cut locus function as the geodesic distance of the cut point to M:

$$c : M \to \mathbb{R}^+, \quad p \mapsto c(p), \quad \text{such that } \gamma(p, c(p)) \text{ is the cut point.} \tag{1.1}$$

If we want to stress the dependence on f we decorate γ and c with index f. It is known [7, Lemma 2.1] that a cut point is either the first focal point on an inward M-geodesic, or is the intersection point of two shortest inward M-geodesics of equal length. For mean-convex surfaces in $\mathbb{S}^3$ the first focal point has distance not greater than $\frac{\pi}{2}$ to M. All cut locus functions of mean-convex Alexandrov embeddings are uniformly bounded from above by $\frac{\pi}{2}$, since otherwise a sphere with negative principal curvatures would touch M inside of N contradicting Hopf's maximum principle. Therefore the cut locus function is bounded by $\frac{\pi}{2}$.

For a mean-convex Alexandrov embedding $\mathrm{f} : M \to \mathbb{S}^3$, the inward M-geodesics give us a parametrisation of the 3-manifold N with boundary $M = \partial N$, which we call generalised cylinder coordinates:

$$\gamma_{\mathrm{f}} : \{(p, t) \in M \times \mathbb{R} \mid 0 \le t < c_{\mathrm{f}}(p)\} \to N\,. \tag{1.2}$$

These coordinates define a diffeomorphism onto the complement of the cut locus. The cut locus is homeomorphic to the quotient space $M/\sim_{\mathrm{f}}$ with the following equivalence relation on M:

$$\begin{aligned} p \sim_{\mathrm{f}} q \quad &\Longleftrightarrow \quad c_{\mathrm{f}}(p) = c_{\mathrm{f}}(q) \quad \text{and} \quad \gamma_{\mathrm{f}}(p, c_{\mathrm{f}}(p)) = \gamma_{\mathrm{f}}(q, c_{\mathrm{f}}(q)) \\ &\Longleftrightarrow \quad \gamma_{\mathrm{f}}(p, c_{\mathrm{f}}(p)) = \gamma_{\mathrm{f}}(q, c_{\mathrm{f}}(q)). \end{aligned}$$

For each $p \in M$ we denote the corresponding equivalence class by

$$[p]_{\mathrm{f}} = \{q \in M \mid \gamma_{\mathrm{f}}(p, c_{\mathrm{f}}(p)) = \gamma_{\mathrm{f}}(q, c_{\mathrm{f}}(q))\}. \tag{1.3}$$

Proposition 1.1. *Let* $\mathrm{f} : M \to \mathbb{S}^3$ *be a mean-convex Alexandrov embedding with constant mean curvature and principal curvatures bounded by* $\kappa_{\max} > 0$ *and assume that the immersion is conformally parabolic. Then the cut locus function is bounded from below by* $\arctan(\kappa_{\max}^{-1})$.

Proof. For the hypersurface $M_t = \cup_{p \in M} \gamma(p, t)$, the mean curvature

$$H(t) = \tfrac{1}{2}\left(\cot\left(\arctan(\kappa_1^{-1}) - t\right) + \cot\left(\arctan(\kappa_2^{-1}) - t\right)\right) \tag{1.4}$$

is positive for all $t \in (0,\, t_{\mathrm{foc}})$, and strictly increasing since

$$H'(t) = \tfrac{1}{2}\left(\sin^{-2}(\arctan(\kappa_1^{-1}) - t) + \sin^{-2}(\arctan(\kappa_2^{-1}) - t)\right) > 0\,.$$

Let c_{f} denote the cut locus function (1.1). If there exists a point $p \in M$ for which $c_{\mathrm{f}}(p) < \arctan(\kappa_{\max}^{-1}) \le t_{\mathrm{foc}}$, then two inward M-geodesics $\gamma(p, \cdot)$, $\gamma(q, \cdot)$ through p, $q \in M$ respectively, have to intersect at a distance of $c_{\mathrm{f}}(p)$ from M, and thus $c_{\mathrm{f}}(p) = c_{\mathrm{f}}(q)$. Hence, if there exists a point $p \in M$ with $c_{\mathrm{f}}(p) < \arctan(\kappa_{\max}^{-1})$ then M_t intersects itself for a value of $t < \arctan(\kappa_{\max}^{-1})$ over two points $p,\, q \in M$. Let

$$c_0 = \inf\{t \mid M_t \text{ intersects over two points of } M\}.$$

Since over all points $p \in M$ the mean curvature of M_t is positive for all $0 < t < \arctan(\kappa_{\max}^{-1})$ with respect to the inner normals, the surfaces M_t cannot intersect itself with opposite sign of the mean curvature vector over two points of M for $0 < t < \arctan(\kappa_{\max}^{-1})$.This imply that the cut locus function has no local minima less than $\arctan(\kappa_{\max}^{-1})$.

Now let $(p_k)_{k\in\mathbb{N}}$ be a sequence in M with

$$\lim_{k\to\infty} c_{\mathrm{f}}(p_k) = c_0 = \inf\{c_{\mathrm{f}}(p) \mid p \in M\}\,.$$

Then there exists a sequence Θ_k of isometries of $\mathbb{S}^3$ which transform each point p_k into a fixed reference point $p_0 \in \mathbb{S}^3$, and the tangent plane of M at p_k into the tangent plane of a fixed geodesic sphere $\mathbb{S}^2_{p_0} \subset \mathbb{S}^3$ which contains p_0. This sequence of isometries transforms neighbourhoods U_k of $p_k \in M$ into normal CMC graphs $\Theta_k[U_k]$ over $B(p_0,\, r) \subset \mathbb{S}^2_{p_0}$. Due to Arzelà-Ascoli, and the a-priori gradient bound from Proposition 4.1 in [3], this bounded sequence of normal CMC graphs over $B(p_0,\, r) \subset \mathbb{S}^2_{p_0}$ has a convergent subsequence. By passing to a subsequence we may achieve that these graphs converge to a normal CMC graph U over $B(p_0,\, r) \subset \mathbb{S}^2_{p_0}$, which is tangent to $\mathbb{S}^2_{p_0}$ at p_0. For $c_0 < \arctan(\kappa_{\max}^{-1})$ the sets $[p_k]_{\mathrm{f}}$ contain besides p_k another point q_k for large k. Furthermore, the sequence of isometries Θ_k transforms the sequence of geodesic 2-spheres tangent to M at q_k into a converging sequence of spheres with limit $\mathbb{S}^2_{q_0}$. This sphere contains the limit $q_0 = \lim \Theta_k(q_k)$ with distance $\mathrm{dist}(p_0,\, q_0) \le 2c_0$. For large k the points q_k have neighbourhoods V_k, whose transforms $\Theta_k[V_k]$ are normal CMC graphs over $B(q_0,\, r) \subset \mathbb{S}^2_{q_0}$. By passing again to a subsequence the normal CMC graphs $\Theta_k[V_k]$ converge to a normal CMC graph V tangent to $\mathbb{S}^2_{q_0}$ at q_0. The transformed inward M-geodesics nearby p_k and q_k converge to normal geodesics of these two limiting CMC surfaces U and V in $\mathbb{S}^3$. Let U_{c_0} and V_{c_0} denote the surfaces U and V shifted by c_0 along these normal geodesics. If we shift both sequences $\Theta_k[U_k]$ and $\Theta_k[V_k]$ by c_0 along the transformed M-geodesics, they converge to U_{c_0} and V_{c_0}. Therefore these surfaces U_{c_0} and V_{c_0} touch each other at a first point of contact with opposite sign of the mean curvature at the limit of the transformed cut points $\lim \Theta_k(\gamma_{\mathrm{f}}(p_k,\, c(p_k))) = \lim \Theta_k(\gamma_{\mathrm{f}}(q_k,\, c(q_k)))$, contradicting Hopf's maximum principle. Hence the shifted surfaces cannot have positive mean curvature with respect to the inner normal and this implies $c_0 = 0$ and $H = 0$.

This case contradicts the maximum principle at infinity [8, Theorem 7]: the generalised cylinder coordinates 1.2 define an immersion from the 3-manifold $(x,t) \in M \times [0, \arctan(\kappa_{\max}^{-1})]$ with boundary components $M \times \{0\}$ and $M \times \{\arctan(\kappa_{\max}^{-1})\}$. Each surface $M \times \{t\}$ has uniform bounded curvature, parabolic conformal type and are mean-convex with mean curvature $H > 0$ pointing away from the minimal surface $M \times \{0\}$. If we assume the infimum $c > 0$ of the cut locus function smaller than $\arctan(\kappa_{\max}^{-1})$, some pieces of $M \times \{c\}$ sit inside of this 3-manifold with possible boundaries in the second boundary component $M \times \{\arctan(\kappa_{\max}^{-1})\}$. All assumptions of [8, Theorem 7] are fulfilled and these pieces contradict the maximum principle since the mean curvature point to $M \times \{0\}$, this give a contradiction. □

2. Chord-arc bound

Proposition 2.1. *Let* $\mathrm{f} : M \to \mathbb{S}^3$ *be a mean-convex Alexandrov embedding with second fundamental form* $\mathfrak{h}$ *with respect to the inner normal* $\mathfrak{N}$*. If* $c > 0$ *is a lower bound on the cut locus function* $c_{\mathrm{f}} \ge c > 0$ (1.1) *and* C' *a bound on the covariant derivative of* $\mathfrak{h}$*:*

$$|(\nabla_X \mathfrak{h})(Y,Z)| \le C' \cdot |X| \cdot |Y| \cdot |Z| \qquad \textit{for all } p \in M \textit{ and } X, Y, Z \in T_pM,$$

then there exists a constant $C > 0$ *depending only on* c *and* C' *such that*

$$\mathrm{dist}_N(p,q) \le \mathrm{dist}_M(p,q) \le C\,\mathrm{dist}_N(p,q) \quad \textit{for all } p,q \in M. \tag{2.1}$$

Proof. For all $p, q \in M$ we have $\mathrm{dist}_N(p,q) \le \mathrm{dist}_M(p,q)$. In general, these distances do not coincide. We shall construct a path from p to q of length at most $C\,\mathrm{dist}_N(p,q)$. Due to (Rinow [11], pages 172 and 141) the points p and q are joined by a shortest path in N. In case this shortest path touches at some points the boundary [1, Theorem 1.], we decompose it into pieces. The boundary points of a shortest path might have accumulation points. But any point of a shortest path, which is not a boundary point, belongs to a unique geodesic piece in N, which has only two boundary points at both ends. Hence it suffices to construct such a path for two points p and q, which are connected by a geodesic in N with only two boundary points at both ends. Let $\chi_p,\, \chi_q \in [0, \frac{\pi}{2}]$ denote the angles in T_pN and T_qN between the inward geodesic γ connecting p and q and the inward normal to M, respectively.

Due to [7, Lemma 2.1] the cut locus function $c_{\mathrm{f}}(p)$ (1.1) is for all $p \in M$ not larger than the first focal point $\gamma(p, t_{\mathrm{foc}})$, where $t_{\mathrm{foc}} = \arctan\big((\max\{\kappa_1,\, \kappa_2\})^{-1}\big) \ge \arctan(\kappa_{\max}^{-1})$. Hence both principal curvatures are uniformly bounded by $\kappa_{\max} = \cot(c)$ with $0 < c \le \frac{\pi}{2}$.

Claim: Suppose $p,\, q \in M$ are connected in N by a geodesic of length $\ge \pi$. Then there exists $\tilde{q} \in M$ with

$$\mathrm{dist}_N(p,\tilde{q}) < \pi \quad \text{and} \quad \mathrm{dist}_N(p,q) = \mathrm{dist}_N(p,\tilde{q}) + \mathrm{dist}_N(\tilde{q},q).$$

To prove this claim, note that all geodesics γ in N starting at $p \in M$ which do not meet M in distances $d \in (0,\pi)$, meet each other at the antipode of p. The tangent space T_pN contains a unique half space of initial directions of geodesics starting at $p \in M$. If the pre-image of $B(p,\pi) \subset N$ with respect to $\exp_p$ contains the intersection of $B(0,\pi) \subset T_pN$ with this half space, then due to Hopf's maximum principle (see e.g. [2]), $B(p,\pi) \subset M$ is a geodesic sphere in $\mathbb{S}^3$ and the claim is obvious. Otherwise there starts at p a geodesic which touches M for some $t \in (0,\pi)$, and the claim follows in this case. This proves the claim.

In the sequel we consider points $p,\, q \in M$ connected by a geodesic in N with $\mathrm{dist}_N(p,q) < \pi$. In the following discussion we use the construction in the proof of Lemma 3.2 of a larger 3-manifold $\hat{N} \supset N$ without boundary, such that the generalised cylinder coordinates (1.2) extend to a diffeomorphism $\hat{\gamma}_{\mathrm{f}} : M \times (-c,c) \to \hat{N}$ and the immersion $\mathrm{f} : N \to \mathbb{S}^3$ extends to an immersion $\hat{\mathrm{f}} : \hat{N} \to \mathbb{S}^3$. Along any geodesic in $\hat{N}$, which starts at some point $p \in M$ the distance to M can only increase with the same speed as the distance to p, if the geodesic is an M-geodesic. Therefore such geodesics exist in $\hat{N}$ at least up to distances smaller than c from the initial point $p \in M$. This shows that in $\hat{N}$ the absolute value of the second coordinate of $\hat{\gamma}_{\mathrm{f}}$ is the distance to M. Let (p,q) be any pair of points in M, which are connected in $\hat{N}$ by a geodesic. We distinguish between the following three cases:

- **(A)** The geodesic stays inside $\{\hat{\gamma}_{\mathrm{f}}(p,t) \mid (p,t) \in M \times [-\frac{c}{2}, \frac{c}{2}]\}$.
- **(B)** The geodesic stays inside N with distance $> \frac{c}{2}$ to M and $(\sin^2(\chi_p) + \sin^2(\chi_q))^{\frac{1}{2}} \le \epsilon$.
- **(C)** The geodesic stays inside N with distance $> \frac{c}{2}$ to M and $(\sin^2(\chi_p) + \sin^2(\chi_q))^{\frac{1}{2}} > \epsilon$.

In case **(A)** the first entries of the cylinder coordinates of the geodesic yields a smooth path in M from p to q. We estimate the length of the derivative $d\hat{\gamma}_{\mathrm{f}}(p,t)(p',t')$ of the generalised cylinder coordinates at $(p,t) \in M \times [-\frac{c}{2}, \frac{c}{2}]$ in direction of $(p',t') \in T_{(p,t)}M \times (-c,c)$ from below. For this purpose we decompose this derivative in components parallel to the geodesic and orthogonal to the geodesic. The length of component orthogonal to the geodesic is a lower bound of the length of the derivative and does not depend on t'. The vector field on $\mathbb{S}^2$ of rotations around the z-axis has length proportional to sin of the geodesic distance to the poles. Therefore on a geodesic sphere of radius $c = \operatorname{arccot}(\kappa_{\max})$ the length of the orthogonal component is equal to

$$|d\hat{\gamma}_{\mathrm{f}}(p,t)(p',0)| = \tfrac{\sin(c-t)}{\sin(c)}|p'| = (\cos(t) - \cot(c)\sin(t))|p'| = (\cos(t) - \kappa_{\max}\sin(t))|p'|.$$

Therefore for $t \leq \frac{c}{2} < \frac{\pi}{2}$ we obtain the lower bound

$$|d\hat{\gamma}_{\mathrm{f}}(p,t)(p',t')| \geq \left(\cos(\tfrac{c}{2}) - \cot(c)\sin(\tfrac{c}{2})\right)|p'| = \frac{1}{2\cos(\frac{c}{2})}|p'| \geq \tfrac{1}{2}|p'|.$$

The integral along the geodesic of this inequality yields for all such pairs (p,q)

$$\operatorname{dist}_M(p,q) \leq 2\operatorname{dist}_{\hat{N}}(p,q). \tag{2.2}$$

This includes all (p,q) which are connected in N by a geodesic with $\operatorname{dist}_N(p,q) < c$.

In case **(B)** we shall consider smooth families of geodesics γ connecting two smooth paths $s \mapsto p(s)$ and $s \mapsto q(s)$ in M parameterised by a real parameter s. For fixed s the geodesic is parameterised by the real parameter t. The derivatives with respect to s are denoted by prime and the derivatives with respect to t by dot. For example $p' \in T_{p(s)}M$ and $q' \in T_{q(s)}M$ denotes the tangent vectors along the paths $s \mapsto p(s)$ and $s \mapsto q(s)$. The geodesic γ extends in $\mathbb{S}^3$ to a closed geodesic. For any $(p',q') \in T_pM \times T_qM$ there exists a Killing field ϑ on $\mathbb{S}^3$, which moves the closed geodesic γ in such a way, that the intersection points at p and q moves along p' and q', respectively. Conversely, all Killing fields ϑ generate a one-dimensional group of isometries of $\mathbb{S}^3$. Let $s \mapsto \gamma_\vartheta(s,\cdot)$ denote the corresponding family of geodesics and $s \mapsto (p(s),q(s))$ the corresponding intersection points with M. To proceed, we need the following Lemma, whose proof we defer to the appendix.

Lemma 2.2. *There exist $\epsilon, \delta > 0$ and $0 < s_0 < \min\{\frac{c}{2}, \frac{2}{3}\}$ depending only on c and C' with the following property: Let $p = p(0)$, $q = q(0) \in M$ be connected by a geodesic in N obeying* **(B)**. *Then there exists a non-trivial Killing field ϑ, such that $d : s \mapsto d(s) = \operatorname{dist}_{\mathbb{S}^3}(p(s),q(s))$ obeys*

$$d'(s) \leq 0, \quad d''(s) \leq -\delta\cos\left(\tfrac{d(s)}{2}\right), \quad |p'(s)| + |q'(s)| \leq 3\cos\left(\tfrac{d(s)}{2}\right) \quad \text{for all } s \in [0,s_0], \tag{2.3}$$

with $s_0 > 0$ small enough such that $|d(s_0) - d(0)| \leq \frac{c}{2}$.

Continuation of the proof of Proposition 2.1. For pairs (p,q) connected in N by a geodesic obeying **(B)** there exists by Lemma 2.2 a Killing field ϑ and two paths $s \mapsto p(s)$ and $s \mapsto q(s)$ along which the length d is reduced for $0 \leq s \leq s_0$. We consider the function $f(s) = \cos(\frac{d(s)}{2})$ and its derivative $f'(s) = -\frac{d'(s)}{2}\sin(\frac{d(s)}{2})$ with the inequality

$$-d'(s) \leq |p'(s)| + |q'(s)| \leq 3\cos(\tfrac{d(s)}{2}) \leq 3\cos(\tfrac{d(s_0)}{2})$$

to derive $\cos(\frac{d(0)}{2}) \geq (1 - \frac{3}{2}s_0)\cos(\frac{d(s_0)}{2})$. Twice integration of (2.3) implies the following inequality together with the separate inequalities for the numerator and the denominator:

$$\frac{d(0) - d(s_0)}{\operatorname{dist}_M(p(s_0),p(0)) + \operatorname{dist}_M(q(s_0),q(0))} \geq \frac{\delta\frac{s_0^2}{2}\cos\left(\frac{d(0)}{2}\right)}{3s_0\cos\left(\frac{d(s_0)}{2}\right)} \geq \delta\frac{s_0}{6}\left(1 - \tfrac{3}{2}s_0\right). \tag{2.4}$$

The constant s_0 is chosen so small such that the geodesic connecting $p(s)$ and $q(s)$ stays inside $\{\hat{\gamma}_{\mathrm{f}}(p,t) \mid p \in M, -\frac{c}{2} \leq t \leq c_{\mathrm{f}}(p)\} \subset \hat{N}$. We divide the geodesic from $p(s_0)$ to $q(s_0)$ in $\hat{N}$ into segments outside of N and inside of N. In this way an application of Lemma 2.2 transforms the geodesic from p to q obeying **(B)** into two paths $s \mapsto p(s)$ from $p(0) = p$ to $p(s_0)$ and $s \mapsto q(s)$ from $q(0) = q$ to $q(s_0)$ in M and several geodesic segments in $\hat{N}$ from $p(s_0)$ to $q(s_0)$ obeying either **(A)**, **(B)** or **(C)**. Due to (2.4) the sum of the lengths of the paths in M times $\delta\frac{s_0}{6}(1 - \frac{3}{2}s_0)$ plus the lengths of the geodesic segments is smaller than the length of the original geodesic.

In case **(C)** we apply the gradient flow of $\mathrm{dist}_{\hat{N}}$. As long as $p \neq q$ are connected in $\hat{N}$ by a geodesic, the function $\mathrm{dist}_{\hat{N}}$ is smooth. The Riemannian metric on $M \times M$ identifies the negative of the gradient of $\mathrm{dist}_{\hat{N}}(p,q)$ with a unique vector field (p', q'). We follow this flow as long as $(\sin^2(\chi_p) + \sin^2(\chi_q))^{\frac{1}{2}} > \epsilon$ holds and the geodesic in $\hat{N}$ connecting p and q stays inside $\{\hat{\gamma}_{\mathrm{f}}(p,t) \mid p \in M, -\frac{c}{2} < t \leq c_{\mathrm{f}}(p)\}$. Along the corresponding integral curves $s \mapsto (p(s), q(s))$ we have $|p'| = \sin(\chi_p)$ and $|q'| = \sin(\chi_q)$. Using Cauchy Schwarz inequality, $\mathrm{dist}_{\hat{N}}(p(s), q(s))$ decreases with derivative

$$-\frac{d}{ds}\,\mathrm{dist}_{\hat{N}}(p(s), q(s)) = \sin(\chi_p)|p'(s)| + \sin(\chi_q)|q'(s)| \geq \frac{\epsilon}{\sqrt{2}}(|p'(s)| + |q'(s)|). \tag{2.5}$$

For such (p, q) the vector field (p', q') is smooth with length bounded from above and below. For finite $s = s_0$ either the geodesic shrinks to point and p becomes equal to q, or we reach a pair (p, q) connected by a geodesic in $\hat{N}$ such that either the geodesic touches the boundary of $\{\hat{\gamma}_{\mathrm{f}}(p,t) \mid p \in M, -\frac{c}{2} \leq t \leq c_{\mathrm{f}}(p)\}$ or $(\sin^2(\chi_p) + \sin^2(\chi_q))^{\frac{1}{2}} \leq \epsilon$ holds. We integrate (2.5) to

$$\mathrm{dist}_M(p(s_0), p(0)) + \mathrm{dist}_M(q(s_0), q(0)) \leq \frac{\sqrt{2}}{\epsilon}\left(\mathrm{dist}_{\hat{N}}(p(0), q(0)) - \mathrm{dist}_{\hat{N}}(p(s_0), q(s_0))\right). \tag{2.6}$$

Again we decompose the final geodesic in geodesic segments obeying either **(A)**, **(B)** or **(C)**. To sum up the application of the gradient flow transforms the geodesic from p to q obeying **(C)** into two paths $s \mapsto p(s)$ from $p(0) = p$ to $p(s_0)$ and $s \mapsto q(s)$ from $q(0) = q$ to $q(s_0)$ in M and several geodesic segments in $\hat{N}$ from $p(s_0)$ to $q(s_0)$ obeying either **(A)**, **(B)** or **(C)**. Due to (2.6) the sum of the lengths of paths in M times $\frac{\epsilon}{\sqrt{2}}$ plus the lengths of the geodesic segments is smaller than the length of the original geodesic.

We iterate the applications of (2.2) to pairs (p, q) obeying **(A)**, the applications of Lemma 2.2 with (2.4) to pairs (p, q) obeying **(B)** and the applications of the gradient flow with (2.6) to pairs obeying **(C)**. Any application of Lemma 2.2 reduces the length $\mathrm{dist}_{\hat{N}}(p, q)$ by a number $\geq \frac{1}{2}\delta s_0^2 \cos(\frac{d}{2})$, where $0 < d < \pi$ denotes the length of the original geodesic. Any application of the gradient flow which ends at a pair obeying **(B)** results in an application of Lemma 2.2. All other applications of the gradient flow reduces $\mathrm{dist}_{\hat{N}}(p, q)$ by a number $\geq \frac{\epsilon c}{2\sqrt{2}}$. Therefore finitely many applications of Lemma 2.2 and the gradient flow transform the geodesic connecting p and q into finitely many paths in M, which connect p with q and whose total length is bounded by $C\,\mathrm{dist}_N(p, q)$ with $C = \max\{2, \frac{12}{\delta(2s_0 - 3s_0^2)}, \frac{\sqrt{2}}{\epsilon}\}$. □

3. Collar pertubation

In this section we localize the concept of a mean-convex Alexandrov embedding. We consider in the following open subsets V of M and open bounded 3-dimensional manifolds W with boundary V. More precisely, we denote by $\partial W = V$ the boundary as defined within the concept of manifolds with boundary. It is in general a subset of the topological boundary.

Definition 3.1. *We call the restriction* $\mathrm{f}|_V$ *of* $\mathrm{f} : M \to \mathbb{S}^3$ *to an open subset* $V \subset M$ *a local mean-convex Alexandrov embedding if* $\mathrm{f}|_V$ *extends as an immersion to an open 3-manifold* W *with* $V = \partial W$ *such that the following hold:*

(i) *The mean curvature of* V *in* $\mathbb{S}^3$ *with respect to the inward normal is non-negative everywhere.*
(ii) *All inward* V*-geodesics exist in* W *until they reach the cut locus* (1.1) *(for* $t \leq \frac{\pi}{2}$*).*
(iii) $W = \{\gamma_{\mathrm{f}}(p,t) \mid p \in V$ *and* $0 < t \leq c_{\mathrm{f}}(p)\}$*.*

If $\mathrm{f} : M \to \mathbb{S}^3$ is a mean-convex Alexandrov embedding which satisfies the chord-arc bound (2.1), then all open subsets $V \subset M$ such that for all $p \in V$ the classes $[p]_{\mathrm{f}} \subset V$ are examples of local mean-convex Alexandrov embeddings. To see that we only have to show that

$$W = \{\gamma_{\mathrm{f}}(p,t) \in N \mid p \in V \text{ and } 0 < t \le c_{\mathrm{f}}(p)\}$$

is open in N. Since V is open, W is open at all points away from the cut locus. Consider a cut point $c_{\mathrm{f}}(p)$ and a sequence of $(q_n) \in N$ converging to $c_{\mathrm{f}}(p)$. We prove that $(q_n) \in W$ for n large enough. Let $q_n = \gamma_{\mathrm{f}}(p_n, t) \in N$ with $p_n \in M$. By the chord-arc bound the set $[p]_{\mathrm{f}}$ is bounded in M, and there is a subsequence of p_n converging to an element of $[p]_{\mathrm{f}} \subset V$. This proves that $p_n \in V$ for n large enough and $q_n \in W$. Hence W is an open neighborhood around cut points. Since W is an open subset of N with $V = \partial W$, $\mathrm{f}_{|V}$ extends naturally as an immersion to W by restriction of the extension of f to W.

We shall prove that 'mean-convex Alexandrov embeddedness' is an open condition, which will allow us to study deformation families of mean-convex Alexandrov embeddings. The main tool is a general perturbation technique of Alexandrov embeddings, which we call collar perturbation. We consider local perturbations $\tilde{\mathrm{f}}$ of a given smooth immersion $\mathrm{f} : M \to \mathbb{S}^3$, which are 'small' with respect to the C^1-topology on the space of immersions from M into $\mathbb{S}^3$.

Lemma 3.2. *For given $c > 0$ and $C' > 0$ there exist $\epsilon > 0$ and $R > 0$ with the following property: Let $\mathrm{f} : M \to \mathbb{S}^3$ be a mean-convex Alexandrov embedding with cut locus function c_{f}* (1.1) *bounded from below by c and second fundamental form obeying* (0.1)*, and let $\tilde{\mathrm{f}} : M \to \mathbb{S}^3$ be an immersion with non negative mean curvature and principal curvatures bounded by $\kappa_{\max} =$* $\cot(c)$*. Furthermore, let both immersions* f *and* $\tilde{\mathrm{f}}$ *obey*

$$\|\mathrm{f} - \tilde{\mathrm{f}}\|_{\mathcal{C}^3(B_p)} < \epsilon \quad \text{with } B_p = B(p,r) \tag{3.1}$$

Here $p \in M$ is some point and $B(p,r) \subset M$ a ball with respect to the metric induced by $\tilde{\mathrm{f}}$. Then the restriction $\tilde{\mathrm{f}}|_V$ of $\tilde{\mathrm{f}}$ to an open neighbourhood $V \subset M$ of p extends to a local mean-convex Alexandrov embedding with $V \subset \partial W$.

Proof. The generalized cylinder coordinates (1.2) define a diffeomorphism γ_{f} of $M \times [0, c)$ onto an open subset of N, which is a collar. Any lower bound on the cut locus function (1.1) is also a lower bound on the distance to the first focal point on the inward M-geodesics. Since f is a mean-convex Alexandrov embedding, the absolute values of the negative principal curvatures are smaller than the positive principal curvatures. Consequently, due to the formula (1.4), the distances to the first focal points on the outward M-geodesics are not smaller than the distances to the first focal points on the inward M-geodesics. Hence the normal variation defines an immersion of $M \times (-c, c)$ into $\mathbb{S}^3$. In particular, the induced metric makes $M \times (-c, c)$ into a Riemannian manifold with constant sectional curvature equal to one. For all elements of this manifold the cylinder coordinates, that is the distances to $M \simeq M \times \{0\}$ and the nearest point in M are uniquely defined. Hence we can glue $M \times (-c, c)$ along $\gamma_{\mathrm{f}}(M \times [0, c))$ to N, and obtain a larger 3-manifold $\hat{N} \supset N$ without boundary, such that the generalised cylinder coordinates (1.2) extend to a diffeomorphism $\hat{\gamma}_{\mathrm{f}} : M \times (-c, c) \to \hat{N}$ and the immersion $\mathrm{f} : N \to \mathbb{S}^3$ extends to an immersion $\hat{\mathrm{f}} : \hat{N} \to \mathbb{S}^3$.

An immersion $\tilde{\mathrm{f}}|_{B(p,r)}$ which satisfies the inequality (3.1) is a normal exponential graph over $\mathrm{f}|_{B(p,r)}$. Hence $\tilde{\mathrm{f}}|_{B(p,r)}$ is an embedded submanifold in $\widehat{N} \cap \hat{\gamma}_{\mathrm{f}}(M \times (-\epsilon, \epsilon))$. We denote this graph by O_1. We shall identify $\tilde{\mathrm{f}}|_{B(p,r)}$ with the immersion $\hat{\mathrm{f}}|_{O_1}$ and we can smooth $\hat{\mathrm{f}}|_{O_1}$ with $\mathrm{f} : M \to \widehat{N} \cap \hat{\gamma}_{\mathrm{f}}(M \times (-\epsilon, \epsilon))$ in the intermediate region $B(p,r) \backslash B(p,R)$ with $R = r/2$. This define a new graph O_2 which extend O. By identification with $\hat{\mathrm{f}}$ we obtain an extension of $\hat{\mathrm{f}}|_O$ to a local perturbation of $\mathrm{f} : M \to \mathbb{S}^3$ wich extend to an immersion of a restriction of $\hat{N}$. We denote the restriction $\hat{N}$ by U.

Due to (3.1) $\hat{\mathrm{f}}$ induces on O a Riemannian metric denoted by dist_O and $\tilde{\mathrm{f}}|_{B(p,R)}$ extends as an immersion to U by this identification with $\hat{\mathrm{f}}$.

For all $q \in O$ let $t \mapsto \gamma_{\tilde{\mathrm{f}}}(q,t)$ denote the inward O-geodesics in $\hat{N}$. These inward O-geodesics exist for $0 \le t \le c - \epsilon$. The union

$$\{\gamma_{\tilde{\mathrm{f}}}(q,t) \mid q \in O_1 \text{ and } 0 \le t \le c-\epsilon\} \cup \{\gamma_{\mathrm{f}}(q,t) \mid q \in M \text{ and } \epsilon < t \le c_{\mathrm{f}}(q)\} \subset U$$

For sufficiently small ϵ/c this subset of U is a connected manifold. Its boundary contains O. Let dist_U denote the distance function of this Riemannian manifold. Due to Proposition 2.1 the immersion f obeys a chord-arc bound with constant C depending only on c and C'. Due to the second inequality of (3.1) the Riemannian metrics induced by f and $\tilde{\mathrm{f}}$ are bounded in terms of each other with a constant depending only on ϵ. For sufficiently small ϵ we obtain the bound

$$\mathrm{dist}_O(q,q') \le C''(\mathrm{dist}_U(q,q') + 2\epsilon) \quad \text{for all } q,q' \in O$$

with C'' depending only on c and C' (and $C'' \ge C$). When $\mathrm{dist}_U(q,q') \ge 2\epsilon$, this inequality gives a chord-arc bound. Now assume that $\mathrm{dist}_U(q,q') \le 2\epsilon$. Since the curvature is uniformly bounded on the cylinder, the surface is locally a bounded graph on a totally geodesic disc of radius $\delta > 0$ tangent at q to the graph. The gradient of this graph is uniformly bounded on the δ-disc. Hence for small enough $\epsilon > 0$, there is a constant C_0 depending on the bound of the curvature such that if $P : O \to \mathbb{S}^2$ is the orthogonal projection on this totally geodesic sphere, we have $\mathrm{dist}(P(q), P(q')) \le (1 + C_0\epsilon)\,\mathrm{dist}_U(q,q')$. Hence for some $\epsilon > 0$ small enough, gradient estimate imply

$$\mathrm{dist}_O(q,q') \le 2\,\mathrm{dist}_U(q,q') \quad \text{for all } q,q' \in O \quad \text{with } \mathrm{dist}_U(q,q') \le 2\epsilon.$$

These two bounds imply the chord-arc bound

$$\mathrm{dist}_O(q,q') \le \tilde{C}\,\mathrm{dist}_U(q,q') \quad \text{for all } q,q' \in O$$

with $\tilde{C} = 2C'' \ge 2$. Let $R = 3\tilde{C}(\pi + \epsilon)$ and the open geodesic balls $O'' = B(p, \frac{R}{3}) \subset O' = B(p, \frac{2R}{3}) \subset O$.

All cut locus functions of mean-convex Alexandrov embeddings are uniformly bounded from above by $\frac{\pi}{2}$, since otherwise a sphere with negative principal curvatures would touch M inside of N contradicting Hopf's maximum principle. The distance of two points of O', whose inward O-geodesics intersect at distances not larger than $\frac{\pi}{2}$, is not larger than $\tilde{C}\pi$. For all $q \in O''$, we have

$$\left\{q' \in O \mid \exists t \in [0,\, \tfrac{\pi}{2}] \text{ with } \mathrm{dist}_U(\gamma_{\tilde{\mathrm{f}}}(q,\, t),\, q') \le t\right\} \subset \left\{q' \in O \mid \mathrm{dist}_U(q,\, q') \le \pi\right\} \subset O'\,.$$

Therefore, for all $q \in O''$ the cut locus function $c_{\tilde{\mathrm{f}}}$ is well defined. For all such $q \in O''$, let $[q]_{\tilde{\mathrm{f}}}$ denote the set

$$[q]_{\tilde{\mathrm{f}}} = \left\{q' \in O \mid \mathrm{dist}_{\hat{N}}\left(\gamma_{\tilde{\mathrm{f}}}(q, c_{\tilde{\mathrm{f}}}(q)),\, q'\right) = c_{\tilde{\mathrm{f}}}(q)\right\}.$$

For any closed subset $A \subset O$ the set $\{q \in O'' \mid [q]_{\tilde{\mathrm{f}}} \cap A \neq \emptyset\}$ is a closed subset of O''. Hence $V = \{q \in O'' \mid [q]_{\tilde{\mathrm{f}}} \subset O''\}$ is an open subset of O. Furthermore $W = \{\gamma_{\tilde{\mathrm{f}}}(q,t) \mid q \in V \text{ and } 0 \le t \le c_{\tilde{\mathrm{f}}}(q)\}$ is a subset of $\hat{N}$ with boundary V. By construction $\hat{\mathrm{f}}_{|V}$ is a local mean-convex Alexandrov embedding with $[q]_{\tilde{\mathrm{f}}} \subset V$. By choice of R, V is an open neighbourhood of p in O. □

4. From local to global mean-convex Alexandrov embeddings

In this section we consider an immersion $\tilde{\mathrm{f}} : M \to \mathbb{S}^3$ which is covered by a family of local mean-convex Alexandrov embedded neighborhoods $\tilde{\mathrm{f}}_p : V_p \to \mathbb{S}^3$ of $p \in M$. We explain how to extend $\tilde{\mathrm{f}} : M \to \mathbb{S}^3$ to the 3-manifold $N = \cup_{p \in M} W_p$, with $\partial N = M$. The set of local immersions $\tilde{\mathrm{f}}_p$ are constructed by local collar perturbations.

Proposition 4.1. *For given $c > 0$ and $C' > 0$ there exist $\epsilon > 0$ and $R > 0$ with the following property: Let $\tilde{\mathrm{f}} : M \to \mathbb{S}^3$ be an immersion with non negative mean curvature and principal curvatures bounded by $\kappa_{\max} = \cot(c)$ such that for all $p \in M$ there exists a mean-convex Alexandrov embedding $\mathrm{f}_p : M \to \mathbb{S}^3$ with cut locus function* (1.1) *bounded from below by c and second fundamental form obeying* (0.1)*. If $\tilde{\mathrm{f}}$ and f_p obey* (3.1) *on $B(p, R) \subset M = \partial N_p$ for all $p \in M$, then $\tilde{\mathrm{f}}$ extends to a mean-convex Alexandrov embedding $\tilde{\mathrm{f}} : \tilde{N} \to \mathbb{S}^3$ with $\partial\tilde{N} = M$.*

Proof. Let C and $\tilde{C}$ denote the constants as in Lemma 3.2. We apply Lemma 3.2 to all $p \in M$ with the same $R = 3\tilde{C}(\pi + \epsilon)$ and decorate the corresponding objects with an index p. The balls $B(p, R)$ are embedded as Riemannian submanifolds $O_p \hookrightarrow \hat{N}_p$. We obtain a covering of M by open subsets $V_p = \{q \in O''_p \mid [q]_{\tilde{\mathrm{f}}} \subset O''_p\}$ of the balls $O''_p = B(p, \frac{R}{3})$ with respect to the metric induced by $\tilde{\mathrm{f}}$. Furthermore, the restrictions $\tilde{\mathrm{f}}|_{V_p}$ of $\tilde{\mathrm{f}}$ to the members V_p of this covering extend to local mean-convex Alexandrov embeddings $\tilde{\mathrm{f}}_p : W_p \to \mathbb{S}^3$ with open subsets $W_p \subset U_p$.

We shall glue the manifolds $(W_p)_{p\in M}$ to obtain a 3-manifold $\tilde{N}$ with boundary M. It suffices to consider $p, q \in M$ with $\mathrm{dist}_M(p, q) < \epsilon$. The intersection $O''_{pq} = O''_p \cap O''_q$ is a connected subset of the Riemannian manifold M with the metric induced by $\tilde{\mathrm{f}}$. The immersions $\tilde{\mathrm{f}}_p$ and $\tilde{\mathrm{f}}_q$ define on $U_p \subset \hat{N}_p$ and $U_q \subset \hat{N}_q$ chord-arc distance $\mathrm{dist}_{U_p}(p', q')$ and $\mathrm{dist}_{U_q}(p', q')$. Now define $R(t) = \tilde{C}(3(\pi + \epsilon) - t)$, and set the related shrunk open sets $O_p(t), O_q(t)$ and $O_{pq}(t) = O_p(t) \cap O_q(t)$. Note that $O_{pq}(t) \subset O_p \cap O_q$. Denote

$$A_p(t) = \{(p', q') \in \bar{O}_{pq}(t) \times \bar{O}_{pq}(t) \mid \mathrm{dist}_{U_p}(p', q') = \mathrm{dist}_{\mathbb{S}^3}(\tilde{\mathrm{f}}(p'), \tilde{\mathrm{f}}(q')) = t\}\,,$$
$$A_q(t) = \{(p', q') \in \bar{O}_{pq}(t) \times \bar{O}_{pq}(t) \mid \mathrm{dist}_{U_q}(p', q') = \mathrm{dist}_{\mathbb{S}^3}(\tilde{\mathrm{f}}(p'), \tilde{\mathrm{f}}(q')) = t\}\,.$$

If $(p', q') \in A_p(t)$ then p' and q' are connected by a geodesic segment lying in U_p which is mapped by $\tilde{\mathrm{f}}_p$ to the unique shortest geodesic in $\mathbb{S}^3$ from $\tilde{\mathrm{f}}(p')$ to $\tilde{\mathrm{f}}(q')$. By Lemma 3.2 the points in O_p obey in U_p a chord-arc bound with constant $\tilde{C}$. If the geodesic segment in U_p from p' to q' meets a point $p'' \in M$, then $p'' \in O_{pq}(\mathrm{dist}_{U_p}(p', p''))$. The chord-arc bound implies that $\mathrm{dist}_{O_p}(p', p'') \le \tilde{C}\,\mathrm{dist}_{U_p}(p', p'')$ and $\mathrm{dist}_{O_p}(p'', q') \le \tilde{C}\,\mathrm{dist}_{U_p}(p'', q')$ and

$$\mathrm{dist}_{O_p}(p, p'') \le \mathrm{dist}_{O_p}(p, q') + \mathrm{dist}_{O_p}(q', p'') \le R(\mathrm{dist}_{U_p}(p', p'')).$$

By definition of the sets $A_p(t)$ this implies that $(p', p'') \in A_p(\mathrm{dist}_{U_p}(p', p''))$ and $(p'', q') \in A_p(\mathrm{dist}_{U_p}(p'', q'))$.

Claim 1. $A_p(t) = A_q(t)$ for all $t \in [0, \pi]$.

We shall prove this claim later, and first show that it implies that $c_{\tilde{\mathrm{f}}_p}$ and $c_{\tilde{\mathrm{f}}_q}$ coincide on $V_p \cap V_q$. Let us assume on the contrary $c_{\tilde{\mathrm{f}}_p}(p') < c_{\tilde{\mathrm{f}}_q}(p')$ for $p' \in V_p \cap V_q$. The domain U_p contains a ball of radius $c_{\tilde{\mathrm{f}}_p}(p')$ centered at $\gamma_{\tilde{\mathrm{f}}_p}(p', c_{\tilde{\mathrm{f}}_p}(p'))$ (analogously with U_q and $c_{\tilde{\mathrm{f}}_q}(p')$).

Then there exists $p' \ne q' \in [p']_{\tilde{\mathrm{f}}_p} \subset O''_p \cap O'_q$. Then p' and q' are connected by a segment of a geodesic in U_p, which meets the boundary only at the end points p' and q' by strict convexity of a geodesic ball centered at the cut point. Therefore there does not exists a point $p'' \in O_p$ with $\mathrm{dist}_{U_p}(p', q') = \mathrm{dist}_{U_p}(p', p'') + \mathrm{dist}_{U_p}(p'', q')$. Due to the claim 1 there does not exist $p'' \in O'_q$ with $\mathrm{dist}_{U_q}(p', q') = \mathrm{dist}_{U_q}(p', p'') + \mathrm{dist}_{U_q}(p'', q')$. Therefore the shortest path in U_q connecting p' and q' is also a segment of a geodesic, which meets the boundary only at the end point p' and q'. Furthermore, both immersions $\hat{\mathrm{f}}_p$ and $\hat{\mathrm{f}}_q$ map these segments onto the same geodesic in $\mathbb{S}^3$ connecting $\tilde{\mathrm{f}}(p')$ and $\tilde{\mathrm{f}}(q')$. Since p' and q' both belong to $[p']_{\tilde{\mathrm{f}}_p}$ there exists a unique geodesic 2-sphere in $\mathbb{S}^3$, which intersects the image of $\tilde{\mathrm{f}}$ orthogonally at $\tilde{\mathrm{f}}(p')$ and at $\tilde{\mathrm{f}}(q')$. Hence there exists in U_p a geodesic 2-sphere, which intersects O_p orthogonally at p' and q'. The pre-image of this 2-sphere in U_q intersects O_q orthogonally at p' and q', and contains a segment of a geodesic connecting p' and q'. This segment of a geodesic is in U_q, because it is contained in a geodesic ball of larger radius and centered at the cut point of p' in U_q. This implies that the inward O_q-geodesics at p' and q' also is contained in the larger geodesic ball of U_q and they meet each other at distance $c_{\tilde{\mathrm{f}}_p}(p') = c_{\tilde{\mathrm{f}}_q}(q')$ in contradiction to $c_{\tilde{\mathrm{f}}_p}(p') < c_{\tilde{\mathrm{f}}_q}(p')$. Interchanging p and q we get the other inequality, and thus both cut locus functions $c_{\tilde{\mathrm{f}}_p}$ and $c_{\tilde{\mathrm{f}}_q}$ coincide on $V_p \cap V_q$.

We claim that the open set $V_{pq} = V_p \cap V_q$ extends in two different but isometric extensions $Z_p \subset W_p$ and $Z_q \subset W_q$ as local mean-convex Alexandrov embeddings with

$$Z_p = \{\gamma_{\tilde{\mathrm{f}}_p}(p', t) \in W_p \mid p' \in V_{pq} \text{ and } 0 < t \leq c_{\tilde{\mathrm{f}}_p}(p')\},$$
$$Z_q = \{\gamma_{\tilde{\mathrm{f}}_q}(q', t) \in W_q \mid q' \in V_{pq} \text{ and } 0 < t \leq c_{\tilde{\mathrm{f}}_q}(q')\}.$$

We identify Z_p with Z_q by a diffeomorphism $\psi : Z_p \to Z_q$ such that its restriction $\psi|_{V_{pq}}$ to V_{pq} is the identity and $\tilde{\mathrm{f}}_p|_{Z_p} = \tilde{\mathrm{f}}_q|_{Z_q} \circ \psi$. A distance preserving bijection $\psi : Z_p \to Z_q$ between two Riemannian manifold is a diffeomorphism by a theorem of Myers-Steenrod [9] (see also [10], Theorem 18, p 147).

We define ψ by $\psi \circ \gamma_{\tilde{\mathrm{f}}_p}(p', t) = \gamma_{\tilde{\mathrm{f}}_q}(p', t)$ for $p' \in V_{pq}$. This map is well-defined if $c_{\tilde{\mathrm{f}}_p}(p') = c_{\tilde{\mathrm{f}}_q}(p')$ for all $p' \in V_{pq}$. This indeed implies that for $p' \in V_{pq}$ the equivalence classes $[p']_{\tilde{\mathrm{f}}_p}$ and $[p']_{\tilde{\mathrm{f}}_q}$ (1.3) of the local mean-convex Alexandrov embeddings $\tilde{\mathrm{f}}_p : W_p \to \mathbb{S}^3$ and $\tilde{\mathrm{f}}_q : W_q \to \mathbb{S}^3$ coincide. Hence we identify the class $[p']_{\tilde{f}} \in V_{pq}$ and then ψ is a bijection. By construction the exponential coordinates obey $\tilde{\mathrm{f}}_{p|Z_p} = \tilde{\mathrm{f}}_{q|Z_q} \circ \psi$. Since the immersions $\tilde{\mathrm{f}}_p$ and $\tilde{\mathrm{f}}_q$ induce the metrics of Z_p and Z_q, the map ψ preserves distances.

The union of mean-convex Alexandrov embeddings extend to a manifold N. It remains to show that N is complete with respect to the Riemannian metric induced by $\tilde{\mathrm{f}}$. For all $r < c$, the submanifolds

$$\{\gamma_{\tilde{\mathrm{f}}_p}(q, t) \in \hat{N}_p \mid q \in M \text{ and } -r \leq t \leq c_{\tilde{\mathrm{f}}_p}(q)\} \subset \hat{N}_p$$

are complete with respect to the Riemannian metric induced by $\hat{\mathrm{f}}_p$. By construction, every point of the Riemannian manifold N with the metric induced by $\tilde{\mathrm{f}}$ is the center of an ϵ-ball contained in one of these complete submanifolds of $\hat{N}_p$. Therefore N is complete.

Proof of Claim 1. We consider the set B of all $t_0 \in [0, \pi]$ such that $A_p(t) = A_q(t)$ holds for all $t \in [0, t_0]$. We prove that B is both closed and open in $[0, \pi]$, and thus $B = [0, \pi]$. Due to the lower bound c of the cut locus function, B contains the set $[0, c - \epsilon)$.

Set $t_0 = \sup B$, and suppose $(p', q') \in A_p(t_0)$. We need to consider two cases. In the first case we assume that the unique geodesic in U_p from p' to q' goes through a point $p'' \in O_p \setminus \{p', q'\}$. This implies that

$$(p', p'') \in A_p(\mathrm{dist}_{U_p}(p', p'')) = A_q(\mathrm{dist}_{U_p}(p', p'')) \text{ and}$$
$$(p'', q') \in A_p(\mathrm{dist}_{U_p}(p'', q')) = A_q(\mathrm{dist}_{U_p}(p'', q')).$$

Both geodesics in U_q which connect p' to p'', and p'' to q', are mapped by $\tilde{\mathrm{f}}_q$ onto the unique geodesic in $\mathbb{S}^3$ connecting $\tilde{\mathrm{f}}(p')$ to $\tilde{\mathrm{f}}(q')$. This implies that p' and q' are connected in U_q by a smooth geodesic, and hence $(p', q') \in A_q(t_0)$. This proves $A_p(t_0) \subset A_q(t_0)$, and analogously also the other inclusion, and therefore $A_p(t_0) = A_q(t_0)$.

For the second case we may assume that there is no point $p'' \in O_p \setminus \{p', q'\}$ on the geodesic in U_p from p' to q'. The mean-convexity of the surface implies that (p', q') is not a local minimum of the function dist_{U_p} on $O_p \times O_p$ (see Lemma 2.2). Then there exists a sequence of (p_n, q_n) which converges to (p', q') such that $\mathrm{dist}_{U_p}(p_n, q_n) < t_0$. The continuity of dist_{U_p} and dist_{U_q}

imply that

$$\mathrm{dist}_{U_q}(p',q') = \lim_{n\to\infty} \mathrm{dist}_{U_q}(p_n,\, q_n) = \lim_{n\to\infty} \mathrm{dist}_{U_p}(p_n,\, q_n) = \mathrm{dist}_{U_p}(p',q')\,.$$

Hence $(p',\, q') \in A_q(t_0)$, and thus $A_p(t_0) \subset A_q(t_0)$. The other inclusion is obtained by switching the roles of p and q. Therefore $A_p(t_0) = A_q(t_0)$ in both cases, which shows $t_0 \in B$, and proves that B is closed.

We now show that B is open. If the maximum t_0 is smaller than π, then there exists a sequence $t_n \in (t_0,\, \pi]$ which converges to t_0 such that $A_p(t_n) \neq A_q(t_n)$. By passing to a subsequence we may assume without loss of generality that there exists $(p_n,\, q_n) \in A_p(t_n)$ but $(p_n,\, q_n) \notin A_q(t_n)$. A subsequence of $(p_n,\, q_n)$ converges to $(p',\, q') \in A_p(t_0) = A_q(t_0)$. Therefore p' and q' are connected by smooth geodesic segments in both U_p and U_q. Both of these geodesic segments are mapped by $\tilde{\mathrm{f}}_p$ and $\tilde{\mathrm{f}}_q$ to the unique shortest geodesic in $\mathbb{S}^3$ from $\tilde{\mathrm{f}}(p')$ to $\tilde{\mathrm{f}}(q')$. Furthermore, both these geodesic segments meet the boundaries O_p and O_q in the same points. The balls of radii $c-\epsilon$ around each of such boundary points in U_p and U_q are isometric. In the complement of these balls both geodesic segments have positive distances to the boundaries O_p and O_q. Hence two tubular neighbourhoods of these geodesic segments in U_p and U_q are isometric as well.

For large n the geodesic segments in U_p connecting p_n with q_n belong to the tubular neighbourhood in U_p. They are isometric to geodesic segments in U_q connecting p_n with q_n. This implies $(p_n,\, q_n) \in A_q(t_n)$ for large n. This contradicts the assumption that $t_0 < \pi$. Hence we have that $B = [0,\, \pi]$, and the claim is proven. □

5. Appendix

We conclude with the proof of Lemma 2.2.

Proof. We shall construct a Killing field ϑ with the desired properties, which rotates γ around two antipodes of γ. The corresponding rotated geodesics $\gamma_\vartheta(s,\cdot)$ belong to a unique geodesic 2-sphere $\mathbb{S}^2 \subset \mathbb{S}^3$. The corresponding paths $s \mapsto p(s)$ and $s \mapsto q(s)$ move along the intersection of this 2-sphere $\mathbb{S}^2$ with M. Hence we can calculate all derivatives on this sphere.

We parameterise this 2-sphere by the real parameter s of the family $s \mapsto \gamma_\vartheta(s,\cdot)$ of rotated geodesics, and the real arc length parameter t of these geodesics. We choose the equator as the points corresponding to $t = 0$ with distance $\frac{\pi}{2}$ to the rotation axis. Let t_p and t_q denote the values of this parameter t at the points $p(s)$ and $q(s)$. Hence the distance $\mathrm{dist}_N(p,q)$ is equal to $\mathrm{dist}_N(p,q) = |t_p - t_q|$. The vector fields ϑ and the geodesic vector field $\dot\gamma$ along the geodesics $\gamma_\vartheta(s,\cdot)$ form an orthogonal basis of the tangent spaces of this 2-sphere away from the zeroes of ϑ. The vector fields ϑ and $\dot\gamma$ have at (s,t) the scalar products

$$g(\vartheta,\vartheta) = \cos^2(t)\,, \qquad g(\vartheta,\dot\gamma) = 0\,, \qquad g(\dot\gamma,\dot\gamma) = 1\,.$$

Since $\dot\gamma$ is a geodesic vector field the derivative $\nabla_{\dot\gamma}\dot\gamma$ vanishes. Moreover, the geodesic curvature in $\mathbb{S}^2$ of the integral curve of ϑ starting at (s,t) is equal to $\tan(t)$. Therefore at (s,t) we have

$$\nabla_\vartheta\vartheta = \cos^2(t)\tan(t)\dot\gamma = \cos(t)\sin(t)\dot\gamma\,, \qquad \nabla_\vartheta\dot\gamma = -\tan(t)\vartheta\,,$$
$$\nabla_{\dot\gamma}\vartheta = -\tan(t)\vartheta\,, \qquad \nabla_{\dot\gamma}\dot\gamma = 0\,.$$

We parameterise a neighbourhood of the geodesic from p to q in such a way that the corresponding vector field $\dot\gamma$ points inward to N at p and outward of N at q, respectively. The derivatives of $s \mapsto p(s)$ and $s \mapsto q(s)$ are equal to

$$p' = \vartheta(p) - \dot\gamma(p)\frac{g(\mathfrak{N}(p),\vartheta(p))}{g(\mathfrak{N}(p),\dot\gamma(p))} \quad \text{and} \quad q' = \vartheta(q) - \dot\gamma(q)\frac{g(\mathfrak{N}(q),\vartheta(q))}{g(\mathfrak{N}(q),\dot\gamma(q))}.$$

The lengths $|p'|$ and $|q'|$ depend on the angles $\sphericalangle(\mathfrak{N}(p),\vartheta(p))$ and $\sphericalangle(\mathfrak{N}(q),\vartheta(q))$. Since ϑ is orthogonal to $\dot\gamma$ and $\sphericalangle(\mathfrak{N}(p),\dot\gamma(p))=\chi_p$ and $\sphericalangle(\mathfrak{N}(q),\dot\gamma(q))=\chi_q$ these angles obey $\sphericalangle(\mathfrak{N}(p),\vartheta(p))\in[\frac{\pi}{2}-\chi_p,\frac{\pi}{2}+\chi_p]$ and $\sphericalangle(\mathfrak{N}(q),\vartheta(q))\in[\frac{\pi}{2}-\chi_q,\frac{\pi}{2}+\chi_q]$. Then we have

$$|p'|\le\frac{|\cos(t_p)|}{\cos(\chi_p)}\qquad\qquad |q'|\le\frac{|\cos(t_q)|}{\cos(\chi_q)}. \tag{5.1}$$

$$d'=\frac{g(\mathfrak{N}(p),\,\vartheta(p))}{g(\mathfrak{N}(p),\,\dot\gamma(p))}-\frac{g(\mathfrak{N}(q),\,\vartheta(q))}{g(\mathfrak{N}(q),\,\dot\gamma(q))}=\frac{g(\mathfrak{N}(p),\,\vartheta(p))}{\cos(\chi_p)}-\frac{g(\mathfrak{N}(q),\,\vartheta(q))}{\cos(\chi_q)}.$$

Along the paths p and q with $X=p'$ and $X=q'$, respectively, we have at (s,t)

$$\begin{aligned}\nabla_X\frac{g(\mathfrak{N},\,\vartheta)}{g(\mathfrak{N},\,\dot\gamma)}&=\frac{g(\nabla_X\mathfrak{N},\,\vartheta)+g(\mathfrak{N},\,\nabla_X\vartheta)}{g(\mathfrak{N},\,\dot\gamma)}-\frac{g(\mathfrak{N},\vartheta)(g(\nabla_X\mathfrak{N},\,\dot\gamma)+g(\mathfrak{N},\,\nabla_X\dot\gamma))}{(g(\mathfrak{N},\,\dot\gamma))^2}\\&=\frac{g(\nabla_X\mathfrak{N},X)+g(\mathfrak{N},\nabla_X\vartheta)}{g(\mathfrak{N},\dot\gamma)}-\frac{g(\mathfrak{N},\vartheta)g(\mathfrak{N},\nabla_X\dot\gamma)}{g(\mathfrak{N},\dot\gamma)^2}\\&=-\frac{\mathfrak{h}(X,\,X)}{g(\mathfrak{N},\dot\gamma)}+\cos(t)\sin(t)+2\tan(t)\left(\frac{g(\mathfrak{N},\,\vartheta)}{g(\mathfrak{N},\,\dot\gamma)}\right)^2.\end{aligned}$$

Hence the second derivative is equal to

$$\begin{aligned}d''=&-\frac{\mathfrak{h}(p',\,p')}{\cos(\chi_p)}-\frac{\mathfrak{h}(q',\,q')}{\cos(\chi_q)}+\frac{\sin(2t_p)-\sin(2t_q)}{2}\\&+2\tan(t_p)\left(\frac{g(\mathfrak{N}(p),\,\vartheta(p))}{g(\mathfrak{N}(p),\,\dot\gamma(p))}\right)^2-2\tan(t_q)\left(\frac{g(\mathfrak{N}(q),\,\vartheta(q))}{g(\mathfrak{N}(q),\,\dot\gamma(q))}\right)^2.\end{aligned}$$

If along the rotation of the geodesic for $s\in[0,s_0]$ the following inequalities are satisfied

$$-\tfrac{\pi}{2}\le t_p\le 0,\quad 0\le t_q\le\tfrac{\pi}{2},\quad \tfrac{c}{2}\le d=t_q-t_p,\quad\text{and}\quad (\sin^2(\chi_p)+\sin^2(\chi_q))^{\frac12}\le\tfrac12, \tag{5.2}$$

then $\min\{\cos(\chi_p),\cos(\chi_q)\}\ge\frac{\sqrt3}{2}$ implies the third inequality of (2.3):

$$|p'|+|q'|\le\frac{\cos(t_p)}{\cos(\chi_p)}+\frac{\cos(t_p)}{\cos(\chi_p)}\le\tfrac{2}{\sqrt3}(\cos(t_p)+\cos(t_q))=\tfrac{4}{\sqrt3}\cos\left(\tfrac{d}{2}\right)\cos\left(\tfrac{t_p+t_q}{2}\right)\le 3\cos(\tfrac{d}{2}).$$

Furthermore, the last two terms of d'' are bounded by

$$\begin{aligned}\left|\tan(t_p)\left(\frac{g(\mathfrak{N}(p),\,\vartheta(p))}{g(\mathfrak{N}(p),\,\dot\gamma(p))}\right)^2\right|&\le\sin(|t_p|)\cos(t_p)\tan^2(\chi_p)\le\frac{\sin(2|t_p|)}{2\cdot3}\\\left|\tan(t_q)\left(\frac{g(\mathfrak{N}(q),\,\vartheta(q))}{g(\mathfrak{N}(q),\,\dot\gamma(q))}\right)^2\right|&\le\sin(|t_q|)\cos(t_q)\tan^2(\chi_q)\le\frac{\sin(2|t_q|)}{2\cdot3}.\end{aligned}$$

Due to $\sin(2t_q)-\sin(2t_p)=2\sin(t_q-t_p)\cos(t_p+t_q)$ and we arrive at

$$d''(s)\le-\frac{\mathfrak{h}(p',\,p')}{\cos(\chi_p)}-\frac{\mathfrak{h}(q',\,q')}{\cos(\chi_q)}-\sin(d)\cos(t_p+t_q)\left(1-\tfrac23\right). \tag{5.3}$$

Now we claim that the second inequality of (2.3) is implied by (5.2) and the existence of δ which satisfy

$$\delta\le\frac19\sin\left(\tfrac{c}{2}\right)\cos(t_p+t_q),\qquad -\frac{\mathfrak{h}(p',\,p')}{|p'|^2}\le\frac{\delta}{2}\quad\text{and}\quad -\frac{\mathfrak{h}(q',\,q')}{|q'|^2}\le\frac{\delta}{2}. \tag{5.4}$$

The assumption (5.2) implies $t_p\le-d+\frac{\pi}{2}$ and $d-\frac{\pi}{2}\le t_q$.

For $d\in[\frac{\pi}{2},\pi)$ we use $\cos(t_p)\le\sin(d)$ and $\cos(t_q)\le\sin(d)$ and for $d\in[\frac{c}{2},\frac{\pi}{2})$ we use $\cos(t_p)\le1$ and $\cos(t_q)\le1$ to obtain

$$\tfrac{1}{2\cdot9}\sin(\tfrac{c}{2})\max\left\{\tfrac{|p'|^2}{\cos(\chi_p)},\tfrac{|q'|^2}{\cos(\chi_q)}\right\}\le\tfrac{3\sqrt3}{8\cdot9}\sin(\tfrac{c}{2})\max\left\{\tfrac{|p'|^2}{\cos(\chi_p)},\tfrac{|q'|^2}{\cos(\chi_q)}\right\}\le\tfrac19\sin(d).$$

Together with (5.4) we can estimate the first two terms in (5.3):

$$-\tfrac{\mathfrak{h}(p',p')}{\cos(\chi_p)} \leq \tfrac{\delta}{2}\tfrac{|p'|^2}{\cos(\chi_p)} \leq \tfrac{1}{9}\sin(d)\cos(t_p+t_q) \qquad -\tfrac{\mathfrak{h}(q',q')}{\cos(\chi_q)} \leq \tfrac{\delta}{2}\tfrac{|q'|^2}{\cos(\chi_q)} \leq \tfrac{1}{9}\sin(d)\cos(t_p+t_q).$$

The third inequality of (5.2) implies $\sin(\frac{c}{2})\cos(\frac{d}{2}) \leq 2\sin(\frac{c}{4})\cos(\frac{d}{2}) \leq 2\sin(\frac{d}{2})\cos(\frac{d}{2}) = \sin(d)$. Thus the second inequality of (2.3) indeed follows with the help of (5.3) from (5.2) and (5.4).

We shall show first that there exists a vector field ϑ obeying at $s=0$

$$\delta \leq \frac{1}{18}\sin\left(\tfrac{c}{2}\right)\cos(t_p+t_q), \qquad -\frac{\mathfrak{h}(p',p')}{|p'|^2} \leq \frac{\delta}{4} \quad \text{and} \quad -\frac{\mathfrak{h}(q',q')}{|q'|^2} \leq \frac{\delta}{4}.$$

The Killing field ϑ is uniquely determined by two choices: firstly, the choice of a geodesic sphere $\mathbb{S}^2 \subset \mathbb{S}^3$, which contains the closed geodesic from p to q, and secondly, a choice of the zeroes of ϑ, or equivalently a choice of the coordinates t_p and t_q with $t_q - t_p = d \mod \pi$. We start with $t_q = -t_p = \frac{d}{2}$ and set $\delta = \frac{1}{18}\sin(\frac{c}{2})$.

Now we choose the 2-sphere $\mathbb{S}^2 \subset \mathbb{S}^3$ which contain the geodesic which connect p and q. This 2-sphere intersect M along curves at p and q. It is uniquely determined either by the line in T_pM tangent to $\mathbb{S}^2$ or by a line in T_qM, which is tangent to $\mathbb{S}^2$. Since f is a mean-convex Alexandrov embedding and both principal curvatures are uniformly bounded by $\kappa_{\max}$, the cone angles of the double cones $\{X \in T_pM \mid \mathfrak{h}(X,X) \geq -\frac{1}{4}\delta|X|^2\}$ and $\{X \in T_qM \mid \mathfrak{h}(X,X) \geq -\frac{1}{4}\delta|X|^2\}$ are not smaller than $\frac{\pi}{2} + \mathrm{O}(\delta)$. For sufficiently small $\epsilon \geq (\sin^2(\chi_p)+\sin^2(\chi_q))^{\frac{1}{2}}$ the tangent direction in the plane orthogonal to $\dot\gamma(p)$ in T_pN, and in the plane orthogonal to $\dot\gamma(q)$ in T_qN of the corresponding spheres build two double cones with cone angles not smaller than $\frac{\pi}{2}$. Hence the intersection of both double cones is non-empty and there exists a 2-sphere where the intersecting curves of $\mathbb{S}^2 \cap M$ at p and q has tangent vectors which satisfy both second and third conditions of (5.4).

Secondly we shall show that the inequalities (5.2) and (5.4) are satisfied for $s \in [0,\, s_0]$ with some $s_0 > 0$. Since the curvature is bounded by $\cot(c)$ and due to the assumption $(\sin^2(\chi_p) + \sin^2(\chi_q))^{\frac{1}{2}} \leq \epsilon$ there exists s_0 such that t_p and t_q do not reach the roots of ϑ for $s \in [0,\, s_0]$. Since the derivatives of $\cos(\chi_p)$, $\cos(\chi_q)$, t_p and t_q with respect to s are uniformly bounded, there exists $s_0 > 0$ such that the inequalities (5.2) and the first inequality of (5.4) are satisfied for $s \in [0,\, s_0]$. Due to (0.1) also the derivatives of $\mathfrak{h}(p',p')$ and $\mathfrak{h}(q',q')$ are uniformly bounded. Hence there exists $s_0 > 0$ only depending on c and C', such that the second and the third inequality of (2.3) are satisfied for $s \in [0,\, s_0]$. We choose $s_0 > 0$ small such that $|d(s) - d(s_0)| \leq c/2$.

Finally we have to satisfy the first inequality of (2.3). At the start point $s = 0$ this is always the case for one choice of the sign of ϑ. Now the second inequality of (2.3) implies the first. □

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