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Minimal surfaces in Riemannian manifolds

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Summary of the thesis

This thesis is devoted to the solution of some problems of various nature about the minimal surfaces.

The study of minimal surfaces in $\mathbb{R}^3$ started with Lagrange in 1762. He studied the problem of determining a graph over an open set W in $\mathbb{R}^2$, with the least possible area among all surfaces that assume given values on the boundary of W.

In 1776, Meusnier supplied a geometric interpretation of the minimal graph equation: the mean curvature H vanishes. Nowadays it has become customary to use the term minimal surface for any surface satisfying $H = 0$, notwithstanding the fact that such surfaces often do not provide a minimum for the area.

In all of the questions I dealed with in this work, one minimal surface plays the key role. It is the Costa-Hoffmann-Meeks surface, the most famous minimal surface. The discovery of the Costa surface was responsible for the rekindling of interest in minimal surfaces in 1982. In that year C. Costa showed the existence of a complete (i.e., it has no boundary) minimal surface of finite topology. It has genus 1 and three ends. D. Hoffman and W. H. Meeks III showed the embeddedness of this surface (i.e. it does not intersect itself). Until that moment the only other known embeddable complete minimal surfaces in $\mathbb{R}^3$ were the plane, the catenoid and the helicoid. They were discovered over two hundred years ago, and it was conjectured that these were the only embedded complete minimal surfaces. Later D. Hoffman and W. H. Meeks III generalized the work of C. Costa showing the existence of a family of complete embedded minimal surfaces with three ends and genus $k \geqslant 1$. We denote by M_k the surface of genus k. It is known as Costa-Hoffman-Meeks of genus k.

An important property of the minimal surfaces is the non degeneracy. The non degeneracy is defined in terms of the space of the Jacobi functions on the surface, that is the functions which belong to the kernel of the Jacobi operator. This operator is defined as the linearized of the mean curvature operator.

J. Pérez and A. Ros showed that the set of the non degenerate properly embedded minimal surfaces with finite total curvature and fixed topology in $\mathbb{R}^3$, has a structure of finite dimensional real-analytic manifold. As application they showed that for M_k with $2 \leqslant k \leqslant 37$, there exists a family of minimal surfaces with three horizontal ends that are obtained by infinitesimal deformations by M_k. This result is based on a work of S. Nayatani which assures the non degeneracy of the Costa-Hoffman-Meeks surface only if its genus assumes the values described above. In his work S. Nayatani computed the dimension of the kernel and the index (i. e. the number of the negative eigenvalues) of

the Jacobi operator about M_k but only if $1 \leqslant k \leqslant 37$. He showed that the dimension of the kernel equals 4. From that it follows the non degeneracy of M_k. In chapter 1 I show that it is possible to extend the result of S. Nayatani for bigger values of k. To be more precise I show that for $k \geqslant 38$ the dimension of the kernel and the index of the Jacobi operator about M_k, are respectively equal to 4 and $2k+3$. That allows us to state that the surface M_k is non degenerate also for $k \geqslant 38$.

The non degeneracy of the surface M_k is one of the essential ingredients of the proof due to L. Hauswirth and F. Pacard of the existence of a new family of examples of minimal surfaces. Thanks to result described in 1, their construction extends automatically to higher values of k. The same result is used in the other sections of the thesis. Without it the constructions that I will describe briefly in the following, would hold only for $k \leqslant 37$.

In chapter 2 following J. Pérez and A. Ros, I show the existence of a family of immersed minimal deformations of M_k for $k \geqslant 1$ having three embedded ends. In difference with M_k, the generic element of this family does not enjoy any property of symmetry. In fact, the admitted deformations are: the rotation about a vertical axis, the translation and dilation of any of the three ends of the surface. In addition, it is possible to bend the two catenoidal type ends and change the type of the middle end obtaining a catenoidal type end.

Here it is the statement of the main result.

Theorem 1. *For each possible choice of the limit values of the normal vectors of the three ends, there is, up to isometries, a 1-dimensional real analytic family of smooth minimal deformations of M_k, for $k \geqslant 1$, letting the middle planar end horizontal.*

The last two chapters of the thesis are devoted to the construction of new families of examples of minimal surfaces. Their construction is based on a gluing procedure which involves the surface M_k.
In chapter 3 I show the existence in the space $\mathbb{H}^2 \times \mathbb{R}$, where $\mathbb{H}^2$ denotes the hyperbolic plane, of a family of minimal examples inspired to M_k. The statement of the main theorem is

Theorem 2. *For all $k \geqslant 1$ there exists in $\mathbb{H}^2 \times \mathbb{R}$ a minimal surface of genus k with three horizontal ends: two catenoidal type ends and a middle planar end.*

I glue the image by a homothety of parameter 2ϵ, with ϵ sufficiently small, of a compact part of M_k along its three boundary curves to two minimal graphs that are respectively asymptotic to an upper half catenoid and a lower half catenoid defined in $\mathbb{H}^2 \times \mathbb{R}$ and to a minimal graph asymptotic to $\mathbb{H}^2 \times \{0\}$.

The chapter 4 is devoted to the construction of two new families of examples of periodic minimal surfaces with genus bigger than 1. This result has been obtained in collaboration with Laurent Hauswirth and M. Magdalena Rodríguez Pérez.

We denote by $\mathcal{K} = \{M_{\sigma,\alpha,\beta}\}_{\sigma,\alpha,\beta}$ the family of minimal surfaces called Karcher-Meeks-Rosenberg examples or toroidal halfplane layers. They have been classified as the only doubly periodic minimal surfaces in $\mathbb{R}^3$ with genus one and finitely many parallel Scherk-type ends in the quotient. We denote by $\widetilde{M}_{\sigma,\alpha,\beta}$ the lifting of $\{M_{\sigma,\alpha,\beta}\}_{\sigma,\alpha,\beta}$ to $\mathbb{S}^1 \times \mathbb{R}^2$ by forgetting its non horizontal period.

The construction is based on a gluing procedure which involves a compact part of the surface M_k with the catenoidal type ends slightly bent, which is glued with a minimal graph about a strip of finite breadth along its middle boundary curve, to one half of $\widetilde{M}_{\sigma,\alpha,\beta}$, (one time with $\alpha = 0$ and a second time with $\beta = 0$) along the upper boundary curve and a Scherk type surface along the lower boundary curve.

We obtain two families of properly embedded minimal surfaces in $\mathbb{S}^1 \times \mathbb{R}^2$ with genus $k \geqslant 1$, infinitely many parallel Scherk-type and two limit ends.

Chapter 1

Index and nullity of the Gauss map of the Costa-Hoffman-Meeks surfaces

Introduction

In the years 80's and 90's the study of the index of minimal surfaces in Euclidean space has been quite active. D. Fisher-Colbrie in [8], R. Gulliver and H. B. Lawson in [10] proved independently that a complete minimal surface M in $\mathbb{R}^3$ with Gauss map G has finite index if and only if it has finite total curvature. D. Fisher-Colbrie also observed that if M has finite total curvature its index coincides with the index of an operator $L_{\bar{G}}$ (that is the number of its negative eigenvalues) associated to the extended Gauss map $\bar{G}$ of $\bar{M}$, the compactification of M. Moreover $N(\bar{G})$, the null space of $L_{\bar{G}}$, if restricted to M consists of the bounded solutions of the Jacobi equation. The nullity, $\mathrm{Nul}(\bar{G})$, that is the dimension of $N(\bar{G})$, and the index are invariants of $\bar{G}$ because they are independent of the choice of the conformal metric on $\bar{M}$.

The computation of the index and of the nullity of the Gauss map of the Costa surface and of the Costa-Hoffman-Meeks surface of genus $g = 2, \ldots, 37$ appeared respectively in the works [30] and [29] of S. Nayatani. The aim of this work is to extend his results to the case where $g \geqslant 38$.

In [30] he studied the index and the nullity of the operator L_G associated to an arbitrary holomorphic map $G : \Sigma \to S^2$, where Σ is a compact Riemann surface. He considered a deformation $G_t : \Sigma \to S^2$, $t \in (0, +\infty)$, with $G_1 = G$ (see equation (1.2)) and gave lower and upper bounds for the index of G_t, $\mathrm{Ind}(G_t)$, and its nullity, $\mathrm{Nul}(G_t)$, for t near to 0 and $+\infty$ and $t = 1$. The computation of the index and the nullity in the case of the Costa surface is based on the fact that the Gauss map of this surface is a deformation for a particular value of t of the map G defined by $\pi \circ G = 1/\wp'$, that is its stereographic projection is equal to the inverse of the derivative of the Weierstrass $\wp$-function for an unit square lattice. S. Nayatani computed $\mathrm{Ind}(G_t)$ and $\mathrm{Nul}(G_t)$ for $t \in (0, +\infty)$, where G is the map defined above. So the result concerning the Costa surface follows as a simple consequence from that. He obtained that for this surface the index and the nullity are equal respectively to 5 and 4.

In [29] S. Nayatani extended the last result treating the case of the Costa-Hoffman-Meeks surface of genus g but only for $2 \leqslant g \leqslant 37$. He obtained that the index is equal to $2g+3$ and the nullity is equal to 4. Here we will show that these results continue to hold also for $g \geqslant 38$.

J. Pérez and A.Ros in [34] call a minimal surface non degenerate if the bounded Jacobi functions about the surface are induced by the isometries of the ambient space. As consequence of the works [29] and [30], the Costa-Hoffman-Meeks surface was known to be non degenerate with respect to this definition, but only for $1 \leqslant g \leqslant 37$.

The result of S. Nayatani about the nullity of the Gauss map of the Costa-Hoffman-Meeks surface is essential for the construction due to L. Hauswirth and F. Pacard [11] of a family of minimal surfaces with two limit ends asymptotic to half Riemann minimal surfaces and of genus g with $1 \leqslant g \leqslant 37$. Their construction is based on a gluing procedure which involves the Costa-Hoffman-Meeks surface of genus g and two half Riemann minimal surfaces. In particular the authors needed show the existence of a family of minimal surfaces close to the Costa-Hoffman-Meeks surface, invariant under the action of the symmetry with respect to the vertical plane $x_2 = 0$, having one horizontal end asymptotic to the plane $x_3 = 0$ and having the upper and the lower end asymptotic (up to translation) respectively to the upper and the lower end of the standard catenoid whose axis of revolution is directed by the vector $\sin\theta\, e_1 + \cos\theta\, e_3$, $\theta \leqslant \theta_0$ with θ_0 sufficiently small. That was obtained by Schauder fixed point theorem and using the fact that the nullity of the Gauss map of the surface is equal to 4. In [11] the authors refer to this last result as a non degeneracy property of the Costa-Hoffman-Meeks surface. It is necessary to remark that here the choice of working with symmetric deformations of the surface with respect to the plane $x_2 = 0$, has a key role. Because of the restriction on the value of the genus which affects the result of S. Nayatani, it was not possible to prove the existence of this family of minimal surfaces for higher values of the genus.

So one of the consequences of our work is the proof of the non degeneracy of the Costa-Hoffman-Meeks surface for $g \geqslant 1$ in the sense of the definition given in [34] and also, only in a symmetric setting, in [11]. So we can state that the family of examples constructed by L. Hauswirth and F. Pacard exists for all the values of the genus. Moreover our result allows us to show in chapter 2 the existence of a family of minimal deformations of the Costa-Hoffman-Meeks surface for each value of the genus.

The author wishes to thank S. Nayatani for having provided the background computations on which are based the results about the Costa-Hoffman-Meeks surfaces contained in [29].

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1.1 Preliminaries

Let M be a complete oriented minimal surface in $\mathbb{R}^3$. The Jacobi operator of M is

$$L = -\triangle + 2K$$

where $\triangle$ is the Laplace-Beltrami operator and K is the Gauss curvature. Moreover we suppose that M has finite total curvature. Then M is conformally equivalent to a compact Riemann surface with finitely many punctures and the Gauss map $G : M \to S^2$ extends to the compactified surface holomorphically. So in the following we will pay attention to a generic compact Riemann surface, denoted by Σ and $G : \Sigma \to S^2$ a not constant holomorphic map, where S^2 is the unit sphere in $\mathbb{R}^3$ endowed with the complex structure induced by the stereographic projection from the north pole (denoted by π). We fix a conformal metric ds^2 on Σ and consider the operator $L_G = -\triangle + |dG|^2$, acting on functions on Σ.

We denote by $N(G)$ the kernel of L_G. We define $\mathrm{Nul}(G)$, the nullity of G, as the dimension of $N(G)$. Since $L(G) = \{a \cdot G \,|\, a \in \mathbb{R}^3\}$ is a three dimensional subspace of $N(G)$, then $\mathrm{Nul}(G) \geqslant 3$. We denote the index of G, that is the number of negative eigenvalues of L_G, by $\mathrm{Ind}(G)$. The index and the nullity are invariants of the map G: they are independent of the metric on the surface Σ. So we can consider on Σ the metric induced by G from S^2.

N. Ejiri and M. Kotani in [6] and S. Montiel and A. Ros in [25] proved that a non linear element of $N(G)$ is expressed as the support function of a complete branched minimal surface with planar ends whose extended Gauss map is G. In the following we will review briefly some results contained in [25] used by S. Nayatani in [30].

We will use some definitions and concepts of the algebraic geometric. They are recalled in subsection 1.5.1.

Let γ be the meromorphic function defined by $\pi \circ G$. Let p_j and r_i be respectively the poles and the branch points of γ. We denote by $P(G) = \sum_{i=1}^{\nu} n_j p_j$, $S(G) = \sum_{i=1}^{\mu} m_i r_i$ respectively the polar and ramification divisor of γ. Here n_j, m_i denote, respectively, the multiplicity of the pole p_j and the multiplicity with which γ takes its value at r_i. We define on the surface Σ the divisor

$$D(G) = S(G) - 2P(G)$$

and introduce the vector space $\bar{H}(G)$ (see [25], theorem 4)

$$\bar{H}(G) = \Big\{ \omega \in H^{0,1}(k_\Sigma + D(G)) \,|\, Res_{r_i}\omega = 0, 1 \leqslant i \leqslant \mu,$$
$$Re \int_\alpha (1-\gamma^2, i(1+\gamma^2), 2\gamma)\omega = 0,\ \forall \alpha \in H_1(\Sigma, \mathbb{Z}) \Big\},$$

where k_Σ is a canonical divisor of Σ and $H_1(\Sigma, \mathbb{Z})$ is the first group of homology of Σ. Suppose that the divisor D has an expression of the form $\sum n_j v_j - \sum m_i u_i$, with $n_j, m_i \in \mathbb{N}$. An element of $H^{0,1}(D)$ can be expressed as fdz, where f is a meromorphic function on Σ with poles of order not bigger than n_j at v_i and zeroes of order not smaller than m_i at u_i. Equivalently, if gdz, where g is a meromorphic function, is the differential form associated with the divisor D, the product fg must be holomorphic.

For $\omega \in \bar{H}(G)$, let $X(\omega) : \Sigma \setminus \{r_1, \ldots, r_\mu\} \to \mathbb{R}^3$ be the conformal immersion defined by

$$X(\omega)(p) = Re \int^p (1-\gamma^2, i(1+\gamma^2), 2\gamma)\omega.$$

Then $X(\omega) \cdot G$, the support function of $X(\omega)$, extends over the ramification points $r_1, \ldots, r_\mu$ smoothly and thus gives an element of $N(G)$. Conversely, every element of $N(G)$ is obtained in this way. In fact the map

$$\begin{array}{rcl} i : \bar{H}(G) & \to & N(G)/L(G) \\ \omega & \to & [X(\omega) \cdot G] \end{array} \tag{1.1}$$

is an isomorphism. This result, used in association with the Weierstrass representation formula, gives a description of the space $N(G)$. To obtain the dimension of $N(G)$ it is sufficient to compute the dimension of $\bar{H}(G)$. Since the dimension of $L(G)$ is equal to 3, then $\mathrm{Nul}(G) = 3 + \dim \bar{H}(G)$.

We denote by A_t a one parameter family $(0 < t < +\infty)$ of conformal diffeomorphisms of the sphere S^2 defined by

$$\pi \circ A_t \circ \pi^{-1} w = tw, \quad w \in \mathbb{C} \cup \{\infty\}.$$

We define for $0 < t < \infty$

$$G_t = A_t \circ G. \tag{1.2}$$

S. Nayatani in [30] gave lower and upper bounds for the index and, applying the method recalled above, for the nullity of G_t, $t \in (0, \infty)$, a deformation of an arbitrary holomorphic map $G : \Sigma \to S^2$, where Σ is a compact Riemann surface. In the same work, choosing appropriately the map G and the surface Σ, he computed the index and the nullity for the Gauss map of the Costa surface. In fact the extended Gauss map of this surface is a deformation of G for a particular value of t. We describe briefly the principal steps to get this result.

Firstly it is necessary to study the vector space $\bar{H}(G_t)$. A differential $\omega \in H^{0,1}(k_\Sigma + D(G))$ with null residue at the ramification points, is an element of $\bar{H}(G_t)$ if and only if the pair $(t\gamma, \omega)$ defines a branched minimal surface by the Weierstrass representation. If one sets $\gamma = 1/\wp'$ then there exist only two values of t, denoted by $t' < t''$, for which the condition above is verified and moreover $\dim H(G_t) = 1$. In other words, thanks to the characterization of the non linear elements of $N(G_t)$ by the isomorphism described by (1.7), if $t = t', t''$, $\mathrm{Nul}(G_t) = 4$. As for the index, if $t = t', t''$ then $\mathrm{Ind}(G_t) = 5$. Since $G_{t''}$ is the extended Gauss map of the Costa surface, one can state:

Theorem 3. *Let $\bar{G}$ be the extended Gauss map of the Costa surface. Then*

$$\mathrm{Nul}(\bar{G}) = 4, \quad \mathrm{Ind}(\bar{G}) = 5.$$

The same author in [29] treated the more difficult case of the Costa-Hoffman-Meeks surfaces of genus $2 \leqslant g \leqslant 37$. That is the subject of next section.

1.2 The case of the Costa-Hoffman-Meeks surface of genus smaller than 38

In this section we expose some of the background details at the base of section 3 of the work [29]. S. Nayatani provided them to us in [31].

We denote by M_g the Costa-Hoffman-Meeks surface of genus g. Let Σ_g be the compact Riemann surface

$$\Sigma_g = \{(z, w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^{g+1} = z^g(z^2 - 1)\} \tag{1.3}$$

and let $Q_0 = (0, 0), P_+ = (1, 0), P_- = (-1, 0), P_\infty = (\infty, \infty)$. It is known that $M_g = \Sigma_g \setminus \{P_+, P_-, P_\infty\}$.

The following result describes the properties of symmetry of M_g and Σ_g.

Lemma 4. *([14]) Consider the conformal mappings*

$$\kappa(z,w) = (\bar{z},\bar{w}) \quad \lambda(z,w) = (-z,\rho w), \tag{1.4}$$

where $\rho = e^{\frac{i\pi g}{g+1}}$ *of* $(\mathbb{C}\cup\{\infty\})^2$. *The map* κ *is of order* 2 *and* λ *is of order* $2g+2$. *The group generated by* κ *and* λ *is the dihedral group* D_{2g+2}. *This group of conformal diffeomorphisms leaves* M_g *invariant, fixes both* Q_0 *and* P_∞ *and extend to* Σ_g. *Also* κ *fixes the points* $P_\pm$ *while* λ *interchanges them.*

We set $\gamma(w) = w$. Let $G : \Sigma_g \to S^2$ be the holomorphic map defined by

$$\pi \circ G(z,w) = \gamma(w). \tag{1.5}$$

We denote by r_i, $i = 1,\dots,\mu$, the ramification points of γ and by $R(G)$ the divisor $\sum_{i=1}^{\mu} r_i$. Theorem 5 of [25] shows that the space $N(G)/L(G)$, that we have introduced in previous section, is also isomorphic to a space of meromorphic quadratic differentials. This alternative description of $N(G)/L(G)$ that we present in the following, was adopted by S. Nayatani in [29]. We start defining the vector spaces $\hat{H}(G)$ and $H(G)$.

$$\hat{H}(G) = \left\{\sigma \in H^{0,2}(2k_\Sigma + R(G)) \,|\, Res_{r_i}\frac{\sigma}{d\gamma} = 0, i = 1,\dots,\mu\right\}, \tag{1.6}$$

$$H(G) = \left\{\sigma \in \hat{H}(G) \,|\, Re\int_\alpha (1-\gamma^2, i(1+\gamma^2), 2\gamma)\frac{\sigma}{d\gamma} = 0,\ \forall\alpha \in H_1(\Sigma,\mathbb{Z})\right\},$$

where k_Σ is a canonical divisor of Σ. We remark that the elements of $H^{0,2}(2k_\Sigma + R(G))$ are quadratic differentials (see subsection 1.5.1). Since hereafter we will work only with quadratic differentials, we can set $H^0(\cdot) = H^{0,2}(\cdot)$ to simplify the notation. If we suppose that the divisor $2k_\Sigma + R(G)$ has an expression of the form $\sum n_j v_j - \sum m_i u_i$, with $n_j, m_i \in \mathbb{N}$, an element of $H^0(2k_\Sigma + R(G))$ can be expressed as $f(dz)^2$, where f is a meromorphic function on Σ with poles of order not bigger than n_j at v_i and zeroes of order not smaller than m_i at u_i. Equivalently, if $g(dz)^2$, where g is a meromorphic function, is the differential form associated with the divisor $2k_\Sigma + R(G)$, the product fg must be holomorphic.

For $\sigma \in H(G)$, let $X(\sigma) : \Sigma \setminus \{r_1,\dots,r_\mu\} \to \mathbb{R}^3$ be the conformal immersion defined by

$$X(\sigma)(p) = Re\int^p (1-\gamma^2, i(1+\gamma^2), 2\gamma)\frac{\sigma}{d\gamma}.$$

Then $X(\sigma)\cdot G$, the support function of $X(\sigma)$, extends over the ramification points $r_1,\dots,r_\mu$ smoothly and thus gives an element of $N(G)$. Conversely, every element of $N(G)$ is obtained in this way. In fact the map

$$\begin{array}{rcl} i : H(G) & \to & N(G)/L(G) \\ \sigma & \to & [X(\sigma)\cdot G] \end{array} \tag{1.7}$$

is an isomorphism. So to obtain the dimension of $N(G)$ it is sufficient to compute the dimension of $H(G)$. We recall that the dimension of $L(G)$ is equal to 3, so $\mathrm{Nul}(G) = 3 + \dim H(G)$.

Since the extended Gauss map of the Costa-Hoffman-Meeks surfaces is a deformation in the sense of the definition (1.2) of the map G, we need to study the space $H(G_t)$. From (1.6) and (1.2) it is clear that $\hat{H}(G) = \hat{H}(G_t)$ and

$$H(G_t) = \left\{ \sigma \in \hat{H}(G_t) \,|\, Re \int_\alpha (1 - t^2\gamma^2, i(1 + t^2\gamma^2), 2t\gamma) \frac{\sigma}{d\gamma} = 0,\, \forall \alpha \in H_1(\Sigma_g, \mathbb{Z}) \right\}.$$

Long computations ([31], see subsection 1.5.2 for some details) show that a basis of the differentials of the form $\sigma/d\gamma$, where $\sigma \in \hat{H}(G) = \hat{H}(G_t)$, and whose residue at the ramification points of $\gamma(w) = w$ is zero, is formed by

$$\omega_k^{(1)} = \frac{z^{k-1}}{w^k}\frac{dz}{w}, \quad \text{with} \quad k = 0, \dots, g-1,$$

$$\omega_k^{(2)} = \frac{((k-2)z^2 - kA^2)}{(z^2 - A^2)^2} \left(\frac{z}{w}\right)^{k-1} \frac{dz}{w}, \quad \text{with} \quad k = 0, \dots, g,$$

$$\omega_k^{(3)} = \frac{((k-2)z^2 - kA^2)}{w(z^2 - A^2)^2} \left(\frac{z}{w}\right)^{k-1} \frac{dz}{w}, \quad \text{with} \quad k = 0, \dots, g-1,$$

where $A = \sqrt{\frac{g}{g+2}}$.

Now we put attention to the space $H(G_t)$. We recall that we are interested in the computation of its dimension. By the definition of $H(G_t)$, a differential $\sigma \in \hat{H}(G_t)$ belongs to $H(G_t)$ if and only if $\forall \alpha \in H_1(\Sigma_g, \mathbb{Z})$ the differential form $\omega = \frac{\sigma}{d\gamma} = \frac{\sigma}{dw}$ satisfies

$$\int_\alpha \omega = t^2 \overline{\int_\alpha \gamma^2(w)\omega}, \tag{1.8}$$

$$Re \int_\alpha \gamma(w)\omega = 0. \tag{1.9}$$

If these two conditions are satisfied then (γ, w) are the Weierstrass data of a branched minimal surface. Of course, it is sufficient to impose that these equations are satisfied when α varies between the elements of a basis of $H_1(\Sigma_g, \mathbb{Z})$. The convenient basis of $H_1(\Sigma_g, \mathbb{Z})$ is constructed as follows. Let $\beta(t) = \frac{1}{2} + e^{i2\pi t}$, $0 \leqslant t \leqslant 1$. Let $\tilde{\beta}(t) = (\beta(t), w(\beta(t)))$ be a lift of β to Σ_g such that, for example, $\tilde{\beta}(0) = (\frac{3}{2}, w(0))$, with $w(0) \in \mathbb{R}$. As stated in lemma 4 the group of conformal diffeomorphisms of Σ_g is isomorphic to the dihedral group D_{2g+2}. The collection $\{\lambda^l \circ \tilde{\beta}, l = 0, \dots, 2g-1\}$, where λ is the generator of D_{2g+2} of order $2g+2$, is a basis of $H_1(\Sigma_g, \mathbb{Z})$.

Now we must impose (1.8) and (1.9) for $\alpha = \lambda^l \circ \tilde{\beta}$, with $l = 0, \dots, 2g-1$. To do that we collapse β to the unit interval. In other terms we deform continously β in such a way the limit curve is the union of two line segments lying on the real line. We set

$$\omega = \sum_0^{g-1} c_k^{(1)} \omega_k^{(1)} + \sum_0^{g} c_k^{(2)} \omega_k^{(2)} + \sum_0^{g-1} c_k^{(3)} \omega_k^{(3)},$$

where $c_k^{(i)} \in \mathbb{C}$.

Taking into account these assumptions, it is possible to show that the equation (1.8), if the genus g is 2, is equivalent to the following system of four equations (see subsection 1.5.3)

$$\begin{cases} f_0 = -t^2 \bar{h}_0 \\ f_1 = 0 \\ p_1 = -t^2 \bar{q}_1 \\ p_2 = -t^2 \bar{q}_0. \end{cases} \tag{1.10}$$

If $g \geqslant 3$ there are the following additional $2g-4$ equations to consider

$$\begin{cases} f_k = -t^2 \bar{q}_{g-k+2} \\ p_{g-k+2} = -t^2 \bar{h}_k \end{cases} \tag{1.11}$$

where $k = 2, \dots, g-1$ and

$$f_0 = \frac{(g+2)^2}{2(g+1)} c_0^{(3)} \sin\left(\frac{\pi}{g+1}\right) K_0,$$

$$f_k = \left(-c_k^{(1)} + \frac{(g+2)(g+2+k)}{2(g+1)} c_k^{(3)}\right) \sin\left(\frac{(k+1)\pi}{g+1}\right) K_k, \quad k = 1, \dots, g-1,$$

$$h_0 = \frac{(g+2)^2}{2(g+1)} c_0^{(3)} \sin\left(\frac{-\pi}{g+1}\right) J_0,$$

$$h_k = \left(c_k^{(1)} + \frac{(g+2)(g+2-k)}{2(g+1)} c_k^{(3)}\right) \sin\left(\frac{(k-1)\pi}{g+1}\right) J_k, \quad k = 2, \dots, g-1,$$

$$p_k = -\frac{(g+2)k}{2(g+1)} c_k^{(2)} \sin\left(\frac{k\pi}{g+1}\right) I_k, \quad k = 1, \dots, g,$$

$$q_k = \frac{(g+2)(2g+4-k)}{2(g+1)} c_k^{(2)} \sin\left(\frac{(k-2)\pi}{g+1}\right) L_k, \quad k = 0, 1, 3, \ldots, g,$$

and

$$I_m = \frac{g+1}{m} \frac{\Gamma\left(1 + \frac{m}{2(g+1)}\right) \Gamma\left(1 - \frac{m}{g+1}\right)}{\Gamma\left(1 - \frac{m}{2(g+1)}\right)},$$

$$J_m = \frac{g+1}{g-m+2} \frac{\Gamma\left(\frac{1}{2} + \frac{m-1}{2(g+1)}\right) \Gamma\left(1 - \frac{m-1}{g+1}\right)}{\Gamma\left(\frac{1}{2} - \frac{m-1}{2(g+1)}\right)},$$

$$K_m = J_{m+2},$$

$$L_m = \frac{m-2}{2g-m+4} I_{m-2}.$$

The equation (1.9) if the genus g is 2, is equivalent to the following system of two equations (see subsection 1.5.3)

$$\begin{cases} d_1 = 0 \\ e_2 = \bar{e}_0. \end{cases} \tag{1.12}$$

If $g \geqslant 3$ there are the following additional $g-2$ equations to consider

$$d_k = \bar{e}_{g-k+2} \tag{1.13}$$

where $k = 2, \ldots, g-1$, and

$$d_k = \left(c_k^{(1)} - \frac{k(g+2)}{2(g+1)} c_k^{(3)}\right) \sin\left(\frac{k\pi}{g+1}\right) I_k, \quad k = 1, \ldots, g-1,$$

$$e_k = \frac{(g+2)(g+2-k)}{2(g+1)} c_k^{(2)} \sin\left(\frac{(k-1)\pi}{g+1}\right) J_k, \quad k = 0, 2, \ldots, g.$$

We are looking for the values of t such that the previous systems have non trivial solutions in terms of $c_i^{(j)}$. Only for these special values of t it holds $\dim H(G_t) > 0$ or equivalently $\mathrm{Nul}(G_t) > 3$.

We start with the analysis of the system (1.10). This system admits non trivial solutions if and only if t assumes three values denoted by t_1, t_2, t_3. Obviously they are functions of g.

If we set $s = \frac{1}{g+1}$ then we can write

$$t_1 = \sqrt{\frac{K_0}{J_0}} = \frac{\sqrt{1-s^2}}{2}\sqrt{\frac{\Gamma(1-s)}{\Gamma(1+s)}\frac{\Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(1+\frac{s}{2}\right)}},$$

$$t_2 = \sqrt{\frac{I_1}{(2g+3)L_1}} = \sqrt{\frac{\Gamma(1-s)}{\Gamma(1+s)}\frac{\Gamma\left(1+\frac{s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}},$$

$$t_3 = \sqrt{\frac{I_2 J_0}{g L_0 K_0}} = \frac{2}{1-s}\sqrt{\left(\frac{\Gamma(1+s)}{\Gamma(1-s)}\right)^3}\sqrt{\frac{\Gamma(1-2s)}{\Gamma(1+2s)}\frac{\Gamma(3/2-s/2)}{\Gamma(1/2+s/2)}}.$$

We recall that if $g \geqslant 3$ there are other equations to consider. They are

$$\begin{cases} f_k = -t^2 \bar{q}_{g-k+2} \\ p_{g-k+2} = -t^2 \bar{h}_k \\ d_k = \bar{e}_{g-k+2} \end{cases}$$

where $k = 2, \ldots, g-1$. Thanks to the particular structure of the equations, it is possible to study separately for each set of three equations the existence of solutions. Each set of three equations admits non trivial solutions if and only if the following matrix has determinant equal to zero

$$\begin{pmatrix} -K_k & (g+2+k)K_k & (g+2+k)t^2 L_{g-k+2} \\ t^2 J_k & (g+2-k)t^2 J_k & (g+2-k)I_{g-k+2} \\ I_k & -kI_k & -kJ_{g-k+2} \end{pmatrix}.$$

After the change of variable $l = g-k+1$ so that $2 \leqslant l \leqslant g-1$, it is possible to show that the determinant is

$$-(g+2)(at^4 + bt^2 + c), \tag{1.14}$$

with

$$\begin{aligned} a &= (2g-l+3)I_{g-l+1}J_{g-l+1}L_{l+1} \\ b &= -2(g-l+1)J_{l+1}J_{g-l+1}K_{g-l+1} \\ c &= (l+1)I_{g-l+1}I_{l+1}K_{g-l+1}. \end{aligned}$$

We are interested in finding the positive values of t such that

$$at^4 + bt^2 + c = 0. \tag{1.15}$$

To simplify the notation we introduce the following three functions

$$F(v) = \left(\frac{\Gamma(\frac{1}{2}+\frac{v}{2})}{\Gamma(\frac{1}{2}-\frac{v}{2})}\right)^2 \frac{\Gamma(1-v)}{\Gamma(1+v)},$$

$$I(v) = \left(\frac{\Gamma(1-\frac{v}{2})}{\Gamma(1+\frac{v}{2})}\right)^2 \frac{\Gamma(1+v)}{\Gamma(1-v)},$$

$$L(v) = \left(\frac{\Gamma(1+\frac{v}{2})}{\Gamma(1-\frac{v}{2})}\right)^2 \frac{\Gamma(1-v)}{\Gamma(1+v)} = \frac{1}{I(v)}.$$

The discriminant $b^2 - 4ac$ of the equation (1.15), seen like an equation of degree two in the variable t^2, is negative if and only if $X = b^2/4ac < 1$. It is possible to show that

$$X = \frac{l^2}{l^2-1} F^2\left(\frac{l}{g+1}\right) I\left(\frac{l-1}{g+1}\right) I\left(\frac{l+1}{g+1}\right). \tag{1.16}$$

S. Nayatani in [31] showed that if $2 \leqslant g \leqslant 37$, then $X < 1$ and as consequence the equation (1.15) has not any solution since its discriminant is negative. Then $\dim H(G_t) > 0$ only for $t = t_1, t_2, t_3$. Summarizing we can state (see [29] for other details):

Theorem 5. *If* $2 \leqslant g \leqslant 37$ *and* $t \in (0, +\infty)$, *then*

$$\mathrm{Nul}(G_t) = \begin{cases} 4 & \textit{if} \quad t = t_1, t_2 \\ 5 & \textit{if} \quad t = t_3 \\ 3 & \textit{elsewhere.} \end{cases}$$

Since the extended Gauss map of the Costa-Hoffman-Meeks surfaces is exactly G_{t_2}, it is possible to state that the null space of the Jacobi operator of M_g has dimension equal to 4 for $2 \leqslant g \leqslant 37$.

Other values of t for which $\mathrm{Nul}(G_t) > 3$ are admitted only if $g \geqslant 38$. In [29] S. Nayatani conjectured these values were bigger than t_3. The proof of the conjecture and its consequences will be showed in sections 1.3 and 1.4.

1.3 The case $g \geqslant 38$

Thanks the previous observations it is clear that if we assume $g \geqslant 38$ It is possible to prove that X is a decreasing function in the variables

$$x = \frac{l}{g+1}, y = \frac{l+1}{g+1}, z = \frac{l-1}{g+1}$$

with $2 \leqslant l \leqslant g-1$. We recall that we have set $s = \frac{1}{g+1}$. We know that for $l = 2$ and $g = 37$ the discriminant of the equation (1.15) is negative. For these values of l and g the variables x, y, z, s are respectively equal to $x_{max} = 2s_{max}$, $y_{max} = 3s_{max}$, $z_{max} = s_{max} = 1/38$.

Then we will study the solutions of (1.15) for $i \in [0, i_{max}]$ (we call these admissible values) where i denotes x, y, z, s because for bigger values of the three variables the discriminant continues to be negative and so the equation (1.15) does not admit solutions. Numerical tests show that the value i_{max}, become smaller as g is bigger. Since it is not possible to explicit the dependence of i_{max} on g we shall work with constant quantities.

All the solutions of (1.15), that we denote by $t_\pm(l, g)$, satisfy $t_\pm^2(l, g) = T_1 \pm T_2$, with

$$T_1 = \frac{l}{l-1} F(x) I(z) \tag{1.17}$$

and

$$T_2 = \sqrt{\left(\frac{l}{l-1}\right)^2 F^2(x) I^2(z) - \frac{l+1}{l-1} L(y) I(z)}. \tag{1.18}$$

We will prove that, for $0 \leqslant \frac{l}{g+1} \leqslant x_{max} = \frac{2}{38}$, with $2 \leqslant l \leqslant g-1$ and $g \geqslant 38$, it holds

$$t_3^2(s) < t_-^2(l, g). \tag{1.19}$$

We need study the behaviour of the functions F, I, L, F^2, I^2 that appear in (1.17) and (1.18). This aim is pursued by the use of first order series of these functions.
The Mac-Laurin series of the functions $F(x), G(z), L(y), F^2(x), I^2(z)$ for admissible values of x, y, z are

$$F(x) = 1 + R_F(d_1x)x, \quad I(z) = 1 + R_I(d_2z)z, \quad L(y) = 1 + R_L(d_3y)y, \tag{1.20}$$
$$F^2(x) = 1 + R_{F^2}(c_1x)x, \quad I^2(x) = 1 + R_{I^2}(c_2x)x,$$

where $c_i, d_i \in (0, 1)$. So we can write

$$F(x)I(z) = 1 + R_{FI}(x, z),\ F^2(x)I^2(z) = 1 + R_{F^2I^2}(x, z),\ L(y)I(z) = 1 + R_{LI}(y, z),$$

with

$$\begin{aligned} R_{FI}(x, z) &= R_F(d_1x)x + R_I(d_2z)z + R_F(d_1x)R_I(d_2z)xz, \\ R_{F^2I^2}(x, z) &= R_{F^2}(c_1x)x + R_{I^2}(c_2z)z + R_{F^2}(c_1x)R_{I^2}(c_2z)xz, \\ R_{LI}(y, z) &= R_L(d_3y)y + R_I(d_2z)z + R_I(d_2z)R_L(d_3y)zy. \end{aligned}$$

In the following $\psi(x)$ the digamma function. It is related to $\Gamma(x)$, the gamma function, by

$$\psi(x) = \frac{d}{dx}\left(\ln \Gamma(x)\right).$$

For the properties of this special function we refer to [1].

The following proposition gives useful properties of the functions just introduced.

Proposition 6. *If $x \in [0, x_{max}]$, $z \in [0, z_{max}]$, and $y \in [0, y_{max}]$, the following assertions hold:*

1. $R_F(x) < 0$
2. $R_I(z) \leqslant 0$
3. $R_L(y) \geqslant 0$
4. $(R_F)'_x(x) > 0$
5. $min(R_I)'_z(z) = -0.095\cdots$
6. $R_{FI}(x, z) \geqslant Cx$ *with* $C = -4\ln 2$
7. $R_{LI}(y, z) \geqslant 0$
8. $R_{I^2}(z) \leqslant 0$
9. $W(x) = R_{F^2}(x) < 0$
10. $W'_x(x) > 0$, *so* $R_{F^2}(x)$ *is an increasing function*
11. $W''_{xx}(x) < 0$
12. $W'''_{xxx}(x) > 0$
13. *If we set* $Y(x) = x\,W(x)$, *then* $Y'_x(x) \leqslant 0$
14. $Y''_{xx}(x) \geqslant 0$
15. $Y'''_{xxx}(x) \leqslant 0$.

Proof.

1. $R_F(x) = F'_x(x) = F(x)\Psi_F(x)$, where

$$\Psi_F(x) = -\psi(1-x) - \psi(1+x) + \psi\left(\frac{1}{2} - \frac{x}{2}\right) + \psi\left(\frac{1}{2} + \frac{x}{2}\right).$$

We observe that

$$\Psi_F(x) = 2\sum_{k=0}^{\infty} \frac{1}{(2k)!}\left(\frac{1}{2^{2k}}\psi^{(2k)}\left(\frac{1}{2}\right) - \psi^{(2k)}(1)\right)x^{2k}.$$

Since $\Psi_F(0) = 2\psi\left(\frac{1}{2}\right) - 2\psi(1) = -4\ln 2$, $\psi^{(2k)}(1) < 0$ and $\psi^{(2k)}\left(\frac{1}{2}\right) = (2^{2k+1} - 1)\psi^{(2k)}(1) < 0$, if $k \geqslant 1$ (see formulas 6.4.2 and 6.4.4 of [1]), we can conclude that $\Psi_F(x) < 0$ and it is a decreasing function. Since $F(x) > 0$ then $R_F(x) < 0$ and $F(x)$ is a decreasing function.

2. $R_I(z) = I'_z(z) = I(z)\Psi_I(z)$, where

$$\Psi_I(z) = \psi(1-z) + \psi(1+z) - \psi\left(1-\frac{z}{2}\right) - \psi\left(1+\frac{z}{2}\right).$$

We observe that

$$\Psi_I(z) = 2\sum_{k=1}^{\infty} \frac{1}{(2k)!}\psi^{(2k)}(1)\left(1-\frac{1}{2^{2k}}\right) z^{2k}.$$

Since $\psi^{(2k)}(1) < 0$ for $k \geqslant 1$ then $\Psi_I(z) \leqslant 0$ and it is a decreasing function. Since $I(z) > 0$ then $R_I(z) \leqslant 0$.

3. $R_L(y) = L'_y(y) = L(y)\Psi_L(y)$, where $\Psi_L(y) = -\Psi_I(y)$. Then $\Psi_L(y) \geqslant 0$ and it is an increasing function. Since $L(y) = 1/I(y) > 0$, then $R_L(y) \geqslant 0$.

4. The derivative of R_F is $F''_{xx}(x) = F(x)(\Psi_F^2(x) + (\Psi_F)'_x(x))$. Since $\Psi_F(x) < 0$ and it is a decreasing function, $\Psi_F^2(x) > 0$ and increasing. It holds $\Psi_F^2(x) \geqslant \Psi_F^2(0) = 16\ln^2 2$.

$$(\Psi_F)'_x(x) = 2\sum_{k=1}^{\infty} \frac{1}{(2k-1)!}\left(\frac{1}{2^{2k}}\psi^{(2k)}\left(\frac{1}{2}\right) - \psi^{(2k)}(1)\right) x^{2k-1}.$$

All the coefficients of the series are negative (see the point 1) so $(\Psi_F)'_x(x) \leqslant 0$ and it is a decreasing function. In particular $(\Psi_F)'_x(x) \geqslant (\Psi_F)'_x(x_{max}) = -0.19\cdots$. Since $F(x) > 0$ and it is a decreasing function we can conclude that

$$F''_{xx}(x) \geqslant F(x_{max})(\Psi_F^2(0) + (\Psi_F)'_x(x_{max})) = 6.4\cdots.$$

5. The derivative of R_I is $I''_{zz}(z) = I(z)(\Psi_I^2(z) + (\Psi_I)'_z(z))$. Since $\Psi_I(z) \leqslant 0$ and it is a decreasing function (see the point 2), $\Psi_I^2(z) \geqslant 0$ and increasing. It holds $\Psi_I^2(z) \leqslant \Psi_I^2(z_{max}) = 1.5\cdots 10^{-6}$.

$$(\Psi_I)'_z(z) = 2\sum_{k=1}^{\infty} \frac{1}{(2k-1)!}\psi^{(2k)}(1)\left(1-\frac{1}{2^{2k}}\right) z^{2k-1}.$$

All the coefficients of the series are negative so $(\Psi_I)'_z(z) \leqslant 0$ and it is a decreasing function. In particular $(\Psi_I)'_z(z) \geqslant (\Psi_I)'_z(z_{max}) = -0.095\cdots$. Since $I(z) > 0$ and it is a decreasing function we can conclude that

$$I''_{zz} \geqslant I(z_{max})(\Psi_I^2(0) + (\Psi_I)'_z(z_{max})) = -0.095\cdots.$$

6. Since $R_F < 0$ and $R_I \leqslant 0$, it holds that

$$R_{FI}(x, z) \geqslant R_F(d_1 x)x + R_I(d_2 z)z,$$

where $d_i \in (0, 1)$. In the point 4 we have proved that R_F is an increasing function and we have computed the minimum value of its derivative. Moreover it is also clear thanks to the point 5 that this value is always bigger than the maximum of the absolute value of the derivative of R_I. Now it is sufficient to remember that the variables t and z are not independent. We can conclude that R_{FI} is an increasing function. Then $R_{FI} \geqslant R_F(0)x + R_I(0)z = Cx$.

7. We recall that $R_{LI}(y, z) = L(y)I(z) - 1$, $L(t) = 1/I(t)$ and

$$y = \frac{l+1}{g+1} > \frac{l-1}{g+1} = z.$$

We want to prove that $L(y)I(z) - 1 \geqslant 0$ or equivalently $L(y) \geqslant 1/I(z)$. But thanks to the point 3, L is an increasing function, so

$$L(y) > L(z) = \frac{1}{I(z)}.$$

8. $R_{I^2}(z) = (I^2)'_z(z) = 2I^2(z)\Psi_I(z)$. From the proof of the point 2, $\Psi_I(z) \leqslant 0$ and it is a decreasing function. Since $2I^2(z) > 0$, then also $R_{I^2}(z) \leqslant 0$.

9. $W(x) = (F^2)'_x(x) = 2F^2(x)\Psi_F(x)$. In the point 1 we have observed that $\Psi_F(x)$ is a negative and decreasing function. Since $2F^2(x) > 0$, then also $W(x)$ is a negative function.

10. $W'_x(x) = F^2\left(4\Psi_F^2(x) + 2(\Psi_F)'_x(x)\right)$. Since $\Psi_F(x) < 0$ and it is a decreasing function, $\Psi_F^2(x)$ is a positive and increasing function. In the proof of the point 4 we observed that $(\Psi_F)'_x(x) \leqslant 0$ and it is a decreasing function. Since $2(\Psi_F)'_x(x_{max}) = -0.38\cdots$ and $4\Psi_F^2(x) \geqslant 4\Psi_F^2(0) = 64\ln^2 2 = 30.74\cdots$, we can conclude that $W'_x(x) > 0$.

11. The explicit expression of W''_{xx} is

$$W''_{xx} = \frac{1}{2}F^2(x)\left(16\Psi_F^3(x) + 24\Psi_F(x)(\Psi_F)'_x(x) + 4(\Psi_F)''_{xx}(x)\right).$$

In the proof of the point 1 we observed that $\Psi_F(x)$ is a negative and decreasing function. So $16\Psi_F^3(x) \leqslant 16\Psi_F^3(0) = -1024\ln^3 2 = -341.\cdots$. Thanks to the proof of the point 10 we know that $(\Psi_F)'_x(x) \leqslant 0$ and it is a decreasing function. In particular $0 \geqslant (\Psi_F)'_x(x) \geqslant (\Psi_F)'_x(x_{max}) = -0.19\cdots$. We can conclude that

$$24\Psi_F(x)(\Psi_F)'_x(x) \leqslant 24(\Psi_F)'_x(x_{max})\Psi_F(x_{max}) = 12.\cdots.$$

As for the last summand, it is negative. In fact

$$(\Psi_F)''_{xx}(x) = 2\sum_{k=1}^{\infty} \frac{1}{(2k-2)!}\left(\frac{1}{2^{2k}}\psi^{(2k)}\left(\frac{1}{2}\right) - \psi^{(2k)}(1)\right)x^{2k-2}.$$

Since all the coefficients of the series are negative, we get

$$4(\Psi_F)''_{xx}(x) \leqslant 4(\Psi_F)''_{xx}(0) = -12\zeta(3) = -14.4\cdots,$$

where $\zeta(\cdot)$ denotes the Riemann zeta function.

Summarizing we can conclude that

$$16\Psi_F^3(x) + 24\Psi_F(x)(\Psi_F)'_x(x) + 4(\Psi_F)''_{xx}(x) \leqslant$$

$$\leqslant 16\Psi_F^3(0) + 24\Psi_F(x_{max})(\Psi_F)'_x(x_{max}) + 4(\Psi_F)''_{xx}(0) = -342.7\cdots.$$

That assures $W''_{xx} < 0$.

12. The explicit expression of W'''_{xxx} is

$$W'''_{xxx} = \frac{1}{4}F^2(x)\left(64\Psi_F^4 + 192\Psi_F^2(\Psi_F)'_x + 48((\Psi_F)'_x)^2 + \right.$$

$$\left. + 64\Psi_F(\Psi_F)''_{xx} + 8(\Psi_F)'''_{xxx}\right).$$

We start observing that, since Ψ_F is a negative decreasing function,

$$64\Psi_F^4(x) \geqslant 64\Psi_F^4(0) = 64(4\ln 2)^4 = 3782.\cdots.$$

Since $(\Psi_F)'_x(x)$ is a not positive and decreasing function (point 10), then $192\Psi_F^2(\Psi_F)'_x$ enjoys the same property. In particular

$$192\Psi_F^2(\Psi_F)'_x \geqslant 192\Psi_F^2(x_{max})(\Psi_F)'_x(x_{max}) = -282.\cdots.$$

From the previous observations it follows that $64\Psi_F(\Psi_F)''_{xx} \geqslant 0$, $48((\Psi_F)'_x)^2 \geqslant 0$ and they are increasing functions.

As for the last summand which appears in the expression of W'''_{xxx}, we observe that

$$(\Psi_F)'''_{xxx} = 2\sum_{k=1}^{\infty} \frac{1}{(2k-1)!}\left(\frac{1}{2^{(2k+2)}}\psi^{(2k+2)}\left(\frac{1}{2}\right) - \psi^{(2k+2)}(1)\right)x^{2k-1}.$$

It is a not positive and decreasing function. We can conclude that

$$8(\Psi_F)'''_{xxx}(x) \geqslant 8(\Psi_F)'''_{xxx}(x_{max}) = -19.9\cdots.$$

Summarizing we can state $W'''_{xxx}(x) > 0$. Furthermore from our observations it follows that

$$W'''_{xxx}(x) \leqslant \big(16\Psi_F^4(x_{max}) + 24((\Psi_F)'_x)^2(x_{max})$$
$$+16\Psi_F(x_{max})(\Psi_F)''_{xx}(x_{max})) < C_W$$

with $C_W = 1125$.

13. It holds that $Y'_x(x) = W(x) + x\,W'_x(x)$. From the points 9, 10 and 11 we know that $W(x)$ is a negative increasing function and $W'_x(x)$ is positive and decreasing for $x \in [0, x_{max}]$. So we can write $W(x) \leqslant W(x_{max}) = -4.1\cdots$ and $W'_x(x) \leqslant W'_x(0) = 64\ln^2 2 = 30.7\cdots$. Then $Y'_x(x) \leqslant W(x_{max}) + x_{max}W'_x(0) < 0$.

14. It holds that $Y''_{xx}(x) = 2W'_x(x) + x\,W''_{xx}(x)$. From the points 10, 11 and 12 we know that $W'_x(x)$ is a positive decreasing function and $W''_{xx}(x)$ is negative and increasing. So we can write $W'_x(x) \geqslant W'_x(x_{max}) = 22.\cdots$. and $W''_{xx}(x) \geqslant W''_{xx}(0) = -64\ln^3 4 - 6\zeta(3) = -177.\cdots$. Then $Y'_x(x) \geqslant 2W'_x(x_{max}) + x_{max}W''_{xx}(0) > 0$.

15. It holds that $Y'''_{xxx}(x) = 3W''_{xx}(x) + x\,W'''_{xxx}(x)$. From the points 11 and 12 we know that $W''_{xx}(x)$ is a negative increasing function and $W'''_{xxx}(x) < C_W$ is positive. Then $Y'''_{xxx}(x) \leqslant 3W''_{xx}(x_{max}) + x_{max}C_W < 0$.

□

Proposition 7. *For all the admissible values of x, y, z it holds that*

$$T_2 \leqslant \frac{1 + Cl^2 x}{l - 1},$$

where $C = -4\ln 2$.

Proof. The epression of T_2 is given by (1.18). We rewrite it in the following way

$$T_2 = \frac{1}{l-1}\sqrt{l^2 F^2(x)\, I^2(z) - (l^2 - 1)L(y)I(z)}.$$

If $1 + \bar{R}(x, y, z, l)$ is the Mac-Laurin series of the function under the square root then we can write

$$T_2 = \frac{1}{l-1}\sqrt{1 + \bar{R}(x, y, z, l)},$$

where $\bar{R}(x, y, z, l) = l^2(R_{F^2}(c_1 x)x(1 + R_{I^2}(c_2 z)z) + R_{I^2}(c_2 z)z) - (l^2 - 1)R_{LI}(y, z)$, and $c_1, c_2 \in (0, 1)$. Thanks to the points 7,8,9 and 10 of proposition 6, we know that $R_{LI}(y, z) \geqslant 0$, $R_{I^2}(x) \leqslant 0$ and that $R_{F^2}(x)$ is a negative increasing function, so $R_{F^2}(c_1 x) \leqslant R_{F^2}(x)$. We can conclude that, if we set

$$R(x, z, l) = l^2 R_{F^2}(x)x(1 + R_{I^2}(c_2 z)z),$$

$$\bar{R}(x,y,z,l) \leqslant l^2 R_{F^2}(c_1 x)x(1+R_{I^2}(c_2 z)z) \leqslant R(x,z,l),$$

then

$$T_2 = \frac{1}{l-1}\sqrt{1+\bar{R}(x,z,l)} \leqslant \frac{1}{l-1}\sqrt{1+R(x,z,l)}.$$

We know that

$$\sqrt{1+f(x)} = \sqrt{1+f(0)} + \frac{f'_t(t)}{2\sqrt{1+f(t)}}|_{t=cx} x,$$

where $c \in (0,1)$. If we apply this result to the function $f(x) = R(x,z,l)$, we get

$$T_2 \leqslant \frac{\sqrt{1+R(x,z,l)}}{l-1} = \frac{1}{l-1}\left(\sqrt{1+R(0,z,l)} + \frac{R'_t(t,z,l)}{2\sqrt{1+R(t,z,l)}}|_{t=cx} x\right),$$

where $c \in (0,1)$. We observe that $R(0,z,l) = 0$. Then

$$T_2 \leqslant \frac{1}{l-1}\left(1 + \frac{R'_t(t,z,l)}{2\sqrt{1+R(t,z,l)}}|_{t=cx} x\right).$$

The proof will be completed after having proved the following result. □

Proposition 8. *For all the admissible values of t, y, z,*

$$\frac{R'_t(t,z,l)}{2\sqrt{1+R(t,z,l)}} \leqslant Cl^2,$$

where $C = -4\ln 2$.

Proof. We set $H(z,l) = l^2(1+R_{I^2}(c_2 z)z) \leqslant l^2$ and $Y(t) = R_{F^2}(t)t$. From the expression of $R(t,z,l) = H(z,l)Y(t)$, it follows that $R'_t(t,z,l) = H(z,l)Y'_t(t)$. Furthermore we can write

$$\frac{R'_t(t,z,l)}{2\sqrt{1+R(t,z,l)}} = \frac{H(z,l)Y'_t(t)}{2\sqrt{1+H(z,l)Y(t)}}.$$

We know from proposition 6 that $Y(t) \leqslant 0$ and $Y'_t(t) < 0$, then $R'_t(t,z,l) = H(z,l)Y'_t(t) \geqslant l^2 Y'_t(t)$, and

$$-\frac{1}{2\sqrt{1+R(t,z,l)}} \geqslant -\frac{1}{2\sqrt{1+l^2 Y(t)}}.$$

We can conclude that

$$-\frac{R'_t(t,z,l)}{2\sqrt{1+R(t,z,l)}} \geqslant -\frac{l^2 Y'_t(t)}{2\sqrt{1+l^2 Y(t)}}.$$

We shall show that this last function is increasing with respect to the variable t. The derivative with respect to the variable t of this function is

$$D(t,l) = -\frac{l^2}{2}\frac{Y''_{tt}\sqrt{1+l^2Y} - l^2\frac{(Y'_t)^2}{2\sqrt{1+l^2Y}}}{1+l^2Y}.$$

We want to show that $D(t,l) \geqslant 0$. We start observing that $1+l^2Y > 0$. So it is sufficient to prove that the quantity

$$E(t,l) = 2Y''_{tt}(1+l^2Y) - l^2(Y'_t)^2$$

is always not positive. It holds that

$$Y'_t(t) = R_{F^2}(t) + t(R_{F^2})'_t(t)$$

and

$$Y''_{tt}(t) = 2(R_{F^2})'_t(t) + t(R_{F^2})''_{tt}(t).$$

Then $Y(0) = 0$, $Y'_t(0) = R_{F^2}(0) = 2C$ and $Y''_{tt}(0) = 2(R_{F^2})'_t(0) = 8\Psi_F(0)^2 = 8C^2$. Furthermore we observe that $l \geqslant 2$. So

$$E(0,l) = 16C^2 - 4l^2C^2 \leqslant 0$$

and the equality holds if $l = 2$. The next step is to show that $E'_t(t,l) \leqslant 0$. It is possible to find the following relation

$$E'_t(t,l) = Y'''_{ttt}(1+l^2Y)$$

Observing that $1+l^2Y > 0$ and $Y'''_{ttt} \leqslant 0$ (see the point 15 of proposition 6), we can conclude that $D(t,l) > 0$. We have showed that

$$-\frac{l^2Y'_t(t)}{2\sqrt{1+l^2Y(t)}}$$

is an increasing function. It gets the minimum for $t=0$ and its value is $-Cl^2$. Then

$$-\frac{R'_t(t,z,l)}{2\sqrt{1+R(t,z,l)}} \geqslant -Cl^2,$$

and the proof is completed. □

As for the first summand which appears in the expression of t_-^2, that is T_1, the following result holds.

Proposition 9. *For all the admissible values of x, z, it holds that*

$$T_1 \geqslant \frac{l}{l-1}(1+Cx)$$

where $C = -4\ln 2$.

Proof. We recall that

$$T_1 = \frac{l}{l-1}F(x)I(z) = \frac{l}{l-1}(1+R_{FI}(x,z)).$$

Thanks to the point 6 of proposition 6 we have $R_{FI}(x,z) \geqslant Cx$. Then the result is immediate. □

The following result gives the estimate of t_-^2.

Proposition 10. *For all the admissible values of x, y, z*

$$t_-^2 \geqslant 1 - Clx,$$

where $C = -4\ln 2$.

We recall that $t_-^2 = T_1 - T_2$. Thanks to propositions 7 and 9 we get

$$t_-^2 \geqslant \frac{l}{l-1}(1+Cx) + \frac{1}{l-1}(-1-Cl^2x) =$$

$$1 + \left(\frac{Cl}{l-1} - \frac{Cl^2}{l-1}\right)x = 1 + \left(\frac{-Cl}{l-1}(l-1)\right)x = 1 - Clx.$$

□

Now we turn our attention to the function t_3. We recall that $s_{max} = \frac{1}{38}$.

Proposition 11. *For $s \in [0, s_{max}]$*

$$t_3^2(s) \leqslant 1 + \frac{7}{2}s.$$

Proof. We recall that

$$t_3^2(s) = T(s) = \frac{4}{(1-s)^2}\left(\frac{\Gamma(1+s)}{\Gamma(1-s)}\right)^3 \frac{\Gamma(1-2s)}{\Gamma(1+2s)} \left(\frac{\Gamma(3/2-s/2)}{\Gamma(1/2+s/2)}\right)^2.$$

It holds that

$$T_s'(s) = \frac{1}{(1-s)}T(s)B(s),$$

where

$$B(s) = 2 + (1-s)\left(-2\psi(1-2s) - 2\psi(1+2s) + 3\psi(1-s) + 3\psi(1+s) - \right.$$

$$\left. -\psi\left(\frac{3}{2} - \frac{s}{2}\right) - \psi\left(\frac{1}{2} + \frac{s}{2}\right)\right).$$

To complete the proof we need the following result.

Proposition 12. *If* $s \in [0, s_{max}]$ *then* $1 < B(s) < 3$.

Proof. We observe that for $s \in [0, s_{max}]$

$$0 < \psi\left(\frac{3}{2} - \frac{s}{2}\right) < \psi\left(\frac{3}{2}\right) = 0.036\cdots, \quad \frac{3}{2} < -\psi\left(\frac{1}{2} + \frac{s}{2}\right) < -\psi\left(\frac{1}{2}\right) < 2.$$

We can conclude that

$$1 < -\psi\left(\frac{1}{2} + \frac{s}{2}\right) - \psi\left(\frac{3}{2} - \frac{s}{2}\right) < 2.$$

Furthermore

$$\psi(1-s) + \psi(1+s) = 2\sum_{k \geqslant 0} \frac{\psi^{(2k)}(1)}{(2k)!} s^{2k},$$

from which it follows that

$$D(s) = -2\psi(1-2s) - 2\psi(1+2s) + 3\psi(1-s) + 3\psi(1+s) = 2\sum_{k \geqslant 0} \frac{\psi^{(2k)}(1)}{(2k)!} s^{2k}(3 - 2^{2k+1}).$$

If $k \geqslant 1$ then $3 - 2^{2k+1} < 0$ and $\psi^{(2k)}(1) < 0$ (see formula 6.4.2 of [1]) then

$$2\psi(1) = -2\gamma_{EM} = D(0) \leqslant D(s) \leqslant D(s_{max}) = -1.146\cdots,$$

where $\gamma_{EM} = 0.577\cdots$ is the Euler-Mascheroni constant. So

$$1 < B(s) \leqslant 2 + (1-s)(2 + D(s_{max})) < 4 + D(s_{max}) < 3.$$

□

Since $B(s) > 0$ then $T(s)$ is an increasing function and we can deduce that

$$T'(s) = \frac{1}{1-s} T(s) B(s) \leqslant \frac{3}{1 - s_{max}} T(s_{max}) < 7/2.$$

The Mac-Laurin series of order 1 of $T(s)$ is $1 + T'_s(cs)s$, where $c \in (0,1)$. So it is immediate to conclude that

$$T(s) \leqslant 1 + \frac{7}{2}s.$$

□

We want to remark that with more work it is possible to show that

$$T(s) \leqslant 1 - Cs.$$

The following proposition shows that the eventual solutions of the equation (1.15) are always bigger than t_3.

Proposition 13. $t_3(s) < t_-(l,g)$ *for* $g \geqslant 1$.

Proof. From our observations, it is sufficient to show that $t_3^2(s) < t_-^2(l,g)$ holds for $g \geqslant 38$. The propositions 10 and 11 assure that

$$t_-^2 \geqslant 1 - Clx,$$

$$t_3^2(s) \leqslant 1 + \frac{7}{2}s.$$

We recall that $x = ls$ and $2 \leqslant l \leqslant g-1$. Then the result is obvious. □

1.4 The index and the nullity of the Costa-Hoffman-Meeks surfaces

We start recalling some results described in previous sections. We denoted by G_t, $t \in (0,+\infty)$, a deformation of the map G defined by (1.5). Thanks to theorem 5 $\mathrm{Nul}(G_t) > 3$ only if t assumes special values. If $2 \leqslant g \leqslant 37$ these values are t_1, t_2, t_3. If $g \geqslant 38$ there are additional values. They are the positive solutions of the equation (1.15). We denoted them by $t_\pm(l,g)$, where $2 \leqslant l \leqslant g-1$, and for definition $t_+ \geqslant t_-$. In previous section we have proved that the inequality $t_3(s) < t_-(l,g)$ holds. S. Nayatani showed in [29] that $t_3 > t_2$ for $g \geqslant 2$. We can conclude that no one of the $t_\pm$ can be equal to t_2. As consequence $\mathrm{Nul}(G_{t_2})$ continues to be equal to 4 also for $g \geqslant 38$, because $\dim H(G_{t_2})$ is equal to 1 for all $g \geqslant 2$.

We recall that M_g denotes the Costa-Hoffman-Meeks surface of genus g. Since the extended Gauss map of M_g is exactly G_{t_2}, and taking into account the result of S. Nayatani about the Costa surface (theorem 3) showed in [30] we have proved the following result.

Theorem 14. *The null space of the Jacobi operator of* M_g *has dimension equal to* 4 *for all* $g \geqslant 1$.

Using the definition of non degeneracy given in [34], we can also rephrase this result giving the following statement.

Corollary 15. *The surface* M_g *is non degenerate for all* $g \geqslant 1$.

Now we turn our attention to the results relative to the index of the map G_t. We recall that Σ_g denotes the compactification of M_g. S. Nayatani proved in [29] the following result.

Theorem 16. *Let* $G : \Sigma_g \to S^2$ *be the holomorphic map defined by (1.5). If* $2 \leqslant g \leqslant 37$, *then*

$$\mathrm{Ind}(G_t) = \begin{cases} 2g+3 & if\ t \leqslant t_1,\ t_2 \leqslant t < t_3,\ t > t_3, \\ 2g+4 & if\ t_1 < t < t_2, \\ 2g+2 & if\ t = t_3. \end{cases}$$

For $t = t_1, t_2, t_3$ we have $\mathrm{Nul}(G_t) > 3$, that is the kernel of L_{G_t} contains at least one non linear element. The eigenvalue associated to this function is zero. The proof of theorem 16 is based on the analysis of the behaviour of these null eigenvalues under a variation of the value of t. Let's suppose that $t \neq t_1, t_2, t_3$ but remaining in a neighbourhood of one of these values. For example we choose t_1. Then the eigenvalue E that before the variation was associated to a non linear element of $N(G_{t_1})$, is not more equal to zero. To compute the index, it was necessary to understand which is the sign assumed by E, respectively for $t > t_1$ and $t < t_1$. Similar considerations are applicable to the eigenvalues associated with t_2 and t_3. See [29] for the details.

If $g \geqslant 38$, we have just proved that the other values for which $\mathrm{Nul}(G_t) > 3$ are bigger than t_3. The presence of these additional values $t_\pm$ does not influence the value of $\mathrm{Ind}(G_t)$ if $t \leqslant t_3$. In other terms theorem 16 continues to hold for $g \geqslant 38$ if we consider $0 < t \leqslant t_3$. Taking into account also the result of S. Nayatani about the Costa surface ($g = 1$) showed in [30], we can give the following statement

Theorem 17. *For all $g \geqslant 1$ the index of the Gauss map of M_g is equal to $2g + 3$.*

1.5 Appendix

This section contains some additional details of the computations made by S. Nayatani.

1.5.1 Divisors and Riemann-Roch theorem

Here we introduce some definitions and concepts of the algebraic geometry. See for example [5].

Let Σ_g be a compact Riemann surface of genus g. A divisor on Σ_g is a finite formal sum of integer multiples of points of Σ_g,

$$D = \sum_{x \in \Sigma_g} n_x x, \quad n_x \in \mathbb{Z}, n_x = 0 \quad \text{for almost all} \quad x.$$

The set of the divisors on Σ_g is denoted by $\mathrm{Div}(\Sigma_g)$. The degree of a divisor is the integer $\deg(D) = \sum n_x$.

Let $\mathbb{C}(\Sigma_g)$ be the field of the meromorphic functions on Σ_g and let $\mathbb{C}(\Sigma_g)^*$ be its multiplicative group of nonzero elements. Every $f \in \mathbb{C}(\Sigma_g)^*$ has a divisor

$$\mathrm{div}(f) = \sum \nu_x(f) x,$$

where $\nu_x(f)$ denotes the order of f at x.

Let ω be a nonzero meromorphic differential n-form on Σ_g. Then ω has a local representation $\omega_x = f_x(z)(dz)^n$ about each point x of Σ_g, where z is the local coordinate about x and $f_x(z) \in \mathbb{C}(\Sigma_g)^*$. So we can define in a natural way $\nu_x(\omega) = \nu_0(f_x)$ and also associate a divisor with a differential form:

$$\operatorname{div}(\omega) = \sum \nu_x(\omega)x.$$

A canonical divisor on Σ_g is a divisor of the form $\operatorname{div}(\omega)$ where ω is a nonzero meromorphic differential form.

Let $D \in \operatorname{div}(\Sigma_g)$. We denote by $H^{0,n}(D)$ the vector space of the meromorphic differential n-forms ω such that

$$\operatorname{div}(\omega) + D \geqslant 0.$$

In other terms, if $D = \operatorname{div}(\eta)$, with η differential form with local representation $\eta_x = g_x(z)(dz)^n$, then the elements of $H^{0,n}(D)$ are the differential forms ω having a local representation $\omega_x = f_x(z)(dz)^n$ with $f_x \in \mathbb{C}(\Sigma_g)$ vanishing to high enough order to make the product $f \cdot g$ holomorphic. We set $\dim H^{0,n}(D) = \ell(D)$.

We are ready to state the following result.

Theorem 18 (Riemann-Roch). *Let Σ_g be a compact Riemann surface of genus g. Let k_{Σ_g} be a canonical divisor on Σ. Then for any divisor $D \in \operatorname{Div}(\Sigma_g)$,*

$$\ell(D) = \deg(D) - g + 1 + \ell(k_{\Sigma_g} - D).$$

The next result gives information about the canonical divisor and a simpler version of Riemann-Roch theorem for divisors of large enough order.

Corollary 19. *Let $\Sigma_g, g, D, k_{\Sigma_g}$ as above.*

- $\deg(k_{\Sigma_g}) = 2g - 2$,
- *If* $\deg(D) > 2g - 2$ *then* $\ell(k_{\Sigma_g} - D) = 0$. *Equivalently* $\ell(D) = \deg(D) - g + 1$.

1.5.2 The determination of a basis of differential forms with null residue at the ramification points

The ramification points (or branch points) of $\gamma(w) = w$ are the zeroes of

$$\frac{dw}{dz} = (g+2)\frac{z^{g-1}(z^2 - A^2)}{(z^g(z^2-1))^{\frac{g}{g+1}}},$$

with $A = \frac{g}{g+2}$, where g denotes the genus. They are given by $Q_0 = (0,0)$, $P_\infty = (\infty, \infty)$, $P_m = (A, B_m)$ and $S_m = (-A, C_m)$ for $m = 0, \ldots, g$, where B_m, C_m denote, respectively, the m-th complex value of $\sqrt[g+1]{A^g(A^2-1)}$ and $\sqrt[g+1]{(-A)^g(A^2-1)}$. We have set $P_\pm = (\pm 1, 0)$. We recall that

$$\hat{H}(G) = \left\{ \sigma \in H^0(2k_{\Sigma_g} + R(G)) \,|\, Res_{r_i} \frac{\sigma}{dw} = 0, i = 1, \ldots, \mu \right\}, \tag{1.21}$$

where k_{Σ_g} is a canonical divisor of Σ_g and $R(G) = \sum_1^\mu r_i$ where r_i are the branch points of $\gamma(w) = w$. In our case it is given by $R(G) = Q_0 + P_\infty + \sum_{m=0}^g (P_m + S_m)$. Furthermore it holds $\hat{H}(G) = \hat{H}(G_t)$.

As for the canonical divisor k_{Σ_g}, we consider $k_{\Sigma_g} = (g-1)P_+ + (g-1)P_-$. We observe that $\deg(k_{\Sigma_g}) = 2g-2$ like stated by corollary 19.

To study the space $\hat{H}(G_t)$ we need understand which are the members of the space $H^0(2k_{\Sigma_g} + R(G))$. Taking into account the definitions of k_{Σ_g} and $R(G)$, then $2k_{\Sigma_g} + R(G) = 2(g-1)P_+ + 2(g-1)P_- + Q_0 + P_\infty + \sum_{m=0}^g P_m + \sum_{m=0}^g S_m$. We deduce that the quadratic differentials σ that are in $H^0(2k_{\Sigma_g} + R(G))$ can have two possible structures:

$$z^k w^j \left(\frac{dz}{w}\right)^2, \tag{1.22}$$

$$z^k w^j \frac{1}{z \pm A} \left(\frac{dz}{w}\right)^2. \tag{1.23}$$

In fact from the definition of H^0, it follows that the quadratic differentials to consider can have a pole of order 0 (differentials of the type (1.22)) or of order 1 (differentials of the type (1.23)) at P_m and S_m for $k = 0, \ldots, g$. Furthermore we observe that the quadratic differential $\left(\frac{dz}{w}\right)^2$ has

- a zero of order 1 at Q_0
- a zero of order 1 at P_∞
- a zero of order $2(g-1)$ at P_+ and P_-.

The quadratic differential $\frac{1}{z \pm A}\left(\frac{dz}{w}\right)^2$ has

- a zero of order 1 at Q_0
- a zero of order $2(g-1)$ at P_+ and P_-
- a zero of order $g+2$ at P_∞

- a pole of order 1 at P_m if we consider sign "-" or at S_m if we consider the sign "+".

We will determine separately the interesting differential forms of type (1.22) and (1.23). To find the differential forms of type (1.23) it is convenient to introduce an auxiliar divisor.

$$D = Q_0 + (g+2)P_\infty + 2(g-1)P_+ + 2(g-1)P_-.$$

We observe that the elements of the vector space $H^0(D)$ after the multiplication by the factor $z \pm A$ are members of $H^0(2k_{\Sigma_g} + R(G))$. It is necessary to remark that to obtain a basis of $H^0(2k_{\Sigma_g}+R(G))$, we will not take into account the differentials of $H^0(2k_{\Sigma_g}+R(G))$ that can be constructed from an element of $H^0(D)$ as described above. Otherwise the number of the founded differential forms exceeds the dimension of $H^0(2k_{\Sigma_g}+R(G))$, that we can compute as follows. We observe that $\deg(2k_{\Sigma_g} + R(G)) = 6g$. Then thanks to corollary of Riemann-Roch theorem 19 we conclude that $\dim H^0(2k_{\Sigma_g} + R(G)) = 5g + 1$. So the basis we are looking for counts $5g+1$ elements. From the observations made above we can deduce that between the forms of type (1.22), we will consider the ones which satisfy the following conditions.

$$k(g+1) + jg \geqslant -1,$$

$$j \geqslant -2(g-1),$$
$$-k(g+1) - j(g+2) \geqslant -1.$$

These relations assure that the selected differentials of type $z^k w^j \left(\frac{dz}{w}\right)^2$, are holomorphic, respectively, at the points Q_0, $P_\pm$ and P_∞. These differentials can be classified in three families. Each family is characterized by particular values of l and k. That is

1. $j = -g+1, \dots, 0, 1$ and $k = -j$,
2. $j = 2-2g, \dots, -g$ and $k = -j$,
3. $j = 2-2g, \dots, -g$ and $k = -j-1$.

As for the forms of type (1.23) we shall consider only the ones which satisfy

$$k(g+1) + jg \geqslant -1,$$

$$j \geqslant -2(g-1),$$
$$-k(g+1) - j(g+2) \geqslant -(g+2).$$

These relations assure that the selected differentials of type $z^k w^j \frac{1}{z \pm A} \left(\frac{dz}{w}\right)^2$, are holomorphic, respectively, at the points Q_0, $P_\pm$ and P_∞. We obtain that $j = -g+1, \dots, 0, 1$ and $k = -j+1$.

Since we are looking for a basis of a vector space we can replace each couple of differentials $\frac{f}{z-A}\left(\frac{dz}{w}\right)^2, \frac{f}{z+A}\left(\frac{dz}{w}\right)^2$ by an appropriate linear combination. We observe that

$$\frac{1}{z-A} \pm \frac{1}{z+A} = \begin{cases} \eta_1 = \frac{z}{z^2-A^2} \\ \eta_2 = \frac{1}{z^2-A^2}. \end{cases}$$

So in the following we will work with the forms $f\eta_1\left(\frac{dz}{w}\right)^2$ and $f\eta_2\left(\frac{dz}{w}\right)^2$, where $f = z^k w^j$ as described above.

The $5g+1$ quadratic differentials we have found forms a basis of $\hat{H}(G_t)$. The last step is to divide each elements of this basis by dw. After simple algebraic manipulations, we obtain the following $5g+1$ differential 1-forms:

$$\begin{array}{lll} \frac{w^k}{z^{k-1}}\frac{dz}{(z^2-A^2)^2} & \text{for} & k=-1,0,\ldots,g-1, \\ \frac{w^k}{z^k}\frac{dz}{(z^2-A^2)^2} & \text{for} & k=-1,0,\ldots,g-1, \\ \frac{w^k}{z^{k+1}}\frac{dz}{(z^2-A^2)} & \text{for} & k=-1,0,\ldots,g-1, \\ \frac{z^k}{w^{k+1}}\frac{dz}{(z^2-A^2)} & \text{for} & k=1,\ldots,g-1, \\ \frac{z^{k-1}}{w^{k+1}}\frac{dz}{(z^2-A^2)} & \text{for} & k=1,\ldots,g-1. \end{array} \tag{1.24}$$

Now it is necessary to select the 1-forms having residue equal to zero at the points Q_0, P_m and S_m with $m = 0, \ldots, g$. Thanks to the properties of symmetry of the surface it is sufficient to verify the null residue condition at the points Q_0, $P_1 = (A, e^{\frac{2\pi i}{g+1}}\sqrt[g+1]{A^g(A^2-1)})$. In fact from the coordinates of the points P_m and S_m, we can deduce that for each $Q \in \{P_m, S_m, k = 0, \ldots, g\}$ there exists $n \in \{0, \ldots, 2g+1\}$ such that $Q = \lambda^n(P_1)$, where λ is the conformal diffeomorphism described in lemma 4. So we can state that the residue of an arbitrary form ω at the point Q is related to the residue at P_1 by

$$\text{Res}_Q\omega = \text{Res}_{P_1}(\lambda^{n-1})^*\omega.$$

Applying this result to the differential forms of the list (1.24) and using the the definition (1.4) of λ, it is easy to obtain that $\text{Res}_Q\omega$ is equal to $\text{Res}_{P_1}\omega$ times a power of $\pm\rho$. So if $\text{Res}_{P_1}\omega = 0$ then $\text{Res}_Q\omega = 0$.

Thanks to algebraic manipulations inspired by the simpler cases where $g = 2, 3$, it is possible to find $3g$ linear independent differential forms satisfying the null residue condition. They constitute the wanted basis.

$$\omega_k^{(1)} = \frac{z^{k-1}}{w^k}\frac{dz}{w} \quad \text{for} \quad k = 1, \ldots, g-1,$$

$$\omega_k^{(2)} = \frac{z^{k-1}((k-2)z^2 - kA^2)}{w^k(z^2-A^2)^2}dz \quad \text{for} \quad k = 0, \ldots, g,$$

$$\omega_k^{(3)} = \frac{z^{k-1}((k-2)z^2 - (k)A^2)}{w^{k+1}(z^2 - A^2)^2} dz \quad \text{for} \quad k = 0, \ldots, g-1.$$

1.5.3 The equations equivalent to the condition of existence of a branched minimal surface.

Let ω_1 and ω_2 two meromorphic differential forms on Σ_g. We write $\omega_1 \sim \omega_2$ if there exists a meromorphic function f on Σ_g such that $\omega_2 = \omega_1 + df$. It is possible to prove that:

$$\omega_k^{(2)} \sim -\frac{k(g+2)}{2(g+1)} \frac{z^{k-1}}{w^k} dz \quad \text{for} \quad k = 0, \ldots, g,$$

$$\omega_k^{(3)} \sim -\frac{(g+2)(g+k+2)}{2(g+1)} \frac{z^{k-1}}{w^{k+1}} dz \quad \text{for} \quad k = 0, \ldots, g-1.$$

Using these relations we get:

$$\int_{\tilde\beta} \omega_k^{(1)} = -2i \sin\frac{(k+1)\pi}{g+1} K_k, \quad \int_{\tilde\beta} \omega_k^{(2)} = -\frac{(g+2)k}{2(g+1)} 2i \sin\frac{k\pi}{g+1} I_k,$$

$$\int_{\tilde\beta} \omega_k^{(3)} = \frac{(g+2)(g+2-k)}{2(g+1)} 2i \sin\frac{(k+1)\pi}{g+1} K_k,$$

$$\int_{\tilde\beta} \gamma\omega_k^{(1)} = 2i \sin\frac{k\pi}{g+1} I_k, \quad \int_{\tilde\beta} \gamma\omega_k^{(2)} = \frac{(g+2)(g+2-k)}{2(g+1)} 2i \sin\frac{(k-1)\pi}{g+1} J_k,$$

$$\int_{\tilde\beta} \gamma\omega_k^{(3)} = -\frac{(g+2)k}{2(g+1)} 2i \sin\frac{k\pi}{g+1} I_k,$$

$$\int_{\tilde\beta} \gamma^2\omega_k^{(1)} = 2i \sin\frac{(k-1)\pi}{g+1} J_k, \quad \int_{\tilde\beta} \gamma^2\omega_k^{(2)} = \frac{(g+2)(2g+4-k)}{2(g+1)} 2i \sin\frac{(k-2)\pi}{g+1} L_k,$$

$$\int_{\tilde\beta} \gamma^2\omega_k^{(3)} = \frac{(g+2)(g+2-k)}{2(g+1)} 2i \sin\frac{(k-1)\pi}{g+1} J_k.$$

We recall that we must impose that $\omega = \sum_0^{g-1} c_k^{(1)}\omega_k^{(1)} + \sum_0^{g} c_k^{(2)}\omega_k^{(2)} + \sum_0^{g-1} c_k^{(3)}\omega_k^{(3)}$, where $c_k^{(i)} \in \mathbb{C}$, satisfies

$$\int_\alpha \omega = t^2 \overline{\int_\alpha \gamma^2(w)\omega}, \quad Re \int_\alpha \gamma(w)\omega = 0$$

for $\alpha = \lambda^l \circ \tilde\beta$ for $l = 0, \ldots, 2g-1$. Now it is convient to introduce some additional notation.

Let

$$\mathcal{L} = \begin{bmatrix} \mathcal{R}_\theta & 0 \\ 0 & 1 \end{bmatrix} \tag{1.25}$$

where $\mathcal{R}_\theta$ is the rotation in the plane by $\theta = g\pi/(g+1)$.
If we denote $\Phi(\omega) = (1-\gamma^2, i(1+\gamma^2), 2\gamma)\omega$, then it is possible to prove

$$\int_{\lambda^l \circ \tilde{\beta}} \Phi(\omega) = \int_{\tilde{\beta}} \lambda^* \Phi(\omega).$$

Since we want to apply this last relation to the differential form ω, it is convenient to remark that:

$$\lambda^*\Phi(\omega_k^{(1)}) = (-1)^k \rho^{-k} \mathcal{L}\Phi(\omega_k^{(1)}),$$
$$\lambda^*\Phi(\omega_k^{(2)}) = (-1)^k \rho^{-k+1} \mathcal{L}\Phi(\omega_k^{(2)}),$$
$$\lambda^*\Phi(\omega_k^{(3)}) = (-1)^k \rho^{-k} \mathcal{L}\Phi(\omega_k^{(3)}),$$

where $\rho = e^{i\frac{g\pi}{g+1}}$. Then the equations

$$\mathrm{Re} \int_{\lambda^l \circ \tilde{\beta}} (1 - t^2\gamma^2, i(1+t^2\gamma^2))\omega = 0, \quad \text{for} \quad l = 0, \dots, 2g-1,$$

are equivalent to:

$$Im\left[\sum_{k=0}^{g-1}\{(-1)^k\rho^{-k}\}^l f_k + \sum_{k=1}^{g}\{(-1)^k\rho^{-(k-1)}\}^l p_k\right] =$$

$$t^2 Im\left[\sum_{k=0,k\neq 1}^{g-1}\{(-1)^k\rho^{-k}\}^l h_k + \sum_{k=0,k\neq 2}^{g}\{(-1)^k\rho^{-(k-1)}\}^l q_k\right]$$

$$Re\left[\sum_{k=0}^{g-1}\{(-1)^k\rho^{-k}\}^l f_k + \sum_{k=1}^{g}\{(-1)^k\rho^{-(k-1)}\}^l p_k\right] =$$

$$-t^2 Re\left[\sum_{k=0,k\neq 1}^{g-1}\{(-1)^k\rho^{-k}\}^l h_k + \sum_{k=0,k\neq 2}^{g}\{(-1)^k\rho^{-(k-1)}\}^l q_k\right],$$

$l = 0, \dots, 2g-1$. These last equations can be arranged as in the systems (1.10) and (1.11).
The equations

$$\mathrm{Re} \int_{\lambda^l \circ \tilde{\beta}} 2t\gamma\omega = 0, \quad \text{for} \quad l = 0, \dots, 2g-1,$$

are equivalent to:

$$Im\left[\sum_{k=1}^{g-1}\{(-1)^k\rho^{-k}\}^l d_k + \sum_{k=0,k\neq 1}^{g}\{(-1)^k\rho^{-(k-1)}\}^l e_k\right] = 0,$$

$l = 0, \dots, 2g-1$. These last equations can be arranged as in the systems (1.12) and (1.13).

Chapter 2

About a family of deformations of the Costa-Hoffman-Meeks surfaces

2.1 Introduction

C. Costa in [3, 4] described a genus one minimal surface with two ends asymptotic to the two ends of a catenoid and a middle end asymptotic to a plane. D. Hoffman and W.H. Meeks in [14], [15] and [16] proved the global embeddedness for the Costa surface, and generalized it for higher genus. We will denote the Costa-Hoffman-Meeks surface of genus $k \geqslant 1$ by M_k. For each $k \geqslant 1$ is a properly embedded minimal surface and has three ends of finite total curvature.

J. Pérez and A. Ros in [34] studied the space $\mathcal{M}$ of minimal surfaces of finite total curvature, genus k and r ends, properly immersed in $\mathbb{R}^3$ and with embedded horizontal ends. Given $M \in \mathcal{M}$, the infinitesimal deformations of M are generated by the elements of the $J(M)$, the space of the Jacobi functions u on M, that is functions such that $Lu = 0$, where L denotes the Jacobi operator of M, which have logarithmic growth at the ends. They showed that $\dim J(M) \geqslant r+3$. They denoted by $\mathcal{M}^* = \{M \in \mathcal{M} : \dim J(M) = r+3\}$ the subspace of non degenerate surfaces and founded that it is an open subset of $\mathcal{M}$ and a $(r+3)$-dimensional real-analytic manifold.

The dimension of the space $J(M)$ just introduced is known for $M = M_k$ for $k \geqslant 1$. In fact thanks to the works [29] and [30] of S. Nayatani, $\dim J(M_k) = 6$, since $r = 3$, but only for $1 \leqslant k \leqslant 37$. Now this result has been extended also for $k \geqslant 38$ (see chapter 1). The elements of $J(M_k)$ are the Jacobi fields associated to the horizontal translations, the rotation about the vertical direction and three functions (one for each end) whose form in a neighbourhood of an end is $a \log |w|$, being a the logarithmic growth. Thus, the one parameter family of deformations of these surfaces described by D. Hoffman and H.

Karcher in [13], contains all the embedded surfaces nearby M_k with a symmetry group generated by k vertical planes, up to dilations preserving the vertical direction.

In this work, following [34], we show the existence of a bigger family of immersed minimal deformations of M_k for $k \geqslant 1$ having three embedded ends. These surfaces do not enjoy any property of symmetry. In fact we admit the possibility to rotate, translate and dilate any of the three ends of the surface and, in addition, to bend the two catenoidal type ends and to change the type of the middle end from a planar type end into a catenoidal type end (we recall that the planar end can be thought to be as a catenoidal type end with null vertical flux). Admitting a bigger number of deformations of M_k, one has, as consequence, the rise of $\dim J(M_k)$. Actually it rises from the value 6 of [34] to 8.

To be more precise we will prove the following result.

Theorem 20. *For for each possible choice of the limit values of the normal vectors of the three ends, there is, up to isometries, a* 1*-dimensional real analytic family of smooth minimal deformations of* M_k*, for* $k \geqslant 1$*, letting the middle planar end horizontal.*

Our result is a consequence of the moduli space theory and of the implicit function theorem. We do not treat the case where also the middle planar end is not horizontal because it can be reconduced to the previous one by an isometry.

The family of surfaces described in the statement of the theorem here, contains the 1-parameter family of deformations of M_k, for $1 \leqslant k \leqslant 37$, obtained by L. Hauswirth and F. Pacard in [11] bending the top and the bottom end and letting horizontal the middle planar end. All the surfaces of this family are not embedded and are symmetric with respect to the vertical plane $x_2 = 0$ that in particular contains the axis of the catenoidal type ends (it is assumed to be the same for the two ends). The parameter is the angle between this axis and the vertical direction. This family is used in the same work to construct some new examples of minimal surfaces by a gluing technique.

One important property of the Costa-Hoffman-Meeks surface, is the non degeneracy. In section 2.3.2 we will prove that M_k is non degenerate for all $k \geqslant 1$ with respect to the definition given in [34].

In [11] F. Pacard and L. Hauswirth studied the mapping properties of the Jacobi operator of M_k acting on the space of the $\mathcal{C}^{2,\alpha}_\delta$ functions defined on M_k and that are invariant under the action of the symmetry with respect to the plane $x_2 = 0$. In particular if $f \in \mathcal{C}^{2,\alpha}_\delta(M_k)$, then $f = O(e^{\delta s})$ on the catenoidal type ends. The mapping properties of the Jacobi operator (denoted by L_δ) acting on functions of $\mathcal{C}^{2,\alpha}_\delta(M_k)$ depend on the choice of δ. The authors give another definition of non degeneracy. They define the surface M_k to be non degenerate if the operator L_δ is injective for all $\delta < -1$.

Thanks to the works [29] and [30] of S. Nayatani and result of chapter 1, the space $K \subset J(M_k)$ of the bounded Jacobi functions, is known to be generated by the functions $\langle N, e_1\rangle$, $\langle N, e_2\rangle$ and $\langle N, e_3\rangle$, $\langle N, e_3 \times p\rangle$, where N denotes the normal vector field about M_k and (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$. These functions are associated to 4 isometries of the ambient space: the three translations and the rotation about the e_3-axis. In [11] the authors remark that the Jacobi function $\langle N, e_3 \times p\rangle$ associated to the rotation about the e_3-axis and the translation along the e_2-axis do not respect the mirror symmetry described above, that is they are not invariant with respect to the action of the map $(x_1, x_2, x_3) \rightarrow (x_1, -x_2, x_3)$. So they did not taken into account them and could conclude that M_k are non degenerate, in the sense of their definition.

The surfaces of the family described in our work do not enjoy any property of symmetry, since we admit to bend the catenoidal type ends in arbitrary directions. Then the Jacobi functions described above must be taken into account. Since the Jacobi function $\langle N, e_3 \times p\rangle$ belongs to the space $\mathcal{C}^{2,\alpha}_\delta(M_k)$ for $\delta = -k-1 \leqslant -2$, the property of non degeneracy does not hold any more. Actually the operator L_δ acting on $\mathcal{C}^{2,\alpha}_\delta(M_k)$ is no more injective for all $\delta < -1$. As consequence, we can state that for all $k \geqslant 1$ the Costa-Hoffman-Meeks surface M_k is degenerate in the sense of the definition given in [11].

2.2 Preliminaries and notation

We denote by $X : M_k \rightarrow \mathbb{R}^3$ the conformal minimal immersion of the Costa-Hoffman-Meeks surface M_k in $\mathbb{R}^3$. If g and η are the Weierstrass data of M_k, we can write:

$$X(z) = \left(\frac{1}{2}\overline{\int g^{-1}\eta} - \frac{1}{2}\int g\eta, \mathrm{Re}\int \eta\right) \in \mathbb{C} \times \mathbb{R} = \mathbb{R}^3.$$

The meromorphic function g is the stereographic projection from the north pole of the Gauss map $N : M_k \rightarrow \mathbb{S}^2$. The total curvature is finite and M_k is conformally diffeomorphic to $\overline{M_k} \setminus \{p_t, p_b, p_m\}$, being $\overline{M_k}$ a compact surface and p_i three points. The Weierstrass data extend in a meromorphic way at each puncture p_i. In particular the Gauss map of $X(z)$ is well defined at p_i. The points p_i are identified with the ends and a neighbourhood of a puncture will parametrize the corresponding end. In the following we will refer to various quantities related to the three ends of the surface using the index t for the top end, the index b for the bottom end and the index m for the planar end. The Gauss map N takes the limit values $(0,0,1)$ at the ends p_t and p_b and $(0,0,-1)$ at the end p_m.

We parametrize the ends p_i in the graph coordinate $x = x_1 + ix_2$ on $D_i^*(\epsilon_i) = \{x \in \mathbb{C}; 0 < |x| \leqslant \epsilon_i\}$ by the immersions

$$X_i(x) = \left(\frac{1}{x}, -\tilde{a}_i \ln |x| + h_i(x)\right) \in \mathbb{C} \times \mathbb{R} = \mathbb{R}^3$$

for $i = t, b$, where h_i is a smooth real valued function on $D_i^*(\epsilon_i)$. The quantities $\tilde{a}_i$ and $h_i(0)$ are called the logarithmic growth and the height of the end. We can observe that, for the null flux condition, $\tilde{a}_t = -\tilde{a}_b$. The Gauss map for an end with logarithmic growth a is given by (see p. 182, [34])

$$N(x) = Q^{-\frac{1}{2}} \left(-a\bar{x} + \bar{x}^2 \nabla_0 h, 1\right), \tag{2.1}$$

where $Q = 1 + |x|^2(a^2 + |x|^2|\nabla_0 h|^2 - 2a\langle x, \nabla_0 h\rangle)$ and $\bar{x}^2\nabla_0 h$ means the product of the complex number $\bar{x}^2$ with the gradient $\nabla_0 h$ respect to the flat metric ds_0^2 of the x-plane.

As for the planar end p_m, we will use the following parametrization

$$X_m(x) = \left(\frac{1}{x}, h_m(x)\right) \quad \text{on} \quad D_m^*(\epsilon_m).$$

So its logarithmic growth is zero.

2.3 The deformation of the surface and its Jacobi operator

In this section we describe how we deform the surface M_k and we give the expressions of the immersions in $\mathbb{R}^3$ of the three ends of the deformed surface. In subsection 2.3.1 we introduce the Jacobi operator of M_k and we study its kernel and its range.

We deform the surface M_k in the following way. Using a smooth cut-off function we glue $X : M_k \backslash (D_t^* \cup D_b^* \cup D_m^*) \to \mathbb{R}^3$ with the parametrizations of the three ends with a different value of the logarithmic growths (that we denote with a_t, a_b, a_m). Furthermore we rotate the ends p_t and p_b, that is we change the directions of their axes of revolution. We denote with $F(\theta_{1,i}, \theta_{2,i})$ the frame defined by the following unit vectors:

$$e_1(\theta_{1,i}, \theta_{2,i}) = \cos\theta_{1,i} e_1 + \sin\theta_{1,i}\sin\theta_{2,i} e_2 + \sin\theta_{1,i}\cos\theta_{2,i} e_3, \tag{2.2}$$

$$e_2(\theta_{1,i}, \theta_{2,i}) = \cos\theta_{2,i} e_2 - \sin\theta_{2,i} e_3,$$

$$e_3(\theta_{1,i},\theta_{2,i}) = -\sin\theta_{1,i}e_1 + \cos\theta_{1,i}\sin\theta_{2,i}e_2 + \cos\theta_{1,i}\cos\theta_{2,i}e_3,$$

where (e_1, e_2, e_3) denotes the canonical base of $\mathbb{R}^3$.

The immersions of the rotated catenoidal type ends on $D_i^*(\epsilon_i)$ are given by

$$X_{i,\theta_{1,i},\theta_{2,i}}(x) = \frac{x_1}{|x|^2}e_1(\theta_{1,i},\theta_{2,i}) - \frac{x_2}{|x|^2}e_2(\theta_{1,i},\theta_{2,i}) + (-a_i \ln|x| + h_i(x))e_3(\theta_{1,i},\theta_{2,i}),$$

for $i = t, b$. As for the planar end, we consider on $D_m^*(\epsilon_m)$ in the canonical frame (e_1, e_2, e_3) the immersion

$$X_{m,0,0}(x) = \left(\frac{1}{x}, -a_m \ln|x| + h_m(x)\right).$$

We define $y = (a_t, a_b, a_m, \theta_{1,t}, \theta_{1,b}, \theta_{2,t}, \theta_{2,b})$. Thanks to the deformation we obtain a family of immersions that we denote with $X_y : M_k \to \mathbb{R}^3$, not necessarily minimal and depending smoothly on y. Now let $\widetilde{N}(y) \in \mathcal{C}^\infty(\overline{M_k}, \mathbb{R}^3)$ be a vector field such that $\langle \widetilde{N}(y), N\rangle = 1$ on $\overline{M_k}\backslash(D_t^* \cup D_b^*)$ and

$$\widetilde{N}(y) = \frac{e_3(\theta_{1,i},\theta_{2,i})}{\langle N, e_3(\theta_{1,i},\theta_{2,i})\rangle}$$

on D_i^* for $i = t, b$. We remark that we do not modify the normal vector field on D_m^* because we keep the the middle planar end horizontal. Let $\mathcal{A}$ be a neighbourhood of $(\tilde{a}_t, \tilde{a}_b, 0)$ (the logarithmic growths of the ends of M_k), $\mathcal{U}$ a neighbourhood of zero in $\mathcal{C}^{2,\alpha}(\overline{M_k})$. For $y \in \mathcal{A} \times [-\epsilon, \epsilon]^4$ and a function $u \in \mathcal{U}$, we consider the family of immersions

$$X_y + u\widetilde{N}(y) : M_k \longrightarrow \mathbb{R}^3. \tag{2.3}$$

Such a family depends analytically on (y, u).

Let $\lambda \in \mathcal{C}^\infty(M_k)$ be a positive function which in terms of the graph coordinate is defined by

$$\lambda(x) = \begin{cases} \frac{1}{|x|^4} & \text{on } D_t^*(\epsilon_t),\ D_b^*(\epsilon_b),\ D_m^*(\epsilon_m), \\ 1 & \text{on } M_k\backslash(D_t^*(2\epsilon_t) \cup D_b^*(2\epsilon_b) \cup D_m^*(2\epsilon_m)). \end{cases} \tag{2.4}$$

We know from [34] that $d\bar{s}^2 = (1/\lambda)\, ds_0^2$ is a Riemannian metric on $\overline{M_k}$. We denote the associated area mesure by $d\bar{A}$. If $H(y, u)$ is the mean curvature function of the immersion $X_y + u\widetilde{N}(y)$, we consider the operator $\overline{H}(y, u) = \lambda H(y, u)$. Since at the ends p_t and p_b, the rotation does not change the value of the mean curvature, we can apply lemma 6.4 proved in [34] at each end to conclude that the operator

$$\overline{H} : \mathcal{A} \times \mathcal{U} \to \mathcal{C}^{0,\alpha}(\overline{M_k})$$

is real-analytic.

2.3.1 The Jacobi operator

We define $\tilde{y} = (\tilde{a}_t, \tilde{a}_b, 0, 0, 0, 0, 0)$ and $\dot{y} = (\dot{a}_t, \dot{a}_b, \dot{a}_m, \dot{\theta}_{1,t}, \dot{\theta}_{2,t}, \dot{\theta}_{1,b}, \dot{\theta}_{2,b})$. and consider a smooth curve

$$\gamma(t) = (a_t(t), a_b(t), a_m(t), \theta_{1,t}(t), \theta_{2,t}(t), \theta_{1,b}(t), \theta_{2,b}(t), u(t)), \tag{2.5}$$

with $|t| < \epsilon$, passing by $(\tilde{y}, 0)$ with acceleration $\gamma'(0) = (\dot{y}, u'(0))$.

To introduce the Jacobi operator we need consider the continuous family of smooth deformations $X_{\gamma(t)} + u(t)\widetilde{N}(\gamma(t)) : M_k \to \mathbb{R}^3$ of the Costa-Hoffman-Meeks surface. Denote by $H(t)$ the mean curvature of $X_{\gamma(t)} + u(t)\widetilde{N}(\gamma(t))$.

If $w = \langle \frac{d}{dt}_{|t=0} \left(X_{\gamma(t)} + u(t)\widetilde{N}(\gamma(t)) \right), N \rangle$ is the variation field, we have:

$$\frac{d}{dt}_{|t=0} H(t) = \frac{1}{2} L w = \frac{1}{2} \left(\Delta_{ds_0^2} w + |A|^2_{ds_0^2} w \right) \tag{2.6}$$

where $|A|_{ds_0^2}$ denotes the norm of the second fundamental form computed with respect to the metric ds_0^2. L is the Jacobi operator of M_k. It can be "compactified" to obtain the operator $\overline{L} = \Delta_{d\bar{s}^2} + |A|^2_{d\bar{s}^2} = \lambda L$ on $\overline{M_k}$. The function λ is defined by (2.4). It is related to the differential of $\overline{H}(t) = \lambda H(t)$ by a relation similar to (2.6).

In the following we will give the expression of a Jacobi function. To express it we need introduce additional notation. Let f_1, f_2, f_3 be the functions defined by:

$$f_1(x,i) = \frac{x_1}{|x|^2} \langle N, e_3 \rangle - (-\tilde{a}_i \ln|x| + h_i(x)) \langle N, e_1 \rangle,$$

$$f_2(x,i) = \frac{x_2}{|x|^2} \langle N, e_3 \rangle + (-\tilde{a}_i \ln|x| + h_i(x)) \langle N, e_2 \rangle,$$

$$f_3(x, \dot{a}_i) = -\dot{a}_i \ln|x| \langle N, e_3 \rangle$$

in $D_i^*(\epsilon_i)$ for $i = t, b, m$ and $f_n(x,i) = 0$, $n = 1, 2, 3$, in $M_k \backslash D_i^*(2\epsilon_i)$.We recall that $\tilde{a}_t, \tilde{a}_b$ are the logarithmic growths of the top and of the bottom end of M_k, and being the middle planar end horizontal, we assume $\tilde{a}_m = 0$.

Proposition 21. *The Jacobi functions about M_k have, in $D_i^*(\epsilon_i)$, the following expression*

$$\theta_{1,i} f_1(x,i) + \theta_{2,i} f_2(x,i) + f_3(x, \dot{a}_i) + u_i$$

for $i = t, b, m$, with $\theta_{1,m} = \theta_{2,m} = 0$, $\tilde{a}_m = 0$ and $u_i \in \mathcal{C}^{2,\alpha}(D_i(\epsilon_i))$.

Proof. A Jacobi function is defined by

$$\langle \frac{d}{dt}_{|t=0} \left(X_{\gamma(t)} + u(t)\widetilde{N}(\gamma(t)) \right), N \rangle. \tag{2.7}$$

We start observing that $X_{\gamma(t)}$ in $D_i^*(\epsilon_i)$ is given by

$$\frac{x_1}{|x|^2} e_1(\theta_{1,i}(t), \theta_{2,i}(t)) - \frac{x_2}{|x|^2} e_2(\theta_{1,i}(t), \theta_{2,i}(t)) + (-a_i(t) \ln |x| + h_i(x)) e_3(\theta_{1,i}(t), \theta_{2,i}(t)),$$

for $i = t, b$ and in $D_m^*(\epsilon_m)$ by

$$\left(\frac{1}{x}, -a_m(t) \ln |x| + h_m(x) \right).$$

We recall (see (2.5)) that $a_i(0) = \tilde{a}_i$ with $\tilde{a}_m = 0$, $\gamma'(0) = (\dot{y}, u'(0))$.
To obtain $\frac{d}{dt}_{|t=0} X_{\gamma(t)}$ we need compute

$$\dot{e}_i = \frac{d}{dt} e_i(\varphi(t), \phi(t)) = \frac{\partial e_i(\varphi, \phi)}{\partial \varphi} \frac{d\varphi}{dt} + \frac{\partial e_i(\varphi, \phi)}{\partial \phi} \frac{d\phi}{dt}$$

and to evaluate it for $t = 0$. We will apply the result in the case $\varphi(t) = \theta_{1,i}(t), \phi(t) = \theta_{2,i}(t)$. So we suppose that $\varphi(0) = \phi(0) = 0$. To this aim it is useful to observe that from the equation (2.2) and the following ones we obtain

$$\left(\frac{\partial e_1(\varphi, \phi)}{\partial \varphi} \right)_{|\varphi=\phi=0} = - \left(\frac{\partial e_2(\varphi, \phi)}{\partial \phi} \right)_{|\varphi=\phi=0} = e_3,$$

$$\left(\frac{\partial e_1(\varphi, \phi)}{\partial \phi} \right)_{|\varphi=\phi=0} = \left(\frac{\partial e_2(\varphi, \phi)}{\partial \varphi} \right)_{|\varphi=\phi=0} = 0,$$

$$\left(\frac{\partial e_3(\varphi, \phi)}{\partial \varphi} \right)_{|\varphi=\phi=0} = -e_1, \quad \left(\frac{\partial e_3(\varphi, \phi)}{\partial \phi} \right)_{|\varphi=\phi=0} = e_2.$$

Then it is easy to obtain

$$\dot{e}_1(0) = \varphi'(0) e_3,$$
$$\dot{e}_2(0) = -\phi'(0) e_3,$$
$$\dot{e}_3(0) = -\varphi'(0) e_1 + \phi'(0) e_2.$$

Then in $D_i^*(\epsilon_i)$

$$\frac{d}{dt}_{|t=0} X_{\gamma(t)} = \frac{x_1}{|x|^2} \theta'_{1,i}(0) e_3 - \frac{x_2}{|x|^2} (-\theta'_{2,i}(0) e_3) +$$
$$+ (-a_i(0) \ln |x| + h_i(x))(-\theta'_{1,i}(0) e_1 + \theta'_{2,i}(0) e_2) + (-a'_i(0) \ln |x|) e_3.$$

Reordering the summands we get

$$\langle \frac{d}{dt}_{|t=0} X_{\gamma(t)}, N\rangle = \theta'_{1,i}(0) f_1(x,i) + \theta'_{2,i}(0) f_2(x,i) + f_3(x, \dot{a}_i).$$

As for the last term of (2.7), we recall that $u(0) = 0$ (see (2.5)) and, on $D_i^*(\epsilon_i)$, we defined

$$\widetilde{N}(\gamma(t)) = \frac{e_3(\theta_{1,i}(t), \theta_{2,i}(t))}{\langle N, e_3(\theta_{1,i}(t), \theta_{2,i}(t))\rangle}.$$

Then $\frac{d}{dt} u(t)\widetilde{N}(\gamma(t))$, evaluated in $t = 0$, is equal to

$$u'(0)\widetilde{N}(\gamma(0)) + u(0)\frac{d}{dt}\widetilde{N}(\gamma(t))|_{t=0} = u'(0).$$

If u_i denotes the restriction of $u'(0)$ to $D_i^*(\epsilon_i)$ for $i = t, b, m$, then the result is obvious. □

The following lemma shows the existing relation between the logarithmic growths and the Jacobi operator.

Lemma 22. *Let $U, V \in \mathcal{C}^{2,\alpha}(M_k)$ be the functions defined in $D_i^*(\epsilon_i)$, for $i \in \{t, b, m\}$, by*

$$U_i(x) = \theta_{1,i} f_1(x,i) + \theta_{2,i} f_2(x,i) + f_3(x, a_i) + u_i(x)$$

and

$$V_i(x) = \varphi_{1,i} f_1(x,i) + \varphi_{2,i} f_2(x,i) + f_3(x, b_i) + v_i(x),$$

with $\theta_{j,i}, \varphi_{j,i} \in \mathbb{R}$ and $\tilde{a}_m = 0, \theta_{j,m} = \varphi_{j,m} = 0$, $j = 1, 2$, and $u_i, v_i \in \mathcal{C}^{2,\alpha}(D_i(\epsilon_i))$. Then we have

$$\int_{\overline{M_k}} (U\bar{L}V - V\bar{L}U) d\bar{A} = \int_{M_k} (ULV - VLU) dA =$$

$$= 2\pi \sum_{i\in\{t,b\}} [\langle \Phi_i, \nabla u_i(0)\rangle - \langle \Theta_i, \nabla v_i(0)\rangle] + 2\pi \sum_{i\in\{t,b,m\}} [b_i u_i(0) - a_i v_i(0)],$$

with $\Theta_i = (\theta_{1,i}, \theta_{2,i})$, $\Phi_i = (\varphi_{1,i}, \varphi_{2,i})$ and $\nabla \cdot = (\partial_{x_1} \cdot, \partial_{x_2} \cdot)$.

Proof. In the following we will need use the Green identity, so we must use the conformal coordinate $z = re^{i\alpha}$ around the ends p_t, p_b and p_m. Thus for the catenoidal type ends, we have

$$\frac{1}{x} = \frac{\tilde{a}_i}{2z}(1 + |z|^2 + z t_i(z)) = \frac{s_i(z)}{z}$$

with $s_i(0) = \frac{\tilde{a}_i}{2} \in \mathbb{R}$, where $\tilde{a}_t$, $\tilde{a}_b$ are the logarithmic growths of the top end and of the bottom end of M_k.

As for the planar end

$$\frac{1}{x} = \frac{s_m(z)}{z}$$

where $s_m(0) = c$. We can assume that $c = 1$.

Since we want to find the expressions of U and V near the ends in terms of coordinates (r, α), it is useful to observe that from (2.1) for a generic end with logarithmic growth a it holds that:

$$\langle N, e_3 \rangle = Q^{-\frac{1}{2}} = \left(1 + |x|^2(a^2 + |x|^2|\nabla_0 h|^2 - 2a\langle x, \nabla_0 h\rangle)\right)^{-\frac{1}{2}},$$

$$\langle N, e_1 \rangle = \langle N, e_3 \rangle \mathrm{Re}(-a\bar{x} + \bar{x}^2 \nabla_0 h),$$

$$\langle N, e_2 \rangle = \langle N, e_3 \rangle \mathrm{Im}(-a\bar{x} + \bar{x}^2 \nabla_0 h).$$

Then in a neighbourhood of each end we can write:

$$\langle N, e_3 \rangle = \left(1 + O(|x|^2)\right)^{-\frac{1}{2}} = 1 + O(|x|^2), \tag{2.8}$$

$$\langle N, e_1 \rangle = \left(1 + O(|x|^2)\right)\left(-ax_1 + O(\bar{x}^2)\right) = -ax_1 + O(|x|^2), \tag{2.9}$$

$$\langle N, e_2 \rangle = \left(1 + O(|x|^2)\right)\left(ax_2 + O(\bar{x}^2)\right) = ax_2 + O(|x|^2). \tag{2.10}$$

In the coordinates (r, α) U_i and V_i have the following expression:

$$U_i(r) = \theta_{1,i} f_1(r, i) + \theta_{2,i} f_2(r, i) + f_3(r, a_i) + u_i(r),$$

$$V_i(r) = \varphi_{1,i} f_1(r, i) + \varphi_{2,i} f_2(r, i) + f_3(r, b_i) + v_i(r)$$

where

$$f_1(r, i) = \frac{\tilde{a}_i \cos\alpha}{2r} + O(r \ln r),$$

$$f_2(r, i) = \frac{\tilde{a}_i \sin\alpha}{2r} + O(r \ln r),$$

$$f_3(r, a) = -a \ln r + O(r).$$

If $D_i(0, r)$ are conformal disks and $M(r) = M \backslash (\cup_{i \in \{t,b,m\}} D_i(0, r))$, then the conformal invariance of the integral implies:

$$I(r) = \int_{M(r)} (ULV - VLU)\, dA = \int_{\partial M(r)} \left(U \frac{\partial V}{\partial \eta} - V \frac{\partial U}{\partial \eta}\right) ds_0 =$$

$$- \sum_{i \in \{t,b,m\}} \int_{\partial D_i(0,r)} \left(U_i \frac{\partial V_i}{\partial r} - V_i \frac{\partial U_i}{\partial r}\right) |dz|, \tag{2.11}$$

where dA is the area mesure associated to ds_0, η is the exterior conormal field to the immersion along $\partial M(r)$ and $|dz| = rd\alpha$. To get the lemma it will be sufficient to let r go to zero.
Of course we have for $i \in \{t, b, m\}$:

$$\frac{\partial U_i}{\partial r} = \theta_{1,i}\frac{\partial f_1(r)}{\partial r} + \theta_{2,i}\frac{\partial f_2(r)}{\partial r} + \frac{\partial f_3(r, a_i)}{\partial r} + \frac{\partial u_i(r)}{\partial r}$$

and a similar expression for $\frac{\partial V_i}{\partial r}$:

$$\frac{\partial V_i}{\partial r} = \varphi_{1,i}\frac{\partial f_1(r)}{\partial r} + \varphi_{2,i}\frac{\partial f_2(r)}{\partial r} + \frac{\partial f_3(r, b_i)}{\partial r} + \frac{\partial v_i(r)}{\partial r}.$$

As for the functions u_i and v_i, we will adopt in the neighbourhood of the origin an expression of the form (we recall that $z = z_1 + iz_2 = re^{i\alpha}$):

$$l = l(0) + r\cos\alpha(\partial_{z_1} l)(0) + r\sin\alpha(\partial_{z_2} l)(0) + O(r^2). \tag{2.12}$$

Now we proceed with the evaluation of each summand that appears in (2.11). For $i \in \{t, b, m\}$ we have (to simplify the notation, we will omit the dependence on r and i)

$$\lim_{r\to 0}\int_{\partial D_i(0,r)} \left(U_i\frac{\partial V_i}{\partial r} - V_i\frac{\partial U_i}{\partial r}\right)|dz| =$$

$$\lim_{r\to 0}\left(\int_{\{|z|=r\}} \left(\varphi_{1,i}u_i(z)\frac{\partial f_1}{\partial r} - \theta_{1,i}v_i(z)\frac{\partial f_1}{\partial r}\right) + \left(\frac{\partial v_i}{\partial r}\theta_{1,i}f_1 - \frac{\partial u_i}{\partial r}\varphi_{1,i}f_1\right) + \right.$$

$$+\left(\varphi_{2,i}u_i(z)\frac{\partial f_2}{\partial r} - \theta_{2,i}v_i(z)\frac{\partial f_2}{\partial r}\right) + \left(\frac{\partial v_i}{\partial r}\theta_{2,i}f_2 - \frac{\partial u_i}{\partial r}\varphi_{2,i}f_2\right) +$$

$$\left.\left(u_i(z)\frac{\partial f_3(b_i)}{\partial r} - v_i(z)\frac{\partial f_3(a_i)}{\partial r}\right) + \left(u_i\frac{\partial v_i}{\partial r} - v_i\frac{\partial v_i}{\partial r}\right)|dz|\right).$$

We define (the expression of l is given by (2.12)):

$$G(l) = \lim_{r\to 0}\int_{\{|z|=r\}} l(r)\frac{\partial f_1}{\partial r}|dz| =$$

$$-\lim_{r\to 0}\int_{\{|z|=r\}} \big(l(0) + r(\cos\alpha(\partial_{z_1} l)(0) + \sin\alpha(\partial_{z_2} l)(0)) + O(r^2)\big)\cdot\left(\frac{\tilde{a}_i\cos\alpha}{2r^2} + O(\ln r)\right) rd\alpha =$$

$$= -\lim_{r\to 0}\int_{\{|z|=r\}} \frac{\tilde{a}_i\cos^2\alpha}{2}(\partial_{z_1} l)(0)d\alpha = -\frac{\pi}{2}\tilde{a}_i(\partial_{z_1} l)(0)$$

and in a similar way:

$$T(l) = \lim_{r\to 0}\int_{\{|z|=r\}} l(r)\frac{\partial f_2}{\partial r}|dz| =$$

$$-\lim_{r\to 0}\int_{\{|z|=r\}} \big(l(0) + r(\cos\alpha(\partial_{z_1}l)(0) + \sin\alpha(\partial_{z_2}l)(0)) + O(r^2)\big)\cdot\left(\frac{\tilde{a}_i\sin\alpha}{2r^2} + O(\ln r)\right) r d\alpha =$$

$$= -\lim_{r\to 0}\int_{\{|z|=r\}} \frac{\tilde{a}_i\sin^2\alpha}{2}(\partial_{z_2}l)(0)d\alpha = -\frac{\pi}{2}\tilde{a}_i(\partial_{z_2}l)(0).$$

Then we can conclude that for $i \in \{t, b, m\}$:

$$\lim_{r\to 0}\int_{\{|z|=r\}} (\varphi_{1,i}u_i(z) - \theta_{1,i}v_i(z))\frac{\partial f_1}{\partial r})|dz| =$$

$$\varphi_{1,i}G(u_i) - \theta_{1,i}G(v_i) = \frac{\pi}{2}\tilde{a}_i(\theta_{1,i}(\partial_{z_1}v_i)(0) - \varphi_{1,i}(\partial_{z_1}u_i)(0)).$$

In the same way we get

$$\lim_{r\to 0}\int_{\{|z|=r\}} (\varphi_{2,i}u_i(z) - \theta_{2,i}v_i(z))\,\frac{\partial f_2}{\partial r}|dz| =$$

$$\varphi_{2,i}T(u_i) - \theta_{2,i}T(v_i) = \frac{\pi}{2}\tilde{a}_i(\theta_{2,i}(\partial_{z_2}v_i)(0) - \varphi_{2,i}(\partial_{z_2}u_i)(0)).$$

We define another couple of functions:

$$R(l) = \lim_{r\to 0}\int_{\{|z|=r\}} \frac{\partial l}{\partial r} f_1|dz| =$$

$$\lim_{r\to 0}\int_{\{|z|=r\}}(\cos\alpha(\partial_{z_1}l)(0) + \sin\alpha(\partial_{z_2}l)(0) + O(r))\cdot\left(\frac{\tilde{a}_i\cos\alpha}{2r} + O(r\ln r)\right) r d\alpha =$$

$$\lim_{r\to 0}\int_{\{|z|=r\}} \frac{\tilde{a}_i\cos^2\alpha}{2}(\partial_{z_1}l)(0)d\alpha = \frac{\pi}{2}\tilde{a}_i(\partial_{z_1}l)(0)$$

and

$$F(l) = \lim_{r\to 0}\int_{\{|z|=r\}} \frac{\partial l}{\partial r} f_2|dz| =$$

$$\lim_{r\to 0}\int_{\{|z|=r\}}(\cos\alpha(\partial_{z_1}l)(0) + \sin\alpha(\partial_{z_2}l)(0) + O(r))\cdot\left(\frac{\tilde{a}_i\sin\alpha}{2r} + O(r\ln r)\right) r d\alpha =$$

$$\lim_{r\to 0}\int_{\{|z|=r\}} \frac{\tilde{a}_i \sin^2\alpha}{2}(\partial_{z_2} l)(0)d\alpha = \frac{\pi}{2}\tilde{a}_i(\partial_{z_2} l)(0).$$

Then we find:

$$\lim_{r\to 0}\int_{\{|z|=r\}} \left(\frac{\partial v_i}{\partial r}\theta_{1,i} f_1 - \frac{\partial u_i}{\partial r}\varphi_{1,i} f_1\right)|dz| =$$

$$\theta_{1,i}R(v_i) - \varphi_{1,i}R(u_i) = \frac{\pi}{2}\tilde{a}_i(\theta_{1,i}(\partial_{z_1} v_i)(0) - \varphi_{1,i}(\partial_{z_1} u_i)(0)).$$

Analogously:

$$\lim_{r\to 0}\int_{\{|z|=r\}} \left(\frac{\partial v_i}{\partial r}\theta_{2,i} f_2 - \frac{\partial u_i}{\partial r}\varphi_{2,i} f_2\right)|dz| =$$

$$\theta_{2,i}F(v_i) - \varphi_{2,i}F(u_i) = \frac{\pi}{2}\tilde{a}_i(\theta_{2,i}(\partial_{z_2} v_i)(0) - \varphi_{2,i}(\partial_{z_2} u_i)(0)).$$

As for the fifth summand, we have

$$\lim_{r\to 0}\int_{\{|z|=r\}} \left(u_i(z)\frac{\partial f_3(b_i)}{\partial r} - v_i(z)\frac{\partial f_3(a_i)}{\partial r}\right)|dz| =$$

$$-\lim_{r\to 0}\int_{\{|z|=r\}} \Bigg(\left(u_i(0) + r\cos\alpha(\partial_{z_1} u_i)(0) + r\sin\alpha(\partial_{z_2} u_i)(0) + O(r^2)\right)\frac{b_i}{r} -$$

$$-\left(v_i(0) + r\cos\alpha(\partial_{z_1} v_i)(0) + r\sin\alpha(\partial_{z_2} v_i)(0) + O(r^2)\right)\frac{a_i}{r}\Bigg) r d\alpha =$$

$$-2\pi(b_i u_i(0) - a_i v_i(0)).$$

To finish we show that

$$\lim_{r\to 0}\int_{\{|z|=r\}} \left(u_i\frac{\partial v_i}{\partial r} - v_i\frac{\partial v_i}{\partial r}\right)|dz| = 0.$$

In fact:

$$\lim_{r\to 0}\int_{\{|z|=r\}} u_i\frac{\partial v_i}{\partial r}|dz| = \lim_{r\to 0}\int_{\{|z|=r\}} ((u_i(0) + O(r))(\cos\alpha(\partial_{z_1} v_i)(0) +$$

$$\sin\alpha(\partial_{z_2} v_i)(0) + O(r)))\, r d\alpha = 0.$$

A similar result is true for the other summand.

If we collect the previous results, we find that for $i = t, b, m$:

$$\lim_{r\to 0}\int_{\partial D_i(0,r)} \left(U_i\frac{\partial V_i}{\partial r} - V_i\frac{\partial U_i}{\partial r}\right)|dz| =$$

$$-\pi\tilde{a}_i\left[(\varphi_{1,i}(\partial_{z_1}u_i)(0)-\theta_{1,i}(\partial_{z_1}v_i)(0))+(\varphi_{2,i}(\partial_{z_2}u_i)(0)-\theta_{2,i}(\partial_{z_2}v_i)(0))\right]$$
$$-2\pi(b_iu_i(0)-a_iv_i(0))=0,$$

with $\theta_{j,m}=\varphi_{j,m}=0$ for $j=1,2$.

In conclusion we have:

$$\lim_{r\to 0}\int_{M(r)}(ULV-VLU)\,dA=\pi\tilde{a}_i\sum_{i\in\{t,b\}}[\varphi_{1,i}(\partial_{z_1}u_i)(0)-\theta_{1,i}(\partial_{z_1}v_i)(0)]+$$

$$\pi\tilde{a}_i\sum_{i\in\{t,b\}}[\varphi_{2,i}(\partial_{z_2}u_i)(0)-\theta_{2,i}(\partial_{z_2}v_i)(0)]+2\pi\sum_{i\in\{t,b,m\}}[b_iu_i(0)-a_iv_i(0)].$$

We must do a change of variables to return in the graph coordinate. It is sufficient to observe that

$$\frac{1}{x}=\frac{\tilde{a}_i}{2z}+O(1)$$

at each catenoidal type end. Then we get

$$\partial_{z_1}u_i(0)=\frac{2}{\tilde{a}_i}\partial_{x_1}u_i(0)\quad \partial_{z_2}u_i(0)=\frac{2}{\tilde{a}_i}\partial_{x_2}u_i(0)$$

and the same equations involving the functions v_i. After a change of sign we can conclude

$$\int_{M_k}(ULV-VLU)dA=$$

$$2\pi\sum_{i\in\{t,b\}}[\varphi_{1,i}(\partial_{z_1}u_i)(0)-\theta_{1,i}(\partial_{z_1}v_i)(0)]+$$

$$+2\pi\sum_{i\in\{t,b\}}[\varphi_{2,i}(\partial_{z_2}u_i)(0)-\theta_{2,i}(\partial_{z_2}v_i)(0)]+2\pi\sum_{i\in\{t,b,m\}}[b_iu_i(0)-a_iv_i(0)].$$

Reordering the terms, we get the statement of the lemma. □

2.3.2 The properties of the kernel and of the range of the Jacobi operator

Let $B=B(M_k)\subset\mathcal{C}^{2,\alpha}(M_k)$ the space of functions v such that their expression in a neighbourhood of p_i, with $i=t,b,m$, is

$$\theta_{1,i}f_1(x,i)+\theta_{2,i}f_2(x,i)+f_3(x,a_i)+v_i(x), \tag{2.13}$$

in the graph coordinate x (here we use the same notation of the lemma 22) with $v_i \in \mathcal{C}^{2,\alpha}(\{|x| \leqslant \epsilon\})$. We recall that $\tilde{a}_m = 0, \theta_{j,m} = 0$ for $j = 1, 2$.

We are interested to study the kernel and the image of the "compactified" Jacobi operator $\bar{L} : B(M_k) \to \mathcal{C}^{0,\alpha}(\overline{M_k})$. We define the following subspaces of the Banach space B :

$$J = J(M_k) = \ker(\bar{L}), \quad K = K(M_k) = J \cap \mathcal{C}^{2,\alpha}(\overline{M_k}), \quad K_0 = K_0(\overline{M_k}) = \bar{L}(B)^{\perp}.$$

The elements of the space K are the Jacobi functions on M_k bounded at the ends. From the previous definitions it follows that

$$\bar{L} : B(M_k) = J \oplus J^{\perp} \to \bar{L}(B) \oplus K_0.$$

Lemma 23. *In the situation described above, it holds that:*

1. $K_0 = \{v \in K; \partial_{x_i} v_j(0) = 0,$ *for* $i = 1, 2,$ $j = t, b,$ *and* $v_j(0) = 0,$ *for* $j = t, b, m\} = \{\langle N, e_3 \times p\rangle\}$
2. $\dim J = 7 + \dim K_0 = 8.$

Proof.

1. Given $v \in K$, we have $v \in K_0$ if and only if $\int_{\overline{M_k}} v\bar{L}U d\bar{A} = 0 \; \forall U \in B$. Here we continue to use the notation of lemma 22. Then we suppose that U on a neighbourhood of the end p_i, $i = t, b, m$ has the following expression:

$$\varphi_{1,i} f_1(x, i) + \varphi_{2,i} f_2(x, i) + f_3(x, b_i) + u_i(x),$$

with $\tilde{a}_m = 0$, $\varphi_{j,m} = 0$, $j = 1, 2$. Then by the lemma 22 we get

$$\int_{\overline{M_k}} U\bar{L}v d\bar{A} = \int_{M_k} ULv dA =$$

$$= 2\pi \sum_{i\in\{t,b\}} [\langle \Phi_i, \nabla u_i(0)\rangle - \langle \Theta_i, \nabla v_i(0)\rangle] + 2\pi \sum_{i\in\{t,b,m\}} [a_i u_i(0) - b_i v_i(0)]$$

for each u_i. This is equivalent to

$$2\pi \sum_{i\in\{t,b\}} \langle \Theta_i, \nabla v_i(0)\rangle + 2\pi \sum_{i\in\{t,b,m\}} b_i v_i(0) = 0.$$

This gives K_0. Now we have to determine the Jacobi fields that are the generators of the space K. Thanks to the works [29] and [30] of S. Nayatani and result contained in 1, for all $k \geqslant 1$, the bounded Jacobi fields are associated to the following isometries of the ambient space: the three translations along the coordinate axes and the rotation about the vertical axis e_3. The space K_0 is generated only by the Jacobi functions which satisfy the conditions just proved. Making use of (2.8), (2.9) and (2.10) with the appropriate values of the logarithmic growths, we want to determine which of the following functions belongs to $K_0 : \langle N, e_3\rangle, \langle N, e_1\rangle, \langle N, e_2\rangle$. We find the following relations:

$$\partial_{x_j}\langle N, e_3\rangle(p_i) = 0, \quad \partial_{x_j}\langle N, e_1\rangle(p_i) = -\tilde{a}_i\delta_{1,j}, \quad \partial_{x_j}\langle N, e_2\rangle(p_i) = \tilde{a}_i\delta_{2,j},$$

with $j = 1, 2$. So we can conclude that these functions do not belong to the space K_0.

Now we consider the Jacobi function associated to the rotation about the vertical axis, that is $\langle N, e_3 \times p\rangle = det(e_3, p, N)$, where $p = (s_1, s_2, s_3)$ denotes the position vector. We observe that its expression is given by

$$s_1\langle N, e_2\rangle - s_2\langle N, e_1\rangle = \frac{x_1}{|x|^2}(a_i x_2 + O(|x|^2)) - \left(-\frac{x_2}{|x|^2}\right)(-a_i x_1 + O(|x|^2)) = O(x).$$

Then it is clear that K_0 is generated by this Jacobi function.

2. We consider the space $V \subset B$ of the functions defined on the disks $D_i^*(\epsilon_i)$ by

$$\theta_{1,i} f_1(x, i) + \theta_{2,i} f_2(x, i) + f_3(x, a_i)$$

It is a 7-dimensional space: in fact a function in V is determined by the values of the following parameters: $a_t, a_b, a_m, \theta_{1,t}, \theta_{1,b}, \theta_{2,t}, \theta_{2,b}$. The spaces B and V can be decomposed in the following way:

$$B = V \oplus \mathcal{C}^{2,\alpha} \quad V = V_1 \oplus V_2,$$

where $V_1 = \{f \in V : \bar{L}f \in \bar{L}(\mathcal{C}^{2,\alpha})\}$ and V_2 is a supplementary space. Then we have $\bar{L}(B) = \bar{L}(V_2) \oplus \bar{L}(\mathcal{C}^{2,\alpha})$. Since $K_0 = \bar{L}(B)^{\perp}$ and $K = \bar{L}(\mathcal{C}^{2,\alpha})$ we deduce

$$\dim K_0 = \text{codim}\bar{L}(B) = \text{codim}\bar{L}(\mathcal{C}^{2,\alpha}) - \dim\bar{L}(V_2) =$$

$$\dim K - \dim V_2 = \dim K - \dim V + \dim V_1,$$

that is

$$\dim K_0 = \dim K - 7 + \dim V_1. \tag{2.14}$$

Now we consider the restriction to $J = \ker(\bar{L})$ of the projection $\pi : B \to V$. It is clear that $\ker(\pi_{|J}) = K = J \cap \mathcal{C}^{2,\alpha}$, then given a function $f \in J$ such that $f = v + u$ with $v \in V$ and $u \in \mathcal{C}^{2,\alpha}$, we have

$$0 = \bar{L}f = \bar{L}v + \bar{L}u$$

and $\pi(f) = v \in V_1$. Furthermore, for any $v \in V_1$ there exists $v' \in \mathcal{C}^{2,\alpha}$ such that $\bar{L}v = \bar{L}v'$, that is $v - v' \in J$. Then $\pi(J) = V_1$ and

$$\dim J = \dim \ker(\pi_{|J}) + \dim \operatorname{Im}(\pi_{|J}) = \dim K + \dim V_1. \tag{2.15}$$

From the equations (2.14) and (2.15) we get

$$\dim J = 7 + \dim K_0 = 8.$$

□

Remark 24. *In [34] a minimal surface is defined to be non degenerate if the space K of the Jacobi fields induced by the isometries of the ambient space contains the space K_0. Thanks to the result of lemma 23 we can conclude that the Costa-Hoffman-Meeks surface M_k is non degenerate for all $k \geqslant 1$ with respect to this definition.*

2.4 The proof of the main result

We consider again the immersion $X_y + u\widetilde{N}(y)$ (see (2.3)) and its mean curvature function $H(y,u)$, where $y = (a_t, a_b, a_m, \theta_{1,t}, \theta_{1,b}, \theta_{2,t}, \theta_{2,b})$. We denote with $e_3(y)$ the unit vector defined in $D_i^*(\epsilon_i)$, for $i = t, b, m$, by $e_3(\theta_{1,i}, \theta_{2,i})$ for $i = t, b$ and by $e_3(0,0)$ for $i = m$. We denote with $ds^2_{y,u}$ the metric induced by $X_y + u\widetilde{N}(y)$. In the following, the subindex $\cdot_{y,u}$ denotes that the corresponding object is computed with respect to the metric $ds^2_{y,u}$. We recall that $\mathcal{A}$ and $\mathcal{U}$ denote, respectively a neighbourhood of $(\tilde{a}_t, \tilde{a}_b, 0)$ and a neighbourhood of zero in $\mathcal{C}^{2,\alpha}(\overline{M_k})$.

Lemma 25. *The mean curvature function $H(y,u)$, with $(y,u) \in \mathcal{A} \times [-\epsilon, \epsilon]^4 \times \mathcal{U}$, is orthogonal to $\langle N_{y,u}, e_3(y) \times p_{y,u}\rangle$, where $N_{y,u}$ is the Gauss map of $X_y + u\widetilde{N}(y)$ and $p_{y,u}$ is the position vector.*

Proof. Given $\epsilon > 0$ small we consider a compact domain $M_k(\epsilon)$ of M_k obtained by removing the disks $D_i(\epsilon)$ which parametrize the ends. We have:

$$2\int_{M(\epsilon)} H(y,u)\langle N_{y,u}, e_3(y) \times p_{y,u}\rangle dA_{y,u} =$$

$$-\int_{M(\epsilon)} \Delta_{y,u} \det(N_{y,u}, X_y + u\widetilde{N}(y), e_3(y)) dA_{y,u} =$$

$$-\int_{\partial M(\epsilon)} \det(\eta_{y,u}, X_y + u\widetilde{N}(y), e_3(y)) ds_{y,u},$$

where $\eta_{y,u}$ is the exterior conormal field to $X_y + u\widetilde{N}(y)$.

We parametrize $\partial M_k(\epsilon)$ by three disjoint copies of the boundary of a disk of radius ϵ. On this boundary we assume $x = \epsilon e^{i\psi}$, $\psi \in [0, 2\pi]$. In each neighbourhood of the ends we will use the appropriate frame to simplify the computation. That is, in the neighbourhood of the end p_l, $l = t, b, m$, we will adopt the frame $F(\theta_{1,l}, \theta_{2,l})$ with $\theta_{1,m} = \theta_{2,m} = 0$. We set

$$\alpha_{\epsilon,l}(y) = (X_y + u\widetilde{N}(y))(\epsilon e^{i\psi}) = \left(\frac{1}{x}, v_l\right)$$

with $v_l = -a_l \ln|x| + h_l(x)$. Then

$$\eta_{y,u,l} ds_{y,u} = -\alpha'_{\epsilon,l} \times N_{y,u} d\psi,$$

where $\alpha'_{\epsilon,l} = \left(\frac{-i}{x}, \langle ix, \nabla_0 v_l\rangle\right)$, for $l = t, b, m$, is the derivative of $\alpha_{\epsilon,l}$ with respect to ψ, $i = \sqrt{-1}$ and ∇_0 denotes the gradient computed with respect to ds_0^2, the flat metric of the x-plane. It is possible to prove that (see p. 194 [34])

$$\eta_{y,u,l} ds_{y,u} = \left[\left(\frac{1}{x}, a_l\right) + \left(\frac{O(\epsilon^2)}{x} + O(\epsilon^2), O(\epsilon)\right)\right] d\psi.$$

Then

$$\int_{|x|=\epsilon} \det(X_y + u\widetilde{N}(y), e_3(y), \eta_{y,u}) ds_{y,u} =$$

$$\sum_{l\in\{t,b,m\}} \int_{|x|=\epsilon} \det\left(\left(\frac{1}{x}, v_l\right), (0,1), \left(\frac{1}{x}, a_l\right) + \left(\frac{O(\epsilon^2)}{x} + O(\epsilon^2), O(\epsilon)\right)\right) d\psi.$$

It is easy to conclude that this expression converges to zero as ϵ goes to zero. □

And to finish, here it is our main result.

We set $c = (0, \tilde{a}_t, \tilde{a}_b, 0, 0, 0, 0, 0, 0)$ and we consider the function $v \in B$ given by (2.13). We recall that $\tilde{y} = (\tilde{a}_t, \tilde{a}_b, 0, 0, 0, 0, 0)$.

Theorem 26. *For each possible choice of the limit values of the normal vectors of the three ends, there is, up to isometries, a 1-dimensional real-analytic family of smooth minimal deformations of M_k, for $k \geqslant 1$, letting the planar end horizontal.*

Proof. We consider the map

$$\begin{array}{rcl} F : \mathbb{R} \times \mathcal{A} \times [-\epsilon, \epsilon]^4 \times \mathcal{U} & \longrightarrow & \mathcal{C}^{0,\alpha}(\overline{M_k}) \\ (r, y, u) & \longrightarrow & \overline{H}(y, u) + r\langle N_{y,u}, e_3(y) \times p_{y,u}\rangle. \end{array}$$

where $N_{y,u}$ is the Gauss map of $X_y + u\widetilde{N}(y)$. The map F is real analytic. Since the values $r = 0, y = \tilde{y}, u = 0$ parametrize the Costa-Hoffman-Meeks surface M_k, the differential of F at c,

$$DF_c : \mathbb{R}^8 \times B(M_k) \longrightarrow \mathcal{C}^{0,\alpha}(\overline{M_k})$$

is given by

$$DF_c(r,y,u) = \frac{1}{2}\bar{L}(v_y + u) + r\langle N_{y,u}, e_3(y) \times p_{y,u}\rangle.$$

Since $\bar{L}(B)^{\perp} = K_0$, we have $\mathrm{Ker}DF = \{0\} \times J$. The differential for $(r,y,u) = c$ is surjective and its kernel has dimension 8. Using the implicit function theorem we find a neighborhood $\mathcal{W}$ of c in $\mathbb{R}^8 \times \mathcal{U}$ such that $\mathcal{V} = F^{-1}(0) \cap \mathcal{W}$ is a real analytic 8-dimensional manifold. Thanks to the orthogonality between the mean curvature function and $\langle N_{y,u}, e_3(y) \times p_{y,u}\rangle$ proved in the lemma 25, we can conclude that $\mathcal{V}$ contains only minimal immersions, hence $\mathcal{V} \subset \{0\} \times \mathcal{A} \times [-\epsilon, \epsilon]^4 \times \mathcal{U}$.

To complete the proof, it remains to observe that up to now we have considered the choice of the parameters $a_t, a_b, a_m, \theta_{1,t}, \theta_{2,t}, \theta_{1,b}, \theta_{2,b}$ arbitrary. But it's necessary that the null flux condition is satisfied. In our case the flux is given by the sum of the flux of three catenoidal ends. So we have that the sum of three vectors must be the null vector. The direction and the length of each vector are respectively given by the direction of axis of revolution and by the logarithmic growth of the respective catenoidal end. It's easy to understand that these three vectors belong to a same vertical plane, that is we must have always $\theta_{2,t} = \theta_{2,b}$. The common value of these angles determines the orientation of this plane (see (2.2)). Furthermore the flux triangle described by the three vectors is uniquely determined by three of the remaining parameters (the logarithmic growths a_t, a_b, a_m and the angles $\theta_{1,t}, \theta_{1,b}$). It is clear that the choice of the limit values of the normal vectors (in other words of the angles $\theta_{1,t}, \theta_{1,b}$) of the three ends determines in unique way, up to a dilation, the flux triangle. So we can conclude that for each possible choice of the flux triangle, there exists a smooth 1-parameter family of minimal surfaces that are deformations of the surface M_k. □

Chapter 3

A Costa-Hoffman-Meeks type surface in $\mathbb{H}^2 \times \mathbb{R}$

3.1 Introduction

In the last years the study of the minimal surfaces in the product spaces $M \times \mathbb{R}$ with $M = \mathbb{H}^2, \mathbb{S}^2$ has been becoming more and more active. The development of the theory of the minimal surfaces in these spaces started with [36] by H. Rosenberg and continued with [25] and [26] by W. H. Meeks and H. Rosenberg. In [32] B. Nelli and H. Rosenberg showed the existence in $\mathbb{H}^2 \times \mathbb{R}$ of a rich family of examples including helicoids, catenoids and, solving Plateau problems, of higher topological type examples inspired by the theory of minimal surfaces in $\mathbb{R}^3$.

C. Costa in [3, 4] and D. Hoffman and W.H. Meeks in [14], [15] and [16] described in $\mathbb{R}^3$ a minimal surface of genus $k \geqslant 1$ with two ends asymptotic to the two ends of a catenoid and a middle end asymptotic to a plane. We will denote the Costa-Hoffman-Meeks surface of genus $k \geqslant 1$ by M_k.

The aim of this work is to show the existence in the space $\mathbb{H}^2 \times \mathbb{R}$ of a family of surfaces inspired to M_k. We shall prove the following result

Theorem 27. *For all $k \geqslant 1$ there exists in $\mathbb{H}^2 \times \mathbb{R}$ a minimal surface of genus k with three horizontal ends: two catenoidal type ends and a middle planar end.*

The construction is based on a gluing procedure. We consider a rescaled version of a compact part of a Costa-Hoffman-Meeks type surface, such that it can be contained in a cylindrical neighbourhood of $\{0, 0\} \times \mathbb{R} \subset \mathbb{H}^2 \times \mathbb{R}$ of sufficiently small radius. Actually it's possible to prove that, in the same set, the mean curvature of such a surface with respect the standard metric of $\mathbb{H}^2 \times \mathbb{R}$, up to an infinitesimal term, can be expressed in terms of the euclidean one. The main result is proved by a gluing procedure (see for example [11]) usually adopted to construct in $\mathbb{R}^3$ new examples starting from known minimal surfaces. In particular we glue the surface described above along its three boundary curves to two minimal graphs that are respectively asymptotic to an upper half catenoid and a lower half catenoid defined in $\mathbb{H}^2\times\mathbb{R}$ and to a minimal graph asymptotic to $\mathbb{H}^2\times\{0\}$.

The author wishes to thank L. Hauswirth for invaluable conversations.

3.2 Minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$

In this work we shall consider the disk model for $\mathbb{H}^2$. Let x_1, x_2 denote the coordinates in $\mathbb{H}^2$ and x_3 the coordinate in $\mathbb{R}$. Then the space $\mathbb{H}^2 \times \mathbb{R}$ is endowed with the metric

$$d\sigma^2 = \frac{dx_1^2 + dx_2^2}{F} + dx_3^2,$$

where

$$F = \frac{1}{4}\left(1 - x_1^2 - x_2^2\right)^2 = \frac{1}{4}\left(1 - r^2\right)^2.$$

We denote with H_u the mean curvature of the graph of the function u over a domain in $\mathbb{H}^2$. Its expression is

$$2H_u = F\text{div}\left(\frac{\nabla u}{\sqrt{1 + F|\nabla u|^2}}\right), \tag{3.1}$$

where div denotes the divergence in $\mathbb{R}^2$. For the details of the computation see [32].

Let Σ_u be the graph of the function u. In this section we want to obtain the expression of the mean curvature of the surface Σ_{u+v} that is the normal graph of the function v over Σ_u and close to it. We shall show how it follows from (3.1) that the linearized mean curvature operator, that we denote with L_u, is given by:

$$F\text{div}\left(\frac{\nabla v}{\sqrt{1 + F|\nabla u|^2}} - F\nabla u\frac{\nabla u \cdot \nabla v}{\sqrt{(1 + F|\nabla u|^2)^3}}\right). \tag{3.2}$$

Furthermore we shall give the expression of H_{u+v}, the mean curvature of the graph of the function $u+v$, in terms of the mean curvature of Σ_u, that is H_u. In the following we shall restrict our attention to two cases: the plane, that is $u = 0$, and (in section 3.6) a part of catenoid defined on the domain $\{(r,\theta) \in \mathbb{H}^2 | r \in [r_\epsilon, 1)\}$, where $r_\epsilon = \epsilon^{\frac{1}{2}}/2$.

Here we shall show that:

$$2H_{u+v} = 2H_u + L_u v + FQ_u(\sqrt{F}\nabla v, \sqrt{F}\nabla^2 v), \tag{3.3}$$

where Q_u is an operator with bounded coefficients if $r \in [r_\epsilon, 1)$ which satisfies

$$Q_u(0,0) = \nabla Q_u(0,0) = 0.$$

To show this, we start observing that:

$$\frac{1}{\sqrt{1+F|\nabla(u+v)|^2}} = \frac{1}{\sqrt{1+F|\nabla u|^2}} - F\frac{\nabla u \cdot \nabla v}{\sqrt{(1+F|\nabla u|^2)^3}} + Q_{u,1}(v). \tag{3.4}$$

The operator $Q_{u,1}(v)$ has the following expression

$$\frac{-F|\nabla v|^2}{(1+F|\nabla(u+\bar{t}v)|^2)^{3/2}} + \frac{3F^2\left(\nabla u \cdot \nabla v + \bar{t}|\nabla v|^2\right)^2}{(1+F|\nabla(u+\bar{t}v)|^2)^{5/2}}, \tag{3.5}$$

with $\bar{t} \in (0,1)$, and it satisfies $Q_1(0) = \nabla Q_1(0) = 0$. To prove (3.4) it's sufficient to set

$$f(t) = \frac{1}{\sqrt{1+F|\nabla(u+tv)|^2}}$$

and to write down the Taylor's series of order one of this function and to evaluate it in $t=1$. That is $f(1) = f(0) + f'(0) + \frac{1}{2}f''(\bar{t})$, with $\bar{t} \in (0,1)$. We insert (3.4) in the expression that defines $2H_{u+v}$ to get

$$F\mathrm{div}\left(\frac{\nabla(u+v)}{\sqrt{1+F|\nabla u|^2}} - F\nabla(u+v)\frac{\nabla u \cdot \nabla v}{\sqrt{(1+F|\nabla u|^2)^3}} + \nabla(u+v)Q_{u,1}(v)\right) =$$

$$2H_u + F\mathrm{div}\left(\frac{\nabla v}{\sqrt{1+F|\nabla u|^2}} - F\nabla u\frac{\nabla u \cdot \nabla v}{\sqrt{(1+F|\nabla u|^2)^3}}\right) + FQ_u(\sqrt{F}\nabla v, \sqrt{F}\nabla^2 v).$$

Since we assume that Σ_u is a minimal surface, we will consider $H_u = 0$.

3.3 The linearized operator about the hyperbolic plane

Now we restrict our attention to the case of the minimal surfaces close to $\mathbb{H}^2 \times \{0\}$, that is the graph of the function $u = 0$. In this case we obtain immediately from (3.2) that $L_{u=0} = F\Delta_{eucl}$, where Δ_{eucl} denotes the Laplace operator in the euclidean metric. In this section we will study the mapping properties of the operator $\mathcal{L} = \Delta_{eucl}$. In the following we will use the polar coordinates r, θ. In particular our aim is to solve in an unique way the problem:

$$\begin{cases} \mathcal{L}w = f & \text{in } S^1 \times [r_0, 1) \\ w|_{r=r_0} = \varphi \end{cases}$$

with $r_0 \in (0,1)$, considering a convenient normed functions space for w, f and φ, so that the norm of w is bounded by the one of f.
Since

$$r^2\mathcal{L} = \left(r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2} \right)$$

has separated variables, let us firstly consider the operator

$$L_\theta = \partial^2_{\theta\theta},$$

which acts on 2π-periodic even functions. It is uniformly elliptic and self-adjoint. In particular, L_θ has discrete spectrum $(\lambda_i)_{i\geqslant 0} = i^2$. Each eigenvalue λ_i has multiplicity one. We denote by $e_i(\theta)$ the eigenfunction associated to λ_i, normalized so that

$$\int_0^{2\pi} (e_i(\theta))^2\, d\theta = 1.$$

The Hilbert basis $\{e_i\}_{i\in\mathbb{N}}$ of the space of 2π-periodic even functions in $L^2(S^1)$ induces the following Fourier decomposition of L^2 functions $g = g(\theta, r)$ which are 2π-periodic and even in the variable θ,

$$g(\theta, r) = \sum_{i\geqslant 0} g_i(r)\, e_i(\theta).$$

From this, we deduce that the operator $\mathcal{L}$, can be decomposed as $\mathcal{L} = \sum_{i\geqslant 0} L_i$, being

$$r^2 L_i = \left(r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r} - \lambda_i \right), \quad \text{for every } i \geqslant 0.$$

From the observations made above we get that the potential of $r^2 L_i$ is

$$P_i := -\lambda_i = -i^2. \tag{3.6}$$

Now we can prove a lemma which assures that $\mathcal{L}$ is injective.

Lemma 28. *Given $0 < r_0 < r_1 < 1$, let w be a solution of $\mathcal{L}w = 0$ on $S^1 \times [r_0, r_1]$ such that $w(\cdot\,, r_0) = w(\cdot\,, r_1) = 0$. Then $w = 0$.*

Proof: We can decompose $w = \sum_{i\geqslant 0} w_i(r)\, e_i(\theta)$. Since the potential P_i of the operator L_i is negative or zero for every $i \geqslant 0$ (see (3.6)) and the operator L_i is uniformly elliptic, the maximum principle holds. Then we get the lemma 28 from the hypothesis on the boundary conditions. □

3.3.1 The mapping properties of the Laplace operator

Definition 29. *Given $\ell \in \mathbb{N}$, $\alpha \in (0,1)$, $\mu \in \mathbb{R}$ and the closed interval I, we define*

$$\mathcal{C}^{\ell,\alpha}_{\mu}(S^1 \times I)$$

to be the space of functions $w := w(\theta, r)$ in $\mathcal{C}^{\ell,\alpha}_{loc}(S^1 \times I)$ for which the following norm is finite

$$\|w\|_{\mathcal{C}^{\ell,\alpha}_{\mu}} := \|(1-r^2)^{-\mu} w\|_{\mathcal{C}^{\ell,\alpha}(S^1\times I)}.$$

Proposition 30. *Given $\mu > 0$, $r_0 \in (0,1)$, there exists an operator*

$$\begin{array}{rccc} G_{r_0}: & \mathcal{C}^{0,\alpha}_{\mu}(S^1 \times [r_0,1)) & \longrightarrow & \mathcal{C}^{2,\alpha}_{\mu}(S^1 \times [r_0,1)) \\ & f & \longmapsto & w := G_{r_0}(f) \end{array}$$

satisfying the following statements

(i) $\mathcal{L}\, w = f$ *on* $S^1 \times [r_0, 1)$,

(ii) $w = 0$ *on* $S^1 \times \{r_0\}$,

(iii) $||w||_{\mathcal{C}^{2,\alpha}_{\mu}(S^1\times[r_0,1))} \leqslant c\, ||f||_{\mathcal{C}^{0,\alpha}_{\mu}(S^1\times[r_0,1))}$, *for some constant $c > 0$ which does not depend on r_0.*

Proof: As consequence of the lemma 28, the operator $\mathcal{L} = \Delta_{eucl}$ is injective. Hence, Fredholm alternative let us assure that there exists, an unique $w \in \mathcal{C}^{2,\alpha}_{\mu}(S^1 \times [r_0, r_1])$, with $w(\theta, r)$ satisfying:

$$\begin{cases} \Delta_{eucl}\, w = f \quad \text{on } S^1 \times [r_0, r_1] \\ w(\cdot, r_0) = w(\cdot, r_1) = 0. \end{cases} \tag{3.7}$$

We want to prove the following assertion.

Assertion 31. *For every $0 < r_0 < r_1 < 1$, $f \in \mathcal{C}^{0,\alpha}_{\mu}(S^1\times[r_0, r_1])$ and $w \in \mathcal{C}^{2,\alpha}_{\mu}(S^1\times[r_0, r_1])$ satisfying (3.7), it exists a constant c such that*

$$\|w\|_{\mathcal{C}^{0,\alpha}_{\mu}(S^1\times[r_0,r_1])} \leqslant c\, \|f\|_{\mathcal{C}^{0,\alpha}_{\mu}(S^1\times[r_0,r_1])}.$$

We suppose by contradiction that the assertion 31 is false, that is it does not exist a universal costant for which the previous estimate holds. Then, for each $n \in \mathbb{N}$, there exist $r_{1,n} > r_{0,n}$ and f_n, w_n satisfying (3.7) (with $r_{0,n}, r_{1,n}, f_n, w_n$ instead of r_0, r_1, f, w) such that

$$\|f_n\|_{\mathcal{C}^{0,\alpha}_{\mu}(S^1\times[r_{0,n},r_{1,n}])} = 1 \quad \text{and} \quad \|w_n\|_{\mathcal{C}^{0,\alpha}_{\mu}(S^1\times[r_{0,n},r_{1,n}])} \to +\infty, \text{ when } n \to \infty.$$

Since $\Omega_n := S^1 \times [r_{0,n}, r_{1,n}]$ is a compact set, $A_n := \sup_{S^1\times[r_{0,n},r_{1,n}]} (1-r^2)^{-\mu}\,|w_n|$ is achieved at a point $(\theta_n, r_n) \in S^1 \times [r_{0,n}, r_{1,n}]$. We define

$$\widetilde{w}_n(\theta, r) := \frac{(1-r_n^2)^{-\mu}}{\|w_n\|_{\mathcal{C}^{0,\alpha}_\mu(\Omega_n)}}\, w_n(\theta, r r_n),$$

and

$$\widetilde{f}_n(\theta, r) := \frac{(1-r_n^2)^{-\mu}}{\|f_n\|_{\mathcal{C}^{0,\alpha}_\mu(\Omega_n)}}\, f_n(\theta, r r_n),$$

for all $(\theta, r) \in S^1 \times I_n$, with $I_n = [r_{0,n}/r_n, r_{1,n}/r_n]$. Clearly, $A_n \leqslant \|w_n\|_{\mathcal{C}^{0,\alpha}_\mu(\Omega_n)}$, and

$$|\widetilde{w}_n(\theta, r)| \leqslant (1-r^2)^{\mu}\, \frac{(1-(rr_n)^2)^{-\mu}|w_n(\theta, rr_n)|}{A_n} \leqslant (1-r^2)^{\mu}.$$

On the other hand, $(1-r^2)^{-\mu}|\nabla\widetilde{w}_n| \leqslant \|\widetilde{w}_n\|_{\mathcal{C}^{2,\alpha}_\mu(S^1\times I_n)} \leqslant \frac{\|w_n\|_{\mathcal{C}^{2,\alpha}_\mu(\Omega_n)}}{\|w_n\|_{\mathcal{C}^{0,\alpha}_\mu(\Omega_n)}}$. Thanks to Schauder estimates, we obtain

$$||w_n||_{\mathcal{C}^{2,\alpha}_\mu(\Omega_n)} \leqslant c\, \Big(||f_n||_{\mathcal{C}^{0,\alpha}_\mu(\Omega_n)} + ||(1-r^2)^{-\mu} w_n||_{C^0(\Omega_n)} \Big) = c\,(1 + A_n).$$

Hence,

$$|\nabla\widetilde{w}_n| \leqslant c\,(1-r^2)^{\mu} \frac{1 + \|w_n\|_{\mathcal{C}^{0,\alpha}_\mu}(\Omega_n)}{\|w_n\|_{\mathcal{C}^{0,\alpha}_\mu(\Omega_n)}} \leqslant c\,(1-r^2)^{\mu}.$$

The sets I_n converge to a nonempty interval I_∞. Since the sequences $(\tilde{w}_n)_n$ and $(\nabla\tilde{w}_n)_n$ are uniformly bounded, Ascoli-Arzelà theorem assures that a subsequence of $(\tilde{w}_n)_n$ converges on compact sets of $S^1 \times I_\infty$ to a function w_∞ that vanishes on $S^1 \times \partial I_\infty$.
We note that $\sup_{S^1\times I_n} (1-r^2)^{-\mu}|\widetilde{w}_n| = \frac{A_n}{\|w_n\|_{\mathcal{C}^{0,\alpha}_\mu(\Omega_n)}}$, which does not converge to zero. In fact $A_n \to \infty$, $\|w_n\|_{\mathcal{C}^{0,\alpha}_\mu(\Omega_n)} \leqslant \|w_n\|_{\mathcal{C}^{2,\alpha}_\mu(\Omega_n)} \leqslant c'(1 + A_n)$ then

$$\frac{A_n}{\|w_n\|_{\mathcal{C}^{0,\alpha}_\mu(\Omega_n)}} \geqslant \frac{A_n}{c'(1+A_n)} \to \frac{1}{c'} > 0.$$

In particular, it holds

$$0 < \sup_{S^1\times I_\infty} (1-r^2)^{-\mu}\,|w_\infty| \leqslant 1. \tag{3.8}$$

In the same way it's possible to prove that a subsequence of $(\tilde{f}_n)_n$ converges on compact sets of $S^1 \times I_\infty$ to the function $f_\infty \equiv 0$ since, if $n \to \infty$,

$$\sup_{S^1\times I_n} (1-r^2)^{-\mu}|\widetilde{f}_n| = \frac{1}{A_n} \to 0.$$

Then the limit function w_∞ must satisfy the differential equation

$$\Delta_{eucl} w_\infty = 0$$

on $S^1 \times I_\infty$ with null boundary conditions on ∂I_∞. Furthermore $r = 1 \in I_\infty$, then $w_\infty(\theta, 1) = 0$. So, also in the case $\partial I_\infty = \emptyset$, we can conclude that $w_\infty(\theta, r) = 0$. This function does not satisfy (3.8), a contradiction. This proves the assertion 31.

Thanks to Schauder estimates, we know that the function introduced in the assertion 31 is uniformly bounded and likewise its gradient. Hence Ascoli-Arzelà theorem assure us that we can take the limit as $r_1 \to 1$ in a sequence of solutions which are defined on $S^1 \times [r_0, r_1]$. This proves the existence of a solution of $\Delta_{eucl} w = f$ defined on $S^1 \times [r_0, 1)$ for which it holds

$$\|w\|_{\mathcal{C}^{0,\alpha}_\mu(S^1\times[r_0,1])} \leqslant c\,\|f\|_{\mathcal{C}^{0,\alpha}_\mu(S^1\times[r_0,1])}.$$

Now it is sufficient to use again Schauder estimates to obtain the estimates for the derivatives.

□

3.4 A family of minimal surfaces close to the hyperbolic plane

In this section we will show the existence of normal minimal graphs over the plane, C_m, of equation $x_3 = 0$ which are asymptotic to it. We will reformulate the problem to use Schauder fixed point theorem. We recall that $r_\epsilon = \epsilon^{1/2}/2$. We know already that the graph of the function v, denoted with Σ_v, is minimal, if and only if the function v is a solution of

$$F\left(\mathcal{L}\, v + Q_0\left(\sqrt{F}\nabla v, \sqrt{F}\nabla^2 v\right)\right) = 0. \tag{3.9}$$

This equation is a simplified version (since $u = 0$) of (3.3) introduced in section 3.2. The operator Q_0 has bounded coefficients for $r \in [r_\epsilon, 1)$. Its expression is $\operatorname{div}(\nabla v Q_{0,1})$ where $Q_{u,1}$ is given by (3.5).

Now let's consider a function $\varphi \in \mathcal{C}^{2,\alpha}(S^1)$ which is even with respect to θ, L^2-orthogonal to e_0 and e_1 and such that $\|\varphi\|_{\mathcal{C}^{2,\alpha}} \leqslant k\epsilon$. We define

$$w_\varphi(\cdot,\cdot) := \mathcal{H}_{r_\epsilon,\varphi}(\cdot,\cdot),$$

where $\mathcal{H}$ is the operator of harmonic extension introduced in proposition 46. In order to solve the equation (3.9), we choose $\mu \in (0,1)$ and look for v of the form $v = w_\varphi + w$ where

$w \in \mathcal{C}^{2,\alpha}_{\mu}(S^1 \times [r_\epsilon, 1))$ and $v = \varphi$ on $S^1 \times \{r_\epsilon\}$. Using proposition 30, we can rephrase this problem as a fixed point problem

$$w = S(\varphi, w) \tag{3.10}$$

where the nonlinear mapping S which depends on ϵ and φ is defined by

$$S(\varphi, w) := -G_{r_\epsilon} \left(\mathcal{L}\, w_\varphi + Q_0\left(w_\varphi + w\right)\right),$$

where the operator G is defined in proposition 30. To prove the existence of a fixed point for (3.10) we need the following result that states that S is a contracting mapping:

Lemma 32. *There exist some constants $c_k > 0$ and $\epsilon_k > 0$, such that*

$$\|S(\varphi, 0)\|_{\mathcal{C}^{2,\alpha}_{\mu}} \leqslant c_k \epsilon^2 \tag{3.11}$$

and, for all $\epsilon \in (0, \epsilon_k)$

$$\|S(\varphi, v_2) - S(\varphi, v_1)\|_{\mathcal{C}^{2,\alpha}_{\mu}} \leqslant \frac{1}{2}\|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}_{\mu}}$$

for all $v_1, v_2 \in \mathcal{C}^{2,\alpha}_{\mu}(S^1 \times [r_\epsilon, 1))$ such that $\|v_i\|_{\mathcal{C}^{2,\alpha}_{\mu}} \leqslant 2c_k\epsilon^2$.

Proof. We know from proposition 30 that $\|G_{r_\epsilon}(f)\|_{\mathcal{C}^{2,\alpha}_{\mu}} \leqslant c\|f\|_{\mathcal{C}^{0,\alpha}_{\mu}}$, then

$$\|S(\varphi, 0)\|_{\mathcal{C}^{2,\alpha}_{\mu}} \leqslant c\|\mathcal{L}\, w_\varphi + Q_0\left(w_\varphi\right)\|_{\mathcal{C}^{0,\alpha}_{\mu}} \leqslant$$

$$\leqslant c\left(\|\mathcal{L}\, w_\varphi\|_{\mathcal{C}^{0,\alpha}_{\mu}} + \|Q_0\left(w_\varphi\right)\|_{\mathcal{C}^{0,\alpha}_{\mu}}\right).$$

So we need find the estimates for the two summands above. We recall that $\|\varphi\|_{2,\alpha} \leqslant k\epsilon$. Since $\mu \in (0, 1)$ and thanks to proposition 46 we obtain

$$\|w_\varphi\|_{\mathcal{C}^{2,\alpha}_{\mu}} \leqslant c\|w_\varphi\|_{\mathcal{C}^{2,\alpha}_{1}} \leqslant c\|\varphi\|_{\mathcal{C}^{2,\alpha}} \leqslant ck\epsilon.$$

Being w_φ a harmonic function we have $\mathcal{L}\, w_\varphi = 0$. The last term is estimated by

$$\|Q_0\left(w_\varphi\right)\|_{\mathcal{C}^{0,\alpha}_{\mu}} \leqslant c_k\epsilon^2.$$

In fact

$$\|Q_0\left(w_\varphi\right)\|_{\mathcal{C}^{0,\alpha}_{\mu}} \leqslant c\|w\|^2_{2,\alpha,\mu} \leqslant c\|\varphi\|^2_{2,\alpha} \leqslant c_k\epsilon^2.$$

Then we can conclude

$$\|S(\varphi, 0)\|_{\mathcal{C}^{2,\alpha}_{\mu}} \leqslant c_k\epsilon^2.$$

As for the second estimate, we consider

$$S(\varphi, v_2) - S(\varphi, v_1) = G_{r_\epsilon}\left(Q_0\left(w_\varphi + v_2\right) - Q_0\left(w_\varphi + v_1\right)\right)$$

and consequently

$$\|S(\varphi, v_2) - S(\varphi, v_1)\|_{\mathcal{C}^{2,\alpha}_\mu} \leqslant c\|Q_0\left(w_\varphi + v_2\right) - Q_0\left(w_\varphi + v_1\right)\|_{\mathcal{C}^{0,\alpha}_\mu}.$$

We observe that from the considerations made above it follows that

$$||Q_0\left(w_\varphi + v_2\right) - Q_0\left(w_\varphi + v_1\right)||_{\mathcal{C}^{0,\alpha}_\mu} \leqslant c||v_2 - v_1||_{\mathcal{C}^{2,\alpha}_\mu}\|w_\varphi\|_{\mathcal{C}^{2,\alpha}_\mu} \leqslant$$

$$\leqslant c_k\epsilon||v_2 - v_1||_{\mathcal{C}^{2,\alpha}_\mu}.$$

Then

$$\|S(\varphi, v_2) - S(\varphi, v_1)\|_{\mathcal{C}^{2,\alpha}_\mu} \leqslant c_k\epsilon||v_2 - v_1||_{\mathcal{C}^{2,\alpha}_\mu}.$$

□

Theorem 33. *Let be* $B := \{w \in \mathcal{C}^{2,\alpha}_\mu(S^1 \times [r_\epsilon, 1)) \,|\, ||w||_{\mathcal{C}^{2,\alpha}_\mu} \leqslant 2c_k\epsilon^2\}$. *Then the nonlinear mapping* S *defined above has a unique fixed point* v *in* B.

Proof: The previous lemma shows that, if ϵ is chosen small enough, the nonlinear mapping S is a contraction mapping from the ball B of radius $2c_k\epsilon^2$ in $\mathcal{C}^{2,\alpha}_\mu(S^1 \times [r_\epsilon, 1))$ into itself. This value follows from the estimate of the norm of $S(0)$. Consequently thanks to Schauder fixed point theorem, S has an unique fixed point w in this ball. □

We have proved the existence of a minimal surface, denoted with S_m, which is close to $\mathbb{H}^2 \times \{0\}$, and close to its boundaryis the vertical graph over the annulus $B_{2r_\epsilon} - B_{r_\epsilon}$ of a function which can be expanded as

$$\mathcal{H}_{r_\epsilon,\varphi}(r,\theta) + \bar{V}_m(r,\theta), \quad \text{with} \quad ||\bar{V}_m||_{\mathcal{C}^{2,\alpha}_\mu} \leqslant c\epsilon.$$

The function V_m depends non linearly on ϵ, φ. Furthermore it satisfies

$$\|\bar{V}_m(\epsilon,\varphi)(r_\epsilon\cdot) - \bar{V}_m(\epsilon,\varphi')(r_\epsilon\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1 - B_{1/2})} \leqslant c\epsilon\|\varphi - \varphi'\|_{\mathcal{C}^{2,\alpha}(S^1)}.$$

3.5 The catenoid in $\mathbb{H}^2 \times \mathbb{R}$

The catenoid in the space $\mathbb{H}^2 \times \mathbb{R}$ can be obtained by the revolution around the x_3 axis, $\{0,0\} \times \mathbb{R}$, of an appropriate curve γ (see [32]). We consider a vertical geodesic plane containing the origin of $\mathbb{H}^2$ and the curve γ. Let r be the euclidean distance between the point of γ at height t and the x_3 axis: we denote with $r = r(t)$ a parametrization of γ. The surface obtained by revolution of γ, is minimal if and only if $r = r(t)$ satisfies the following differential equation:

$$4r(t)\frac{\partial^2 r}{\partial t^2} - 4\left(\frac{\partial r}{\partial t}\right)^2 - (1 - r(t)^4) = 0. \tag{3.12}$$

A first integral for this equation is:

$$\left(\frac{\partial r}{\partial t}\right)^2 = Cr^2 - \frac{1+r^4}{4} \tag{3.13}$$

with $C > \frac{1}{2}$. It is possible to prove that the function $r(t)$ has a minimum value r_{min} given by:

$$r_{min} = \sqrt{\frac{2C+1}{2}} - \sqrt{\frac{2C-1}{2}} < 1.$$

Since we assume $C = \frac{1}{4\epsilon^2}$, we get

$$r_{min} = \sqrt{\frac{2C+1}{2}} - \sqrt{\frac{2C-1}{2}} = \sqrt{C}\left(1 + \frac{1}{4C} - 1 + \frac{1}{4C} + \mathcal{O}(\frac{1}{C^2})\right) =$$

$$\frac{1}{2\sqrt{C}} + O\left(\frac{1}{C^{3/2}}\right) = \epsilon + \mathcal{O}(\epsilon^3).$$

We denote with C_t and C_b, respectively, the part of the catenoid contained in $\mathbb{H}^2 \times \mathbb{R}^+$ and $\mathbb{H}^2 \times \mathbb{R}^-$.

We set

$$t_\epsilon = -\epsilon \ln \epsilon.$$

We need find the parametrization of C_t and C_b as graphs on the horizontal plane respectively for $t \in [t_\epsilon - \epsilon, t_\epsilon + \epsilon]$ and $t \in [-t_\epsilon - \epsilon, -t_\epsilon + \epsilon]$.

Lemma 34. *For $\epsilon > 0$ small enough, we have*

$$r_\epsilon(t) = \epsilon \cosh \frac{t}{2\epsilon} + \mathcal{O}(\epsilon^3 e^{\frac{t}{2\epsilon}}) \text{ and } \partial_t r_\epsilon(t) = \frac{1}{2} \sinh \frac{t}{2\epsilon} + \mathcal{O}(\epsilon^2 e^{\frac{t}{2\epsilon}})$$

for $t \in [0, t_\epsilon + \epsilon]$. Moreover if $t \in [t_\epsilon - \epsilon, t_\epsilon + \epsilon]$, we derive

$$r_\epsilon(t) = \mathcal{O}(\epsilon^{1/2}) \text{ and } \partial_t r_\epsilon = \mathcal{O}\left(\epsilon^{-1/2}\right).$$

Proof. We define the function $v(t)$ such that $r_\epsilon(t) = r_\epsilon(0) \cosh v(t)$, with $v(0) = 0$ and $r_\epsilon(0)$ which satisfies

$$Cr_\epsilon^2(0) - \frac{1 + r_\epsilon^4(0)}{4} = 0,$$

from which

$$\frac{1}{4} = Cr_\epsilon^2(0) - \frac{r_\epsilon^4(0)}{4}. \tag{3.14}$$

Plugging $r_\epsilon(t)$ in (3.13) and using (3.14), we have

$$(\partial_t v)^2 = C - \frac{r_\epsilon^2(0)}{4}(1+\cosh^2 v(t))$$

and under the hypothesis that $v(t) = g(\frac{t}{2\epsilon})$ and

$$\frac{t}{2\epsilon} \leqslant g\left(\frac{t}{2\epsilon}\right) \leqslant \frac{t}{2\epsilon} + 1$$

we obtain that $(\partial_t v)^2 = C + \mathcal{O}(\epsilon^2 e^{\frac{t}{2\epsilon}})$ and then $v(t) = \sqrt{C}t + \mathcal{O}(\epsilon^3 e^{\frac{t}{2\epsilon}})$. We remark a posteriori that $\frac{t}{2\epsilon} \leqslant v(t) \leqslant \frac{t}{2\epsilon} + 1$ holds for $t \in [0, t_\epsilon + \epsilon]$, $\epsilon > 0$ small enough and then

$$r_\epsilon(t) = r_\epsilon(0)\cosh v(t) = \epsilon \cosh\left(\frac{t}{2\epsilon}\right) + \mathcal{O}(\epsilon^3 e^{\frac{t}{2\epsilon}}) \tag{3.15}$$

where we use that $r_\epsilon(0) = r_{min} = \epsilon + \mathcal{O}(\epsilon^3)$. Now we assume that $t \in [t_\epsilon - \epsilon, t_\epsilon + \epsilon]$, then $r_\epsilon(t) = \mathcal{O}(\epsilon^{1/2})$ and $\partial_t r_\epsilon(t) = \frac{1}{2}\sinh\left(\frac{t}{2\epsilon}\right) + \mathcal{O}(\epsilon^2 e^{\frac{t}{2\epsilon}}) = \mathcal{O}(\epsilon^{-1/2})$. □

Now we can prove a lemma that give us the parametrization of the part of catenoid whose height t belongs to a neighbourhood of t_ϵ.

Lemma 35. *For $\epsilon > 0$, small enough and $t \in [t_\epsilon - \epsilon, t_\epsilon + \epsilon]$, the catenoid can be parametrized on an annulus of $C_\epsilon = \left\{re^{i\theta}; \frac{r_\epsilon}{\sqrt{\epsilon}} \leqslant r \leqslant r_\epsilon\sqrt{\epsilon}\right\}$ by the graph of the function $U_t(r,\theta)$ which satisfies*

$$U_t(r,\theta) = 2\epsilon \ln\frac{2r}{\epsilon} + v_t(r,\theta), \tag{3.16}$$

$$\partial_r U_t(r,\theta) = \frac{2\epsilon}{r} + v_{t,d}(r,\theta).$$

with $v_t(r,\theta) = \mathcal{O}(\epsilon^2)$ and $v_{t,d}(r,\theta) = \mathcal{O}(\epsilon^{5/2})$.

Proof: The results follow easily from the hypothesis and the equation (3.15).

□

It's easy to understand that the parametrization of the catenoid for values of t in a neighbourhood of $-t_\epsilon$ is obtained by a change of the sign in the expression 3.16. Similar consideration are true for the derivative. Then we obtain

$$U_b(r,\theta) = -2\epsilon \ln\frac{2r}{\epsilon} + v_b(r,\theta),$$

$$\partial_r U_b(r,\theta) = -\frac{2\epsilon}{r} + v_{b,d}(r,\theta)$$

with $v_b(r,\theta) = \mathcal{O}(\epsilon^2)$ and $v_{b,d}(r,\theta) = \mathcal{O}(\epsilon^{5/2})$.

3.6 A family of minimal surfaces close to a catenoid on $S^1 \times [r_\epsilon, 1)$

In this section we want to show the existence of minimal normal graphs over the catenoid defined in $S^1 \times [r_\epsilon, 1) \subset \mathbb{H}^2$ and asymptotic to it. We know that the graph of the function $u + v$ is minimal, being u the function whose graph is the catenoid, if and only if v is a solution of the equation

$$H_{u+v} = 0 \tag{3.17}$$

whose expression is given by (3.3). The explicit expression of $L_u v$ is

$$F\left(\frac{1}{\sqrt{A}}\Delta v + \frac{\partial}{\partial r}\left(\frac{1}{\sqrt{A}}\right)\frac{\partial v}{\partial r} - \frac{1}{A^{\frac{3}{2}}}\frac{\partial u}{\partial r}\frac{\partial}{\partial r}\left(F\frac{\partial u}{\partial r}\right)\frac{\partial v}{\partial r} - F\frac{\partial u}{\partial r}\frac{\partial}{\partial r}\left(\frac{1}{A^{\frac{3}{2}}}\frac{\partial u}{\partial r}\frac{\partial v}{\partial r}\right)\right), \tag{3.18}$$

where $F = \frac{1}{4}(1-r^2)^2$,

$$A = 1 + F|\nabla u|^2 = \frac{(4C-2)r^2}{4Cr^2 - 1 - r^4}$$

and

$$\frac{\partial u}{\partial r} = \pm\frac{2}{\sqrt{4Cr^2 - 1 - r^4}},$$

as it is easy to obtain using (3.13). It's useful to observe that since we assume $C = \frac{1}{4\epsilon^2}$ and $r_\epsilon = \epsilon^{\frac{1}{2}}/2$, we have that, for $r \in [r_\epsilon, 1)$, $A = 1 + \mathcal{O}(\epsilon)$, $\frac{\partial u}{\partial r} = \mathcal{O}(\sqrt{\epsilon})$,

$$\frac{\partial A}{\partial r} = \frac{(8C-4)(-r+r^5)}{(4Cr^2-1-r^4)^2} = \mathcal{O}(\sqrt{\epsilon})$$

and

$$\frac{\partial^2 u}{\partial r^2} = \frac{(8Cr - 4r^3)}{\sqrt{(4Cr^2-1-r^4)^3}} = \mathcal{O}(1).$$

Taking into account these estimates, we can conclude that it holds

$$\bar{L}_u v := \sqrt{A}\left(\frac{\partial}{\partial r}\left(\frac{1}{\sqrt{A}}\right)\frac{\partial v}{\partial r} - \frac{1}{A^{\frac{3}{2}}}\frac{\partial u}{\partial r}\frac{\partial}{\partial r}\left(F\frac{\partial u}{\partial r}\right)\frac{\partial v}{\partial r} - F\frac{\partial u}{\partial r}\frac{\partial}{\partial r}\left(\frac{1}{A^{\frac{3}{2}}}\frac{\partial u}{\partial r}\frac{\partial v}{\partial r}\right)\right) =$$

$$\mathcal{O}(\sqrt{\epsilon})\frac{\partial v}{\partial r} + \mathcal{O}(\sqrt{\epsilon})\frac{\partial^2 v}{\partial^2 r}. \tag{3.19}$$

Then we can write $\sqrt{A}L_u v = F\left(\mathcal{L}v + \bar{L}_u v\right)$.

We remark that we have already studied the mapping properties of the operator $\mathcal{L}$ in section 3.3.

Then the graph of a function v over Σ_u is minimal if and only if v is a solution of the following equation

$$\mathcal{L}v + \bar{L}_u v + \sqrt{A}Q_u(v) = 0. \tag{3.20}$$

Thanks to the observations on the functions A and u_r, we can conclude that Q_u has bounded coefficients in $[r_\epsilon, 1)$. Now we consider a function $\varphi \in \mathcal{C}^{2,\alpha}(S^1)$ which is even with respect to θ, L^2-orthogonal to e_0 and e_1 and such that $\|\varphi\|_{\mathcal{C}^{2,\alpha}} \leqslant k\epsilon$. We define

$$w_\varphi(\cdot,\cdot) := \mathcal{H}_{r_\epsilon,\varphi}(\cdot,\cdot)$$

where the operator $\mathcal{H}$ has been introduced in proposition 46. In order to solve the equation (3.20), we choose $\mu \in (0,1)$ and look for v of the form $v = w_\varphi + w$ where $w \in \mathcal{C}^{2,\alpha}_\mu(S^1 \times [r_\epsilon, 1))$ and $v = \varphi$ on $S^1 \times \{r_\epsilon\}$. We can rephrase this problem as a fixed point problem, that is

$$w = S(\varphi, w) \tag{3.21}$$

where the nonlinear mapping S is defined by

$$S(\varphi, w) := -G_{r_\epsilon}\left(\mathcal{L}w_\varphi + \bar{L}_u(w_\varphi + w) + \sqrt{A}Q_u\left(w_\varphi + w\right)\right),$$

where the operator G is defined in proposition 30. To prove the existence of a solution for (3.21) we need the following result which states that S is a contracting mapping.

Lemma 36. *There exist some constants $c_k > 0$ and $\epsilon_k > 0$, such that*

$$\|S(\varphi, 0)\|_{\mathcal{C}^{2,\alpha}_\mu} \leqslant c_k \epsilon^{3/2} \tag{3.22}$$

and, for all $\epsilon \in (0, \epsilon_k)$

$$\|S(\varphi, w_2) - S(\varphi, w_1)\|_{\mathcal{C}^{2,\alpha}_\mu} \leqslant \frac{1}{2}\|w_2 - w_1\|_{\mathcal{C}^{2,\alpha}_\mu}$$

for all $w_1, w_2 \in \mathcal{C}^{2,\alpha}_\mu(S^1 \times [r_\epsilon, 1))$ such that $\|w_i\|_{\mathcal{C}^{2,\alpha}_\mu} \leqslant 2c_k\epsilon^{\frac{3}{2}}$.

Proof. We know from the proposition 30 that $\|G_{r_\epsilon}(f)\|_{\mathcal{C}^{2,\alpha}_\mu} \leqslant c\|f\|_{\mathcal{C}^{0,\alpha}_\mu}$. Then

$$\|S(\varphi, 0)\|_{\mathcal{C}^{2,\alpha}_\mu} \leqslant c\|\mathcal{L}w_\varphi + \bar{L}_u(w_\varphi) + \sqrt{A}Q_u\left(w_\varphi\right)\|_{\mathcal{C}^{0,\alpha}_\mu} \leqslant$$
$$c\left(\|\mathcal{L}\, w_\varphi\|_{\mathcal{C}^{0,\alpha}_\mu} + \|\bar{L}_u(w_\varphi)\|_{\mathcal{C}^{0,\alpha}_\mu} + \|Q_u\left(w_\varphi\right)\|_{\mathcal{C}^{0,\alpha}_\mu}\right).$$

Here we have used the fact that $A = 1 + \mathcal{O}(\epsilon)$.
So we need to find the estimates of each summand. We recall that $\|\varphi\|_{\mathcal{C}^{2,\alpha}} \leqslant k\epsilon$. For all $\mu \in (0,1)$, $\|w_\varphi\|_{\mathcal{C}^{2,\alpha}_\mu} \leqslant \|w_\varphi\|_{\mathcal{C}^{2,\alpha}_1}$, and thanks to proposition 46 we get that

$$\|w_\varphi\|_{\mathcal{C}^{2,\alpha}_\mu} \leqslant c\|\varphi\|_{\mathcal{C}^{2,\alpha}(S^1)} \leqslant ck\epsilon$$

and $\mathcal{L}\, w_\varphi = 0$. We use (3.19) for finding the estimate of $\bar{L}_u w_\varphi$. We obtain

$$\|\bar{L}_u(w_\varphi)\|_{\mathcal{C}^{0,\alpha}_\mu} \leqslant c\epsilon^{1/2}\|w_\varphi\|_{\mathcal{C}^{2,\alpha}_\mu} \leqslant c_k\epsilon^{3/2}.$$

The last term is estimated observing that

$$\|Q_u\left(w_\varphi\right)\|_{\mathcal{C}^{0,\alpha}_\mu} \leqslant c\|w_\varphi\|^2_{\mathcal{C}^{2,\alpha}_\mu} \leqslant c_k\epsilon^2.$$

Putting together all these estimates we get

$$\|S(\varphi, w_\varphi)\|_{\mathcal{C}^{2,\alpha}_\mu} \leqslant c_k\epsilon^{3/2}.$$

As for the second estimate, we observe that

$$S(\varphi, w_2) - S(\varphi, w_1) = -G_{r_\epsilon}\left(\bar{L}_u(w_\varphi + w_2) + \sqrt{A}Q_u\left(w_\varphi + w_2\right)\right) +$$

$$G_{r_\epsilon}\left(\bar{L}_u(w_\varphi + w_1) + \sqrt{A}Q_u\left(w_\varphi + w_1\right)\right)$$

and

$$\|S(\varphi, w_2) - S(\varphi, w_1)\|_{\mathcal{C}^{2,\alpha}_\mu} \leqslant$$

$$c\|\bar{L}_u(w_\varphi + w_2) - \bar{L}_u(w_\varphi + w_1) + Q_u\left(w_\varphi + w_2\right) - Q_u\left(w_\varphi + w_1\right)\|_{\mathcal{C}^{0,\alpha}_\mu} =$$

$$= c\|\bar{L}_u(w_2 - w_1) + Q_u\left(w_\varphi + w_2\right) - Q_u\left(w_\varphi + w_1\right)\|_{\mathcal{C}^{0,\alpha}_\mu} \leqslant$$

$$\leqslant \|\bar{L}_u(w_2 - w_1)\|_{\mathcal{C}^{0,\alpha}_\mu} + ||Q_u\left(w_\varphi + w_1\right) - Q_u\left(w_\varphi + w_2\right)||_{\mathcal{C}^{0,\alpha}_\mu}.$$

We observe that from the considerations above it follows that

$$\|\bar{L}_u(w_2 - w_1)\|_{\mathcal{C}^{0,\alpha}_\mu} \leqslant c\epsilon^{1/2}||w_2 - w_1||_{\mathcal{C}^{2,\alpha}_\mu},$$

and

$$||Q_u\left(w_\varphi + w_1\right) - Q_u\left(w_\varphi + w_2\right)||_{\mathcal{C}^{0,\alpha}_\mu} \leqslant c||w_2 - w_1||_{\mathcal{C}^{2,\alpha}_\mu}\|w_\varphi\|_{\mathcal{C}^{2,\alpha}_\mu}$$
$$\leqslant c_k\epsilon||w_2 - w_1||_{\mathcal{C}^{2,\alpha}_\mu}.$$

Then

$$\|S(\varphi, w_2) - S(\varphi, w_1)\|_{\mathcal{C}^{2,\alpha}_\mu} \leqslant c\epsilon^{1/2}||w_2 - w_1||_{\mathcal{C}^{2,\alpha}_\mu}.$$

□

Theorem 37. *Let be* $B := \{w \in \mathcal{C}^{2,\alpha}_\mu(S^1 \times [r_\epsilon, 1)) \mid ||w||_{2,\alpha,\mu} \leqslant 2c_k\epsilon^{3/2}\}$. *Then the nonlinear mapping* S *defined above has a unique fixed point* v *in* B.

Proof: The previous lemma shows that, if ϵ is chosen small enough, the nonlinear mapping S is a contraction mapping from the ball B of radius $2c_k\epsilon^{3/2}$ in $\mathcal{C}^{2,\alpha}_\mu(S^1 \times [r_\epsilon, 1))$ into itself. This value follows from the estimate of the norm of $S(\varphi, 0)$. Consequently thanks to Schauder fixed point theorem, S has a unique fixed point w in this ball. □

We have proved the existence of a minimal surface S_t, which is close to the part of catenoid C_t introduced in section 3.5 and close to its boundary is a graph over the annulus $B_{2r_\epsilon} - B_{r_\epsilon}$ of the function which can be expanded as

$$2\epsilon \ln\left(\frac{2r}{\epsilon}\right) + \mathcal{H}_{r_\epsilon,\varphi}(r,\theta) + \bar{V}_t(r,\theta),$$

with $\|\bar{V}_t\|_{\mathcal{C}^{2,\alpha}_\mu} \leqslant c\epsilon^{3/2}$.

Analogously we can show the existence of a minimal surface S_b, which is close to the part of catenoid denoted by C_b introduced in section 3.5 and close to its boundary is a graph over the annulus $B_{2r_\epsilon} - B_{r_\epsilon}$ of the function

$$-2\epsilon \ln\left(\frac{2r}{\epsilon}\right) - \mathcal{H}_{r_\epsilon,\varphi}(r,\theta) - \bar{V}_b(r,\theta),$$

with $\|\bar{V}_b\|_{\mathcal{C}^{2,\alpha}_\mu} \leqslant c\epsilon^{3/2}$. The functions $\bar{V}_t, \bar{V}_b$ depend non linearly on ϵ, φ. Furthermore they satisfy

$$\|\bar{V}(\epsilon,\varphi)(r_\epsilon\cdot) - \bar{V}(\epsilon,\varphi')(r_\epsilon\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1 - B_{1/2})} \leqslant c\epsilon^{\frac{1}{2}}\|\varphi - \varphi'\|_{\mathcal{C}^{2,\alpha}(S^1)}.$$

3.7 The mean curvature in a neighbourhood of $\{0,0\} \times \mathbb{R}$ in $\mathbb{H}^2 \times \mathbb{R}$

We recall that the model we use for $\mathbb{H}^2 \times \mathbb{R}$ the disk model endowed with the metric g_{hyp}. If x_1, x_2 denote the coordinates in $\mathbb{H}^2$ and x_3 the coordinate in $\mathbb{R}$, then

$$g_{hyp} = \frac{dx_1^2 + dx_2^2}{F} + dx_3^2,$$

where

$$F = \frac{1}{4}\left(1 - x_1^2 - x_2^2\right)^2 = \frac{1}{4}\left(1 - r^2\right)^2.$$

In this section we want to express the mean curvature H_{hyp} of a surface, contained in a cylindrical neighbourhood C of $\{0,0\} \times \mathbb{R}$ in $\mathbb{H}^2 \times \mathbb{R}$ in terms of the mean curvature H_e computed with respect to the euclidean metric $g_{eucl} = dx_1^2 + dx_2^2 + dx_3^2$.

Let $\bar{g}$ be the metric defined on $\mathbb{R}^3$ by $4dx_1^2 + 4dx_2^2 + dx_3^2$. We consider the map f : $(\mathbb{R}^3, g_{eucl}) \to (\mathbb{R}^3, \bar{g})$ defined by

$$(x_1, x_2, x_3) \to \left(\frac{x_1}{2}, \frac{x_2}{2}, x_3\right). \tag{3.23}$$

It is easy to see that it is an isometric embedding. That is the pull-back of the metric $\bar{g}$ by f equals g_{eucl}.

We denote by $\bar{\nabla}$ the riemannian connection on $(\mathbb{R}^3, \bar{g})$. Let X, Y be two vector fields on $(\mathbb{R}^3, g_{eucl})$. We define $\bar{X} := f_*(X)$, $\bar{Y} := f_*(Y)$, Then the connection defined on $(\mathbb{R}^3, g_{eucl})$. by

$$\nabla_X Y = \bar{\nabla}_{\bar{X}} \bar{Y}, \tag{3.24}$$

coincides with the riemannian connection relative to the metric g_{eucl}. If N_e denotes the normal vector to a surface Σ with respect to the euclidean metric, the mean curvature $H_e(\Sigma)$ of Σ in $(\mathbb{R}^3, g_{eucl})$ is defined to be half the trace of the mapping

$$X \to -[\nabla_X N_{eucl}]^T,$$

where $[\cdot]^T$ denotes the projection on $T\Sigma$. We set $\bar{\Sigma} = f(\Sigma)$. Similarly if $\bar{N}$ denotes the normal vector to $\bar{\Sigma}$ with respect the metric $\bar{g}$, the mean curvature $\bar{H}(\bar{\Sigma})$ of $\bar{\Sigma}$ in $(\mathbb{R}^3, \bar{g})$ is defined to be half the trace of the mapping

$$X \to -[\bar{\nabla}_X \bar{N}]^T,$$

where $[\cdot]^T$ denotes the projection on $T\bar{\Sigma}$. Thanks to the relation (3.24) and from the fact that $T\bar{\Sigma} = f_* T\Sigma$ it is clear that

$$\bar{H}(\bar{\Sigma}) = H_e(\Sigma). \tag{3.25}$$

Now we turn our attention toward the space $\mathbb{H}^2 \times \mathbb{R}$. Now our aim is to find a relation between the mean curvature with respect to the metric g_{hyp} of a surface contained in a cylindrical neighbourhood C of $\{0, 0\} \times \mathbb{R}$ in $\mathbb{H}^2 \times \mathbb{R}$ and the mean curvature of the same surface seen in the riemannian manifold $(C, \bar{g})$.

The Christoffel symbols, Γ_{ij}^k, associated to g_{hyp} all vanish except

$$\Gamma_{11}^1 = \Gamma_{21}^2 = \Gamma_{12}^2 = -\Gamma_{22}^1 = \frac{2x^1}{1 - r^2},$$

$$\Gamma_{12}^1 = \Gamma_{22}^2 = \Gamma_{21}^1 = -\Gamma_{11}^2 = \frac{2x^2}{1 - r^2}.$$

Let $\partial_1 = \frac{1}{2}\frac{\partial}{\partial x_1}$, $\partial_2 = \frac{1}{2}\frac{\partial}{\partial x_2}$, $\partial_1 = \frac{\partial}{\partial x_3}$ be the elements of a basis of the tangent space. Now, if $X = \sum_i X^i \partial_i$ and $Y = \sum_j Y^j \partial_j$ are two tangent vector fields, the expression of the covariant derivative in $(\mathbb{H}^2 \times \mathbb{R}, g_{hyp})$ is given by

$$\nabla^h_X Y = \sum_k \left(\sum_{i,j} X^i Y^j \Gamma^k_{ij} + X(Y^k) \right) \partial_k.$$

It is clear that

$$\nabla^h_X Y = \bar{\nabla}_X Y + \sum_{k=1}^{2} \sum_{i,j} X^i Y^j \Gamma^k_{ij} \partial_k. \tag{3.26}$$

It is possible to show that $\sum_{k=1}^2 \sum_{i,j} X^i Y^j \Gamma^k_{ij} \partial_k$ is the vector whose components with respect the basis $(\partial_1, \partial_2, \partial_3)$ are given by

$$\begin{bmatrix} \frac{Y^1 x_1}{\sqrt{F}} + \frac{Y^2 x_2}{\sqrt{F}} & \frac{Y^1 x_2}{\sqrt{F}} - \frac{Y^2 x_1}{\sqrt{F}} & 0 \\ -\frac{Y^1 x_2}{\sqrt{F}} + \frac{Y^2 x_1}{\sqrt{F}} & \frac{Y^1 x_1}{\sqrt{F}} + \frac{Y^2 x_2}{\sqrt{F}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}.$$

The mean curvature $H_{hyp}(S)$ of a surface S in $(\mathbb{H}^2 \times \mathbb{R}, g_{hyp})$ is defined to be half the trace of the mapping

$$X \to -[\nabla_X N_{hyp}]^T,$$

where $[\cdot]^T$ denotes the projection onto the tangent bundle of the surface and $N_{hyp} = (N_1, N_2, N_3)$ the normal vector to S. Thanks to (3.26) we get the relation

$$[\nabla^h_X N_{hyp}]^T = [\bar{\nabla}_X N_{hyp}]^T + \sum_{k=1}^{2} \sum_{i,j} X^i N^j \Gamma^k_{ij} \partial_k. \tag{3.27}$$

Now we observe that the normal vector $\bar{N}$ to S with respect the metric $\bar{g}$ does not coincide with N_{hyp}. Since g_{hyp} is asymptotic to the metric $\bar{g}$, it is reasonable to think that $N_{hyp}(S)$ can be related to $\bar{N}(S)$. If we suppose that $N_{hyp}(S) = \frac{(v_1,v_2,v_3)}{|(v_1,v_2,v_3)|_{g_{hyp}}}$ and $\bar{N}(S) = \frac{(v_1,v_2,v_3)}{|(v_1,v_2,v_3)|_{\bar{g}}}$ then it is possible to show that

$$N_{hyp}(S) = 2\sqrt{F}\bar{N}(S)\left(1 + \mathcal{O}(r^2)\right).$$

Now we insert this result into (3.27) and we compute the opposite of half the trace. We obtain

$$H_{hyp}(S) = (\bar{H}(S) - 2(x_1 N_1 + x_2 N_2))(1 + \mathcal{O}(r^2)). \tag{3.28}$$

Thanks to (3.25) we know that $\bar{H}(S) = H_e(f^{-1}(S))$. We have proved the following result

Proposition 38. *Let S be a surface contained in a sufficiently small cylindrical neighbourhood C of $\{0,0\} \times \mathbb{R} \subset \mathbb{H}^2 \times \mathbb{R}$ endowed with the metric g_{hyp}. If $N = \sum_{i=1}^{3} N_i \partial_i$ and $H_{hyp}(\cdot)$ denote respectively the normal vector to S and the mean curvature with respect to g_{hyp}, $H_e(\cdot)$ the mean curvature with respect to g_e and f is the map defined by* (3.23)*, then*

$$H_{hyp}(S) = (H_e(f^{-1}(S)) - 2(x_1 N_1 + x_2 N_2))(1 + \mathcal{O}(r^2)). \tag{3.29}$$

3.8 A rescaled Costa-Hoffman-Meeks type surface

In this section we will describe the surface obtained by rescaling of the Costa-Hoffmann-Meeks surface of genus $k \geqslant 1$, M_k, (see C. Costa [3], [4] and D. Hoffman and W. H. Meeks [15], [16]) and we will study the mapping properties of its Jacobi operator. We denote by $M_{k,\epsilon}$ the image of M_k by an homothety of parameter 2ϵ. We will adapt to our situation some of the analytical tools used in [11] to show the existence of a family of minimal surfaces close to M_k with one planar end and two slightly bent catenoidal ends by an angle $\xi \in (-\xi_0, \xi_0)$, $\xi_0 > 0$ and small enough. We denote an element of this family by $M_k(\xi)$, then $M_k(\xi)|_{\xi=0} = M_k$.

3.8.1 The Costa-Hoffman-Meeks surface.

We start by giving a brief description of the surface M_k. After suitable rotation and translation, M_k enjoys the following properties.

1. It has one planar end E_m asymptotic to the $x_3 = 0$ plane, one top end E_t and one bottom end E_b that are respectively asymptotic to the upper end and to the lower end of a catenoid with x_3-axis of revolution. The planar end E_m is located between the two catenoidal ends.

2. It is invariant under the action of the rotation of angle $\frac{2\pi}{k+1}$ about the x_3-axis, under the action of the symmetry with respect to the $x_2 = 0$ plane and under the action of the composition of a rotation of angle $\frac{\pi}{k+1}$ about the x_3-axis and the symmetry with respect to the $x_3 = 0$ plane.

3. It intersects the $x_3 = 0$ plane in $k + 1$ straight lines, which intersect themselves at the origin with angles equal to $\frac{\pi}{k+1}$. The intersection of M_k with the plane $x_3 = $ *const* $(\neq 0)$ is a single Jordan curve. The intersection of M_k with the upper half space $x_3 > 0$ (resp. with the lower half space $x_3 < 0$) is topologically an open annulus.

We denote with X_i, with $i = t, b, m$, the parametrization of the end E_i and with $X_{i,\epsilon}$ the parametrization of the corresponding end $E_{i,\epsilon}$ of $M_{k,\epsilon}$.

Now we give a local description of the surface $M_{k,\epsilon}$ near its ends and we introduce coordinates that we will use.

The planar end. The planar end $E_{m,\epsilon}$ of the surface $M_{k,\epsilon}$ can be parametrized by

$$X_{m,\epsilon}(x) := \left(\frac{2\epsilon x}{|x|^2}, 2\epsilon u_m(x)\right) \in \mathbb{R}^3, \tag{3.30}$$

where $x \in \bar{B}_{\rho_0}(0) - \{0\} \subset \mathbb{R}^2$ and the function u_m tends to 0 like $u_m(x) = \mathcal{O}(|x|^{k+1})$. Here $\rho_0 > 0$ is fixed small enough. We will assume $\rho_0 = \mathcal{O}(\epsilon^{\frac{1}{4}})$. The minimal surface equation has the following form

$$\frac{|x|^4}{4\epsilon^2} \operatorname{div}\left(\frac{\nabla u}{(1 + |x|^4 |\nabla u|^2)^{1/2}}\right) = 0. \tag{3.31}$$

It can be shown (see [11]) that the function u_m can be extended at the origin continuously using Weierstrass representation. We can prove that $u_m \in \mathcal{C}^{2,\alpha}(\bar{B}_{\rho_0})$.

If we linearize in $u = 0$ the nonlinear equation (3.31) we obtain the expression an operator which is close, up to a multiplication by $4\epsilon^2$, to the Jacobi operator about the plane, that is $\mathcal{L}_{\mathbb{R}^2} = |x|^4\Delta$. To be more precise, the linearization of (3.31) gives

$$L_u = \frac{|x|^4}{4\epsilon^2}\operatorname{div}\left(\frac{\nabla v}{\sqrt{1 + |x|^4|\nabla u|^2}} - |x|^4\nabla u\frac{\nabla u \cdot \nabla v}{\sqrt{(1 + |x|^4|\nabla u|^2)^3}}\right).$$

We will give the expression of H_{u+v}, the mean curvature of the graph of the function $u+v$, in terms of the mean curvature of Σ_u, that is H_u. In the following we shall restrict our attention to the planar case, that is $u = 0$, on a domain of the form $\{(r, \theta) \in B_{r_0}(0) | r \in [r_1, r_2]\}$. Here we shall show that

$$2H_{u+v} = 2H_u + L_u v + \frac{|x|^4}{4\epsilon^2}Q_u(\sqrt{|x|^4}\nabla v, \sqrt{|x|^4}\nabla^2 v), \tag{3.32}$$

where Q_u satisfies

$$Q_u(0, 0) = \nabla Q'_u(0, 0).$$

To show (3.32), we start observing that:

$$\frac{1}{\sqrt{1 + |x|^4|\nabla(u + v)|^2}} = \frac{1}{\sqrt{1 + |x|^4|\nabla u|^2}} - |x|^4\frac{\nabla u \cdot \nabla v}{\sqrt{(1 + |x|^4|\nabla u|^2)^3}} + Q_{u,1}(v) \tag{3.33}$$

where the function $Q_{u,1}$ satisfies $Q_{u,1}(0) = \nabla Q_{u,1}(0) = 0$. The proof of that is very close to the one that appears in section 3.2: it's necessary only to replace F by $|x|^4$. So we can omit some details. Secondly we observe that $2H_{u+v}$ is given by

$$\frac{|x|^4}{4\epsilon^2}\operatorname{div}\left(\frac{\nabla(u + v)}{\sqrt{1 + |x|^4|\nabla u|^2}} - |x|^4\nabla(u + v)\frac{\nabla u \cdot \nabla v}{\sqrt{(1 + |x|^4|\nabla u|^2)^3}} + \nabla(u + v)Q_{u,1}(v)\right).$$

From this it follows the wanted expression.

Since we assume that Σ_u is a minimal surface, we will consider $H_u = 0$.

Following what we have done in section 3.6 replacing F by $|x|^4$ we get the expression of the minimal surfaces equation that we will use in the following sections:

$$\frac{|x|^4}{4\epsilon^2}\left(\mathcal{L}v + \bar{L}_u v + Q_u(\sqrt{|x|^4}\nabla v, \sqrt{|x|^4}\nabla^2 v)\right) = 0. \tag{3.34}$$

The catenoidal ends. We denote by X_c the parametrization of the standard catenoid C whose axis of revolution is the x_3-axis. Its expression is

$$X_c(s,\theta) := (\cosh s\,\cos\theta, \cosh s\,\sin\theta, s) \in \mathbb{R}^3$$

where $(s,\theta) \in \mathbb{R}\times S^1$. The unit normal vector field about C is given by

$$n_c(s,\theta) := \frac{1}{\cosh s}\,(\cos\theta, \sin\theta, -\sinh s).$$

The catenoid C may be divided in two pieces, denoted $C_\pm$, which are defined as the image by X_c of $(\mathbb{R}^\pm \times S^1)$. For any $\epsilon > 0$, we define the catenoid C_ϵ as the image of C by an homothety of parameter 2ϵ. We denote with $X_{c,\epsilon} := 2\epsilon X_c$ its parametrization. Of course, by this transformation, to $C_\pm$, correspond two surfaces denoted $C_{\epsilon,\pm}$.

Up to some dilation, we can assume that the two ends $E_{t,\epsilon}$ and $E_{b,\epsilon}$ of $M_{k,\epsilon}$ are asymptotic to some translated copy of the catenoid parametrized by $X_{c,\epsilon}$ in the vertical direction. Therefore, $E_{t,\epsilon}$ and $E_{b,\epsilon}$ can be parametrized, respectively, by

$$X_{t,\epsilon} := X_{c,\epsilon} + w_t\, n_c + \sigma_{t,\epsilon}\, e_3 \tag{3.35}$$

for $(s,\theta) \in (s_0,\infty)\times S^1$,

$$X_{b,\epsilon} := X_{c,\epsilon} - w_b\, n_c - \sigma_{b,\epsilon}\, e_3 \tag{3.36}$$

for $(s,\theta) \in (-\infty,-s_0)\times S^1$, where $\sigma_{t,\epsilon}, \sigma_{b,\epsilon} \in \mathbb{R}$, functions w_t, w_b tend exponentially fast to 0 as s goes to ∞ reflecting the fact that the ends are asymptotic to a catenoidal end.

In section 3 of [24] it is given the expression of the mean curvature operator about of a surface close to a rescaled standard catenoid. We can adapt this result to our situation. We obtain that the surface parametrized by $X_{c,\epsilon} + w\, n_c$ is minimal if and only if the function w satisfies the minimal surface equation $H_w = 0$, where

$$H_w = -\frac{1}{4\epsilon^2}\,\mathbb{L}_C w + \frac{1}{2\epsilon\cosh^2 s}Q_{2,\epsilon}\left(\frac{w}{2\epsilon\cosh s}, \frac{\nabla w}{2\epsilon\cosh s}, \frac{\nabla^2 w}{2\epsilon\cosh s}\right) +$$

$$\frac{1}{2\epsilon\cosh s}Q_{3,\epsilon}\left(\frac{w}{2\epsilon\cosh s},\frac{\nabla w}{2\epsilon\cosh s},\frac{\nabla^2 w}{2\epsilon\cosh s}\right). \tag{3.37}$$

Here $\mathbb{L}_C$ is the Jacobi operator about the catenoid, that is

$$\mathbb{L}_C w = \frac{1}{\cosh^2 s}\left(\frac{\partial^2 w}{\partial s^2}+\frac{\partial^2 w}{\partial\theta^2}+\frac{2w}{\cosh^2 s}\right)$$

and $Q_{2,\epsilon}$ and $Q_{3,\epsilon}$ are functions which are bounded in $\mathcal{C}^k(\mathbb{R}\times S^1)$ for all k, uniformly in ϵ. They satisfy

$$Q_{2,\epsilon}(0,0,0)=Q_{3,\epsilon}(0,0,0)=0 \qquad \text{and} \qquad \nabla Q_{2,\epsilon}(0,0,0)=\nabla Q_{3,\epsilon}(0,0,0)=0, \tag{3.38}$$

$$\nabla^2 Q_{3,\epsilon}(0,0,0)=0. \tag{3.39}$$

We will write for short

$$Q_\epsilon(w_\Phi)=\frac{1}{2\epsilon\cosh^2 s}Q_{2,\epsilon}\left(\frac{w_\Phi}{2\epsilon\cosh s},\frac{\nabla w_\Phi}{2\epsilon\cosh s},\frac{\nabla^2 w_\Phi}{2\epsilon\cosh s}\right)+$$

$$\frac{1}{2\epsilon\cosh s}Q_{3,\epsilon}\left(\frac{w_\Phi}{2\epsilon\cosh s},\frac{\nabla w_\Phi}{2\epsilon\cosh s},\frac{\nabla^2 w_\Phi}{2\epsilon\cosh s}\right). \tag{3.40}$$

For all $r<\rho_0$ and $s>s_0$, we define

$$M_{k,\epsilon}(s,r):=M_{k,\epsilon}-\left[X_{t,\epsilon}((s,\infty)\times S^1)\cup X_{b,\epsilon}((-\infty,-s)\times S^1)\cup X_{m,\epsilon}(B_r(0))\right]. \tag{3.41}$$

The parametrizations of the three ends of $M_{k,\epsilon}$ induce a decomposition of $M_{k,\epsilon}$ into slightly overlapping components: a compact piece $M_{k,\epsilon}(s_0+1,\rho_0/2)$ and three noncompact pieces $X_{t,\epsilon}((s_0,\infty)\times S^1)$, $X_{b,\epsilon}((-\infty,-s_0)\times S^1)$ and $X_{m,\epsilon}(\bar{B}_{\rho_0}(0))$.

We define a weighted space of functions on $M_{k,\epsilon}$.

Definition 39. *Given $\ell\in\mathbb{N}$, $\alpha\in(0,1)$ and $\Delta\in\mathbb{R}$, the space $\mathcal{C}^{\ell,\alpha}_\delta(M_{k,\epsilon})$ is defined to be the space of functions in $\mathcal{C}^{\ell,\alpha}_{loc}(M_{k,\epsilon})$ for which the following norm is finite*

$$\|w\|_{\mathcal{C}^{\ell,\alpha}_\delta(M_{k,\epsilon})}:=\|w\|_{\mathcal{C}^{\ell,\alpha}(M_{k,\epsilon}(s_0+1,\rho_0/2))}+\|\,w\circ X_{m,\epsilon}\|_{\mathcal{C}^{\ell,\alpha}(B_{\rho_0}(0))}$$

$$+\sup_{s\geqslant s_0}e^{-\delta s}\,\left(\|w\circ X_{t,\epsilon}\|_{\mathcal{C}^{\ell,\alpha}([s,s+1]\times S^1)}+\|w\circ X_{b,\epsilon}\|_{\mathcal{C}^{\ell,\alpha}([-s-1,-s]\times S^1)}\right)$$

and which are invariant under the action of the symmetry with respect to the $x_2=0$ plane.

We remark that there is no weight on the middle end. In fact we compactify this end and we consider a weighted space of functions defined on a two ended surface. We will perturb the surface $M_{k,\epsilon}$ by the normal graph of a function $u\in\mathcal{C}^{2,\alpha}_\delta(M_{k,\epsilon})$. In particular the middle end $E_{m,\epsilon}$ will be just translated in the vertical direction.

3.8.2 The Jacobi operator

The Jacobi operator about $M_{k,\epsilon}$ is

$$\mathbb{L}_{M_{k,\epsilon}} := \Delta_{M_{k,\epsilon}} + |A_{M_{k,\epsilon}}|^2$$

where $|A_{M_{k,\epsilon}}|$ is the norm of the second fundamental form on $M_{k,\epsilon}$.

In the parametrization of the ends introduced above, the volume forms $\mathrm{dvol}_{M_{k,\epsilon}}$ can be written as $\gamma_t\, ds\, d\theta$ and $\gamma_b\, ds\, d\theta$ near the catenoidal type ends and as $\gamma_m\, dx_1\, dx_2$ near the middle end. Now we can define globally on $M_{k,\epsilon}$ a smooth function

$$\gamma : M_{k,\epsilon} \longrightarrow [0,\infty)$$

that is identically equal to $4\epsilon^2$ on $M_{k,\epsilon}(s_0-1, 2\rho_0)$ and equal to γ_t (resp. γ_b, γ_m) on the end $E_{t,\epsilon}$ (resp. $E_{b,\epsilon}$, E_m). They are defined in such a way that on $X_{t,\epsilon}((s_0,\infty)\times S^1)$ and on $X_{b,\epsilon}((-\infty,s_0)\times S^1)$ we have

$$\gamma\circ X_{t,\epsilon}(s,\theta) \sim 4\epsilon^2\cosh^2 s \qquad \text{and} \qquad \gamma\circ X_{b,\epsilon}(s,\theta)\sim 4\epsilon^2\cosh^2 s.$$

Finally on $X_{m,\epsilon}(B_{\rho_0})$, we have

$$\gamma\circ X_m(x)\sim \frac{4\epsilon^2}{|x|^4}.$$

It is possible to check that:

$$\begin{array}{rccc} \mathcal{L}_{\epsilon,\delta}: & \mathcal{C}^{2,\alpha}_\delta(M_{k,\epsilon}) & \longrightarrow & \mathcal{C}^{0,\alpha}_\delta(M_{k,\epsilon}) \\ & w & \longmapsto & \gamma\,\mathbb{L}_{M_{k,\epsilon}}(w) \end{array}$$

is a bounded linear operator. The subscript δ is meant to keep track of the weighted space over which the Jacobi operator is acting. Observe that, the function γ is here to counterbalance the effect of the conformal factor $\frac{1}{\sqrt{|g_{M_{k,\epsilon}}|}}$ in the expression of the Laplacian in the coordinates we use to parametrize the ends of the surface $M_{k,\epsilon}$. This is precisely what is needed to have the operator defined from the space $\mathcal{C}^{2,\alpha}_\delta(M_{k,\epsilon})$ into the target space $\mathcal{C}^{0,\alpha}_\delta(M_{k,\epsilon})$.

To have a better grasp of what is going on, let us linearize the nonlinear equation (3.37) at $w=0$. We get the expression of the Jacobi operator about the rescaled catenoid C_ϵ

$$\mathbb{L}_{C_\epsilon} := \frac{1}{4\epsilon^2\cosh^2 s}\left(\partial_s^2+\partial_\theta^2+\frac{2}{\cosh^2 s}\right).$$

We can observe that the operator $4\epsilon^2\cosh^2 s\,\mathbb{L}_{C_\epsilon}$ maps the space $(\cosh s)^\delta\,\mathcal{C}^{2,\alpha}((s_0,\infty)\times S^1)$ into the space $(\cosh s)^\delta\,\mathcal{C}^{0,\alpha}((s_0,\infty)\times S^1)$.

Similarly, if we linearize the nonlinear equation (3.31) at $u = 0$, we obtain the expression of the Jacobi operator about the plane times $4\epsilon^2$.

$$\mathbb{L}_{\mathbb{R}^2_\epsilon} := \frac{|x|^4}{4\epsilon^2}\,\Delta.$$

Again, the operator $\frac{4\epsilon^2}{|x|^4}\,\mathbb{L}_{\mathbb{R}^2_\epsilon} = \Delta$ clearly maps the space $\mathcal{C}^{2,\alpha}(\bar{B}_{\rho_0})$ into the space $\mathcal{C}^{0,\alpha}(\bar{B}_{\rho_0})$. Now, the function γ plays, for the ends of the surface $M_{k,\epsilon}$, the role played by the function $\cosh^2 s$ for the ends of the standard catenoid and the role played by the function $|x|^{-4}$ for the plane. Since the Jacobi operator about $M_{k,\epsilon}$ is asymptotic to $\mathbb{L}_{\mathbb{R}^2_\epsilon}$ at $E_{m,\epsilon}$ and is asymptotic to $\mathbb{L}_{C_\epsilon}$ at $E_{t,\epsilon}$ and $E_{b,\epsilon}$, we conclude that the operator $\mathcal{L}_{\epsilon,\delta}$ maps $\mathcal{C}^{2,\alpha}_\delta(M_{k,\epsilon})$ into $\mathcal{C}^{0,\alpha}_\delta(M_{k,\epsilon})$.

Now we recall the notion of non degeneracy introduced in [11].

Definition 40. *The surface $M_{k,\epsilon}$ is said to be non degenerate if $\mathcal{L}_{\epsilon,\delta}$ is injective for all $\delta < -1$.*

It useful to observe that a duality argument in the weighted Lebesgue spaces, implies that

$$(\mathcal{L}_{\epsilon,\delta} \quad \text{is injective}) \quad \Leftrightarrow \quad (\mathcal{L}_{\epsilon,-\delta} \quad \text{is surjective})$$

if $\delta \notin \mathbb{Z}$. See [27] and [18] for more details.

The non degeneracy of $M_{k,\epsilon}$ is related to the mapping properties of $\mathcal{L}_{\epsilon,\delta}$ and to the kernel of this operator. From the observations made above, it follows that at the catenoidal type ends and at the middle planar end the Jacobi operators of $M_{k,\epsilon}$ and M_k are respectively asymptotic to $\mathbb{L}_C$ and $\mathbb{L}_{C_\epsilon}$ which coincide up to a multiplication by $4\epsilon^2$. So we could transpose the all results about the surface $M_k(0)$ contained in [11] related to the study of its mean curvature operator, to the surface $M_{k,\epsilon}$, including non degeneracy.

The Jacobi fields. It is known that a smooth one parameter group of isometries containing the identity generates a Jacobi field, that is a solution of the equation $\mathbb{L}_{M_{k,\epsilon}} u = 0$. These solutions are generated by the following one parameter groups of isometries: the vertical translations, the translations along the x_1-axis, the dilations. See [11] for details.

The group of vertical translations generated by the Killing vector field $\Xi(p) = e_3$ gives rise to the Jacobi field

$$\Phi^{0,+}(p) := n(p) \cdot e_3.$$

The vector field $\Xi(p) = p$ that is associated to the one parameter group of dilation generates a Jacobi fields

$$\Phi^{0,-}(p) := n(p) \cdot p.$$

The Killing vector field $\Xi(p) = e_1$ that generates the group of translations along the x_1-axis is associated to a Jacobi field

$$\Phi^{1,+}(p) := n(p) \cdot e_1.$$

Finally, we denote by

$$\Phi^{1,-}(p) := n(p) \cdot (e_2 \times p)$$

the Jacobi field associated to the Killing vector field $\Xi(p) = e_2 \times p$ that generates the group of rotations about the x_2-axis.

The Jacobi equation has other solutions which are not taken into account because in the difference with the four Jacobi fields just introduced they are not invariant under the action of the symmetry with respect to the $x_2 = 0$ plane.

With these notations, we define the deficiency space

$$\mathcal{D} := \text{Span}\{\chi_t \, \Phi^{j,\pm}, \chi_b \, \Phi^{j,\pm} \; : \; j = 0, 1\}$$

where χ_t is a cutoff function that is identically equal to 1 on $X_{t,\epsilon}((s_0 + 1, \infty) \times S^1)$, identically equal to 0 on $M_{k,\epsilon} - X_{t,\epsilon}((s_0, \infty) \times S^1)$ and that is invariant under the action of the symmetry with respect to the $x_2 = 0$ plane. Also, we agree that

$$\chi_b(\cdot) := \chi_t(- \cdot).$$

Clearly

$$\begin{array}{rccc} \tilde{\mathcal{L}}_{\epsilon,\delta} : & \mathcal{C}^{2,\alpha}_\delta(M_{k,\epsilon}) \oplus \mathcal{D} & \longrightarrow & \mathcal{C}^{0,\alpha}_\delta(M_{k,\epsilon}) \\ & w & \longmapsto & \gamma \, \mathbb{L}_{M_{k,\epsilon}}(w) \end{array}$$

is a bounded linear operator. The linear decomposition Lemma proved in [23] for constant mean curvature surfaces (see also [18] for minimal hypersurfaces) can be adapted to our situation and thanks to the results of S. Nayatani contained in [29],[30] and extended in chapter 1, stating that any bounded Jacobi field respecting the mirror symmetry with respect to the $x_2 = 0$ plane is linear combination of $\Phi^{0,\pm}$ and $\Phi^{1,+}$, we get the following result

Proposition 41. *All bounded Jacobi fields on $M_{k,\epsilon}$ that are invariant with respect to the $x_2 = 0$ plane are linear combination of $\Phi^{0,\pm}$ and $\Phi^{1,+}$ and choose $\delta \in (1, 2)$. Then the operator $\tilde{\mathcal{L}}_{\epsilon,\delta}$ is surjective and has a kernel of dimension 4. Moreover, there exists $G_{\epsilon,\delta}$ a right inverse for $\tilde{\mathcal{L}}_{\epsilon,\delta}$ whose norm is bounded.*

3.9 An infinite dimensional family of minimal surfaces which are close to a compact part of a rescaled Costa-Hoffman-Meeks type surface in $\mathbb{H}^2 \times \mathbb{R}$.

We recall that in section 3.7 we found that the mean curvature with respect to the metric g_{hyp} of a surface $\bar{S}$ contained in a cylindrical neighbourhood of $\{0,0\} \times \mathbb{R}$ of sufficiently small radius in $\mathbb{H} \times \mathbb{R}$ can be expressed in terms of the euclidean mean curvature of a surface S, which is related to $\bar{S}$ by the relation $S = f(\bar{S})$, being f the map defined by (3.23).

In this section we will apply this result to prove the existence of a family of minimal surfaces close to the surface $\bar{M}_{k,\epsilon} = f(M_{k,\epsilon})$ contained in a cylindrical neighbourhood of radius $r_\epsilon = \sqrt{\epsilon}/2$ of $\{0,0\} \times \mathbb{R}$. We remark that the image by the map f^{-1} of a surface contained in such a domain is contained in a cylinder of radius $2r_\epsilon$.

We start giving the statement of a result that can be easily obtained by [11], lemma 2.2. It describes the region of the surface $M_{k,\epsilon}$ which can be parametrized by a graph on a annular neighbourhood of $2r_\epsilon$ contained in the $x_3 = 0$ plane.

Lemma 42. *There exists $\epsilon_0 > 0$ such that, for all $\epsilon \in (0, \epsilon_0)$ an annular part of the ends $E_{t,\epsilon}$, $E_{b,\epsilon}$ and $E_{m,\epsilon}$ of $M_{k,\epsilon}$ can be written as vertical graphs over the horizontal plane of the functions*

$$U_t(r,\theta) = \sigma_{t,\epsilon} + 2\epsilon \ln\left(\frac{2r}{\epsilon}\right) + f_t(r,\theta), \tag{3.42}$$

$$\partial_r U_t(r,\theta) = \left(\frac{2\epsilon}{r}\right) + f_{t,d}(r,\theta),$$

$$U_b(r,\theta) = -\sigma_{b,\epsilon} - 2\epsilon \ln\left(\frac{2r}{\epsilon}\right) + f_b(r,\theta), \tag{3.43}$$

$$\partial_r U_b(r,\theta) = -\left(\frac{2\epsilon}{r}\right) + f_{b,d}(r,\theta),$$

where $f_i = \mathcal{O}(\epsilon^2)$, $f_{i,d} = \mathcal{O}(\epsilon^{5/2})$, $i = t, b$. As for the parametrization of the planar end, it satisfies

$$U_m(r,\theta) = f_m(r,\theta) = \mathcal{O}(\left(\frac{r}{2\epsilon}\right)^{-(k+1)}) \tag{3.44}$$

$$\partial_r U_m(r,\theta) = f_{m,d}(r,\theta) = \mathcal{O}(\left(\frac{r}{2\epsilon}\right)^{-k}).$$

Here (r,θ) are the polar coordinates in the $x_3 = 0$ plane. The functions $\mathcal{O}(\epsilon)$ are defined in the annulus $B_{4\,\epsilon^{1/2}} - B_{\epsilon^{1/2}/4}$ and are bounded in $\mathcal{C}^\infty_b$ topology by a constant (independent on ϵ) multiplied by ϵ, where the partial derivatives are computed with respect to the vector fields $r\,\partial_r$ and ∂_θ.

Then $M_{k,\epsilon}$ has two ends $E_{t,\epsilon}$ and $E_{b,\epsilon}$ which are graphs over the $x_3 = 0$ plane of functions U_t and U_b defined on the annulus $B_{4\,\epsilon^{1/2}} - B_{\epsilon^{1/2}/4}$.

Taking into account the definition of $\bar{M}_{k,\epsilon} = f(M_{k,\epsilon})$, it is clear that a lemma with identic statement can be proved also for this surface.

We set $s_\epsilon = -\frac{1}{2}\ln\epsilon$ and we define $M^T_{k,\epsilon}$ to be equal to $M_{k,\epsilon}$ from which we have removed the image of $(s_\epsilon, +\infty) \times S^1$ by $X_{t,\epsilon}$, the image of $(-\infty, -s_\epsilon) \times S^1$ by $X_{b,\epsilon}$ and the image of $B_{\rho_\epsilon}(0)$ by $X_{m,\epsilon}$ with $\rho_\epsilon := 2\epsilon\frac{1}{2r_\epsilon} = 2\epsilon^{1/2}$. We set $\bar{M}^T_{k,\epsilon} = f(M^T_{k,\epsilon})$. In this section we will prove the existence of a family of surfaces close to $\bar{M}^T_{k,\epsilon}$. To this aim we will use proposition 38 and we will follow the work [11].

First, we modify the parametrization of the ends $E_{t,\epsilon}$, $E_{b,\epsilon}$ for appropriates values of s, so that, when $r \in [\epsilon^{1/2}/4, 4\,\epsilon^{1/2}]$ the curves corresponding to the image of

$$\theta \to (2\epsilon r\,\cos\theta, 2\epsilon r\,\sin\theta, U_t(r,\theta)), \quad \theta \to (2\epsilon r\,\cos\theta, 2\epsilon r\,\sin\theta, U_b(r,\theta))$$

correspond, respectively, up to a vertical translation to the curves $s = -2\epsilon\ln(2r)$ and $s = 2\epsilon\ln(2r)$.

The curve $\theta \to (2\epsilon r\,\cos\theta, 2\epsilon r\,\sin\theta, U_m(r,\theta))$ corresponds to $\rho = \frac{2\epsilon}{r}$.

The second step is the modification of unit normal vector field on $M_{k,\epsilon}$ into a transverse unit vector field $\tilde{n}_\epsilon$ in such a way that it coincides with the normal vector field n_ϵ on $M_{k,\epsilon}$, is equal to e_3 on the graph over $B_{2\epsilon^{1/2}} - B_{3\epsilon^{1/2}/8}$ of the functions U_t and U_b and interpolate smoothly between the different definitions of $\tilde{n}_\epsilon$ in different subsets of $M^T_{k,\epsilon}$.
The graph of a function u, using the vector field $\tilde{n}_\epsilon$, will be a minimal surface if and only if u is a solution of a second order nonlinear elliptic equation of the form

$$\mathbb{L}_{M^T_{k,\epsilon}}\, u = \tilde{L}_\epsilon\, u + Q_\epsilon\,(u)$$

where $\mathbb{L}_{M^T_{k,\epsilon}}$ is the Jacobi operator about $M^T_{k,\epsilon}$, Q_ϵ is a nonlinear second order differential operator and $\tilde{L}_\epsilon$ is a linear operator which takes into account the change of the parametrization and of the change of the normal vector field. In [11] it is proved that this last operator has coefficients uniformly bounded by a constant times ϵ^2.

Now, we consider three functions $\varphi_t, \varphi_b, \varphi_m \in \mathcal{C}^{2,\alpha}(S^1)$ which are even, with respect to θ, φ_t, φ_b are L^2 orthogonal to 1 and $\cos\theta$ while φ_m is L^2 orthogonal to 1. Assume that they satisfy

$$\|\varphi_t\|_{\mathcal{C}^{2,\alpha}} + \|\varphi_b\|_{\mathcal{C}^{2,\alpha}} + \|\varphi_m\|_{\mathcal{C}^{2,\alpha}} \leqslant \kappa\,\epsilon.$$

We set $\Phi := (\varphi_t, \varphi_b, \varphi_m)$ and we define w_Φ to be the function equal to

1. $\chi_+ \, H_{\varphi_t}(s_\epsilon - s, \cdot)$ on the image of $X_{t,\epsilon}$ where χ_+ is a cut-off function equal to 0 for $s \leqslant s_0 + 1$ and identically equal to 1 for $s \in [s_0 + 2, s_\epsilon]$
2. $\chi_- \, H_{\varphi_b}(s + s_\epsilon, \cdot)$ on the image $X_{b,\epsilon}$ where χ_- is a cut-off function equal to 0 for $s \geqslant -s_0 - 1$ and identically equal to 1 for $s \in [-s_\epsilon, -s_0 - 2]$
3. $\chi_m \, \tilde{H}_{\rho_\epsilon, \varphi_m}(\cdot, \cdot)$ on the image of $X_{m,\epsilon}$, where χ_m is a cut-off function equal to 0 for $r \geqslant \rho_0$ and identically equal to 1 for $\rho \in [\rho_\epsilon, \rho_0/2]$
4. zero on the remaining part of the surface $M^T_{k,\epsilon}$.

We recall that the operators $\tilde{H}$ and H have been introduced respectively in Propositions 48 and 47.

We would like to prove that, under appropriates hypotheses, the graph about $\bar{M}^T_{k,\epsilon}$ of the function $\bar{u} = f(u)$ with $u = w_\Phi + v$, is a minimal surface. If we denote it by $\bar{\Sigma}_{\bar{u}}$, this is equivalent to solve the equation:

$$H_{hyp}(\bar{\Sigma}_{\bar{u}}) = 0.$$

If we define $\Sigma_u = f^{-1}(\bar{\Sigma}_{\bar{u}})$ and denote by $N_u = (N_1(u), N_2(u), N_3(u))$ the unit normal vector to $\bar{\Sigma}_{\bar{u}}$, thanks to proposition 38 we can write the equation to solve as

$$(H_e(\Sigma_u) - 2(x_1 N_1(u) + x_2 N_2(u)))(1 + \mathcal{O}(r^2)) = 0.$$

$H_e(\Sigma_u)$ is the mean curvature of the graph of the function u about $M^T_{k,\epsilon}$. Taking into account that $u = w_\Phi + v$, its expression is given by

$$\mathbb{L}_{M^T_{k,\epsilon}}(w_\Phi + v) - \tilde{L}_\epsilon(w_\Phi + v) - Q_\epsilon(w_\Phi + v).$$

To simplify the notation we set $-2(x_1 N_1(u) + x_2 N_2(u)) = P(w_\Phi + v)$. The resolution of the previous equation is obtained thanks to the one of the following fixed point problem:

$$v = T(\Phi, v) \tag{3.45}$$

with

$$T(\Phi, v) = G_{\epsilon,\delta} \circ \mathcal{E}_\epsilon \left(\gamma \left(\tilde{L}_\epsilon(w_\Phi + v) + P(w_\Phi + v) - \mathbb{L}_{M^T_{k,\epsilon}} w_\Phi + Q_\epsilon(w_\Phi + v) \right) \right)$$

where $\delta \in (1,2)$, the operator $G_{\epsilon,\delta}$ is defined in proposition 41 and $\mathcal{E}_\epsilon$ is a linear extension operator such that

$$\mathcal{E}_\epsilon : \mathcal{C}^{0,\alpha}_\delta(M^T_{k,\epsilon}) \longrightarrow \mathcal{C}^{0,\alpha}_\delta(M_{k,\epsilon}),$$

where $\mathcal{C}^{0,\alpha}_\delta(M^T_{k,\epsilon})$ denotes the space of functions of $\mathcal{C}^{0,\alpha}_\delta(M_{k,\epsilon})$ restricted to $M^T_{k,\epsilon}$. It is defined by $\mathcal{E}_\epsilon v = v$ in $M^T_{k,\epsilon}$, $\mathcal{E}_\epsilon v = 0$ in the image of $[s_\epsilon + 1, +\infty) \times S^1$ by $X_{t,\epsilon}$, in the image of $(-\infty, -s_\epsilon - 1) \times S^1$ by $X_{b,\epsilon}$ and in the image of $B_{\rho_\epsilon/2} \times S^1$ by $X_{m,\epsilon}$. Finally $\mathcal{E}_\epsilon v$ is an interpolation of these values in the remaining part of $M_{k,\epsilon}$ such that, for example,

$$(\mathcal{E}_\epsilon v) \circ X_{t,\epsilon}(s,\theta) = ((1 + s_\epsilon - s)v) \circ X_{t,\epsilon}(s_\epsilon, \theta),$$

for $(s,\theta) \in [s_\epsilon, s_\epsilon + 1] \times S^1$

$$(\mathcal{E}_\epsilon v) \circ X_{b,\epsilon}(s,\theta) = ((1 + s_\epsilon + s)v) \circ X_{b,\epsilon}(s_\epsilon, \theta),$$

for $(s,\theta) \in [-s_\epsilon - 1, -s_\epsilon] \times S^1$ and

$$(\mathcal{E}_\epsilon v) \circ X_{m,\epsilon}(\rho,\theta) = ((\frac{2}{\rho_\epsilon}\rho - 1)v) \circ X_{m,\epsilon}(\rho_\epsilon, \theta)$$

for $(\rho,\theta) \in [\rho_\epsilon/2, \rho_\epsilon] \times S^1$.

Remark 43. *From the definition of $\mathcal{E}_\epsilon$, if $\operatorname{supp} v \cap (B_{\rho_0} - B_{\rho_\epsilon}) \neq \emptyset$ then*

$$\|(\mathcal{E}_\epsilon v) \circ X_{m,\epsilon}\|_{\mathcal{C}^{0,\alpha}(B_{\rho_0})} \leqslant c\epsilon^{-\alpha} \|v \circ X_{m,\epsilon}\|_{\mathcal{C}^{0,\alpha}(B_{\rho_0})}.$$

This phenomenon of explosion of the norm does not occur near the catenoidal type ends:

$$\|(\mathcal{E}_\epsilon v) \circ X_{t,\epsilon}\|_{\mathcal{C}^{0,\alpha}([s_\epsilon+1,+\infty)\times S^1)} \leqslant c\|v \circ X_{t,\epsilon}\|_{\mathcal{C}^{0,\alpha}([s_\epsilon+1,+\infty)\times S^1)}.$$

A similar equation holds for the bottom end. In the following we will assume $\alpha > 0$ and near to zero.

The existence of a solution $v \in \mathcal{C}^{2,\alpha}_\delta(M^T_{k,\epsilon})$ for the equation (3.45) is a consequence of the following result which proves that T is a contracting mapping.

Lemma 44. *There exist constants $c_\kappa > 0$ and $\epsilon_\kappa > 0$, such that*

$$\|T(\Phi, 0)\|_{\mathcal{C}^{2,\alpha}_\delta} \leqslant c_\kappa\, \epsilon^2 \tag{3.46}$$

and, for all $\epsilon \in (0, \epsilon_\kappa)$

$$\|T(\Phi, v_2) - T(\Phi, v_1)\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\epsilon})} \leqslant \frac{1}{2}\, \|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}_\delta(M^T_{k,\epsilon})}$$

for all $v_1, v_2 \in \mathcal{C}^{2,\alpha}_\delta(M^T_{k,\epsilon})$ and satisfying $\|v\|_{\mathcal{C}^{2,\alpha}_\delta} \leqslant 2\, c_\kappa\, \epsilon^2$

Proof. We recall that the Jacobi operator associated to $M_{k,\epsilon}$, is asymptotic to the operator $\mathbb{L}_{C,\epsilon}$ near the catenoidal ends, and it is asymptotic to the laplacian near of the planar end. The function w_Φ is identically zero far from the ends where the explicit expression of $\mathbb{L}_{M_{k,\epsilon}}$ is not known: this is the reason of our particular choice in the definition of w_Φ. Then from the definition of w_Φ and thanks to proposition 41 we obtain the estimate

$$\|\mathcal{E}_\epsilon\left(\gamma\mathbb{L}_{M^T_{k,\epsilon}}w_\Phi\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\epsilon})} \leqslant ||\gamma_t\mathbb{L}_{C,\epsilon}(w_\Phi\circ X_{t,\epsilon})||_{\mathcal{C}^{0,\alpha}_\delta([s_0+1,s_\epsilon]\times S^1)}+$$

$$+||\gamma_b\mathbb{L}_{C,\epsilon}(w_\Phi\circ X_{b,\epsilon})||_{\mathcal{C}^{0,\alpha}_\delta[-s_\epsilon,-s_0-1]\times S^1)} \leqslant c\left|\left|\frac{2}{\cosh^2 s}w_\Phi\circ X_{t,\epsilon}\right|\right|_{\mathcal{C}^{0,\alpha}_\delta[s_0+1,s_\epsilon]\times S^1)} \leqslant$$

$$c\epsilon\|w_\Phi\circ X_{t,\epsilon}\|_{\mathcal{C}^{2,\alpha}_\delta[s_0+1,s_\epsilon]\times S^1)} \leqslant c_\kappa\,\epsilon^{2+\frac{\delta}{2}}.$$

Using the properties of $\tilde{L}_\epsilon$, we obtain

$$\|\mathcal{E}_\epsilon\left(\gamma\tilde{L}_\epsilon\,w_\Phi\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\epsilon})} \leqslant c\epsilon^2\|w_\Phi\circ X_{t,\epsilon}\|_{\mathcal{C}^{2,\alpha}_\delta([s_0+1,s_\epsilon]\times S^1)}+$$

$$c\epsilon^2\|w_\Phi\circ X_{m,\epsilon}\|_{(\mathcal{C}^{2,\alpha}[\rho_\epsilon,\rho_0/2]\times S^1)} \leqslant c_\kappa\;\epsilon^2.$$

The estimate of $\|\mathcal{E}_\epsilon\left(\gamma P(w_\Phi)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\epsilon})}$ is related to the estimate of the horizontal components of the normal vector to surface at the catenoidal type ends and the middle planar end and to the definition of the function γ. It is possible to show that $\|\mathcal{E}_\epsilon\left(\gamma P(w_\Phi)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\epsilon})} \leq \epsilon^2$.

As for the last term, we recall that the operator Q_ϵ has two different expressions if we consider the catenoidal type end and the middle planar end (see equation (3.37) and (3.32)). In particular we assume that at the middle planar end, Q_ϵ keeps track also of the operator $\bar{L}_u$, for $u = u_m$ (see (3.30)) that appears in the expression of the mean curvature operator given by (3.34). It holds that $\|\bar{L}_{u_m}w_\Phi\|_{\mathcal{C}^{2,\alpha}_\delta([\rho_\epsilon,\rho_0/2]\times S^1)} \leqslant c\epsilon\|w_\Phi\circ X_{m,\epsilon}\|_{\mathcal{C}^{2,\alpha}_\delta([\rho_\epsilon,\rho_0/2]\times S^1)} \leqslant c_k\epsilon^2$. We find the following estimate

$$\|\mathcal{E}_\epsilon\left(\gamma Q_\epsilon\left(w_\Phi\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\epsilon})} \leqslant c_k\epsilon^2.$$

It is convenient to recall that the expression of Q_ϵ at the catenoidal type ends is given by (3.40). So we obtain

$$\|\mathcal{E}_\epsilon\left(\gamma Q_\epsilon\left(w_\Phi\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\epsilon})} \leqslant c\epsilon\|Q_{2,\epsilon}\left(\frac{w_\Phi}{2\epsilon\cosh s},\frac{\nabla w_\Phi}{2\epsilon\cosh s},\frac{\nabla^2 w_\Phi}{2\epsilon\cosh s}\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\epsilon})}$$

$$\leqslant c\epsilon\|\frac{w_\Phi}{2\epsilon\cosh s}\circ X_{t,\epsilon}\|^2_{\mathcal{C}^{2,\alpha}_\delta([s_0+1,s_\epsilon]\times S^1)}+c\epsilon\|\frac{w_\Phi}{2\epsilon\cosh s}\circ X_{b,\epsilon}\|^2_{\mathcal{C}^{2,\alpha}_\delta([-s_\epsilon,-s_0-1]\times S^1)}+$$

$$+c\epsilon^{1-\alpha}\|w_\Phi\circ X_{m,\epsilon}\|^2_{\mathcal{C}^{2,\alpha}([\rho_\epsilon,\rho_0/2]\times S^1)}+\|\bar{L}_{u_m}w_\Phi\|_{\mathcal{C}^{2,\alpha}([\rho_\epsilon,\rho_0/2]\times S^1)} \leqslant c_k\epsilon^2$$

As for the second estimate, we recall that

$$T(\Phi,v):=G_{\epsilon,\delta}\left(\mathcal{E}_\epsilon\left(\gamma\left(\tilde{L}_\epsilon(w_\Phi+v)+P(w_\Phi+v)-\mathbb{L}_{M^T_{k,\epsilon}}w_\Phi+Q_\epsilon\left(w_\Phi+v\right)\right)\right)\right).$$

Then

$$T(\Phi,v_2)-T(\Phi,v_1)=G_{\epsilon,\delta}\left(\mathcal{E}_\epsilon\left(\gamma\left(\tilde{L}_\epsilon(w_\Phi+v_2)+P(w_\Phi+v_2)-\mathbb{L}_{M^T_{k,\epsilon}}w_\Phi+Q_\epsilon\left(w_\Phi+v_2\right)\right)\right)\right)-$$

$$G_{\epsilon,\delta}\left(\mathcal{E}_\epsilon\left(\gamma\left(\tilde{L}_\epsilon(w_\Phi+v_1)+P(w_\Phi+v_1)-\mathbb{L}_{M^T_{k,\epsilon}}w_\Phi+Q_\epsilon\left(w_\Phi+v_1\right)\right)\right)\right)$$

and

$$\|T(\Phi,v_2)-T(\Phi,v_1)\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\epsilon})}\leqslant c\|\mathcal{E}_\epsilon\left(\gamma\left(\tilde{L}_\epsilon(w_\Phi+v_2)+P(w_\Phi+v_2)-\tilde{L}_\epsilon(w_\Phi+v_1)\right.\right.$$

$$\left.\left.-P(w_\Phi+v_1)+Q_\epsilon\left(w_\Phi+v_2\right)-Q_\epsilon\left(w_\Phi+v_1\right)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\epsilon})}\leqslant$$

$$\leqslant\|\mathcal{E}_\epsilon\left(\gamma\left(\tilde{L}(v_2-v_1)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\epsilon})}+\|\mathcal{E}_\epsilon\left(\gamma\left(P(w_\Phi+v_2)-P(w_\Phi+v_1)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\epsilon})}+$$

$$+\|\mathcal{E}_\epsilon\left(\gamma\left(Q_\epsilon\left(w_\Phi+v_1\right)-Q_\epsilon\left(w_\Phi+v_2\right)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\epsilon})}.$$

We observe that from the considerations above it follows that

$$\|\mathcal{E}_\epsilon\left(\gamma\left(\tilde{L}_\epsilon(v_2-v_1)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\epsilon})}\leqslant c\epsilon^2||v_2-v_1||_{\mathcal{C}^{2,\alpha}_\delta(M^T_{k,\epsilon})},$$

$$\|\mathcal{E}_\epsilon\left(\gamma\left(P(w_\Phi+v_2)-P(w_\Phi-v_1)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\epsilon})}\leqslant c\epsilon^2||v_2-v_1||_{\mathcal{C}^{2,\alpha}_\delta(M^T_{k,\epsilon})}$$

and

$$\|\mathcal{E}_\epsilon\left(\gamma\left(Q_\epsilon\left(w_\Phi+v_1\right)-Q_\epsilon\left(w_\Phi+v_2\right)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\epsilon})}$$

$$\leqslant c\epsilon||v_2-v_1||_{\mathcal{C}^{2,\alpha}_\delta(M^T_{k,\epsilon})}\|\frac{w_\Phi}{2\epsilon\cosh s}\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\epsilon})}+\|\bar{L}_{u_m}(v_2-v_1)\|_{\mathcal{C}^{2,\alpha}([\rho_\epsilon,\rho_0/2]\times S^1)}\leqslant$$

$$\leqslant\left(c_k\epsilon^{\frac{3}{2}+\frac{\delta}{2}}+c\epsilon\right)||v_2-v_1||_{\mathcal{C}^{2,\alpha}_\delta(M^T_{k,\epsilon})}.$$

Then

$$\|T(\Phi,v_2)-T(\Phi,v_1)\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\epsilon})}\leqslant c\epsilon||v_2-v_1||_{\mathcal{C}^{2,\alpha}_\delta(M^T_{k,\epsilon})}.$$

□

This argument provides a minimal surface $\bar{M}^T_{k,\epsilon}(\Phi)$ which is close to $\bar{M}^T_{k,\epsilon}$ and has three boundaries. This surface is, close to its upper and lower boundary, a vertical graph over the annulus $B_{r_\epsilon}-B_{r_\epsilon/2}$, whose parametrization is, respectively, given by

$$\bar{U}_t(r,\theta)=\sigma_{t,\epsilon}+2\epsilon\ln(2r)+H_{\varphi_t}(s_\epsilon+\ln 2r,\theta)+V_t(r,\theta),$$

$$\bar{U}_b(r,\theta) = -\sigma_{b,\epsilon} - 2\epsilon\ln(2r) - H_{\varphi_b}(\ln 2r + s_\epsilon, \theta) + V_b(r,\theta),$$

where $s_\epsilon = -\frac{1}{2}\ln\epsilon$. Nearby the middle boundary the surface is a vertical graph whose parametrization is

$$\bar{U}_m(r,\theta) = \tilde{H}_{\rho_\epsilon,\varphi_m}\left(\frac{2\epsilon}{r},\theta\right) + V_m(r,\theta).$$

The boundaries of the surface correspond to $r = r_\epsilon = \epsilon^{1/2}/2$. All the functions V_i, $i = t, b, m$, depend non linearly on ϵ, φ. The functions $V_i(\epsilon, \varphi_i)$, for $i = t, b$, satisfy $\|V_i(\epsilon,\varphi_i)(r_\epsilon\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1 - B_{1/2})} \leqslant c\epsilon$ and

$$\|V_i(\epsilon,\varphi)(r_\epsilon\cdot) - V_i(\epsilon,\varphi')(r_\epsilon\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1 - B_{1/2})} \leqslant c\epsilon^{1-\frac{\delta}{2}}\|\varphi - \varphi'\|_{\mathcal{C}^{2,\alpha}}.$$

The function $V_m(\epsilon,\varphi_m)$ satisfies $\|V_m(\epsilon,\varphi_m)(r_\epsilon\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1 - B_{1/2})} \leqslant c\epsilon$ and

$$\|V_m(\epsilon,\varphi)(r_\epsilon\cdot) - V_m(\epsilon,\varphi')(r_\epsilon\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1 - B_{1/2})} \leqslant c\epsilon\|\varphi - \varphi'\|_{\mathcal{C}^{2,\alpha}}.$$

3.10 The matching of Cauchy data

In this section we shall complete the proof of Theorem 27.

We need introduce additional notation. Given an even function $f \in \mathcal{C}^{2,\alpha}(S^1)$ with the following Fourier expansion

$$f(\theta) = \sum_{n\in\mathbb{N}} a_n e_i(n\theta) = \sum_{n\in\mathbb{N}} a_n \cos(n\theta),$$

then we denote with $\pi''(f)$ the function

$$\sum_{n\geqslant 2} a_n \cos(n\theta)$$

and with $\pi'(f)$ the function

$$a_0 + a_1\cos(\theta).$$

In section 3.6 we have obtained the minimal surfaces S_t, S_b which are graphs on $D = B_1 \setminus B_{r_\epsilon} \subset \mathbb{H}^2$ and are asymptotic, respectively, to the parts of catenoid denoted C_t and C_b introduced in section 3.5. In section 3.4 we have obtained a surface S_m which is a minimal graph on D and is asymptotic to $C_m := \mathbb{H}^2 \times \{0\}$.

In other words we are able to solve, for $\Phi = (\varphi_t, \varphi_b, \varphi_m) \in [\pi'' C^{2,\alpha}(S^1)]^3$ and $\|\varphi_i\|_{\mathcal{C}^{2,\alpha}} \leqslant \epsilon$ a system of minimal surface equations, that we write for short as

$$\begin{cases} L_c u_S &= Q_c(u_S) & \text{on} \quad C_t, C_b \\ L_p u_S &= Q_p(u_S) & \text{on} \quad C_m \\ \pi'' u_S &= (\varphi_t - \pi'' v_t, \varphi_b - \pi'' v_b, \varphi_m) & \text{on} \quad (\partial C_t, \partial C_b, \partial C_m) \end{cases}$$

Here the index p and c mean that the corresponding minimal surface equation is relative, respectively, to a plane or a catenoid. So it is possible to construct three minimal graphs about the plane and two part of a catenoid in $\mathbb{H}^2 \times \mathbb{R}$. The functions $v_t, v_b, v_{t,d}, v_{b,d}$ are introduced in (3.16) and in following equations.

The parametrizations of the surfaces S_t, S_b that are denoted by $X'_{i,\Phi}$ with $i = t, b$, satisfy in a neighbourhood of their boundaries

$$X'_{i,\Phi} = 2\epsilon \ln\left(\tfrac{2r}{\epsilon}\right) + \varphi_i + \pi'(u_S + v_i)$$

$$\partial_r X'_{i,\Phi} = \tfrac{2\epsilon}{r} + v_{i,d} + \partial_r u_S$$

and a surface S_m, whose parametrization $X'_{m,\Phi}$, in a neighbourhood of its boundary, satisfies

$$X'_{m,\Phi} = \varphi'_m + \pi'(u_S)$$

$$\partial_r X'_{m,\Phi} = \partial_r u_S$$

Now we can define

$$\begin{aligned} E_\epsilon : [\pi'' C^{2,\alpha}(S^1)]^3 &\longrightarrow C^{2,\alpha}(S^1)^3 \times C^{1,\alpha}(S^1)^3 \\ \Phi &\longrightarrow [(X'_{t,\Phi}, X'_{b,\Phi}, X'_{m,\Phi}), (\partial_r X'_{t,\Phi}, \partial_r X'_{b,\Phi}, \partial_r X'_{m,\Phi})]_{|r_\epsilon}. \end{aligned}$$

Finally, in section 3.9 we have obtained the surface $\bar{M}^T_{k,\epsilon}(\Phi)$, whose boundary consists in three curves on ∂B_{r_ϵ}. In fact we solved, for $\Phi = (\varphi_t, \varphi_b, \varphi_m) \in [\pi'' C^{2,\alpha}(S^1)]^3$ and $\|\varphi_i\|_{\mathcal{C}^{2,\alpha}} \leqslant \epsilon$ the problem

$$\begin{cases} H_{hyp}(\Sigma_{u_C}) &= 0 & \text{on} \quad \bar{M}_{k,\epsilon} \\ \pi'' u_C &= (\varphi_t - \pi'' f_t, \varphi_b - \pi'' f_b, \varphi_m - \pi'' f_m) & \text{on} \quad \partial \bar{M}_{k,\epsilon}. \end{cases}$$

The functions $f_t, f_b, f_m, f_{t,d}, f_{b,d}, f_{m,d}$ are introduced in (3.42),(3.43) and (3.44).
The parametrization of the surface Σ_{u_C} near the catenoidal type ends are denoted by $X_{i,\Phi}$ with $i = t, b$. In a neighbourhood of the boundaries of Σ_{u_C}, they satisfy

$$X_{i,\Phi} = 2\epsilon \ln \tfrac{2r}{\epsilon} + \varphi_i + \pi'(u_C + f_i)$$

$$\partial_r X_{i,\Phi} = \tfrac{2\epsilon}{r} + f_{i,d} + \partial_r u_C$$

The parametrization of the middle planar end, $X_{m,\Phi}$, in a neighbourhood of its boundary, satisfies

$$X_{m,\Phi} = \varphi_m + \pi'(u_C + f_m)$$

$$\partial_r X_{m,\Phi} = f_{m,d} + \partial_r u_C.$$

We define

$$\begin{array}{rcl} F_\epsilon : [\pi'' C^{2,\alpha}(S^1)]^3 & \longrightarrow & C^{2,\alpha}(S^1)^3 \times C^{1,\alpha}(S^1)^3 \\ \Phi & \longrightarrow & [(X_{t,\Phi}, X_{b,\Phi}, X_{m,\Phi}), (\partial_r X_{t,\Phi}, \partial_r X_{b,\Phi}, \partial_r X_{m,\Phi})]_{|r_\epsilon}. \end{array}$$

We set $C_\epsilon := E_\epsilon - F_\epsilon$.
We want to prove that the surfaces S_t, S_b, S_m and $\bar{M}^T_{k,\epsilon}$ can be glued along their boundaries to obtain a C^∞-surface. Firstly we will show that these surface correspond in a C^1 way along the boundaries curves. This is true if it exists $\Psi = (\psi_1, \psi_2, \psi_3)$ such that $C_\epsilon(\Psi) = 0$. The existence of the appropriates boundary functions is proven in the following theorem. Finally, to show that the surface is C^∞, it is sufficient to apply the regularity theory. That completes the proof of Theorem 27.

Theorem 45. *There exists ϵ_0 such that, for each $0 < \epsilon < \epsilon_0$, there exists $\Psi = (\psi_1, \psi_2, \psi_3) \in [\pi'' C^{2,\alpha}(S^1)]^3$ which solves $C_\epsilon(\Psi) = 0$.*

Proof. We consider the harmonic extensions of ψ_i, $i = 1, 2, 3$, on the ends of $\bar{M}_{k,\epsilon}$, that is

1. $\bar{w}_t = \chi_+ \, H_{\psi_1}(s_\epsilon - s, \cdot)$, on the upper end
2. $\bar{w}_b = \chi_- \, H_{\psi_2}(s + s_\epsilon, \cdot)$, on the lower end
3. $\bar{w}_m = \chi_m \, \tilde{H}_{R_\epsilon, \psi_3}(\cdot, \cdot)$ on the middle end (the definition of the map f is given by (3.23)),

and its harmonic extensions

1. $w_t = \mathcal{H}_{r_\epsilon, \psi_1}$ on S_t
2. $w_b = -\mathcal{H}_{r_\epsilon, \psi_2}$ on S_b
3. $w_m = \mathcal{H}_{r_\epsilon, \psi_3}$ on S_m

(see section 3.9 for the definitions of the cut-off functions). We recall that the operators $\mathcal{H}$, H and $\tilde{H}$ have been introduced respectively in proposition 46, 47 and 48. We consider the following maps

$$\begin{array}{rcl} E_0 : [\pi'' C^{2,\alpha}(S^1)]^3 & \longrightarrow & C^{2,\alpha}(S^1)^3 \times C^{1,\alpha}(S^1)^3 \\ \Psi & \longrightarrow & [(w_t, w_b, w_m), (\partial_r w_t, \partial_r w_b, \partial_r w_m)]_{|r_\epsilon} \end{array}$$

and

$$\begin{array}{rcl} F_0 : [\pi'' C^{2,\alpha}(S^1)]^3 & \longrightarrow & C^{2,\alpha}(S^1)^3 \times C^{1,\alpha}(S^1)^3 \\ \Psi & \longrightarrow & [(\bar{w}_t, \bar{w}_b, \bar{w}_m), (\partial_r \bar{w}_t, \partial_r \bar{w}_b, \partial_r \bar{w}_m)]_{|r_\epsilon}. \end{array}$$

Now using Fourier expansion of the function, we can see that $C_0 = E_0 - F_0$ has an inverse which is bounded independently of ϵ. In particular, the equation $C_0(\Psi) = 0$ has the unique solution $\Psi = (0,0,0)$. Now we consider $(C_\epsilon - C_0)(\Psi)$, whose expression is

$$(\pi'(u_S - u_C) + \pi'(v_t - f_t), \pi'(u_S - u_C) + \pi'(v_b - f_b), \pi'(u_S - u_C) + \pi'(-f_m),$$

$$\partial_r(u_S - w_t) - \partial_r(u_C - \bar{w}_t) + v_{t,d} - f_{t,d},$$

$$\partial_r(u_S - w_b) - \partial_r(u_C - \bar{w}_b) + v_{b,d} - f_{b,d},$$

$$\partial_r(u_S - w_m) - \partial_r(u_C - \bar{w}_m) - f_{m,d}).$$

It is easy to prove that

$$||(C_\epsilon - C_0)(\Psi)||_{C^{2,\alpha}(S^1)^3 \times C^{1,\alpha}(S^1)^3} \leqslant \epsilon.$$

In order to solve $C_\epsilon(\Psi) = 0$, we find a fixed point for the mapping

$$D_\epsilon(\Psi) := C_0^{-1}\left((C_\epsilon - C_0)(\Psi)\right).$$

□

3.11 Appendix

The results contained in this section are about the existence of some harmonic extension operators. The first one gives the harmonic extension of a function on $\mathbb{H}^2 \setminus D_{r_0}$.

Proposition 46. *If $\mu \in (0,1)$ there exists an operator*

$$\mathcal{H}_{r_0} : C^{2,\alpha}(S^1) \longrightarrow C_1^{2,\alpha}(S^1 \times [r_0, 1)),$$

such that for every function $\varphi(\theta) \in C^{2,\alpha}(S^1)$, which is L^2-orthogonal to e_0, e_1, the function $w_\varphi = \mathcal{H}_{r_0,\varphi}$ solves

$$\begin{cases} \Delta_{eucl} w_\varphi = 0 & \text{on} \quad S^1 \times [r_0, 1) \\ w_\varphi = \varphi & \text{on} \quad S^1 \times \{r_0\}. \end{cases}$$

Moreover,

$$||\mathcal{H}_{r_0,\varphi}||_{C_1^{2,\alpha}(S^1 \times [r_0,1))} \leqslant c\,||\varphi||_{C^{2,\alpha}(S^1)}, \tag{3.47}$$

for some constant $c > 0$.

Proof. We observe that

$$\Delta_{eucl}(1-r^2)^\mu = \left(-4\mu(1-r^2)+4\mu(\mu-1)r^2\right)\left(1-r^2\right)^{\mu-2} =$$

$$= \left(-4\mu+4\mu^2r^2\right)\left(1-r^2\right)^{\mu-2} \leqslant 0$$

if $\mu \in (0,1]$. So $(1-r^2)^\mu$ it's a superharmonic function. Then the function $||\varphi||_{C^{2,\alpha}(S^1)}(1-r^2)$ can be used as barrier function and we can apply the Perron method. We can conclude that the solution, that we denote with w_φ, exists and satisfies

$$|w_\varphi| \leqslant c||\varphi||_{C^{2,\alpha}(S^1)}(1-r^2).$$

Using the initial assumption we can write

$$(1-r^2)^{-1}|w_\varphi| \leqslant c|\varphi|.$$

The estimates for the derivatives of w_φ are obtained by Schauder estimates. We can conclude that

$$||w_\varphi||_{C^{2,\alpha}_1} \leqslant c||\varphi||_{C^{2,\alpha}(S^1)}.$$

□

Now we give the statement of a result whose proof is contained in [7]. It gives the harmonic extension of a function on a half catenoid.

Proposition 47. *There exists an operator*

$$H : \mathcal{C}^{2,\alpha}(S^1) \longrightarrow \mathcal{C}^{2,\alpha}_{-2}([0,+\infty)\times S^1),$$

such that for all $\varphi \in \mathcal{C}^{2,\alpha}(S^1)$, even function and orthogonal to e_i, $i=0,1$ in the L^2-sense, the function $w = H(\varphi)$ solves

$$\begin{cases} (\partial_s^2+\partial_\theta^2)\, w &= 0 \quad in \quad S^1\times[0,+\infty) \\ w &= \varphi \quad on \quad S^1\times\{0\} \end{cases}$$

Moreover

$$\|H(\varphi)\|_{\mathcal{C}^{2,\alpha}_{-2}([0,+\infty)\times S^1)} \leqslant c\,\|\varphi\|_{\mathcal{C}^{2,\alpha}(S^1)},$$

for some constant $c>0$.

The following result gives a harmonic extension of a function on $\mathbb{R}^2\setminus D_{\bar\rho}$.

Proposition 48. *There exists an operator*

$$\tilde H_{\bar\rho} : C^{2,\alpha}(S^1) \longrightarrow C^{2,\alpha}_{-2}(S^1\times[\bar\rho,+\infty)),$$

such that for each even function $\varphi(\theta) \in C^{2,\alpha}(S^1)$, which is L^2-orthogonal to the constant function and $\cos\theta$, then $w_\varphi = \tilde{H}_{\bar{\rho},\varphi}$ solves

$$\begin{cases} \Delta w_\varphi = 0 & \text{on} \quad S^1 \times [\bar{\rho}, +\infty) \\ w_\varphi = \varphi & \text{on} \quad S^1 \times \{\bar{\rho}\}. \end{cases}$$

Moreover,

$$||\tilde{H}_{\bar{\rho},\varphi}||_{C^{2,\alpha}_{-2}(S^1 \times [\bar{\rho}, +\infty))} \leqslant c\, ||\varphi||_{C^{2,\alpha}(S^1)}, \tag{3.48}$$

for some constant $c > 0$.

Proof. We consider the decomposition of the function φ with respect to the basis $\{\cos(i\theta)\}$, that is

$$\varphi = \sum_{i=2}^{\infty} \varphi_i \cos(i\theta).$$

Then the solution w_φ is given by

$$w_\varphi(\rho, \theta) = \sum_{i=2}^{\infty} \left(\frac{\bar{\rho}}{\rho}\right)^i \varphi_i \cos(i\theta).$$

Since $\frac{\bar{\rho}}{\rho} \leqslant 1$, then $\left(\frac{\bar{\rho}}{\rho}\right)^i \leqslant \left(\frac{\bar{\rho}}{\rho}\right)$, we can conclude that $|w(r,\theta)| \leqslant c\rho^{-2}|\varphi(\theta)|$ and then $||w_\varphi||_{C^{2,\alpha}_{-2}} \leqslant c||\varphi||_{C^{2,\alpha}}$. □

Chapter 4

Singly periodic minimal surfaces with arbitrary nonzero genus and infinitely many ends

4.1 Introduction

In 1988, H. Karcher [19, 20] defined a family of doubly periodic minimal surfaces, called *toroidal halfplane layers*, with genus one and four horizontal Scherk-type ends[1] in the quotient. In 1989, Meeks and Rosenberg [25] developed a general theory for doubly periodic minimal surfaces having finite topology in the quotient, and used an approach of minimax type to obtain the existence of a family of doubly periodic minimal surfaces, also with genus one and four horizontal Scherk-type ends in the quotient. These Karcher's and Meeks and Rosenberg's surfaces have been generalized in [35], constructing a 3-parameter family $\mathcal{K} = \{M_{\sigma,\alpha,\beta}\}_{\sigma,\alpha,\beta}$ of surfaces, called KMR examples (sometimes, they are also referred in the literature as toroidal halfplane layers). Such examples have been classified by Pérez, Rodríguez and Traizet [33] as the only doubly periodic minimal surfaces with genus one and finitely many parallel (Scherk-type) ends in the quotient. The possible limits of KMR examples are: the catenoid, the helicoid, any singly or doubly periodic Scherk minimal surface, any Riemann minimal example or another KMR example.

Each $M_{\sigma,\alpha,\beta}$ has an horizontal period T_1 (the period at the ends) and a non horizontal period T_2 coming from homology. We denote by $\widetilde{M}_{\sigma,\alpha,\beta}$ the lifting of $M_{\sigma,\alpha,\beta}$ to $\mathbb{S}^1 \times \mathbb{R}^2$ by forgetting the period T_2. The surface $\widetilde{M}_{\sigma,\alpha,\beta}$, that we go on calling KMR example, has genus zero, infinitely many parallel Scherk-type ends, and two limit ends. We consider in this work KMR examples near the catenoidal limit, so $\sigma \to 0$ or equivalently $T_1 \to \infty$

[1] A horizontal *Scherk-type* end is an end asymptotic to a horizontal half-plane, invariant by one of the period vectors of the surface.

and $T_2 \to \infty$, and with $\alpha = 0$ or $\beta = 0$.

In 1982, C. Costa [3, 4] discovered a genus one minimal surface with two catenoidal ends and one planar middle end, which is embedded outside a ball in $\mathbb{R}^3$. D. Hoffmann and W.H. Meeks [14] proved the global embeddedness for this Costa's example, and generalized it for bigger genus. For each $k \geq 1$, we will denote by M_k such Costa-Hoffmann-Meeks' surface, which is a properly embedded minimal surface of genus k and three ends: one middle end asymptotic to the plane $\{x_3 = 0\}$, one top catenoidal end and one bottom catenoidal end. L. Hauswirth and F. Pacard [11] have obtained a deformation of M_k, for $1 \leq k \leq 37$ (in these cases, M_k is nondegenerate [29],[30]), by bending the catenoidal ends. We denote such deformed examples by $M_k(\xi)$. It is known that their construction extends for $k \geq 38$ thanks to the result showed in chapter 1.

V. Ramos Batista [2] has constructed a singly periodic Costa minimal surface, with two catenoidal ends, one Scherk-type middle end and genus one in the quotient. We produce two new families of examples of periodic minimal surfaces of higher genus as follows. We consider a compact part of $M_k(\xi)$ contained in a vertical solid cylinder, $D \times \mathbb{R}$, of radius $1/(2\sqrt{\epsilon})$ and centered at the origin. We glue it to a minimal graph on $(\mathbb{R}^2 - D)/T_1$ asymptotic to $\{x_3 = 0\}/T_1$, to one half of $\widetilde{M}_{\sigma,\alpha,\beta}$, near the catenoidal limit, that is with σ small (one time with $\alpha = 0$ and a second time with $\beta = 0$) and with a Scherk type surface. We obtain two families of properly embedded minimal surfaces in $\mathbb{S}^1 \times \mathbb{R}^2$ with genus $k \geqslant 1$, infinitely many parallel Scherk-type and two limit ends.

4.2 A Costa-Hoffman-Meeks type surface with bent catenoidal ends

In this section we recall the result shown in [11] about the existence of a family of minimal surfaces close to the Costa-Hoffman-Meeks surfaces of genus $k \geqslant 1$, one planar end and two slightly bent catenoidal ends by an angle ξ. We denote one member of the family by $M_k(\xi)$. Then $M_k(0)$ is the family of the Costa-Hoffman-Meeks surface of genus k (see Costa [3], [4] and D. Hoffman and W. H. Meeks [14], [15], [16]).

The family of the Costa-Hoffman-Meeks surfaces. Each member of the family of surfaces $M_k(0)$, after suitable rotation and translation, enjoys the following properties.

1. It has one planar end E_m asymptotic to the $x_3 = 0$ plane, one top end E_t and one bottom end E_b that are respectively asymptotic to the upper end and to the lower end of a catenoid with x_3-axis of revolution. The planar end E_m is located between the two catenoidal ends.

2. It is invariant under the action of the rotation of angle $\frac{2\pi}{k+1}$ about the x_3-axis, under the action of the symmetry with respect to the $x_2 = 0$ plane and under the action

of the composition of a rotation of angle $\frac{\pi}{k+1}$ about the x_3-axis and the symmetry with respect to the $x_3 = 0$ plane.

3. It intersects the $x_3 = 0$ plane in $k + 1$ straight lines, which intersect themselves at the origin with angles equal to $\frac{\pi}{k+1}$. The intersection of M_k with the plane $x_3 =$ *const* $(\neq 0)$ is a single Jordan curve. The intersection of M_k with the upper half space $x_3 > 0$ (resp. with the lower half space $x_3 < 0$) is topologically an open annulus.

Now we give a local description of the surfaces $M_k(0)$ near its ends and we introduce coordinates that we will use.

The planar end. The planar end E_m of the surface M_k can be parametrized by

$$X_m(x) := \left(\frac{x}{|x|^2}, u_m(x) \right) \in \mathbb{R}^3$$

where $x \in \bar{B}_{\rho_0}(0) - \{0\} \subset \mathbb{R}^2$ and the function u_m tends to 0 like $u_m(x) = \mathcal{O}(|x|^{k+1})$. Here $\rho_0 > 0$ is fixed small enough. The minimal surface equation has the following form

$$|x|^4 \operatorname{div} \left(\frac{\nabla u}{(1 + |x|^4 |\nabla u|^2)^{1/2}} \right) = 0. \tag{4.1}$$

It can be shown (see [11]) that the function u_m can be extended at the origin continuously using Weierstrass representation. We can prove that $u_m \in \mathcal{C}^{2,\alpha}(\bar{B}_{\rho_0})$.

The catenoidal ends. We denote by X_c the parametrization of the standard catenoid C whose axis of revolution is the x_3-axis. Its expression is

$$X_c(s, \theta) := (\cosh s \, \cos\theta, \cosh s \, \sin\theta, s) \in \mathbb{R}^3$$

where $(s, \theta) \in \mathbb{R} \times S^1$. The unit normal vector field about C is given by

$$n_c(s, \theta) := \frac{1}{\cosh s} (\cos\theta, \sin\theta, -\sinh s).$$

Up to some dilation, we can assume that the two ends E_t and E_b of M_k are asymptotic to some translated copy of the catenoid parametrized by X_c in the vertical direction. Therefore, E_t and E_b can be parametrized, respectively, by

$$X_t := X_c + w_t \, n_c + \sigma_t \, e_3$$

for $(s, \theta) \in (s_0, \infty) \times S^1$,

$$X_b := X_c - w_b \, n_c - \sigma_b \, e_3$$

for $(s,\theta) \in (-\infty, -s_0,) \times S^1$, where $\sigma_t, \sigma_b \in \mathbb{R}$, functions w_t, w_b tend exponentially fast to 0 as s goes to ∞ reflecting the fact that the ends are asymptotic to a catenoidal end.

We recall that the surface parametrized by by $X := X_c + w\, n_c$ is minimal if and only if the function w satisfies the minimal surface equation which, for normal graphs over a catenoid has the following form

$$\frac{1}{\cosh^2 s}\left(\frac{\partial^2 w}{\partial s^2} + \frac{\partial^2 w}{\partial \theta^2} + \frac{2w}{\cosh^2 s} + Q_2\left(\frac{w}{\cosh s}\right) + \cosh s\, Q_3\left(\frac{w}{\cosh s}\right)\right) = 0, \tag{4.2}$$

where Q_2, Q_3 are linear second order differential operators which are bounded in $\mathcal{C}^k(\mathbb{R}\times S^1)$ for all k. These functions satisfy $Q_2(0) = Q_3(0) = 0$, $\nabla Q_2(0) = \nabla Q_3(0) = 0$,$\nabla^2 Q_3(0) = 0$ and then:

$$\|Q_j(v_2) - Q_j(v_1)\|_{\mathcal{C}^{0,\alpha}([s,s+1]\times S^1)} \leqslant c \left(\sup_{i=1,2} \|v_i\|_{\mathcal{C}^{2,\alpha}([s,s+1]\times S^1)}\right)^{j-1} \|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}([s,s+1]\times S^1)} \tag{4.3}$$

for all $s \in \mathbb{R}$ and all v_1, v_2 such that $\|v_i\|_{\mathcal{C}^{2,\alpha}((s,s+1)\times S^1)} \leqslant 1$. The constant $c > 0$ does not depend on s.

The family of Costa-Hoffman-Meeks surfaces with bent catenoidal ends. Using an elaborate version of the implicit function theorem and following [18] and [23] it is possible to prove the following

Theorem 49 ([11])**.** *There exists $\xi_0 > 0$ and a smooth one parameter family of minimal hypersurfaces $(M_k(\xi))_\xi$, for $\xi \in (-\xi_0, \xi_0)$, with two catenoidal ends and one planar end. In particular $M_k(0) = M_k$, the upper (resp. lower) catenoidal end of $M_k(\xi)$ is, up to a translation along its axis, asymptotic to the upper (resp. lower) end of the standard catenoid whose axis of revolution is directed by $\sin\xi\, e_1 + \cos\xi\, e_3$. Moreover $M_k(\xi)$ has one horizontal planar end and is invariant under the action of the symmetry with respect to the $x_2 = 0$ plane.*

The upper (lower) end of $M_k(\xi)$ is, up to a translation, asymptotic to the upper (lower) end of the same (standard) catenoid. Then the upper end $E_t(\xi)$ and the lower end $E_b(\xi)$ of $M_k(\xi)$, if R_ξ denotes the rotation of angle ξ about the x_2 axis, can be parametrized respectively by

$$X_{t,\xi} = R_\xi\left(X_c + w_{t,\xi}\, n_c\right) + \sigma_{t,\xi}\, e_3 + \varsigma_{t,\xi}\, e_1 \tag{4.4}$$

$$X_{b,\xi} = R_\xi\left(X_c - w_{b,\xi}\, n_c\right) - \sigma_{b,\xi}\, e_3 - \varsigma_{b,\xi}\, e_1 \tag{4.5}$$

where the functions $w_{t,\xi}, w_{b,\xi}$, the numbers $\sigma_{t,\xi}, \varsigma_{t,\xi}, \sigma_{b,\xi}, \varsigma_{b,\xi} \in \mathbb{R}$ depend smoothly on ξ and satisfy $w_{t,0} = w_t$, $w_{b,0} = w_b$, $\sigma_{b,0} = \sigma_b$ $\sigma_{t,0} = \sigma_t$, $\varsigma_{t,0} = 0$ and $\varsigma_{b,0} = 0$.

For all $r < \rho_0$ and $s > s_0$, we define

$$M_k(\xi, s, r) := M_k(\xi) - \left[X_{t,\xi}((s,\infty)\times S^1) \cup X_{b,\xi}((-\infty,-s)\times S^1) \cup X_m(B_\rho(0))\right]. \quad (4.6)$$

The parametrizations of the three ends of $M_k(\xi)$ induce a decomposition of $M_k(\xi)$ into slightly overlapping components: a compact piece $M_k(\xi, s_0+1, \rho_0/2)$ and three noncompact pieces $X_{t,\xi}((s_0,\infty)\times S^1)$, $X_{b,\xi}((-\infty,-s_0)\times S^1)$ and $X_m(\bar{B}_{\rho_0}(0))$.

We define the weighted space of functions on $M_k(\xi)$.

Definition 50. *Given $\ell \in \mathbb{N}$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$, the space $\mathcal{C}^{\ell,\alpha}_\delta(M_k(\xi))$ is defined to be the space of functions in $\mathcal{C}^{\ell,\alpha}_{loc}(M_k)$ for which the following norm is finite*

$$\|w\|_{\mathcal{C}^{\ell,\alpha}_\delta(M_k)} := \|w\|_{\mathcal{C}^{\ell,\alpha}(M_k(\xi,s_0+1,\rho_0/2))} + \|\, w \circ X_m\|_{\mathcal{C}^{\ell,\alpha}(B_{\rho_0}(0))}$$

$$+ \sup_{s \geqslant s_0} e^{-\delta s} \left(\|w \circ X_{t,\xi}\|_{\mathcal{C}^{\ell,\alpha}((s,s+1)\times S^1)} + \|w \circ X_{b,\xi}\|_{\mathcal{C}^{\ell,\alpha}((-s-1,-s)\times S^1)}\right)$$

and which are invariant under the action of the symmetry with respect to the $x_2 = 0$ plane.

We remark that there is no weight on the middle end. In fact we compactify this end and we consider a weighted space of functions defined on a two ended surface. We will perturb the surface $M_k(\xi)$ by the normal graph of a function $u \in \mathcal{C}^{2,\alpha}_\delta$ and the middle end E_m will be just translated in the vertical direction.

The Jacobi operator. The Jacobi operator about $M_k(\xi)$ is

$$\mathbb{L}_{M_k(\xi)} := \Delta_{M_k(\xi)} + |A_{M_k(\xi)}|^2$$

where $|A_{M_k(\xi)}|$ is the norm of the second fundamental form on $M_k(\xi)$.

In the parametrization introduced above of the ends the volume forms $\text{dvol}_{M_k(\xi)}$ can be written as $\gamma_t\, ds\, d\theta$ and $\gamma_b\, ds\, d\theta$ near the catenoidal type ends and as $\gamma_m\, dx_1\, dx_2$ near the middle end. Now we can define globally on $M_k(\xi)$ a smooth function

$$\gamma : M_k(\xi) \longrightarrow [0,\infty)$$

that is identically equal to 1 on $M_k(\xi, s_0-1, 2\rho_0)$ and equal to γ_t (resp. γ_b, γ_m) on the end $E_{t,\xi}$ (resp. $E_{b,\xi}$, E_m). Observe that, on $X_{t,\xi}((s_0,\infty)\times S^1)$ and on $X_{b,\xi}((-\infty,s_0)\times S^1)$ we have

$$\gamma \circ X_{t,\xi}(s,\theta) \sim \cosh^2 s \qquad \text{and} \qquad \gamma \circ X_{b,\xi}(s,\theta) \sim \cosh^2 s.$$

Finally on $X_m(B_{\rho_0})$, we have

$$\gamma \circ X_m(x) \sim |x|^{-4}.$$

Granted the above defined spaces, one can check that:

$$\begin{array}{rccc} \mathcal{L}_{\xi,\delta}: & \mathcal{C}^{2,\alpha}_{\delta}(M_k(\xi)) & \longrightarrow & \mathcal{C}^{0,\alpha}_{\delta}(M_k(\xi)) \\ & w & \longmapsto & \gamma\,\mathbb{L}_{M_k(\xi)}\,(w) \end{array}$$

is a bounded linear operator. The subscript δ is meant to keep track of the weighted space over which the Jacobi operator is acting. Observe that, the function γ is here to counterbalance the effect of the conformal factor $\frac{1}{\sqrt{|g_{M_k(\xi)}|}}$ in the expression of the Laplacian in the coordinates we use to parametrize the ends of the surface $M_k(\xi)$. This is precisely what is needed to have the operator defined from the space $\mathcal{C}^{2,\alpha}_{\delta}(M_k(\xi))$ into the target space $\mathcal{C}^{0,\alpha}_{\delta}(M_k(\xi))$.

To have a better grasp of what is going on, let us linearize the nonlinear equation (4.2) at $w = 0$ we get the expression of the Jacobi operator about the standard catenoid

$$\mathbb{L}_C := \frac{1}{\cosh^2 s}\left(\partial_s^2 + \partial_\theta^2 + \frac{2}{\cosh^2 s}\right).$$

We can observe that the operator $\cosh^2 s\,\mathbb{L}_C$ maps the space $(\cosh s)^\delta\,\mathcal{C}^{2,\alpha}((s_0,\infty)\times S^1)$ into the space $(\cosh s)^\delta\,\mathcal{C}^{0,\alpha}((s_0,\infty)\times S^1)$.

Similarly, if we linearize the nonlinear equation (4.1) at $u = 0$, we obtain the expression of the Jacobi operator about the plane

$$\mathbb{L}_{\mathbb{R}^2} := |x|^4\,\Delta.$$

Again, the operator $|x|^{-4}\,\mathbb{L}_{\mathbb{R}^2} = \Delta$ clearly maps the space $\mathcal{C}^{2,\alpha}(\bar{B}_{\rho_0})$ into the space $\mathcal{C}^{0,\alpha}(\bar{B}_{\rho_0})$. Now, the function γ plays, for the ends of the surface $M_k(\xi)$, the role played by the function $\cosh^2 s$ for the ends of the standard catenoid and the role played by the function $|x|^{-4}$ for the plane. Since the Jacobi operator about $M_k(\xi)$ is asymptotic to $\mathbb{L}_{\mathbb{R}^2}$ at E_m and is asymptotic to $\mathbb{L}_C$ at E_t and E_b, we conclude that the operator $\mathcal{L}_{\xi,\delta}$ maps $\mathcal{C}^{2,\alpha}_{\delta}(M_k(\xi))$ into $\mathcal{C}^{0,\alpha}_{\delta}(M_k(\xi))$.

We recall the notion of non degeneracy introduced in [11]:

Definition 51. *The surface $M_k(\xi)$ is said to be non degenerate if $\mathcal{L}_{\xi,\delta}$ is injective for all* $\delta < -1$.

It useful to observe that a duality argument in the weighted Lebesgue spaces, implies that

$$(\mathcal{L}_{\xi,\delta} \quad \text{is injective}) \quad \Leftrightarrow \quad (\mathcal{L}_{\xi,-\delta} \quad \text{is surjective})$$

if $\delta \notin \mathbb{Z}$. See [27] and [18] for more details.

The non degeneracy of $M_k(\xi)$ follows from the study of the kernel of $\mathcal{L}_{\xi,\delta}$.

The Jacobi fields. It is known that a smooth one parameter group of isometries containing the identity generates a Jacobi field, that is a solution of the equation $\mathbb{L}_{M_k(\xi)} u = 0$. These solutions are generated by the following one parameter groups of isometries: the vertical translations, the translations along the x_1-axis, the dilations. We refer [11] for details.

The group of vertical translations generated by the Killing vector field $\Xi(p) = e_3$ gives rise to the Jacobi field

$$\Phi^{0,+}(p) := n(p) \cdot e_3.$$

The vector field $\Xi(p) = p$ that is associated to the one parameter group of dilation generates a Jacobi fields

$$\Phi^{0,-}(p) := n(p) \cdot p$$

The Killing vector field $\Xi(p) = e_1$ that generates the group of translations along the x_1-axis is associated to a Jacobi field

$$\Phi^{1,+}(p) := n(p) \cdot e_1$$

Finally, we denote by

$$\Phi^{1,-}(p) := n(p) \cdot (e_2 \times p)$$

the Jacobi field associated to the Killing vector field $\Xi(p) = e_2 \times p$ that generates the group of rotations about the x_2-axis.

The Jacobi equation has other solutions which are not taken into account because in the difference with the four Jacobi fields just introduced they are not invariant under the action of the symmetry with respect to the $x_2 = 0$ plane.

With these notations, we define the deficiency space

$$\mathcal{D} := \mathrm{Span}\{\chi_t\, \Phi^{j,\pm}, \chi_b\, \Phi^{j,\pm} \ : \ j = 0, 1\}$$

where χ_t is a cutoff function that is identically equal to 1 on $X_t((s_0+1, \infty) \times S^1)$, identically equal to 0 on $M_k - X_t((s_0, \infty) \times S^1)$ and that is invariant under the action of the symmetry with respect to the $x_2 = 0$ plane. Also, we agree that

$$\chi_b(\cdot) := \chi_t(-\,\cdot).$$

Clearly

$$\begin{array}{rccc} \tilde{\mathcal{L}}_{\xi,\delta}: & \mathcal{C}^{2,\alpha}_{\delta}(M_k(\xi)) \oplus \mathcal{D} & \longrightarrow & \mathcal{C}^{0,\alpha}_{\delta}(M_k) \\ & w & \longmapsto & \gamma\, \mathbb{L}_{M_k(\xi)}\,(w) \end{array}$$

is a bounded linear operator. The linear decomposition Lemma proved in [23] for constant mean curvature surfaces (see also [18] for minimal hypersurfaces) can be adapted to our situation and thanks to a result of S. Nayatani contained in [29],[30] and extended in chapter 1, which states that any bounded Jacobi field respecting the mirror symmetry with respect to the $x_2 = 0$ plane is linear combination of $\Phi^{0,\pm}$ and $\Phi^{1,+}$, we get the following result

Proposition 52. *Assume that all bounded Jacobi fields on M_k that are invariant with respect to the $x_2 = 0$ plane are linear combination of $\Phi^{0,\pm}$ and $\Phi^{1,+}$ and choose $\delta \in (1, 2)$. Then (reducing ξ_0 if this is necessary) the operator $\tilde{\mathcal{L}}_{\xi,\delta}$ is surjective and has a kernel of dimension 4. Moreover, there exists $G_{\xi,\delta}$ a right inverse for $\tilde{\mathcal{L}}_{\xi,\delta}$ that depends smoothly on ξ and in particular whose norm is bounded uniformly as $|\xi| < \xi_0$.*

4.3 An infinite dimensional family of minimal surfaces which are close to $M_k(\xi)$

In this section we consider a truncature of $M_k(\xi)$. First we recall a result of [11] that describes the region of the surface which can be parametrized by a graph on a $x_3 = 0$ plane.

Lemma 53 ([11])**.** *There exists $\epsilon_0 > 0$ such that, for all $\epsilon \in (0, \epsilon_0)$ and all $|\xi| \leqslant \epsilon$ an annular part of the ends $E_t(\xi)$, $E_b(\xi)$ and E_m of $M_k(\xi)$ can be written as vertical graphs over the horizontal plane for the functions*

$$\begin{aligned} U_t(r,\theta) &= \sigma_{t,\xi} + \ln(2r) + \xi\, r\, \cos\theta + \mathcal{O}(\epsilon), \\ U_b(r,\theta) &= -\sigma_{b,\xi} - \ln(2r) - \xi\, r\, \cos\theta + \mathcal{O}(\epsilon), \\ U_m(r,\theta) &= \mathcal{O}(r^{-(k+1)}). \end{aligned}$$

Here (r,θ) are the polar coordinates in the $x_3 = 0$ plane. The functions $\mathcal{O}(\epsilon)$ are defined in the annulus $B_{4\,\epsilon^{-1/2}} - B_{\epsilon^{-1/2}/4}$ and are bounded in $\mathcal{C}^{\infty}_b$ topology by a constant (independent on ϵ) multiplied by ϵ, where the partial derivatives are computed with respect to the vector fields $r\,\partial_r$ and ∂_θ.

Then $M_k(\epsilon/2)$ has two ends $E_t(\epsilon/2)$ and $E_b(\epsilon/2)$ which are graphs over the $x_3 = 0$ plane for functions U_t and U_b defined on the annulus $B_{4\,\epsilon^{-1/2}} - B_{\epsilon^{-1/2}/4}$. We set $s_\epsilon = -\frac{1}{2}\ln\epsilon$ and we define $M_k^T(\epsilon/2)$ to be equal to $M_k(\epsilon/2)$ from which we have removed the image of $(s_\epsilon, +\infty) \times S^1$ by $X_{t,\epsilon/2}$, the image of $(-\infty, -s_\epsilon) \times S^1$ by $X_{b,\epsilon/2}$ and the image of $B_{\rho_\epsilon}(0)$

by X_m with $\rho_\epsilon := 2\epsilon^{1/2}$. In this section we will prove the existence of a family of surfaces close to $M_k^T(\epsilon/2)$. We follow the work [11].

First, we modify the parametrization of the end $E_t(\epsilon/2)$, $E_b(\epsilon/2)$ and E_m, for appropriates values of s, so that, when $r \in [\epsilon^{-1/2}/4, 4\,\epsilon^{-1/2}]$ the curves corresponding to the image of

$$\theta \to (r\,\cos\theta, r\,\sin\theta, U_t(r,\theta)), \quad \theta \to (r\,\cos\theta, r\,\sin\theta, U_b(r,\theta))$$

correspond to the curve $s = \pm\log(2r)$.

The curve $\theta \to (r\,\cos\theta, r\,\sin\theta, U_m(r,\theta))$ corresponds to $\rho = \frac{1}{r}$.

The second step is the modification of unit normal vector field on $M_k(\epsilon/2)$ into a transverse unit vector field $\tilde{n}_{\epsilon/2}$ in such a way that it coincides with the normal vector field $n_{\epsilon/2}$ on $M_k(\epsilon/2)$, is equal to e_3 on the graph over $B_{2\epsilon^{-1/2}} - B_{3\epsilon^{-1/2}/8}$ of the functions U_t and U_b and interpolate smoothly between the different definitions of $\tilde{n}_{\epsilon/2}$ in different subsets of $M_k^T(\epsilon/2)$.
The graph of a function u, using the vector field $\tilde{n}_{\epsilon/2}$, will be a minimal surface if and only if u is a solution of a second order nonlinear elliptic equation of the form

$$\mathbb{L}_{M_k^T(\epsilon/2)}\,u = \tilde{L}_{\epsilon/2}\,u + Q_\epsilon\,(u)$$

where $\mathbb{L}_{M_k^T(\epsilon/2)}$ is the Jacobi operator about $M_k^T(\epsilon/2)$, Q_ϵ is a nonlinear second order differential operator and $\tilde{L}_{\epsilon/2}$ is a linear operator which takes into account the change of the parametrization and of the change of the normal vector field. It is possible to prove that this last operator has coefficients uniformly bounded by a constant times ϵ^2.

Now, we consider three functions $\varphi_t, \varphi_b, \varphi_m \in \mathcal{C}^{2,\alpha}(S^1)$ which are even, with respect to θ, φ_t, φ_b are L^2 orthogonal to 1 and $\cos\theta$ while φ_m is L^2 orthogonal to 1. Assume that they satisfy

$$\|\varphi_t\|_{\mathcal{C}^{2,\alpha}} + \|\varphi_b\|_{\mathcal{C}^{2,\alpha}} + \|\varphi_m\|_{\mathcal{C}^{2,\alpha}} \leqslant \kappa\,\epsilon.$$

We set $\Phi := (\varphi_t, \varphi_b, \varphi_m)$ and we define w_Φ to be the function equal to

1. $\chi_+\,H_{\varphi_t}(s_\epsilon - s, \cdot)$ on the image of $X_{t,\epsilon/2}$ where χ_+ is a cut-off function equal to 0 for $s \leqslant s_0 + 1$ and identically equal to 1 for $s \in [s_0 + 2, s_\epsilon]$

2. $\chi_-\,H_{\varphi_b}(s - s_\epsilon, \cdot)$ on the image $X_{b,\epsilon/2}$ where χ_- is a cut-off function equal to 0 for $s \geqslant -s_0 - 1$ and identically equal to 1 for $s \in [-s_\epsilon, -s_0 - 2]$

3. $\chi_m\,\tilde{H}_{\rho_\epsilon,\varphi_m}(\cdot,\cdot)$ on the image of X_m, where χ_m is a cut-off function equal to 0 for $r \geqslant \rho_0$ and identically equal to 1 for $\rho \in [\rho_\epsilon, \rho_0/2]$

4. zero on the remaining part of the surface $M_k^T(\epsilon/2)$.

We recall that the operators $\tilde{H}$ and H have been introduced respectively in Propositions 79 and 80.

We would like to prove that, under appropriates hypothesis, the graph about $M_k^T(\epsilon/2)$ of the function $u = w_\Phi + v$ is a minimal surface. This is equivalent to solve the equation:

$$\mathbb{L}_{M_k(\epsilon/2)}(w_\Phi + v) = \tilde{L}_{\epsilon/2}(w_\Phi + v) + Q_\epsilon(w_\Phi + v)$$

on $M_k^T(\epsilon/2)$, so that the graph of $u = w_\Phi + v$ will be a minimal surface. The resolution of the previous equation is obtained thanks to the one of the following fixed point problem:

$$v = T(\Phi, v) \tag{4.7}$$

with

$$T(\Phi, v) = G_{\epsilon/2,\delta} \circ \mathcal{E}_\epsilon \left(\gamma\left(\tilde{L}_{\epsilon/2}(w_\Phi + v) - \mathbb{L}_{M_k^T(\epsilon/2)}\, w_\Phi + Q_\epsilon(w_\Phi + v)\right)\right)$$

where $\delta \in (1, 2)$, the operator $G_{\epsilon/2,\delta}$ is defined in proposition 52 and $\mathcal{E}_\epsilon$ is a linear extension operator such that

$$\mathcal{E}_\epsilon : \mathcal{C}_\delta^{0,\alpha}(M_k^T(\epsilon/2)) \longrightarrow \mathcal{C}_\delta^{0,\alpha}(M_k(\epsilon/2)),$$

where $\mathcal{C}_\delta^{0,\alpha}(M_k^T(\epsilon/2))$ denotes the space of functions of $\mathcal{C}_\delta^{0,\alpha}(M_k(\epsilon/2))$ restricted to $M_k^T(\epsilon/2)$. It is defined by $\mathcal{E}_\epsilon v = v$ in $M_k^T(\epsilon/2)$, $\mathcal{E}_\epsilon v = 0$ in the image of $[s_\epsilon + 1, +\infty) \times S^1$ by $X_{t,\epsilon/2}$, in the image of $(-\infty, -s_\epsilon - 1) \times S^1$ by $X_{b,\epsilon/2}$ and in the image of $B_{\rho_\epsilon/2} \times S^1$ by X_m. Finally $\mathcal{E}_\epsilon v$ is an interpolation of these values in the remaining part of $M_k(\epsilon/2)$ such that, for example,

$$(\mathcal{E}_\epsilon v) \circ X_{t,\epsilon/2}(s, \theta) = ((1 + s_\epsilon - s)v) \circ X_{t,\epsilon/2}(s_\epsilon, \theta),$$
$$(\mathcal{E}_\epsilon v) \circ X_{b,\epsilon/2}(s, \theta) = ((1 + s_\epsilon + s)v) \circ X_{b,\epsilon/2}(s_\epsilon, \theta)$$

for $(s, \theta) \in [s_\epsilon, s_\epsilon + 1] \times S^1$ and

$$(\mathcal{E}_\epsilon v) \circ X_m(\rho, \theta) = ((\frac{2}{\rho_\epsilon}\rho - 1)v) \circ X_m(\rho_\epsilon, \theta)$$

for $(\rho, \theta) \in [\rho_\epsilon/2, \rho_\epsilon] \times S^1$.

Remark 54. *As consequence of the properties of $\mathcal{E}_\epsilon$, if $\operatorname{supp} v \cap B_{\rho_0} - B_{\rho_\epsilon} \neq \emptyset$ then*

$$\|(\mathcal{E}_\epsilon v) \circ X_m\|_{\mathcal{C}^{0,\alpha}(B_{\rho_0})} \leqslant c\epsilon^{-\alpha}\|v \circ X_m\|_{\mathcal{C}^{0,\alpha}(B_{\rho_0})}.$$

This phenomenon of explosion of the norm does not occur near the catenoidal type ends:

$$\|(\mathcal{E}_\epsilon v) \circ X_{t,\epsilon/2}\|_{\mathcal{C}^{0,\alpha}([s_\epsilon+1,+\infty)\times S^1)} \leqslant c\|v \circ X_{t,\epsilon/2}\|_{\mathcal{C}^{0,\alpha}([s_\epsilon+1,+\infty)\times S^1)}.$$

A similar equation holds for the bottom end.

In the following we will assume $\alpha > 0$ and near to zero.

The existence of a solution $v \in \mathcal{C}^{2,\alpha}_{\delta}(M_k^T(\epsilon/2))$ for the equation (4.7) is a consequence of the following result which proves that T is a contracting mapping.

Lemma 55. *There exist constants $c_\kappa > 0$ and $\epsilon_\kappa > 0$, such that*

$$\|T(\Phi, 0)\|_{\mathcal{C}^{2,\alpha}_{\delta}} \leqslant c_\kappa\, \epsilon^2 \tag{4.8}$$

and, for all $\epsilon \in (0, \epsilon_\kappa)$

$$\|T(\Phi, v_2) - T(\Phi, v_1)\|_{\mathcal{C}^{2,\alpha}_{\delta}(M_k(\epsilon/2))} \leqslant \frac{1}{2}\,\|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}_{\delta}(M_k^T(\epsilon/2))}$$

for all $v_1, v_2 \in \mathcal{C}^{2,\alpha}_{\delta}(M_k^T(\epsilon/2))$ and satisfying $\|v\|_{\mathcal{C}^{2,\alpha}_{\delta}} \leqslant 2\, c_\kappa\, \epsilon^2$.

Proof. We recall that the Jacobi operator associated to $M_k(\epsilon/2)$, is asymptotic to the operator of the catenoid near the catenoidal ends, and it is asymptotic to the laplacian near of the planar end. The function w_Φ is identically zero far from the ends where the explicit expression of $\mathbb{L}_{M_k(\epsilon/2)}$ is not known: this is the reason of our particular choice in the definition of w_Φ. Then from the definition of w_Φ and thanks to proposition 52 we obtain the estimate

$$\|\mathcal{E}_\epsilon\left(\gamma \mathbb{L}_{M_k^T(\epsilon/2)}\, w_\Phi\right)\|_{\mathcal{C}^{0,\alpha}_{\delta}(M_k(\epsilon/2))} \leqslant \left|\left|\gamma_t \mathbb{L}_C(w_\Phi \circ X_{t,\epsilon/2})\right|\right|_{\mathcal{C}^{0,\alpha}_{\delta}([s_0+1,s_\epsilon]\times S^1)} +$$

$$+\left|\left|\gamma_b \mathbb{L}_C(w_\Phi \circ X_{b,\epsilon/2})\right|\right|_{\mathcal{C}^{0,\alpha}_{\delta}([-s_\epsilon,-s_0-1]\times S^1)} \leqslant c\left|\left|\frac{2}{\cosh^2 s}\, w_\Phi \circ X_{t,\epsilon/2}\right|\right|_{\mathcal{C}^{0,\alpha}_{\delta}([s_0+1,s_\epsilon]\times S^1)} \leqslant$$

$$c\epsilon\|w_\Phi \circ X_{t,\epsilon/2}\|_{\mathcal{C}^{2,\alpha}_{\delta}([s_0+1,s_\epsilon]\times S^1)} \leqslant c_\kappa\, \epsilon^{2+\frac{\delta}{2}}.$$

Using the properties of $\tilde{L}_{\epsilon/2}$, we obtain

$$\|\mathcal{E}_\epsilon\left(\gamma \tilde{L}_{\epsilon/2}\, w_\Phi\right)\|_{\mathcal{C}^{0,\alpha}_{\delta}(M_k(\epsilon/2))} \leqslant c\epsilon^2\|w_\Phi \circ X_{t,\epsilon/2}\|_{\mathcal{C}^{0,\alpha}_{\delta}([s_0+1,s_\epsilon]\times S^1)} + c\epsilon^2\|w_\Phi \circ X_m\|_{\mathcal{C}^{0,\alpha}([\rho_\epsilon,\rho_0/2]\times S^1)} \leqslant$$

$$\leqslant c\epsilon^2\|w_\Phi \circ X_{t,\epsilon/2}\|_{\mathcal{C}^{2,\alpha}_{\delta}([s_0+1,s_\epsilon]\times S^1)} + c\epsilon^2\|w_\Phi \circ X_m\|_{\mathcal{C}^{2,\alpha}([\rho_\epsilon,\rho_0/2]\times S^1)} \leqslant c_\kappa\, \epsilon^2.$$

As for the last term, we recall that the operator Q_ϵ has two different expressions if we consider the catenoidal type end and the planar end. It holds that

$$\|\mathcal{E}_\epsilon\left(\gamma Q_\epsilon\left(w_\Phi\right)\right)\|_{\mathcal{C}^{0,\alpha}_{\delta}(M_k(\epsilon/2))} \leqslant c_k \epsilon^2.$$

In fact

$$\|\mathcal{E}_\epsilon\left(\gamma Q_\epsilon\left(w_\Phi\right)\right)\|_{\mathcal{C}^{0,\alpha}_{\delta}(M_k(\epsilon/2))} \leqslant c\epsilon\|w_\Phi \circ X_{t,\epsilon/2}\|^2_{\mathcal{C}^{2,\alpha}_{\delta}([s_0+1,s_\epsilon]\times S^1)} +$$

$$c\epsilon\|w_\Phi \circ X_{b,\epsilon/2}\|^2_{\mathcal{C}^{2,\alpha}_{\delta}([-s_\epsilon,-s_0-1]\times S^1)} + c\epsilon^{1-\alpha}\|w_\Phi \circ X_m\|^2_{\mathcal{C}^{2,\alpha}([\rho_\epsilon,\rho_0/2]\times S^1)} \leqslant c_k \epsilon^2.$$

As for the second estimate, we recall that

$$T(\Phi, v) := G_{\epsilon/2,\delta}\left(\mathcal{E}_\epsilon\left(\gamma\left(\tilde{L}_{\epsilon/2}(w_\Phi + v) - \mathbb{L}_{M_k^T(\epsilon/2)}\, w_\Phi + Q_\epsilon\left(w_\Phi + v\right)\right)\right)\right).$$

Then

$$T(\Phi, v_2) - T(\Phi, v_1) = G_{\epsilon/2,\delta}\left(\mathcal{E}_\epsilon\left(\gamma\left(\tilde{L}_{\epsilon/2}(w_\Phi + v_2) - \mathbb{L}_{M_k^T(\epsilon/2)}\, w_\Phi + Q_\epsilon\left(w_\Phi + v_2\right)\right)\right)\right) -$$

$$G_{\epsilon/2,\delta}\left(\mathcal{E}_\epsilon\left(\gamma\left(\tilde{L}_{\epsilon/2}(w_\Phi + v_1) - \mathbb{L}_{M_k^T(\epsilon/2)}\, w_\Phi + Q_\epsilon\left(w_\Phi + v_1\right)\right)\right)\right)$$

and

$$\|T(\Phi, v_2) - T(\Phi, v_1)\|_{\mathcal{C}^{2,\alpha}_\delta(M_k(\epsilon/2))} \leqslant c\|\mathcal{E}_\epsilon\left(\gamma\left(\tilde{L}_{\epsilon/2}(w_\Phi + v_2) - \mathbb{L}_{M_k^T(\epsilon/2)}\, w_\Phi + Q_\epsilon\left(w_\Phi + v_2\right) -\right.\right.$$

$$\left.\left. -\tilde{L}_{\epsilon/2}(w_\Phi + v_1) + \mathbb{L}_{M_k^T(\epsilon/2)}\, w_\Phi - Q_\epsilon\left(w_\Phi + v_1\right)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_k(\epsilon/2))} =$$

$$\|\mathcal{E}_\epsilon\left(\gamma\left(\tilde{L}_{\epsilon/2}(w_\Phi + v_2) - \tilde{L}_{\epsilon/2}(w_\Phi + v_1) + Q_\epsilon\left(w_\Phi + v_2\right) - Q_\epsilon\left(w_\Phi + v_1\right)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_k(\epsilon/2))} =$$

$$= \|\mathcal{E}_\epsilon\left(\gamma\left(\tilde{L}_{\epsilon/2}(v_2 - v_1) + Q_\epsilon\left(w_\Phi + v_2\right) - Q_\epsilon\left(w_\Phi + v_1\right)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_k(\epsilon/2))} \leqslant$$

$$\leqslant \|\mathcal{E}_\epsilon\left(\gamma\left(\tilde{L}(v_2 - v_1)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_k(\epsilon/2))} + \|\mathcal{E}_\epsilon\left(\gamma\left(Q_\epsilon\left(w_\Phi + v_1\right) - Q_\epsilon\left(w_\Phi + v_2\right)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_k(\epsilon/2))}.$$

We observe that from the considerations above it follows that

$$\|\mathcal{E}_\epsilon\left(\gamma\left(\tilde{L}_{\epsilon/2}(v_2 - v_1)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_k(\epsilon/2))} \leqslant c\epsilon^2||v_2 - v_1||_{\mathcal{C}^{2,\alpha}_\delta(M_k^T(\epsilon/2))}$$

and

$$\|\mathcal{E}_\epsilon\left(\gamma\left(Q_\epsilon\left(w_\Phi + v_1\right) - Q_\epsilon\left(w_\Phi + v_2\right)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_k(\epsilon/2))}$$
$$\leqslant c||v_2 - v_1||_{\mathcal{C}^{2,\alpha}_\delta(M_k^T(\epsilon/2))}\|w_\Phi\|_{\mathcal{C}^{0,\alpha}_\delta(M_k(\epsilon/2))} \leqslant$$
$$\leqslant \left(c_k\epsilon^{1+\frac{\delta}{2}}\right)||v_2 - v_1||_{\mathcal{C}^{2,\alpha}_\delta(M_k^T(\epsilon/2))}.$$

Then

$$\|T(\Phi, v_2) - T(\Phi, v_1)\|_{\mathcal{C}^{2,\alpha}_\delta(M_k(\epsilon/2))} \leqslant c\epsilon||v_2 - v_1||_{\mathcal{C}^{2,\alpha}_\delta(M_k^T(\epsilon/2))}.$$

□

This argument provides a minimal surface $M_k^T(\epsilon/2, \Phi)$ which is close to $M_k^T(\epsilon/2)$ and has three boundaries. This surface is, close to its upper and lower boundary, a vertical graph over the annulus $B_{\epsilon^{-1/2}/2} - B_{\epsilon^{-1/2}/4}$ whose parametrization is, respectively, given by

$$\bar{U}_t(r, \theta) = \sigma_{t,\epsilon/2} + \ln(2r) + \frac{\epsilon}{2} r\cos\theta + H_{\varphi_t}(s_\epsilon - \ln 2r, \theta) + V_t(r, \theta),$$

$$\bar{U}_b(r,\theta) = -\sigma_{b,\epsilon/2} - \ln(2r) + \frac{\epsilon}{2} r\cos\theta + H_{\varphi_b}(\ln 2r - s_\epsilon, \theta) + V_b(r,\theta),$$

where $s_\epsilon = -\frac{1}{2}\ln\epsilon$. The boundaries of the surface correspond to $r_\epsilon = \frac{1}{2}\epsilon^{-1/2}$. Nearby the middle boundary the surface is a vertical graph over the annulus $B_{2\rho_\epsilon} - B_{\rho_\epsilon/2}$, where $\rho_\epsilon = 2\epsilon^{1/2}$. Its parametrization is

$$\bar{U}_m(r,\theta) = \tilde{H}_{\rho_\epsilon,\varphi_m}(r,\theta) + V_m(r,\theta).$$

All the functions V_i for $i = t, b, m$ depend non linearly on ϵ, φ. The functions $V_i(\epsilon, \varphi_i)$, for $i = t, b$, satisfy $\|V_i(\epsilon,\varphi_i)(r_\epsilon \cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1 - B_{1/2})} \leqslant c\epsilon$ and

$$\|V_i(\epsilon,\varphi)(r_\epsilon\cdot) - V_i(\epsilon,\varphi')(r_\epsilon\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1 - B_{1/2})} \leqslant c\epsilon^{1-\delta/2}\|\varphi - \varphi'\|_{\mathcal{C}^{2,\alpha}}$$

The function $V_m(\epsilon, \varphi_i)$ satisfies $\|V_m(\epsilon,\varphi_m)(\rho_\epsilon\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_2 - B_1)} \leqslant c\epsilon$ and

$$\|V_m(\epsilon,\varphi)(\rho_\epsilon\cdot) - V_m(\epsilon,\varphi')(\rho_\epsilon\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_2 - B_1)} \leqslant c\epsilon\|\varphi - \varphi'\|_{\mathcal{C}^{2,\alpha}}$$

4.4 KMR examples $M_{\sigma,\alpha,\beta}$

In 1988, H. Karcher [19, 20] defined a family of doubly periodic minimal surfaces, called *toroidal halfplane layers*, with genus one and four horizontal Scherk-type ends[2] in the quotient. In 1989, Meeks and Rosenberg [25] developed a general theory for doubly periodic minimal surfaces having finite topology in the quotient, and used an approach of minimax type to obtain the existence of a family of doubly periodic minimal surfaces, also with genus one and four horizontal Scherk-type ends in the quotient. These Karcher's and Meeks and Rosenberg's surfaces have been generalized in [35], constructing a 3-parameter family $\mathcal{K} = \{M_{\sigma,\alpha,\beta}\}_{\sigma,\alpha,\beta}$ of surfaces, called KMR examples (sometimes, they are also referred in the literature as toroidal halfplane layers). Such examples have been classified by Pérez, Rodríguez and Traizet [33] as the only doubly periodic minimal surfaces with genus one and finitely many parallel (Scherk-type) ends in the quotient. The possible limits of KMR examples are: the catenoid, the helicoid, any singly or doubly periodic Scherk minimal surface, any Riemann minimal example or another KMR example. Our aim in this section is to study two subfamilies of KMR examples which are invariants under a reflection symmetry about a vertical plane, near the catenoidal limit.

Firstly, we briefly recall the construction of the KMR examples. For each $\sigma \in (0, \frac{\pi}{2})$, $\alpha \in [0, \frac{\pi}{2}]$ and $\beta \in [0, \frac{\pi}{2}]$ with $(\alpha, \beta) \neq (0, \sigma)$, consider the rectangular torus $\Sigma_\sigma = \left\{(z,w) \in \overline{\mathbb{C}}^2 \mid w^2 = (z^2 + \lambda^2)(z^2 + \lambda^{-2})\right\}$, where $\lambda = \lambda(\sigma) = \cot\frac{\sigma}{2} > 1$. The KMR

[2]A horizontal *Scherk-type* end is an end asymptotic to a horizontal half-plane, invariant by one of the period vectors of the surface.

Figure 4.1: The position of the branch values

example $M_{\sigma,\alpha,\beta}$ is determined by its Gauss map g and the differential of its height function h, which are defined on Σ_σ and given by:

$$g(z,w) = \frac{az+b}{i(\overline{a}-\overline{b}z)}, \qquad dh = \mu\,\frac{dz}{w},$$

where:

- $a = a(\alpha,\beta) = \cos\frac{\alpha+\beta}{2} + i\cos\frac{\alpha-\beta}{2}$;
- $b = b(\alpha,\beta) = \sin\frac{\alpha-\beta}{2} + i\sin\frac{\alpha+\beta}{2}$;
- $\mu = \mu(\sigma) = \frac{\pi\csc\sigma}{\mathcal{K}(\sin^2\sigma)}$, where $\mathcal{K}(m) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-m\sin^2 u}}\,du$, $0 < m < 1$, is the complete elliptic integral of the first kind. Such μ has been chosen so that the vertical part of the flux of $M_{\sigma,\alpha,\beta}$ along any horizontal level section equals 2π.

Remark 56.

(i) $b \to 0$ *if and only if* $\alpha \to 0$ *and* $\beta \to 0$*, in which case* $a \to 1+i$*.*

(ii) $\left|\frac{b}{a}\right| = \tan\frac{\varphi}{2}$*, where* φ *is the angle between the North Pole* $(0,0,1) \in \mathbb{S}^2$ *and the pole of* g *seen in* $\mathbb{S}^2$ *via the inverse of the stereographic projection.*

The KMR example $M_{\sigma,\alpha,\beta}$ can be parametrized on Σ_σ by the immersion $X = (X_1, X_2, X_3) = \Re\int\Phi$, where Φ is the Weierstrass form:

$$\Phi = \left(\frac{1}{2}\left(\frac{1}{g} - g\right)dh, \frac{i}{2}\left(\frac{1}{g} + g\right)dh, dh\right).$$

The ends of $M_{\sigma,\alpha,\beta}$ corresponds to the punctures $\{A, A', A'', A'''\} = g^{-1}(\{0,\infty\})$, and the branch values of g are those with $w = 0$, i.e.

$$D = (-i\lambda, 0),\;\; D' = (i\lambda, 0),\;\; D'' = (\tfrac{i}{\lambda}, 0),\;\; D''' = (-\tfrac{i}{\lambda}, 0). \tag{4.9}$$

Seen in $\mathbb{S}^2$, these points form two pairs of antipodal points: $D'' = -D$ and $D''' = -D'$. (Each KMR example can be given in terms of the branch values of its Gauss map.)

In [35], it is proven that the above Weierstrass data define a properly embedded minimal surface ($M_{\sigma,\alpha,\beta}$) invariant by two independent translations: the translation by the period T_1 at its ends, and the period T_2 along a homology class. Moreover, the group of isometries $\mathrm{Iso}(M_{\sigma,\alpha,\beta})$ of $M_{\sigma,\alpha,\beta}$ always contains a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$,

Figure 4.2: Left: $M_{\sigma,0,0}$, with $\sigma = \frac{\pi}{4}$. Right: $M_{\sigma,\alpha,0}$ for $\sigma = \alpha = \frac{\pi}{4}$.

Figure 4.3: $M_{\sigma,0,\beta}$, where $\sigma = \frac{\pi}{4}$ and $\beta = \frac{\pi}{8}$.

with generators $\mathcal{D}$ (corresponding to the deck transformation $(z,w) \mapsto (z,-w)$), which represents a central symmetry about any of the four branch points of g, and $\mathcal{F}$, which consists of a translation by $\frac{1}{2}(T_1+T_2)$. In particular, the ends of $M_{\sigma,\alpha,\beta}$ are equally spaced.

We are going to focus on the two most symmetric subfamilies of KMR examples: $\{M_{\sigma,\alpha,0}\}_{\sigma,\alpha}$ and $\{M_{\sigma,0,\beta}\}_{\sigma,\beta}$.

1. When $\alpha = \beta = 0$, $M_{\sigma,0,0}$ contains four straight lines parallel to the x_1-axis, and $\mathrm{Iso}(M_{\sigma,0,0})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$ with generators S_1, S_2, S_3, R_D: S_1 is a reflection symmetry in a vertical plane orthogonal to the x_1-axis; S_2 is a reflection symmetry across a plane orthogonal to the x_2-axis; S_3 is a reflection symmetries in a horizontal plane (these three planes can be chosen meeting at a point, which is not contained in the surface); and R_D is the π-rotation around one of the four straight lines contained in the surface, see Figure 4.2 left. In this case, $T_1 = (0, \pi\mu, 0)$.

2. When $0 < \alpha < \frac{\pi}{2}$, $\mathrm{Iso}(M_{\sigma,\alpha,0})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, with generators $\mathcal{D}$, S_2 and R_2, where S_2 represents a reflection symmetry across a plane orthogonal to the x_2-axis, and R_2 is a π-rotation around a line parallel to the x_2-axis that cuts $M_{\sigma,\alpha,0}$ orthogonally, see Figure 4.2 right. Now $T_1 = (0, \pi\mu t_\alpha, 0)$, with $t_\alpha = \frac{\sin\sigma}{\sqrt{\sin^2\sigma\cos^2\alpha + \sin^2\alpha}}$.

3. Suppose that $0 < \beta < \sigma$. Then $M_{\sigma,0,\beta}$ contains four straight lines parallel to the x_1-axis, and $\mathrm{Iso}(M_{\sigma,0,\beta})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, with generators S_1, R_1 and R_D: S_1 represents a reflection symmetry across a plane orthogonal to the x_1-axis; R_1 corresponds to a π-rotation around a line parallel to the x_1-axis that cuts the surface orthogonally; and R_D is the π-rotation around any one of the straight lines contained in the surface, see Figure 4.3. Moreover, $T_1 = (0, \pi\mu t^\beta, 0)$, where $t^\beta = \frac{\sin\sigma}{\sqrt{\sin^2\sigma - \sin^2\beta}}$.

From now on, we will denote by $T = T_1$ the period of the surface at its ends.

Finally, it will be useful to see Σ_σ as a branched 2-covering of $\overline{\mathbb{C}}$ through the map $(z,w) \mapsto z$. Thus Σ_σ can be seen as two copies $\overline{\mathbb{C}}_1, \overline{\mathbb{C}}_2$ of $\overline{\mathbb{C}}$ glued along two common cuts γ_1, γ_2, which we tale along the imaginary axis: γ_1 from D to D' and γ_2 from D'' to D'''.

4.4.1 $M_{\sigma,\alpha,\beta}$ as a graph over $\{x_3 = 0\}/T$

The KMR examples $M_{\sigma,\alpha,\beta}$ converge, as $(\sigma, \alpha, \beta) \to (0, 0, 0)$, to a vertical catenoid, since Σ_σ converges to two pinched spheres, $g(z) \to z$ and $dh \to \pm\frac{dz}{z}$ as $\sigma, \alpha, \beta \to 0$. In fact, we can obtain two catenoids in the limit, depending on the choice of branch for w (for each copy of $\overline{\mathbb{C}}$ in Σ_σ, we obtain one catenoid in the limit). Our aim along this paper consists of gluing KMR examples $M_{\sigma,\alpha,0}$ or $M_{\sigma,0,\beta}$ near this catenoidal limit, to a convenient compact piece of a deformed Costa-Hoffman-Meeks surface $M_k(\epsilon/2)$. In this subsection we express part of $M_{\sigma,\alpha,\beta}$ as a vertical graph over the $\{x_3 = 0\}$-plane when σ, α, β are small.

Consider $M_{\sigma,\alpha,\beta}$ near the catenoidal limit, i.e. σ, α, β close to zero. Without lost of generality, we can assume $dh \sim \frac{dz}{z}$ in $\overline{\mathbb{C}}_1$. We are studying the surface in an annulus about one of its ends, say a zero of its Gauss map.

Lemma 57. *Let σ, α, β be small. Up to translations, $M_{\sigma,\alpha,\beta}$ can be parametrized in the annulus $\{(z, w) \in \Sigma_\sigma \mid z \in \overline{\mathbb{C}}_1,\ |\frac{b}{a}| < z < \nu\}$ (for $\nu > |\frac{b}{a}|$ small) as:*

$$\begin{cases} X_1 + i\,X_2 = \frac{-1}{2}\left(z + \frac{1}{\bar{z}}\right) + \frac{i\,b}{\bar{a}} \ln|z| + \frac{(1+i)\bar{b}}{4\bar{z}^2} + \mathcal{O}(b^2\, z^{-3} + \lambda^{-2} z^{-2}) \\ X_3 = \ln|z| + \mathcal{O}(\lambda^{-2} z^{-2}), \end{cases}$$

Proof. We have assumed $dh \sim \frac{dz}{z}$ in the annulus we are working in. More precisely, we have

$$dh = \mu \frac{dz}{\sqrt{(z^2+\lambda^2)(z^2+\lambda^{-2})}} = \frac{\mu}{\lambda} \frac{dz}{z\sqrt{(\lambda^{-2}+z^{-2})(z^2+\lambda^{-2})}} = \frac{\mu}{\lambda} \frac{dz}{z} \frac{1}{\sqrt{1+\lambda^{-2}z^2+\lambda^{-2}z^{-2}+\lambda^{-4}}}.$$

Since $\frac{\mu(\sigma)}{\lambda(\sigma)} = \frac{\pi}{(1+\cos(\sigma))\mathcal{K}(\sin^2\sigma)} = 1 + \mathcal{O}(\sigma^4) = 1 + \mathcal{O}(\lambda^{-4})$, and $\frac{1}{\sqrt{1+t}} = 1 + \mathcal{O}(t)$ for $t > 0$ small, we get

$$dh = \frac{dz}{z}(1 + \mathcal{O}(\lambda^{-4}))(1 + \mathcal{O}(\lambda^{-2}z^2 + \lambda^{-2}z^{-2} + \lambda^{-4})).$$

Since $|z| < 1$, then both $\lambda^{-2}|z|^2 < \lambda^{-2}|z|^{-2}$ and $\lambda^{-4} < \lambda^{-2}|z|^{-2}$. Therefore,

$$dh = \frac{dz}{z}(1 + \mathcal{O}(\lambda^{-2}z^{-2})).$$

Fix any point $z_0 \in \overline{\mathbb{C}}_1$. Thus $X_1(z) + i\,X_2(z) = \frac{1}{2}\left(\overline{\int_{z_0}^z \frac{dh}{g}} - \int_{z_0}^z g\,dh\right)$. Straightforward computations give us

$$\begin{aligned} \int_{z_0}^z \frac{dh}{g} &= \frac{i}{a}\int_{z_0}^z \frac{\bar{a}-\bar{b}\omega}{\omega(\omega+b/a)}(1 + \mathcal{O}(\sigma^2\omega^{-2}))d\omega \\ &= \frac{i}{a}\left(\frac{|a|^2}{b} \ln\frac{z}{z+b/a} - \bar{b}\ln(z + b/a)\right) + C_1 + \mathcal{O}(\sigma^2 z^{-2} + \sigma^2 b z^{-3}), \end{aligned}$$

where $C_1 = \frac{i}{a}\left(\frac{|a|^2}{b}\ln\frac{z_0+b/a}{z_0} + \bar{b}\,\ln(z_0+b/a)\right)$; and

$$\begin{aligned}\int_{z_0}^{z} g\,dh &= -i\int_{z_0}^{z}\frac{a\omega+b}{\omega(\bar{a}-\bar{b}\omega)}(1+\mathcal{O}(\sigma^2\omega^{-2}))d\omega\\ &= \frac{-i}{\bar{a}}\left(\frac{|a|^2+|b|^2}{-\bar{b}}\ln\frac{\bar{a}-\bar{b}z}{\bar{a}-\bar{b}z_0} + b\,\ln\frac{z}{z_0}\right) + \mathcal{O}(\sigma^2 z^{-1} + \sigma^2 b z^{-2}).\end{aligned}$$

For $|t| < |z|$ we have $\ln\frac{z}{z+t} = \frac{-t}{z} + \frac{t^2}{z^2} + \mathcal{O}(t^2 z^{-3})$ and $\ln(z+t) = \ln z + \frac{t}{z} + \mathcal{O}(t^3 z^{-2})$. Hence taking $|b/a| < |z| < 1$, we obtain:

- $\int_{z_0}^{z}\frac{dh}{g} = -\frac{i\,\bar{b}}{a}\ln z - \frac{i\left(|a|^2+|b|^2\right)}{a^2 z} + \frac{ib\bar{a}}{2a^2z^2} + C_1 + \mathcal{O}(b^2\,z^{-3} + \lambda^{-2}z^{-2})$,
- $\int_{z_0}^{z} g\,dh = -\frac{i\,b}{\bar{a}}\ln z - \frac{i\left(|a|^2+|b|^2\right)}{\bar{a}^2}\,z + C_2 + \mathcal{O}(\lambda^{-2}z^{-1})$,

where $C_1, C_2 \in \mathbb{C}$ verify $\overline{C_1} - C_2 = \frac{z_0(1+|z_0|^2)}{|z_0|^2} + \mathcal{O}(b)$. Therefore,

$$\begin{aligned}X_1 + i\,X_2 &= \frac{1}{2}\left(\overline{\int_{z_0}^{z}\frac{dh}{g}} - \int_{z_0}^{z} g\,dh\right)\\ &= \frac{i\,b}{\bar{a}}\ln|z| + \frac{i(|a|^2+|b|^2)}{2\,\bar{a}^2}\left(z + \frac{1}{\bar{z}}\right) - \frac{i\bar{b}a}{2\bar{a}^2\bar{z}^2} + \frac{z_0(1+|z_0|^2)}{2|z_0|^2} + \mathcal{O}(b^2\,z^{-3} + \lambda^{-2}z^{-2}).\end{aligned}$$

Taking into account that $\frac{i\left(|a|^2+|b|^2\right)}{\bar{a}^2} = -1 + \mathcal{O}(b)$ and $\frac{ia}{2\bar{a}^2} = -\frac{1+i}{4} + \mathcal{O}(b)$, we have

$$X_1 + i\,X_2 = \frac{z_0(1+|z_0|^2)}{2|z_0|^2} + \frac{i\,b}{\bar{a}}\ln|z| + \frac{-1}{2}\left(z + \frac{1}{\bar{z}}\right) + \frac{(1+i)\bar{b}}{4\bar{z}^2} + \mathcal{O}(b^2\,z^{-3} + \lambda^{-2}z^{-2}).$$

Similarly, $\int_{z_0}^{z} dh = \ln z - \ln z_0 + \mathcal{O}(\lambda^{-2}z^{-2})$, hence

$$X_3 = \Re\int_{z_0}^{z} dh = \ln|z| - \ln|z_0| + \mathcal{O}(\lambda^{-2}z^{-2}),$$

which finishes Lemma 57. □

Lemma 58. *Let (r,θ) denote the polar coordinates in the $\{x_3 = 0\}$ plane and define $\varepsilon = b + \lambda^{-1}$. Then a piece of $M_{\sigma,\alpha,\beta}$ can be written as a vertical graph of*

$$\widetilde{U}(r,\theta) = -\ln(2r) + r\left(\eta_1\cos\theta + \eta_2\sin\theta\right) + \mathcal{O}(\varepsilon),$$

for $(r,\theta) \in (\frac{1}{4\sqrt{\varepsilon}}, \frac{4}{\sqrt{\varepsilon}}) \times [0, 2\pi)$, where $\eta_1 = \Re(b) + \Im(b)$ and $\eta_2 = \Re(b) - \Im(b)$.

Remark 59. *Recall that $b = \sin\frac{\alpha-\beta}{2} + i\sin\frac{\alpha+\beta}{2}$. In particular:*

- *When $\beta = 0$, we have $\eta_1 = 2\sin\frac{\alpha}{2}$ and $\eta_2 = 0$, so*

$$\widetilde{U}(r,\theta) = -\ln(2r) + 2\sin\frac{\alpha}{2} r\cos\theta + \mathcal{O}(\varepsilon), \qquad \mathcal{O}(\varepsilon) = \mathcal{O}(\alpha+\sigma).$$

- *When $\alpha = 0$, $\eta_1 = 0$ and $\eta_2 = 2\sin\frac{\beta}{2}$, so*

$$\widetilde{U}(r,\theta) = -\ln(2r) + 2\sin\frac{\beta}{2} r\sin\theta + \mathcal{O}(\varepsilon), \qquad \mathcal{O}(\varepsilon) = \mathcal{O}(\beta+\sigma).$$

Proof. From Lemma 57, we know that $(X_1 + iX_2)(z) = \frac{-1}{2}\left(z + \frac{1}{\overline{z}}\right) + A(z)$, where

$$A(z) = \frac{i\,b}{\overline{a}}\ln|z| + \frac{(1+i)\overline{b}}{4\overline{z}^2} + \mathcal{O}(b^2\,z^{-3} + \lambda^{-2}z^{-2}).$$

Denote $z = |z|e^{i\psi}$. Then $z + \frac{1}{\overline{z}} = \left(|z| + \frac{1}{|z|}\right)e^{i\psi}$, and

$$r\cos\theta = X_1 = -\frac{1}{2}\left(|z| + \frac{1}{|z|}\right)\cos\psi + A_1,$$

$$r\sin\theta = X_2 = -\frac{1}{2}\left(|z| + \frac{1}{|z|}\right)\sin\psi + A_2,$$

where $A_1 = \Re(A)$ and $A_2 = \Im(A)$. Therefore,

$$r^2 = \frac{1}{4}\left(|z| + \frac{1}{|z|}\right)^2 - \left(|z| + \frac{1}{|z|}\right)(A_1\cos\psi + A_2\sin\psi) + A_1^2 + A_2^2. \tag{4.10}$$

From (4.10) we deduce:

$$r^2 = \tfrac{1}{4}\left(|z| + \tfrac{1}{|z|}\right)^2\left(1 - \tfrac{4|z|}{|z|^2+1}(A_1\cos\psi + A_2\sin\psi) + \tfrac{4|z|^2}{(|z|^2+1)^2}(A_1^2 + A_2^2)\right).$$

When $|z| = \mathcal{O}(\sqrt{\varepsilon})$, the functions A_i are bounded, and we get

$$r = \frac{1}{2}\left(|z| + \frac{1}{|z|}\right)(1 + \mathcal{O}(\sqrt{\varepsilon})), \tag{4.11}$$

and so $r = \mathcal{O}(1/\sqrt{\varepsilon})$. Moreover, we get $\frac{r}{\frac{1}{2}\left(|z|+\frac{1}{|z|}\right)} = 1 + \mathcal{O}(\sqrt{\varepsilon})$, from where

$$e^{i\theta}(1 + \mathcal{O}(\sqrt{\varepsilon})) = \frac{X_1 + iX_2}{\frac{1}{2}\left(|z| + \frac{1}{|z|}\right)} = -e^{i\psi} + \frac{2|z|A}{1+|z|^2} = -e^{i\psi} + \mathcal{O}(\sqrt{\varepsilon})).$$

Hence

$$e^{i\psi} = -e^{i\theta}(1 + \mathcal{O}(\sqrt{\epsilon})).$$

From (4.10) and (4.11) we obtain

$$\frac{(1+|z|^2)^2}{4|z|^2} = r^2 + \left(|z| + \frac{1}{|z|}\right)(A_1 \cos\psi + A_2 \sin\psi) - A_1^2 - A_2^2$$

$$= r^2 \left(1 + \frac{1}{r}(1 + \mathcal{O}(\sqrt{\varepsilon}))\,(A_1 \cos\psi + A_2 \sin\psi) - \frac{A_1^2 + A_2^2}{r^2}\right)$$

$$= r^2 \left(1 + \frac{1}{r}\,(A_1 \cos\psi + A_2 \sin\psi) + \mathcal{O}(\varepsilon)\right).$$

Therefore,

$$\frac{1}{|z|^2} = (2r)^2 \left(1 + \frac{1}{r}\,(A_1 \cos\psi + A_2 \sin\psi) + \mathcal{O}(\varepsilon)\right)(1 + \mathcal{O}(\varepsilon)).$$

Since $\ln(1+t) = t + \mathcal{O}(t^2)$, we deduce

$$-\ln|z| = \ln(2r) + \frac{1}{2r}(A_1 \cos\psi + A_2 \sin\psi) + \mathcal{O}(\varepsilon).$$

Finally, it is not very difficult to prove that

$$A_1 \cos\psi + A_2 \sin\psi = \frac{1}{4|z|^2}\,(\eta_1 \cos\psi - \eta_2 \sin\psi)$$

$$= r^2\,(-\eta_1 \cos\theta + \eta_2 \sin\theta)\,(1 + \mathcal{O}(\sqrt{\varepsilon})),$$

from where Lemma 58 follows. □

If we consider small translations of $M_{\sigma,\alpha,\beta}$ dilated by a factor $1+\gamma$, for some small γ, we obtain

$$X_3 = -(1+\gamma)\ln\frac{2r}{1+\gamma} + r\,(-\eta_1 \cos\theta + \eta_2 \sin\theta) + \frac{1}{r}(\kappa_1 \cos\theta + \kappa_2 \sin\theta) + \kappa_3 + \mathcal{O}(\varepsilon),$$

for small $\kappa_1, \kappa_2, \kappa_3 \in \mathbb{R}^+$, where $\eta_1 = b_1 + b_2$ and $\eta_2 = b_1 - b_2$.

4.4.2 Parametrization of the KMR example on the cylinder

In this subsection we want to parametrize the KMR example $M_{\sigma,\alpha,\beta}$ on a cylinder. To this aim, we introduce the sphero-conal coordinates (x, y) on the unit sphere $\mathbb{S}^2$ (see [17]): For any $(x, y) \in \mathbb{S}^1 \times [0, \pi) \equiv [0, 2\pi) \times [0, \pi)$, define

$$F(x, y) = (\cos x \, \sin y, \sin x \, m(y), l(x) \, \cos y) \in \mathbb{S}^2,$$

where

$$m(y) = \sqrt{1 - \cos^2 \sigma \cos^2 y} \quad \text{and} \quad l(x) = \sqrt{1 - \sin^2 \sigma \sin^2 x}.$$

Recall that the conformal compactification Σ_σ of $M_{\sigma,\alpha,\beta}$ only depends on σ. The parameter $\sigma \in (0, \frac{\pi}{2})$ will remains fixed along this subsection, and we will omit the dependence of the functions we are introducing on σ.

Note that, when $\sigma = 0$, the conformal compactification of the limit surface $M_{0,\alpha,\beta}$ is not a torus but a sphere, and the above sphero-conal coordinates reduce to the spherical ones.

Il faut bien ecrire ça: The coordinate surfaces $\{x = \textit{constant}\}$ and $\{y = \textit{constant}\}$ are two elliptic cones with vertex at the origin. The cross section of each of these cones with a vertical plane which is orthogonal to the axis of the cone is an ellipse.

Recall that Σ_σ can be seen as a branched 2-covering of $\overline{\mathbb{C}}$, by gluing $\overline{\mathbb{C}}_1, \overline{\mathbb{C}}_2$ along two common cuts γ_1 and γ_2 along the imaginary axis joining the branch points D, D' and D'', D''' respectively (see (4.9)). If we compose $F(x, y)$ with the stereographical projection and enlarge the domain of definition of the function, we obtain the differentiable map $\mathbf{z}(x, y) : \mathbb{S}^1 \times \mathbb{S}^1 \equiv [0, 2\pi) \times [0, 2\pi) \to \overline{\mathbb{C}}$ given by

$$\mathbf{z}(x, y) = \frac{\cos x \, \sin y + i \, \sin x \, m(y)}{1 - l(x) \, \cos y},$$

which is a branch 2-covering of $\overline{\mathbb{C}}$ with branch values in the four points whose sphero-conal coordinates are $(x, y) \in \left\{\pm\frac{\pi}{2}\right\} \times \{0, \pi\}$, which also correspond to D, D', D'', D'''. Moreover, $\mathbf{z}(x, y)$ maps $\mathbb{S}^1 \times (0, \pi)$ on $\overline{\mathbb{C}} - (\gamma_1 \cup \gamma_2)$. Hence we can parametrize the KMR example by $\mathbf{z}$, by means of its Weierstrass data.

We denote by $\widetilde{M}_{\sigma,\alpha,\beta}$ the lifting of $M_{\sigma,\alpha,\beta}$ to $\mathbb{R} \times \mathbb{S}^1 \times \mathbb{R}$ by forgetting its non horizontal period (i.e. its period in homology, T_2). We can then parametrize $\widetilde{M}_{\sigma,\alpha,\beta}$ on $\mathbb{S}^1 \times \mathbb{R}$ by extending $\mathbf{z}$ to $[0, 2\pi) \times \mathbb{R}$. But such a parametrization is not conformal, since the sphero-conal coordinates $(x, y) \mapsto F(x, y)$ of the sphere are not conformal. As the stereographic projection is a conformal map, it suffices to find new conformal coordinates (u, v) of the sphere defined on the cylinder. In particular, we look for a change of variables $(x, y) \mapsto (u, v)$ for which $|\widetilde{F}_u| = |\widetilde{F}_v|$ and $\langle \widetilde{F}_u, \widetilde{F}_v \rangle = 0$, where $\widetilde{F}(u, v) = F(x(u, v), y(u, v))$.

We observe that

$$|F_x| = \frac{\sqrt{T(x,y)}}{m(x)} \quad \text{and} \quad |F_y| = \frac{\sqrt{T(x,y)}}{l(y)},$$

with $T(x,y) = \sin^2\sigma\,\cos^2 x + \cos^2\sigma\,\sin^2 y$. Then it is natural to consider the change of variables $(x,y) \in [0,2\pi) \times [0,\pi) \mapsto (u,v) \in [0,U_\sigma] \times$**???** defined by

$$u(x) = \int_0^x \frac{1}{l(t)} dt \quad \text{and} \quad v(y) = \int_0^y \frac{1}{m(s)} ds, \tag{4.12}$$

where

$$U_\sigma = u(2\pi) = \int_0^{2\pi} \frac{dt}{\sqrt{1-\sin^2\sigma\sin^2 t}}. \tag{4.13}$$

Note that U_σ is a function on σ that goes to 2π as σ approaches to zero, and that the above change of variables is well defined because $\sigma \in (0, \frac{\pi}{2})$. From all this, we can deduce that $\widetilde{M}_{\sigma,\alpha,\beta}$ is conformally parametrized on $(u,v) \in I_\sigma \times \mathbb{R}$, with $I_\sigma = [0, U_\sigma]$.

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Remark 60. *In lemma 58 assuming $|z| = \mathcal{O}(\sqrt{\epsilon})$ we have found the equation as a graph about an appropriate neighbourhood of the part of the surface $\widetilde{M}_{\sigma,\alpha,\beta}$ along which we will glue it with the Costa-Hoffman-Meeks surface. It is possible to prove that if $|z| = \mathcal{O}(\sqrt{\epsilon})$ and $\sigma = \mathcal{O}(\sqrt{\epsilon})$ then $y = y_\epsilon = \pi - \mathcal{O}(\sqrt{\epsilon})$. Moreover using (4.12) we can obtain the corresponding value of v. It is given by $v_\epsilon = -\frac{1}{2}\ln\epsilon + \mathcal{O}(1)$.*

4.5 The Jacobi operator about $\widetilde{M}_{\sigma,\alpha,\beta}$

The Jacobi operator for $M_{\sigma,\alpha,\beta}$ is given by $\mathcal{J} = \Delta_{ds^2} + |A|^2$, where $|A|^2$ is the squared norm of the second fundamental form on $M_{\sigma,\alpha,\beta}$ and Δ_{ds^2} is the Laplace-Beltrami operator with respect to the metric ds^2 induced on the surface by the immersion X defined in section 4.4. That is

$$ds^2 = \frac{1}{4}\left(|g| + |g|^{-1}\right)^2 |dh|^2.$$

In subsection 4.4.2 we have presented the parametrization of $M_{\sigma,\alpha,\beta}$ and of its lifting on a cylinder. We recall we adopted the coordinates (x,y). In this new frame the Jacobi operator, $\mathcal{J}$, is obtained considering as metric on the rectangular torus Σ_σ, the pull-back of standard metric ds_0^2 of the sphere $\mathbb{S}^2$ by the Gauss map. The following relation holds

$$\mathcal{J} = -K\,(\Delta_{ds_0^2} + 2)$$

(being $K = -\frac{1}{2}|A|^2$ the Gauss curvature) since $\Delta_{ds^2} = -K\ \Delta_{ds_0^2}$ and $dN^*(ds_0^2) = -K\ ds^2$. From [17] and taking into account the parametrization of $\widetilde{M}_{\sigma,\alpha,\beta}$ given in subsection 4.4.2, we can deduce that, in the (x,y)-variables on $\mathbb{S}^2 \setminus \{\gamma_1 \cup \gamma_2\}$

$$\Delta_{ds_0^2} := \frac{l(x)\, m(y)}{T(x,y)} \left[\partial_x \left(\frac{l(x)}{m(y)}\, \partial_x \right) + \partial_y \left(\frac{m(y)}{l(x)}\, \partial_y \right) \right].$$

After the conformal change of coordinates $(x,y) \to (u,v)$ defined by (4.12), we can write $\mathcal{J} = \frac{-K}{T(x(u),y(v))} \mathcal{L}_\sigma$, where

$$\mathcal{L}_\sigma := \partial^2_{uu} + \partial^2_{vv} + 2\sin^2\sigma\ \cos^2(x(u)) + 2\cos^2\sigma\ \sin^2(y(v)) \tag{4.14}$$

is known as Lamé operator.

Remark 61. *In proposition 66, we will take limits as $\sigma \to 0$. For such a limit, the Riemann surface Σ_σ degenerates into a Riemann surface with nodes consisting of two spheres jointed by two common points, and the corresponding Jacobi operator equals $\mathcal{L}_0 = \partial^2_{xx} + \sin y\, \partial_y\, (\sin y\, \partial_y) + 2\sin^2 y$ in the (x,y)-variables. Note that in this case the change of variables $(x,y) \mapsto (u,v)$ is not defined.*

4.5.1 The mapping properties of the Jacobi operator

Our aim along this subsection is to study the mapping properties of the operator $\mathcal{J}$. It is clear that it is sufficient to study the simpler operator $\mathcal{L}_\sigma$ defined by (4.14). So we want to study the possibility to solve in a unique way the problem

$$\begin{cases} \mathcal{L}_\sigma w = f, & \text{in } I_\sigma \times [v_0, +\infty[\\ w|_{v=v_0} = \varphi \end{cases}$$

with $v_0 \in \mathbb{R}$, considering convenient normed functional spaces for w, f and φ, so that the norm of w is bounded by the one of f.

Since $\mathcal{L}_\sigma$ has separated variables, let us firstly consider the operator

$$L_\sigma = \partial^2_{uu} + 2\sin^2\sigma \cos^2(x(u)).$$

The quantity U_σ defined in (4.13) is the period of the function $\cos(x(u))$. It is possible to prove that $U_\sigma \to 2\pi$ as $\sigma \to 0$. We let L_σ act on the U_σ-periodic and even functions. Moreover it is uniformly elliptic and self-adjoint. In particular, L_σ has discrete spectrum $(\lambda_{\sigma,i})_{i\geqslant 0}$, that we assume arranged so that $\lambda_{\sigma,i} < \lambda_{\sigma,i+1}$ for every i. Each eigenvalue $\lambda_{\sigma,i}$ is simple because we only consider even functions. We denote by $e_{\sigma,i}$ the even eigenfunction associated to $\lambda_{\sigma,i}$, normalized so that

$$\int_0^{U_\sigma} (e_{\sigma,i}(u))^2\, du = 1.$$

Lemma 62. *For every $i \geqslant 0$, the eigenvalue $\lambda_{\sigma,i}$ of the operator L_σ and its associated eigenfunctions $e_{\sigma,i}$ satisfy*

$$-2\sin^2\sigma \leqslant \lambda_{\sigma,i} - i^2 \leqslant 0, \qquad |e_{\sigma,i} - e_{0,i}|_{\mathcal{C}^2} \leqslant c_i \sin^2\sigma, \tag{4.15}$$

where $e_{0,i}(u) := \cos(ix(u))$ for every $u \in I_\sigma$, and the constant $c_i > 0$ depends only on i (it does not depend on σ).

Proof. The bound for $\lambda_{\sigma,i} - i^2$ comes from the variational characterization of the eigenvalues,

$$\lambda_{\sigma,i} = \sup_{\operatorname{codim} E = i} \inf_{e\in E,\, ||e||_{L^2}=1} \int_0^{U_\sigma} \left((\partial_u e)^2 - 2\sin^2\sigma\cos^2(x(u))\, e^2\right) du,$$

where E is a subset of the space of U_σ-periodic even functions in $L^2(I_\sigma)$, since it always holds $0 \leqslant 2\sin^2\sigma\cos^2(x(u)) \leqslant 2\sin^2\sigma$.
The bound for the eigenfunctions follows from standard perturbation theory [21].

□

The Hilbert basis $\{e_{\sigma,i}\}_{i\in\mathbb{N}}$ of the space of U_σ-periodic and even functions in $L^2(I_\sigma)$ induces the following Fourier decomposition of L^2 functions $g = g(u,v)$ which are U_σ-periodic and even in the u-variable,

$$g(u,v) = \sum_{i\geqslant 0} g_i(v)\, e_{\sigma,i}(u).$$

From this, we deduce that the operator $\mathcal{L}_\sigma$, can be decomposed as $\mathcal{L}_\sigma = \sum_{i\geqslant 0} L_{\sigma,i}$, being

$$L_{\sigma,i} = \partial^2_{vv} + 2\cos^2\sigma\sin^2(y(v)) - \lambda_{\sigma,i}\,, \quad \text{for every } i \geqslant 0.$$

Since $0 \leqslant 2\cos^2\sigma\sin^2(y(v)) \leqslant 2\cos^2\sigma = 2 - 2\sin^2\sigma$, the lemma 62 give us

$$P_{\sigma,i} := 2\cos^2\sigma\, \sin^2(y(v)) - \lambda_{\sigma,i} \leqslant 2 - i^2. \tag{4.16}$$

This fact allows us to prove the following lemma, which assures that $\mathcal{L}_\sigma$ is injective when restricted to the set of functions that are L^2-orthogonal to $e_{\sigma,0}$ and $e_{\sigma,1}$ in the u-variable.

Lemma 63. *Given $v_0 < v_1$, let w be a solution of $\mathcal{L}_\sigma w = 0$ on $I_\sigma \times [v_0, v_1]$ such that*

(i) $w(\cdot\,, v_0) = w(\cdot\,, v_1) = 0.$

(ii) $\int_0^{T_\sigma} w(u,v) e_{\sigma,i}(u)\, du = 0$ *, for every* $v \in [v_0, v_1]$ *and* $i = 0, 1.$

Then $w = 0.$

Proof. By *(ii)*, $w = \sum_{i\geqslant 2} w_i(v)\, e_{\sigma,i}(u)$. Since the potential $P_{\sigma,i}$ of the operator $L_{\sigma,i}$ is negative for every $i \geqslant 2$ (see (4.16)) and the operator $L_{\sigma,i}$ is elliptic, the maximum principle holds. We can then conclude the lemma 63 from *(i)*. □

Now we can state the following

Lemma 64. *For all $i \geqslant 2$ and $\forall \sigma$ there exists a unique positive solution of*

$$L_{\sigma,i} w_{\sigma,i} = 0 \text{ with } w_{\sigma,i}(0) = 1$$

defined on $I_\sigma = [0, U_\sigma]$ such that

$$\frac{1}{c_{\sigma,i}} e^{\pm\gamma_{\sigma,i} v} \leqslant w_{\sigma,i}(v) \leqslant c_{\sigma,i} e^{\pm\gamma_{\sigma,i} v} \tag{4.17}$$

for some constants $\gamma_{\sigma,i} > 0$ and $c_{\sigma,i} > 1$.

Proof. For $i \geqslant 2$ and for $0 < \sigma < \pi/2$, the potential $P_{\sigma,i}$ of $L_{\sigma,i}$ is negative, hence this operator satisfies the maximum principle. We can choose two constants $\gamma < 0$ close enough to 0 and $\gamma' < 0$ with $|\gamma'|$ large enough so that

$$\gamma^2 + P_{\sigma,i} < 0 \qquad \text{and} \qquad \gamma'^{\,2} + P_{\sigma,i} > 0.$$

This choice allows us to use the Perron method to prove the existence of the solution. In fact the functions $v \to e^{\gamma v}$ and $v \to e^{\gamma' v}$ respectively satisfy

$$L_{\sigma,i} e^{\gamma v} = (\gamma^2 + P_{\sigma,i}) e^{\gamma v} < 0$$

and

$$L_{\sigma,i} e^{\gamma' v} = (\gamma'^2 + P_{\sigma,i}) e^{\gamma' v} > 0$$

that is they are respectively a subsolution and a supersolution and so they can be used like barrier functions.

The existence of the constants $\gamma_{\sigma,i}$ follows from the fact that the potential of $L_{\sigma,i}$ consists of the function $\sin^2(y(v))$ which is periodic in the variable v. We denote the period by R_σ. It is possible to show that $R_\sigma \to +\infty$ if $\sigma \to 0$. Indeed, we can define the linear operator $M_{\sigma,i}$ by

$$M_{\sigma,i}(s_1, s_2) = (s(R_\sigma), \partial_v s(R_\sigma))$$

where s is the unique solution of

$$L_{\sigma,i}\, s = 0$$

which is defined on $[0, R_\sigma]$ and which satisfies $s(0) = s_1$ and $\partial_v s(0) = s_2$.

We claim that $M_{\sigma,i}$ has two positive eigenvalues which satisfy $m_{\sigma,i} < 1 < n_{\sigma,i}$ and $m_{\sigma,i}\, n_{\sigma,i} = 1$. The solution s which has been described above is exponentially decaying and $(s(0), \partial_v s(0))$ corresponds to an eigenvector of $M_{\sigma,i}$ with eigenvalue $m_{\sigma,i} < 1$. Therefore, we have

$$M_{\sigma,i}(s(0), \partial_v s(0)) = m_{\sigma,i}\, (s(0), \partial_v s(0)) = (s(R_\sigma), \partial_v s(R_\sigma))$$

This implies that

$$(s(2\,R_\sigma), \partial_v s(2\,R_\sigma)) = M_{\sigma,i}(s(R_\sigma), \partial_v s(R_\sigma)) = m^2_{\sigma,i}\,(s(0), \partial_v s(0))$$

and so for all $k \in \mathbb{N}$ we have

$$(s(k\,R_\sigma), \partial_v s(k\,R_\sigma)) = m^k_{\sigma,i}(s(0), \partial_v s(0)).$$

Hence we have the relation

$$e^{-\gamma_{\sigma,i} R_\sigma} = m_{\sigma,i}$$

which defines $\gamma_{\sigma,i}$.

Now we prove the claim. Assume that g_1 and g_2 are two solutions of $L_{\sigma,i}s = 0$ with $g_1(0) = \partial_v g_1 = 0$ and $g_2(0) = \partial_v g_1 = 0$. The Wronskian $W(g_1, g_2)$ associated to g_1 and g_2 does not depend on the variable v because the Wronskian of a differential equation of the form

$$\partial_{vv} f + a(v)\partial_v f + b(v) f = 0$$

must satisfy the differential equation $\partial_v W + a(v)W = 0$. In this case $a(v) = 0$ and so we have $\partial_v W = 0$. Then if we compute W at $v = 0$ and $v = R_\sigma$, we can write

$$g_1(0)g_2'(0) - g_1'(0)g_2(0) = g_1(R_\sigma)g_2'(R_\sigma) - g_1'(R_\sigma)g_2(R_\sigma).$$

Now we observe that from the previous assumption it follows that

$$g_1(0)g_2'(0) = det(N_{\sigma,i})g_1(0)g_2'(0),$$

where $N_{\sigma,i}$ is the matrix associated to the operator $M_{\sigma,i}$. But the determinant is equal to the product of its eigenvalues so we have proven that $m_{\sigma,i}n_{\sigma,i} = 1$. Finally the entries of $N_{\sigma,i}$ are real and so it is for its trace. This implies that the eigenvalues are real. We must prove that the eigenvalues cannot be equal to 1. It is sufficient to observe that it is not possible, otherwise it should exist a nontrivial bounded solution of the homogeneous problem $L_{\sigma,i}\, s = 0$. Namely at a point where this solution has a positive maximum (that is its second derivative is negative) we have $0 = L_{\sigma,i}\, s \leqslant P_{\sigma,i}\, s < 0$ and in correspondence of a negative minimum (that is its second derivative is positive) $0 = L_{\sigma,i}\, s \geqslant P_{\sigma,i}\, s > 0$). □

The parameters $\gamma_{\sigma,i}$ are called the indicial roots of the operator $L_{\sigma,i}$.

When $i = 0$ and $i = 1$, the argument used above does not hold since the potential of $L_{\sigma,i}$ is not negative. In this case, the explicit solutions of the equation $L_{\sigma,i}\, s = 0$ are obtained thanks to the existence of the Jacobi fields. The idea is that our surface has many properties of symmetry and this induces 4 independent Jacobi fields that we are going to describe. We recall that in the following we will consider the two subfamilies of $M_{\sigma,\alpha,\beta}$ for which $\alpha = 0$ and $\beta = 0$. We described them in section 4.4. These surfaces are invariant by the symmetry about the plane $\{x_1 = 0\}$ ($\alpha = 0$) and the plane $\{x_2 = 0\}$ ($\beta = 0$). So the set of the Jacobi fields to consider are different in the two cases.

- Two Jacobi fields can be obtained by considering the one parameter families of minimal surfaces which is induced by the translations in the x_3-direction and by
  - the translations in the x_2-direction in the case where $\alpha = 0$,
  - the translations in the x_1-direction in the case where $\beta = 0$.

  These Jacobi fields are clearly periodic and hence bounded.
- A third Jacobi field can be obtained by considering the one parameter family of minimal surfaces which is induced by dilatation from the origin. The so-obtained Jacobi field is not bounded and in fact it grows linearly.
- The last Jacobi field can be obtained by considering the one parameter family of minimal surfaces which is induced by changing the parameter σ. Again, this Jacobi field is not periodic and grows linearly.

The Jacobi operator $\mathcal{L}_\sigma$ becomes a Fredholm operator when restricted to the following functional space.

Definition 65. *Given $\sigma \in (0, \pi/2)$, $\ell \in \mathbb{N}$, $\alpha \in (0, 1)$, $\mu \in \mathbb{R}$ and an interval I, we define $\mathcal{C}^{\ell,\alpha}_\mu(I_\sigma \times I)$ to be the space of functions $w \equiv w(u, v)$ in $\mathcal{C}^{\ell,\alpha}_{loc}(I_\sigma \times I)$ which are even in the variable u and for which the following norm is finite:*

$$\|w\|_{\mathcal{C}^{\ell,\alpha}_\mu} := \sup_{v\in I} e^{-\mu v} \|w\|_{\mathcal{C}^{\ell,\alpha}(I_\sigma\times[v,v+1])}.$$

Proposition 66. *Given $\mu \in (-2, -1)$, there exists a $\sigma_0 \in (0, \pi/2)$ such that, for every $\sigma \in (0, \sigma_0)$ and $v_0 \in \mathbb{R}$, there exists an operator*

$$\begin{array}{rccc} G_{\sigma,v_0} : & \mathcal{C}^{0,\alpha}_\mu(I_\sigma \times [v_0, +\infty)) & \longrightarrow & \mathcal{C}^{2,\alpha}_\mu(I_\sigma \times [v_0, +\infty)) \\ & f & \longmapsto & w := G_{\sigma,v_0}(f) \end{array}$$

satisfying the following statements:

(i) $\mathcal{L}_\sigma\, w = f$ *on* $I_\sigma \times [v_0, +\infty)$;

(ii) $w \in Span\{e_{\sigma,0}, e_{\sigma,1}\}$ *on* $I_\sigma \times \{v_0\}$;

(iii) $||w||_{\mathcal{C}^{2,\alpha}_\mu} \leqslant c\,||f||_{\mathcal{C}^{0,\alpha}_\mu}$, *for some constant $c > 0$ which does not depend on σ, v_0.*

Proof. Every $f \in \mathcal{C}^{0,\alpha}_{\mu}(I_\sigma \times [v_0, +\infty))$ can be decomposed as

$$f = f_0\, e_{\sigma,0} + f_1\, e_{\sigma,1} + \bar{f},$$

where $\bar{f}(\cdot, v)$ is L^2-orthogonal to $e_{\sigma,0}$ and $e_{\sigma,1}$ for each v.

Step 1. Firstly, let's prove the proposition 66 for the functions $f \in \mathcal{C}^{0,\alpha}_{\mu}(I_\sigma \times [v_0, +\infty))$ that are L^2-orthogonal to $\{e_{\sigma,0}, e_{\sigma,1}\}$. As a consequence of the lemma 63, $\mathcal{L}_\sigma$ is injective when it acts on this set of functions. Hence, the Fredholm alternative assures that there exists, for each $v_1 > v_0 + 1$, an unique $w \in \mathcal{C}^{2,\alpha}_{\mu}$, with $w(\cdot, v)$ L^2-orthogonal to $e_{\sigma,0}, e_{\sigma,1}$ satisfying:

$$\begin{cases} \mathcal{L}_\sigma\, w = f & \text{on } I_\sigma \times [v_0, v_1], \\ w(\cdot, v_0) = w(\cdot, v_1) = 0. \end{cases} \tag{4.18}$$

Assertion 67. *There exists a constant c and $\sigma_0 \in (0, \pi/2)$ such that for every $\sigma \in (0, \sigma_0)$, $v_0 \in \mathbb{R}$, $v_1 > v_0 + 1$, $f \in \mathcal{C}^{0,\alpha}_{\mu}(I_\sigma \times [v_0, v_1])$ and $w \in \mathcal{C}^{2,\alpha}_{\mu}(I_\sigma \times [v_0, v_1])$ satisfying the equation (4.18) and*

$$\|w\|_{\mathcal{C}^{0,\alpha}_{\mu}} \leqslant c\, \|f\|_{\mathcal{C}^{0,\alpha}_{\mu}}. \tag{4.19}$$

Suppose by contradiction that the assertion 67 is false. Then, for every $n \in \mathbb{N}$, there exists $\sigma_n \in (0, 1/n)$, $v_{1,n} > v_{0,n} + 1$ and f_n, w_n satisfying (4.18) (for $\sigma_n, v_{0,n}, v_{1,n}$ instead of σ, v_0, v_1) such that

$$\|f_n\|_{\mathcal{C}^{0,\alpha}_{\mu}} = 1 \quad \text{and} \quad \|w_n\|_{\mathcal{C}^{0,\alpha}_{\mu}} \to +\infty, \text{ when } n \to \infty.$$

Since $I_{\sigma_n} \times [v_{0,n}, v_{1,n}]$ is a compact set, $A_n := \sup\limits_{I_{\sigma_n} \times [v_{0,n}, v_{1,n}]} e^{-\mu v}\, |w_n|$ is achieved at a point $(u_n, v_n) \in I_{\sigma_n} \times [v_{0,n}, v_{1,n}]$. We define

$$\widetilde{w}_n(u, v) := \frac{e^{-\mu v_n}}{\|w_n\|_{\mathcal{C}^{0,\alpha}_{\mu}}}\, w_n(u, v + v_n),$$

for all $(u, v) \in I_{\sigma_n} \times I_n$, with $I_n = [v_{0,n} - v_n, v_{1,n} - v_n]$. Clearly, $A_n \leqslant \|w_n\|_{\mathcal{C}^{0,\alpha}_{\mu}}$, and

$$|\widetilde{w}_n(u, v)| \leqslant e^{\mu\, v}\, \frac{e^{-\mu(v+v_n)} |w_n(u, v + v_n)|}{A_n} \leqslant e^{\mu\, v}.$$

On the other hand, $e^{-\mu\, v} |\nabla \widetilde{w}_n| \leqslant \|\widetilde{w}_n\|_{\mathcal{C}^{2,\alpha}_{\mu}} = \frac{\|w_n\|_{\mathcal{C}^{2,\alpha}_{\mu}}}{\|w_n\|_{\mathcal{C}^{0,\alpha}_{\mu}}}$. Thanks to Schauder estimate, we obtain $||w_n||_{C^{2,\alpha}_{\mu}} \leqslant c' \left(||f_n||_{C^{0,\alpha}_{\mu}} + ||e^{-\mu\, v} w_n||_{C^0} \right) = c'\,(1 + A_n)$.
Hence[3],

$$|\nabla \widetilde{w}_n| \leqslant c'\, e^{\mu\, v} \frac{1 + \|w_n\|_{\mathcal{C}^{0,\alpha}_{\mu}}}{\|w_n\|_{\mathcal{C}^{0,\alpha}_{\mu}}} \leqslant c\, e^{\mu\, v}.$$

[3]From now on, c will denote any arbitrary positive constant.

The intervals I_n converge to a nonempty (and possibly unbounded) interval I_∞. Since the sequences $(\tilde{w}_n)_n$ and $(\nabla \tilde{w}_n)_n$ are uniformly bounded, Ascoli-Arzelà theorem assures that a subsequence of $(\tilde{w}_n)_n$ converges for $n \to \infty$ (and $\sigma_n \to 0$) on compact sets of $I_{\sigma_n} \times I_\infty$ to a function w_∞, which is L^2-orthogonal to $\{e_{0,0}, e_{0,1}\}$ for each $v \in I_\infty$, and vanishes on $I_0 \times \partial I_\infty$, when $\partial I_\infty \neq \emptyset$. Note that

$$\sup_{v \in I_n} e^{-\mu\, v} \|\widetilde{w}_n\|_{\mathcal{C}^{0,\alpha}(I_{\sigma_n} \times [v,v+1])} = \frac{A_n}{\|w_n\|_{\mathcal{C}^{0,\alpha}_\mu}}, \tag{4.20}$$

does not converge to zero. In fact $A_n \to \infty$, $\|w_n\|_{\mathcal{C}^{0,\alpha}_\mu} \leqslant \|w_n\|_{\mathcal{C}^{2,\alpha}_\mu} \leqslant c'(1 + A_n)$ then

$$\frac{A_n}{\|w_n\|_{\mathcal{C}^{0,\alpha}_\mu}} \geqslant \frac{A_n}{c'(1 + A_n)} \to \frac{1}{c'} > 0.$$

In particular,

$$\sup_{I_{\sigma_n} \times I_n} e^{-\mu v} |\tilde{w}_n| = 1. \tag{4.21}$$

Since if $n \to \infty$ we have $\sigma_n \to 0$, from (4.12) we can conclude

$$u \to x \quad \text{and} \quad v \to \frac{1}{2} \ln |\tan \frac{y}{2}|.$$

From the last expression we get

$$y(v) = 2 \arctan(e^{2v}) \quad \text{and} \quad e^{2v} = |\tan \frac{y}{2}|.$$

Using well kwown trigonometric formulae also we find

$$\cos y(v) = \frac{1 - e^{4v}}{1 + e^{4v}} \quad \text{and} \quad \sin y(v) = \frac{2e^{2v}}{1 + e^{4v}}. \tag{4.22}$$

It is possible to find the expression of the function w_∞ working with the x, y coordinates and after that to come back to the u, v coordinates. In fact we can observe that, up to subsequence, the function w_∞ satisfies $\mathcal{L}_0 w_\infty = 0$ with

$$\mathcal{L}_0 = \partial^2_{xx} + \sin y\, \partial_y \left(\sin y\, \partial_y\right) + 2 \sin^2 y.$$

Now we consider the eigenfunctions decomposition of w_∞,

$$w_\infty(x, y) = \sum_{j \geqslant 2} a_j(y) \, \cos(j\, x).$$

Each coefficient a_j must satisfy the associate Legendre differential equation

$$\left(\sin^2 y\, \partial^2_{yy} + \cos y\, \sin y\, \partial_y - j^2 + 2 \sin^2 y\right) \, a_j = 0.$$

Since $j \geqslant 2$ the solutions are the associated Legendre functions of second kind (the functions of second kind being zero) $a_j(y) = Q_1^j(\cos y)$, where

$$Q_1^j(t) = (-1)^j \sqrt{(1-t^2)^j}\, \frac{d^j Q_1^0(t)}{dt^j}, \qquad \text{with } \; Q_1^0(t) = \frac{t}{2} \ln\left(\frac{1+t}{1-t}\right) - 1.$$

Now we come back to the variables (u, v). We observe that the function

$$w_\infty(u,v) = \sum_{j \geqslant 2} Q_1^j(\cos y(v)) \cos(j\, u)$$

does not satisfy the inequality (4.21) with $n \to +\infty$ and $\mu \in (-2, -1)$, a contradiction.

This proves the assertion 67, that is, for every $v_1 > v_0 + 1$, there exists a function $\bar{w}$ satisfying (4.19). Let's take the limit as $v_1 \to \infty$. Clearly,

$$e^{-\mu\, v} |\bar{w}| \leqslant \|\bar{w}\|_{\mathcal{C}^{0,\alpha}_\mu} \leqslant c\, \|f\|_{\mathcal{C}^{0,\alpha}_\mu}.$$

And the Schauder estimates assures

$$e^{-\mu\, v} |\nabla \bar{w}| \leqslant \|\bar{w}\|_{\mathcal{C}^{2,\alpha}_\mu} \leqslant c \left(\|f\|_{\mathcal{C}^{0,\alpha}_\mu} + \|\bar{w}\|_{\mathcal{C}^{0}_\mu} \right) \leqslant c \|f\|_{\mathcal{C}^{0,\alpha}_\mu}.$$

Hence Ascoli-Arzelà theorem says to us that a subsequence of $\{w_{v_1}\}_{v_1 > v_0 + 1}$ converges to a function $w \in \mathcal{C}^{2,\alpha}_\mu$ defined on $I_\sigma \times [v_0, \infty)$, which clearly satisfies the statement (iii) of proposition 66.

Step 2 Let's now consider $f \in \mathcal{C}^{0,\alpha}_\mu(I_\sigma \times [v_0, +\infty))$ in $Span\{e_{\sigma,0}, e_{\sigma,1}\}$, i.e.

$$f(u,v) = f_0(v)\, e_{\sigma,0}(u) + f_1(v)\, e_{\sigma,1}(u).$$

We extend the functions $f_0(v), f_1(v)$ for $v \leqslant v_0$ to be equal, respectively, to $f_0(v_0), f_1(v_0)$. Given $v_1 > v_0 + 1$, consider

$$\begin{cases} L_{\sigma,j} w_j = f_j, & v \in (-\infty, v_1] \\ w_j(v_1) = \partial_v w_j(v_1) = 0 \end{cases} \tag{4.23}$$

Peano theorem assures the existence and the uniqueness of the solution w_j. Our aim consists in proving the following

Assertion 68. $\|w_j\|_{\mathcal{C}^{0,\alpha}_\mu} \leqslant c\, \|f_j\|_{\mathcal{C}^{0,\alpha}_\mu}$ *for some constant* c *which does not depend on* v_1.

Suppose by contradiction that, for every $n \in \mathbb{N}$, there exists $\sigma_n \in (0, 1/n)$, $v_{1,n} > v_{0,n} + 1$ and $f_{j,n}, w_{j,n}$ satisfying (4.23) such that

$$\|f_{j,n}\|_{\mathcal{C}^{0,\alpha}_\mu} = 1 \quad \text{and} \quad \|w_{j,n}\|_{\mathcal{C}^{0,\alpha}_\mu} \to +\infty, \text{ when } n \to \infty.$$

The solution $w_{j,n}$ of the previous equation is a linear combination of the two solutions of the homogeneous problem $L_{\sigma_n,j}\, w = 0$. They are the Jacobi fields associated to the isometries and it is known that they are at most linearly growing functions. Hence the supremum

$$A_n := \sup_{(-\infty, v_{1,n}]} e^{-\mu v} |w_{j,n}|$$

is achieved in a point which we call $v_n \in (-\infty, v_{1,n}]$. We define on $I_n := (-\infty, v_{1,n} - v_n]$ the function $\tilde{w}_{j,n}$ by

$$\tilde{w}_{j,n}(v) := \frac{1}{\|w_{j,n}\|_{\mathcal{C}^{0,\alpha}_\mu}} e^{-\mu v_n}\, w_{j,n}(v_n + v).$$

As above, one shows that the sequence $(v_{1,n} - v_n)_n$ remains bounded away from 0, that is $v_{1,n} > v_n$ for each n. Without loss of generality, we can assume that the sequence $(v_{1,n} - v_n)_n$ converges to $\bar{v}_1 \in (0, +\infty]$. We set $I_\infty = (-\infty, \bar{v}_1]$.

As in Step 1, we can also assume that the sequence of functions $(\tilde{w}_{j,n})_n$ converges on compact subsets of I_∞ to a nontrivial function $\tilde{w}_j$. We observe that $\tilde{w}_j(\bar{v}_1) = 0$ if $\bar{v}_1 < +\infty$ and that

$$\sup_{v \in I_\infty} e^{-\mu v}\, |\tilde{w}_j| = 1. \tag{4.24}$$

Secondly $\tilde{w}_j$, in the coordinates x, y is a solution of

$$\left(\sin^2 y\, \partial^2_{yy} + \cos y\, \sin y\, \partial_y - j^2 + 2\sin^2 y\right)\, a_j = 0. \tag{4.25}$$

This is again the associated Legendre differential equation. The solutions of the equation (4.25) are the associated Legendre functions of first $P_1^j(\cos y)$ and second kind $Q_1^j(\cos y)$ with $j = 0, 1$.

We have reached a contradiction because we can observe that these solutions, after the change of coordinates to come back to the u, v coordinates, do not satisfy the equation (4.24) with $\mu \in (-2, -1)$.

So we have proved that

$$\sup_{(-\infty, v_1]} e^{-\mu v}\, |w_j| \leqslant c \sup_{(-\infty, v_1]} e^{-\mu v}\, |f_j|.$$

Now we pass to the limit as v_1 tends to $+\infty$ in a sequence of solutions which are defined on I_∞. This proves the existence of a solution of

$$L_{\sigma,j}\, w_j = f_j$$

which is defined in $[v_0, +\infty)$. In addition, we know that

$$\sup_{[v_0, +\infty)} e^{-\mu v}\, |w_j| \leqslant c \sup_{[v_0, +\infty)} e^{-\mu v}\, |f_j|.$$

Multiplying for the eigenfunctions, taking the supremum on $I_\sigma \times [v_0, +\infty)$ and using a last time elliptic estimates, we get the wanted estimate. So the proof of the result is complete. □

4.6 A family of minimal surfaces close to $\widetilde{M}_{\sigma,0,\beta}$ and $\widetilde{M}_{\sigma,0,\beta}$

The aim of this section is to find a family of minimal surfaces near to a translated and dilated copy of $\widetilde{M}_{\sigma,0,\beta}$ and $\widetilde{M}_{\sigma,0,\beta}$ with given Dirichlet data on the boundary. We start recalling that in subsection 4.4.1 we got that a translated and dilated copy of $\widetilde{M}_{\sigma,\alpha,\beta}$ can be expressed as the graph over the $x_3 = 0$ plane of the function

$$(1+\gamma)\ln\frac{2r}{1+\gamma} + r\left[\eta_1\cos\theta + \eta_2\sin\theta\right] + \frac{1}{r}(\kappa_1\cos\theta + \kappa_2\sin\theta) + \kappa_3 + g_t. \tag{4.26}$$

where $g_t = \mathcal{O}(\epsilon)$, $\kappa_1, \kappa_2, \kappa_3 \in \mathbb{R}^+$ and small, $\eta_1 = b_1 + b_2$, $\eta_2 = b_1 - b_2$, $b_1 = \sin\frac{\alpha-\beta}{2}$, $b_2 = \sin\frac{\alpha+\beta}{2}$, and r belongs to a neighbourhood of $r_\epsilon = \frac{1}{2\sqrt{\epsilon}}$.

We denote by Z the immersion of the surface $\widetilde{M}_{\sigma,\alpha,\beta}$. The following proposition, whose proof is contained in section 4.11, states that the linearized of the mean curvature operator is the Lamé operator introduced in section 4.4.2.

Proposition 69. *The surface parameterized by $Z_f := Z + f\,N$ is minimal if and only if the function f is a solution of*

$$\mathcal{L}_\sigma f = Q_\sigma(f, f_u, f_v, f_{uu}, f_{uv}, f_{vv}).$$

where $\mathcal{L}$ is the Lamé operator and Q_σ is a nonlinear operator which satisfies

$$\|Q_\sigma(f_2) - Q_\sigma(f_1)\|_{\mathcal{C}^{0,\alpha}(I_\sigma\times[v,v+1])} \leqslant c \sup_{i=1,2} \|f_i\|_{\mathcal{C}^{2,\alpha}(I_\sigma\times[v,v+1])} \|f_2 - f_1\|_{\mathcal{C}^{2,\alpha}(I_\sigma\times[v,v+1])}$$

for all f_1, f_2 such that $\|f_i\|_{\mathcal{C}^{2,\alpha}(I_\sigma\times[v,v+1])} \leqslant 1$. Here the constant $c > 0$ does not depend on $v \in \mathbb{R}$, nor on $\sigma \in (0,1)$.

As a consequence of the dilation of factor $1+\gamma$ of the surface the minimal surface equation becomes

$$\mathcal{L}_\sigma\, w = \frac{1}{1+\gamma} Q_\sigma\left((1+\gamma)\cdot\right), \tag{4.27}$$

where hereafter we use a simplified notation for the operator Q_σ.
We now truncate the surfaces $\widetilde{M}_{\sigma,0,\beta}$ and $\widetilde{M}_{\sigma,\alpha,0}$ at the graph of the curve $r = \frac{1}{2\sqrt{\epsilon}}$ of the function (4.26) with, respectively, $\alpha = 0$ and $\beta = 0$, and we consider only the upper half

of these surfaces which we call M_1 and M_2. We are interested in minimal normal graphs over these surfaces which are asymptotic to them. The normal graph of the function w over M_1, M_2 is minimal, if and only if w is a solution of (4.27).
We make the following change of coordinates

$$(r,\theta) = \left(\frac{1}{2}e^v, \theta\right).$$

It is important to remark that though the surfaces M_1, M_2 can be parameterized by (58), their boundary does not correspond to the curve $v = v_\epsilon = -1/2 \ln \epsilon$. We therefore modify the above parametrization so that over the annulus $B_{4r_\epsilon} - B_{r_\epsilon}$ the image of the function (4.26) corresponds to the horizontal curve $v = v_\epsilon$. Finally, we interpolate smoothly the two parametrizations over the annulus $B_{6r_\epsilon} - B_{2r_\epsilon}$. We would like that the normal vector field relative to M_1, M_2 is vertical near the boundary of this surface. This can be achieved by modifying the normal vector field into a transverse vector field $\tilde{N}$ which agrees with the normal vector field N for all $t \geqslant v_\epsilon + \ln 4$ and with the vector e_3 for all $v \in [v_\epsilon, v_\epsilon + \ln 2]$.

Now, we consider a graph over this surface for some function u, using the modified vector field $\tilde{N}$. This graph will be minimal if and only if the function u is a solution of a nonlinear elliptic equation related to (4.27). To get the new equation, we take into account the effects of the change of parameterization and the change in the vector field N into $\tilde{N}$. The new minimal surface equation is

$$\mathcal{L}_\sigma\, w = \tilde{L}_\epsilon\, w + \tilde{Q}_\sigma\,(\cdot)\,. \tag{4.28}$$

Here $\tilde{Q}_\sigma$ enjoys the same properties of Q_σ, since it is obtained by a slight perturbation from it. The operator $\tilde{L}_\epsilon$ is a linear second order operator whose coefficients are supported in $[v_\epsilon, v_\epsilon + \ln 4] \times S^1$ and are bounded by a constant multiplied for $\epsilon^{1/2}$, in $\mathcal{C}^\infty$ topology, where partial derivatives are computed with respect to the vector fields ∂_u and ∂_v.

As a fact, if we take into account the effect of the change of the normal vector field, we would obtain, applying the result of Appendix B of [11], a similar formula where the coefficients of the corresponding operator $\tilde{L}_\epsilon$ are bounded by a constant multiplied for ϵ since

$$\tilde{N}_\epsilon \cdot N_\epsilon = 1 + \mathcal{O}(\epsilon)$$

for $t \in [t_\epsilon, t_\epsilon + \ln 2]$. Instead, if we take into account the effect of the change in the parameterization, we would obtain a similar formula where the coefficients of the corresponding operator $\tilde{L}_\epsilon$ are bounded by a constant multiplied for $\epsilon^{1/2}$. The estimate of the coefficients of $\tilde{L}_\epsilon$ follows from these considerations.

Now, assume that we are given a function $\varphi \in \mathcal{C}^{2,\alpha}(I_\sigma)$ which is even with respect to u, L^2-orthogonal to $e_{\sigma,0}, e_{\sigma,1}$ and such that $\|\varphi\|_{\mathcal{C}^{2,\alpha}} \leqslant k\epsilon$. We define

$$w_\varphi(\cdot,\cdot) := \bar{\mathcal{H}}_{v_\epsilon,\varphi}(\cdot,\cdot),$$

where $v_\epsilon = -1/2 \ln \epsilon + \mathcal{O}(1)$ and $\bar{\mathcal{H}}$ is introduced in proposition 81.
In order to solve the equation (4.28), we choose $\mu \in (-2,-1)$ and look for u of the form $w = w_\varphi + g$ where $g \in \mathcal{C}^{2,\alpha}_\mu(I_\sigma \times [v_\epsilon, \infty))$ and $w = \varphi$ on $I_\sigma \times \{v_\epsilon\}$. Using Proposition 66, we can rephrase this problem as a fixed point problem

$$g = S(\varphi, g) \tag{4.29}$$

where the nonlinear mapping S which depends on ϵ and φ is defined by

$$S(\varphi, g) := G_{\epsilon, v_\epsilon}\left(\tilde{L}_\epsilon(w_\varphi + g) - \mathcal{L}_\sigma\, w_\varphi + \tilde{Q}_\epsilon\,(w_\varphi + g)\right).$$

where the operator G_{ϵ,v_ϵ} is defined in Proposition 66. To prove the existence of a fixed point for (4.29) we need the following

Lemma 70. *There exist some constants $c_k > 0$ and $\epsilon_k > 0$, such that*

$$\|S(\varphi,0)\|_{\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))} \leqslant c_k \epsilon^{1+\mu/2} \tag{4.30}$$

and, for all $\epsilon \in (0, \epsilon_k)$

$$\|S(\varphi,g_2) - S(\varphi,g_1)\|_{\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))} \leqslant c_k \epsilon^{\frac{1}{2}}\, \|g_2 - g_1\|_{\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))}$$

for all $g_1, g_2 \in \mathcal{C}^{2,\alpha}_\mu(I_\sigma \times [v_\epsilon,\infty))$ such that $\|g_i\|_{\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))} \leqslant c_k\epsilon^{1+\mu/2}$.

Proof: We know from Proposition 66 that $\|G_{\epsilon,v_\epsilon}(f)\|_{\mathcal{C}^{2,\alpha}_\mu} \leqslant c\|f\|_{\mathcal{C}^{0,\alpha}_\mu}$, then

$$\|S(\varphi,0)\|_{\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))} \leqslant c\|\tilde{L}_\epsilon(w_\varphi) - \mathcal{L}_\sigma\, w_\varphi + \tilde{Q}_\sigma\,(w_\varphi)\,\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))} \leqslant$$

$$\leqslant c\left(\|\tilde{L}_\epsilon(w_\varphi)\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))} + \|\mathcal{L}_\sigma\, w_\varphi\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))} + \|\tilde{Q}_\sigma\,(w_\varphi)\,\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))}\right).$$

So we need to find the estimates for the three above norms.
We recall that $|\varphi|_{2,\alpha} \leqslant k\epsilon$. For all $\mu \in (-2,-1)$, thanks to Proposition 81 we know that

$$|w|_{2,\alpha;[v,v+1]} \leqslant e^{\mu(v-v_\epsilon)}|\varphi|_{2,\alpha} \tag{4.31}$$

Using the relation $e^{-\mu v_\epsilon} = \epsilon^{\mu/2}$ we know that

$$\|w_\varphi\|_{\mathcal{C}^{2,\alpha}_\mu} = \sup_{v\in[v_\epsilon,\infty]} e^{-\mu v}|w|_{2,\alpha;[v,v+1]} \leqslant \sup_{v\in[v_\epsilon,\infty]} e^{-\mu v}e^{\mu(v-v_\epsilon)}|\varphi|_{2,\alpha} \leqslant$$

$$\leqslant \epsilon^{\mu/2}|\varphi|_{2,\alpha} \leqslant c_k\epsilon^{1+\mu/2}.$$

From this inequality and from the estimates of the coefficients of $\tilde{L}_\epsilon$, it follows that

$$\|\tilde{L}_\epsilon(w_\varphi)\|_{\mathcal{C}^{0,\alpha}_\mu} \leqslant c\epsilon^{1/2}\|w_\varphi\|_{\mathcal{C}^{0,\alpha}_\mu} \leqslant c_k\epsilon^{(3+\mu)/2}.$$

As for $\mathcal{L}_\sigma$ we consider the following relation

$$\mathcal{L}_\sigma\, w_\varphi = 2T\, w_\varphi,$$

where $T \leqslant 1$. It comes from the definition of w_φ and Proposition 81 which gives us the following relation:

$$\partial^2_{uu} w_\varphi + \partial^2_{vv} w_\varphi = 0.$$

Therefore, we conclude that

$$\|\mathcal{L}_\sigma\, w_\varphi\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))} \leqslant 2\|w_\varphi\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))} \leqslant c_k\epsilon^{1+\mu/2}.$$

The last term is estimated by

$$\|\tilde{Q}_\sigma\,(w_\varphi)\,\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))} \leqslant c_k\epsilon^{2+\mu/4}.$$

In fact

$$\|\tilde{Q}_\sigma\,(w_\varphi)\,\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))} \leqslant c \sup_{v\in[v_\epsilon,\infty)} e^{-\mu v}|w|^2_{0,\alpha;I_\sigma\times[v,v+1]} \leqslant c\|w\|^2_{2,\alpha,\mu/2} \leqslant c|\varphi|^2_{2,\alpha} \sup_{[v_\epsilon,\infty)} e^{-\mu v/2}e^{\mu(v-v_\epsilon)}$$

$$\leqslant c|\varphi|^2_{2,\alpha} \sup_{[v_\epsilon,\infty)} e^{+\mu v/2}e^{-\mu v_\epsilon} \leqslant c\epsilon^{+\mu/4}|\varphi|^2_{2,\alpha} \leqslant c_k\epsilon^{2+\mu/4}.$$

Putting together these estimates we get the first result. As for the second estimate, we recall that

$$S(\varphi,g) := G_{\epsilon,v_\epsilon}\left(\tilde{L}_\epsilon(w_\varphi+g) - \mathcal{L}_\sigma\, w_\varphi + \tilde{Q}_\sigma\,(w_\varphi+g)\right).$$

Then

$$S(\varphi,g_2) - S(\varphi,g_1) = G_{\epsilon,v_\epsilon}\left(\tilde{L}_\epsilon(w_\varphi+g_2) - \mathcal{L}_\sigma\, w_\varphi + \tilde{Q}_\sigma\,(w_\varphi+g_2)\right) -$$

$$G_{\epsilon,v_\epsilon}\left(\tilde{L}_\epsilon(w_\varphi+g_1) - \mathcal{L}_\sigma\, w_\varphi + \tilde{Q}_\sigma\,(w_\varphi+g_1)\right)$$

and

$$\|S(\varphi,g_2) - S(\varphi,g_1)\|_{\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))} \leqslant c\|\tilde{L}_\epsilon(w_\varphi+g_2) - \mathcal{L}_\sigma\, w_\varphi + \tilde{Q}_\sigma\,(w_\varphi+g_2) -$$

$$-\tilde{L}_\epsilon(w_\varphi+g_1) + \mathcal{L}_\sigma\, w_\varphi - \tilde{Q}_\sigma\,(w_\varphi+g_1)\,\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))} =$$

$$\|\tilde{L}_\epsilon(w_\varphi+g_2) - \tilde{L}_\epsilon(w_\varphi+g_1) + \tilde{Q}_\sigma\,(w_\varphi+g_2) - \tilde{Q}_\sigma\,(w_\varphi+g_1)\,\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))} =$$

$$= \|\tilde{L}_\epsilon(g_2-g_1) + \tilde{Q}_\sigma\,(w_\varphi+g_2) - \tilde{Q}_\sigma\,(w_\varphi+g_1)\,\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))} \leqslant$$

$$\leqslant \|\tilde{L}_\epsilon(g_2 - g_1)\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))} + \|\tilde{Q}_\sigma\left(w_\varphi + g_1\right) - \tilde{Q}_\sigma\left(w_\varphi + g_2\right)\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))}.$$

We observe that from the considerations above it follows that

$$\|\tilde{L}_\epsilon(g_2 - g_1)\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))} \leqslant c_k\epsilon^{1/2}||g_2 - g_1||_{\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))}$$

and that there is $j + j' \geqslant 1$ such that:

$$\|\tilde{Q}_\sigma\left(w_\varphi + g_1\right) - \tilde{Q}_\sigma\left(w_\varphi + g_2\right)\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))} \leqslant$$

$$\leqslant c||g_2 - g_1||_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))}\left(|w_\varphi + g_2|^j|w_\varphi + g_1|^{j'}\right) \leqslant$$

$$\leqslant c_k\left(\epsilon^{2+\mu/2}\right)^{j+j'}||g_2 - g_1||_{\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))} \leqslant c_k\epsilon^{4+2\mu}||g_2 - g_1||_{\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))}.$$

Then

$$\|S(\varphi, g_2) - S(\varphi, g_1)\|_{\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))} \leqslant c_k\epsilon^{4+2\mu}||g_2 - g_1||_{\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))}.$$

□

Theorem 71. *Let be* $B := \{g \in \mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))\,|\,||g||\mathcal{C}^{2,\alpha}_\mu \leqslant c_k\epsilon^{1+\mu/2}\}$. *Then the nonlinear mapping* S *defined above has a unique fixed point* g *in* B.

Proof. The previous lemma shows that, if ϵ is chosen small enough, the nonlinear mapping S is a contraction mapping[4] from the ball B of radius $c_k\epsilon^{1+\mu/2}$ in $\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\epsilon,\infty))$ into itself. This value comes from the estimate of the norm of $S(\varphi, 0)$. Consequently thanks to the Schauder theorem, S has a unique fixed point v in this ball. □

This argument provides a minimal surface $M_i(\phi)$ which is close to M_i and has one boundary. This surface is, close to its boundary, a vertical graph over the annulus $B_{\epsilon^{-1/2}/2} - B_{\epsilon^{-1/2}/4}$ whose parametrization is given, for $\alpha = 0$ by

$$U_{t,1}(r,\theta) = (1+\gamma)\ln\frac{2r}{1+\gamma} + r\eta_2\sin\theta + \frac{1}{r}(\kappa_1\cos\theta + \kappa_2\sin\theta) + \kappa_3 + \bar{\mathcal{H}}_{v_\epsilon,\varphi}(v_\epsilon - \ln 2r, \theta) + V(r,\theta).$$

and, for $\beta = 0$, by

$$U_{t,2}(r,\theta) = (1+\gamma)\ln\frac{2r}{1+\gamma} + r\eta_1\cos\theta + \frac{1}{r}(\kappa_1\cos\theta + \kappa_2\sin\theta) + \kappa_3 + \bar{\mathcal{H}}_{v_\epsilon,\varphi}(v_\epsilon - \ln 2r, \theta) + V(r,\theta).$$

where $v_\epsilon = -\frac{1}{2}\ln\epsilon$. The boundary of the surface corresponds to $r_\epsilon = \frac{1}{2}\epsilon^{-1/2}$. The function V depends non linearly on ϵ, ϕ. It satisfies $\|V(\epsilon,\phi_i)(r_\epsilon\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1 - B_{1/2})} \leqslant c\epsilon$ and

$$\|V_i(\epsilon,\phi)(r_\epsilon\cdot) - V_i(\epsilon,\phi')(r_\epsilon\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1 - B_{1/2})} \leqslant c\epsilon^{1-\mu/2}\|\phi - \phi'\|_{\mathcal{C}^{2,\alpha}(I_\sigma)}.$$

[4]after the correct choise of the constant k that appears in the estimate of the norm of φ.

4.7 Periodic flat cylinder

In this section we are interested in finding an infinite family of minimal surface close to a horizontal strip from which we have removed a circle. We shall use the following model. We denote by $\Sigma = \{x+iy \in \mathbb{C}; y \in [-\frac{\pi}{\eta}, \frac{\pi}{\eta}]\}$, with $\eta > 0$ and enough small, the horizontal flat cylinder whose embedding in $\mathbb{R}^3/T$ is $X(z) = (z, 0)$, where $T = \frac{2\pi}{\eta}e_2$ is the period. Topologically Σ is equivalent to $\mathbb{R} \times S^1$. We denote B_s the ball of radius s centered in the origin. The equation to consider is

$$\text{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 \tag{4.32}$$

for u defined on $\Sigma - B_s$ and with boundary data on ∂B_s.

We define the subdomains $\Omega_{x_1} = \{z \in \Sigma; |x| \leqslant x_1\}$ and denote the two cylindrical ends with $E_1 = \{z \in \Sigma;\ x \geqslant x_1\}$, $E_2 = \{z \in \Sigma;\ x \leqslant -x_1\}$.

Definition 72. *Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\mu \in \mathbb{R}$, we define the weighted Hölder space $C^{k,\alpha}_\mu(\Sigma)$ to be the space of functions of $C^{k,\alpha}_{loc}(\Sigma)$ for which the following norm is finite*

$$||u||_{C^{k,\alpha}_\mu} := [u]_{k,\alpha;\Omega_{x_1}} + \sup_{x \geqslant x_1} e^{-\mu x}\big([u]_{k,\alpha,([x,x+1]\times S^1)} + [u]_{k,\alpha,([-x-1,-x]\times S^1)}\big)$$

where $[u]_{k,\alpha,\Omega}$ denotes the usual $C^{k,\alpha}$ Hölder norm on the set Ω.

We denote by $C^{k,\alpha}_\mu(\Sigma - B_s)$ the subspace of the functions of $C^{k,\alpha}_\mu(\Sigma)$ restricted to $\Sigma - B_s$ and by $[C^{k,\alpha}_\mu(\Sigma - B_s)]_0$ the subspace of the functions vanishing on the boundary.

There exists an extension operator $\mathcal{E}_s : C^{0,\alpha}_\mu(\Sigma - B_s) \to C^{0,\alpha}_\mu(\Sigma)$ which satisfies $||\mathcal{E}_s(u)|| \leqslant C||u||$. The operator is defined by $\mathcal{E}_s(u) = u$ on $\Sigma - B_s$ and $\mathcal{E}_s(u) = (2|z|/s - 1)u(z)$ on $B_s - B_{s/2}$ and $\mathcal{E}_s(u) = 0$ on $B_{s/2}$.

Now we consider the bounded operator Δ_μ :

$$\Delta : C^{2,\alpha}_\mu(\Sigma) \longrightarrow C^{0,\alpha}_\mu(\Sigma)$$

where $\Delta = \partial^2_y + \partial^2_x$. It follows from the general theory of elliptic partial differential operators that Δ_μ is a Fredholm operator for all $\mu \notin \mathbb{Z}$. It is well known that, if $\mu \in \mathbb{R} - \mathbb{Z}$, then Δ_μ is injective if and only if $\Delta_{-\mu}$ is surjective and the dimension of the kernel of $\Delta_{-\mu}$ is equal to the cokernel of Δ_μ. Furthermore, it is possible to show that the operator Δ_μ is injective for $-1 < \mu < 0$ and so it is surjective for $0 < \mu < 1$.

When the weight parameter is negative, we can still make the operator surjective by considering a finite dimensional extension of the Hölder space defined above. To be more precise we need additional notation. Let us set $x_1 = 2s$ such that $B_s \subset \{-x_1 < x < x_1\}$. We introduce the cut-off functions ξ_1 and ξ_2 satisfying the following conditions:

- $\xi_1 = 0$ for $x < x_1$ and $\xi_1 = 1$ for $x > 2x_1$,
- $\xi_2 = 0$ for $x > -x_1$ and to $\xi_2 = 1$ for $x < -2x_1$.

Now we are ready to define

$$\mathcal{D}_0 := \text{Span}\{\xi_1, \xi_2, \xi_1 x, \xi_2 x\}.$$

This space is identified with $\mathbb{R}^4$ and is endowed with the Euclidean norm. For $-1 < \mu < 0$ we have $\text{Ker}\,\Delta_{-\mu} \subset C^{2,\alpha}_\mu(\Sigma) \oplus \mathcal{D}_0$. In fact if we expand a solution of $\Delta u = 0$ on $\{x > x_1\}$ by

$$u = \sum_{n \in \mathbb{Z}} u_n(x) e^{i \frac{2\pi}{T} ny},$$

where T is the period with respect to the variable y, then $u = u_0 + \bar{u}$, with $\bar{u} \in C^{2,\alpha}_\mu$ with $-1 < \mu < 0$, and u_0 affine function, i.e. a linear combination of the constant function and x near the cylindrical ends E_1 and E_2. The index theory assures that the kernel is two dimensional. We define $K_0 = \text{Span}\{\xi_1, \xi_2\}$. The linear decomposition Lemma proved in [23] for constant mean curvature surfaces (see also [18] for minimal hypersurfaces) can be adapted to our situation. So $D_0 = K_0 \oplus N_0$, where N_0 denotes a complementary space. Then

$$\ker \Delta_{-\mu} \subset C^{2,\alpha}_\mu(\Sigma) \oplus N_0$$

and the application

$$\Delta : C^{2,\alpha}_\mu(\Sigma) \oplus K_0 \longrightarrow C^{0,\alpha}_\mu(\Sigma)$$

is an isomorphism.

Proposition 73. *If $0 < \mu < \epsilon^2$ and $T = \frac{2\pi}{\eta}$ then there exists an operator*

$$F_\mu : \mathcal{C}^{0,\alpha}_\mu(\Sigma) \to \mathcal{C}^{2,\alpha}_\mu(\Sigma) \oplus K_0$$

such that for all $f \in \mathcal{C}^{0,\alpha}_\mu(\Sigma)$, the function $v := F_\mu(f) = w + a_1\xi_1 + a_2\xi_2$ solves $\Delta v = f$ in Σ. Moreover,

$$||F_\mu(f)||_{\mathcal{C}^{2,\alpha}_\mu \oplus K_0} = |a_1| + |a_2| + ||w||_{\mathcal{C}^{2,\alpha}_\mu} \leqslant c\, \|f\|_{\mathcal{C}^{0,\alpha}_\mu},$$

for some constant $c > 0$.

Proof. We solve the equation $\Delta v = f$ at the ends E_1 and E_2 of Σ. For $x > x_1$ we consider the Fourier series of f and v (a similar argument holds for E_2).

$$v = \sum_{n\in\mathbb{Z}} v_n e^{i\frac{2\pi}{T}ny} \text{ and } f = \sum_{n\in\mathbb{Z}} f_n e^{i\frac{2\pi}{T}ny}$$

For $n \neq 0$, applying the barrier functions method we find that the solution v_n satisfies

$$|v_n(x)| \leqslant \frac{||f||_{\mathcal{C}^{0,\alpha}_\mu}}{\left(\frac{2\pi}{T}\right)^2 n^2 - \mu^2} e^{\mu x},$$

and for $n = 0$, $v_n(x)$ is given by

$$v_0(x) = \int_x^\infty \int_t^\infty f_0(s) ds dt.$$

We can conclude that $e^{-\mu x}|v| \leqslant c||f||_{\mathcal{C}^{0,\alpha}_\mu}$ and applying Schauder estimates we obtain

$$||\xi_1 v||_{\mathcal{C}^{2,\alpha}_\mu} \leqslant c||f||_{\mathcal{C}^{0,\alpha}_\mu}.$$

Obviously we can apply the same argument for E_2. We denote by v_1, v_2 the solutions corresponding to E_1, E_2, we define $g = \xi_1 v_1 + \xi_2 v_2$ and we set $\bar{f} = f - \Delta g$. We parametrize Σ conformally on $\mathbb{C}^*$ and we solve $\Delta w = \bar{f}$ on $\mathbb{C} \cup \{\infty\}$. Then the function $v = w + g$ satisfies $\Delta v = f$. □

It is possible to show the following result.

Proposition 74. *Let $\phi \in C^{2,\alpha}(\partial B_s)$ a function L^2-orthogonal to $z \to 1$, then there exists an operator $\mathcal{H}$ such that $w_\phi = \mathcal{H}_\phi \in C^{2,\alpha}_{-2}(\Sigma - B_s) \oplus \mathrm{Span}\{\xi_1, \xi_2\}$ which solves the following problem*

$$\begin{cases} \Delta w_\phi = 0 & on \quad \Sigma - B_s \\ w_\phi = \phi & on \quad \partial B_s. \end{cases}$$

and satisfies $\|\mathcal{H}_\phi\|_{C^{2,\alpha}_{-2}(\Sigma - B_s)\oplus K_0} \leqslant c\|\phi\|_{C^{2,\alpha}}$.

In the following we consider $s = \bar{s} = \frac{1}{2\sqrt{\epsilon}}$. We recall that we have set $x_1 = 2s$.

Let $\phi(\theta)$ be a $C^{2,\alpha}$ function defined on $\partial B_{\bar{s}}$, even and L^2 orthogonal to the costant function and to $\theta \to \cos\theta$ and such that

$$||\phi||_{\mathcal{C}^{2,\alpha}(S^1)} \leqslant \kappa\epsilon \tag{4.33}$$

and $w_\phi = \mathcal{H}_\phi$ the harmonic extension in $C^{2,\alpha}_\mu(\Sigma - B_{\bar{s}}) \oplus \mathrm{Span}\{\xi_1, \xi_2\}$ obtained applying Proposition 74. Our aim is to find a minimal surface close to $\Sigma - B_{\bar{s}}$ and which is the graph of a function whose form is $w_\phi + v$. The equation (4.32) can be written in the following form

$$\Delta v = Q(v + w_\phi) \text{ on } \Sigma - B_{\bar{s}},$$

where

$$Q(v + w_\phi) = \frac{\nabla(v + w_\phi)\nabla|\nabla(v + w_\phi)|^2}{2(1 + |\nabla(v + w_\phi)|^2)}.$$

When $v \in \mathcal{C}^{2,\alpha}_\mu(\Sigma - B_{\bar{s}}) \oplus K_0$ then $\nabla v \in C^{1,\alpha}_\mu(\Sigma - B_{\bar{s}}) \oplus K_0$ and $Q(v) \in C^{0,\alpha}_\mu(\Sigma - B_{\bar{s}}) \oplus K_0$. We rephrase the problem as a fixed point problem, that is

$$v = T(\phi, v) \tag{4.34}$$

where

$$T(\phi, v) = F_\mu \circ \mathcal{E}_{\bar{s}}(Q(v + w_\phi)).$$

To prove the existence of a solution of (4.34) we need the following result which states that T is a contracting mapping.

Lemma 75. *There exist constants $c_\kappa > 0$ and $\epsilon_\kappa > 0$, such that*

$$\|T(\phi, 0)\|_{\mathcal{C}^{2,\alpha}_\mu \oplus K_0} \leqslant c_\kappa\, \epsilon^{\frac{3}{2}} \tag{4.35}$$

and, for all $\epsilon \in (0, \epsilon_\kappa)$

$$\|T(\phi, v_2) - T(\phi, v_1)\|_{\mathcal{C}^{2,\alpha}_\mu \oplus K_0} \leqslant \frac{1}{2}\, \|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}_\mu \oplus K_0}$$

and

$$\|T(\phi_2, v_2) - T(\phi_1, v_1)\|_{\mathcal{C}^{2,\alpha}_\mu \oplus K_0} \leqslant c_\kappa\, \epsilon\, \|\phi_2 - \phi_1\|_{\mathcal{C}^{2,\alpha}(S^1)}$$

for all $v, v_1, v_2 \in \mathcal{C}^{2,\alpha}_\mu(\Sigma) \oplus K_0$ whose norm is bounded by $2\, c_\kappa\, \epsilon^{\frac{3}{2}}$ and for all boundary data ϕ, ϕ_1, ϕ_2 that are orthogonal to the constant function and to $\theta \to \cos\theta$ and satisfy (4.33).

Proof. We use the result of Proposition 74 to obtain the estimate

$$\|\mathcal{E}_{\bar{s}}(Q(w_\phi))\|_{\mathcal{C}^{0,\alpha}_\mu} \leqslant \|w_\phi \circ E_1\|^2_{\mathcal{C}^{0,\alpha}_{\mu/2}(\Sigma)} + c\|w_\phi\|^2_{K_0} \leqslant c\epsilon^{\frac{3}{2}}.$$

As for the second estimate, we recall that

$$T(\varphi, v) := F_\mu \circ \mathcal{E}_{\bar{s}}\left(Q\left(w_\varphi + v\right)\right).$$

Then

$$T(\varphi, v_2) - T(\varphi, v_1) = F_\mu \circ \mathcal{E}_{\bar{s}}\left(Q\left(w_\varphi + v_2\right)\right) - F_\mu \circ \mathcal{E}_{\bar{s}}\left(Q\left(w_\varphi + v_1\right)\right)$$

and

$$\|T(\varphi, v_2) - T(\varphi, v_1)\|_{\mathcal{C}^{2,\alpha}_\mu(\Sigma) \oplus K_0} \leqslant \|\mathcal{E}_{\bar{s}}\left(Q\left(w_\varphi + v_1\right) - Q\left(w_\varphi + v_2\right)\right)\|_{\mathcal{C}^{0,\alpha}_\mu(\Sigma) \oplus K_0} \leqslant$$

$$\leqslant c\|w_\varphi\|_{\mathcal{C}^{0,\alpha}_\mu(\Sigma)\oplus K_0}\|v_2-v_1\|_{\mathcal{C}^{2,\alpha}_\mu(\Sigma)\oplus K_0} \leqslant c_k\epsilon^{\frac{1}{2}}\|v_2-v_1\|_{\mathcal{C}^{2,\alpha}_\mu(\Sigma)\oplus K_0}.$$

□

The previous lemma shows that, provided ϵ is chosen small enough, the nonlinear mapping $T(\phi,\,\cdot)$ is a contraction mapping from the ball of radius $2\,c_\kappa\,\epsilon^{\frac{3}{2}}$ in $\mathcal{C}^{2,\alpha}_\mu(\Sigma-B_{\bar{s}})$ into itself. Consequently the equation (4.34) admits a solution v in this ball. The graph over $\Sigma-B_{\bar{s}}$ for the function $w_\phi+v$ is a minimal surface close to $\Sigma-B_{\bar{s}}$, it has two horizontal ends and one boundary. This surface is by construction a vertical graph over the annulus $\bar{B}_{2\bar{s}}-B_{\bar{s}}$ for some function $\bar{U}_B$ that can be expanded as

$$\bar{U}_B(r,\theta)=\mathcal{H}_\phi(r,\theta)+\tilde{V}_B(r,\theta)$$

where $\tilde{V}_B(\epsilon,\varphi)$ depends nonlinearly on ϵ and ϕ. The boundary of the surface corresponds to $r=r_\epsilon=\frac{1}{2\sqrt{\epsilon}}$. Furthermore the following estimates hold

$$\|\tilde{V}_B(\epsilon,\varphi)(r_\epsilon\,\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_2-B_1)}\leqslant\,c\,\epsilon$$

and

$$\|\tilde{V}_B(\epsilon,\phi)(r_\epsilon\,\cdot)-\tilde{V}_B(\epsilon,\phi')(r_\epsilon\,\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_2-B_1)}\leqslant\,c\,\epsilon^{\frac{1}{2}}\,\|\phi-\phi'\|_{\mathcal{C}^{2,\alpha}(S^1)} \tag{4.36}$$

where the constant $c>0$ does not depend on ϵ.

4.8 The existence of minimal graph close to a Scherk type surface.

In this section we are interested in proving the existence of a family of minimal surfaces close to a Scherk type surface K defined on $\Sigma-B_{r_\epsilon}$ where $r_\epsilon=1/2\sqrt{\epsilon}$. We solve the following equivalent problem. Let $\bar{\Sigma}=\{x+iy\in\mathbb{C};y\in[-\pi,\pi]\}$ the horizontal flat cylinder whose embedding in $\mathbb{R}^3/T$ is $X(z)=(z,0)$, where $T=2\pi e_2$ is the period. Topologically $\bar{\Sigma}$ is equivalent to $\mathbb{R}\times S^1$. We denote by B_s the ball of radius s centered in the origin. We denote by X the solution of

$$\Delta u=0\quad\text{in}\quad\bar{\Sigma}-B_s, \tag{4.37}$$

where δ_0 is the Dirac distribution, such that, up to an additive constant, in a neighbourhood of the point $z=0$ is asymptotic to $\epsilon\ln r$ and in $\{x>x_k\}$, with k enough big, is asymptotic to a function which is linear in the variable x. We want to show the existence of a minimal surface which is the graph of the function given by $X+w$ about $\bar{\Sigma}-B_s$. The equation to consider is

$$\operatorname{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}=0 \tag{4.38}$$

for u defined on $\bar{\Sigma} - B_s$ and with boundary data on ∂B_s.
Let $\phi(\theta)$ be a $C^{2,\alpha}$ function defined on $\partial B_{\bar{s}}$, even and L^2 orthogonal to the costant function and to $\theta \to \cos\theta$ and such that

$$||\phi||_{\mathcal{C}^{2,\alpha}(S^1)} \leqslant \kappa\epsilon^2 \tag{4.39}$$

and $w_\phi = \mathcal{H}_\phi$ the harmonic extension in $C^{2,\alpha}_\mu(\bar{\Sigma} - B_{\bar{s}}) \oplus \mathrm{Span}\{\xi_1, \xi_2\}$ obtained applying Proposition 74. Our aim is to find a minimal surface close to $\bar{\Sigma} - B_{\bar{s}}$ and which is the graph of a function whose form is $X + w_\phi + v$. The equation (4.38) can be written in the following form

$$\Delta v = Q(X + v + w_\phi) \text{ on } \bar{\Sigma} - B_{\bar{s}},$$

where

$$Q(X + w_\phi + v) = \frac{\nabla(X + w_\phi + v)\nabla|\nabla(X + w_\phi + v)|^2}{2(1 + |\nabla(X + w_\phi + v)|^2)}.$$

When $t \in \mathcal{C}^{2,\alpha}_\mu(\bar{\Sigma} - B_{\bar{s}}) \oplus K_0$ then $\nabla t \in C^{1,\alpha}_\mu(\bar{\Sigma} - B_{\bar{s}}) \oplus K_0$ and $Q(t) \in C^{0,\alpha}_\mu(\bar{\Sigma} - B_{\bar{s}}) \oplus K_0$. We rephrase the problem as a fixed point problem, that is

$$v = T(\phi, v) \tag{4.40}$$

where

$$T(\phi, v) = F_\mu \circ \mathcal{E}_{\bar{s}}(Q(X + w_\phi + v)).$$

To prove the existence of a solution of (4.40) we need the following result which states that T is a contracting mapping.

Lemma 76. *There exist constants $c_\kappa > 0$ and $\epsilon_\kappa > 0$, such that*

$$\|T(\phi, 0)\|_{\mathcal{C}^{2,\alpha}_\mu \oplus K_0} \leqslant c_\kappa\, \epsilon^{\frac{3}{2}} \tag{4.41}$$

and, for all $\epsilon \in (0, \epsilon_\kappa)$

$$\|T(\phi, v_2) - T(\phi, v_1)\|_{\mathcal{C}^{2,\alpha}_\mu \oplus K_0} \leqslant \frac{1}{2}\, \|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}_\mu \oplus K_0}$$

and

$$\|T(\phi_2, v_2) - T(\phi_1, v_1)\|_{\mathcal{C}^{2,\alpha}_\mu \oplus K_0} \leqslant c_\kappa\, \epsilon\, \|\phi_2 - \phi_1\|_{\mathcal{C}^{2,\alpha}(S^1)}$$

for all $v, v_1, v_2 \in \mathcal{C}^{2,\alpha}_\mu(\Sigma) \oplus K_0$ whose norm is bounded by $2\, c_\kappa\, \epsilon^{\frac{3}{2}}$ and for all boundary data ϕ, ϕ_1, ϕ_2 that are orthogonal to the constant function and to $\theta \to \cos\theta$ and satisfy (4.39).

Proof. We use the result of Proposition 74 to obtain the estimate

$$\|\mathcal{E}_{\bar{s}}(Q(X + w_\phi))\|_{\mathcal{C}^{0,\alpha}_\mu} \leqslant \|(X + w_\phi) \circ E_1\|^2_{\mathcal{C}^{0,\alpha}_{\mu/2}(\Sigma)} + c\|(X + w_\phi)\|^2_{K_0} \leqslant c\epsilon^{\frac{3}{2}}.$$

As for the second estimate, we recall that

$$T(\varphi, v) := F_\mu \circ \mathcal{E}_{\bar{s}}\left(Q\left(X + w_\phi + v\right)\right).$$

Then

$$T(\phi, v_2) - T(\phi, v_1) = F_\mu \circ \mathcal{E}_{\bar{s}} \left(Q \left(X + w_\phi + v_2 \right) \right) - F_\mu \circ \mathcal{E}_{\bar{s}} \left(Q \left(X + w_\phi + v_1 \right) \right)$$

and

$$\| T(\phi, v_2) - T(\varphi, v_1) \|_{\mathcal{C}^{2,\alpha}_\mu(\Sigma) \oplus K_0} \leqslant \| \mathcal{E}_{\bar{s}} \left(Q \left(X + w_\phi + v_1 \right) - Q \left(X + w_\phi + v_2 \right) \right) \|_{\mathcal{C}^{0,\alpha}_\mu(\Sigma) \oplus K_0} \leqslant$$

$$\leqslant c \| X + w_\phi \|_{\mathcal{C}^{0,\alpha}_\mu(\Sigma) \oplus K_0} \| v_2 - v_1 \|_{\mathcal{C}^{2,\alpha}_\mu(\Sigma) \oplus K_0} \leqslant c_k \epsilon^{\frac{1}{2}} \| v_2 - v_1 \|_{\mathcal{C}^{2,\alpha}_\mu(\Sigma) \oplus K_0}.$$

□

The previous lemma shows that, provided ϵ is chosen small enough, the nonlinear mapping $T(\phi, \cdot)$ is a contraction mapping from the ball of radius $2\, c_\kappa\, \epsilon^{\frac{3}{2}}$ in $\mathcal{C}^{2,\alpha}_\mu(\bar{\Sigma} - B_{\bar{s}})$ into itself. Consequently the equation (4.40) admits a solution v in this ball. The graph over $\bar{\Sigma} - B_{\bar{s}}$ for the function $X + w_\phi + v$ is a minimal surface close to the graph of the function X about $\bar{\Sigma} - B_{\bar{s}}$, it has a Scherk type end and one boundary. This surface is by construction a vertical graph over the annulus $\bar{B}_{2\bar{s}} - B_{\bar{s}}$ for some function $\bar{U}_K$ that can be expanded as

$$\bar{U}_K(r, \theta) = \epsilon \ln r + \mathcal{H}_\phi(r, \theta) + \tilde{V}_K(r, \theta)$$

where $\tilde{V}_K(\epsilon, \phi)$ depends nonlinearly on ϵ and ϕ. The boundary of the surface corresponds to $r = \sqrt{\epsilon}/2$. Furthermore the following estimates hold

$$\| \tilde{V}_K(\epsilon, \phi)(r_\epsilon \, \cdot) \|_{\mathcal{C}^{2,\alpha}(\bar{B}_2 - B_1)} \leqslant \, c\, \epsilon$$

and

$$\| \tilde{V}_K(\epsilon, \phi)(r_\epsilon \, \cdot) - \tilde{V}_K(\epsilon, \phi')(r_\epsilon \, \cdot) \|_{\mathcal{C}^{2,\alpha}(\bar{B}_2 - B_1)} \leqslant \, c\, \epsilon^{\frac{1}{2}} \, \| \phi - \phi' \|_{\mathcal{C}^{2,\alpha}(S^1)} \tag{4.42}$$

where the constant $c > 0$ does not depend on ϵ.

By the equations written above it is easy to get the solution of our initial problem: that is finding a Scherk type minimal graph S_K about $\Sigma - B_{r_\epsilon}$ with Dirichlet condition on ∂B_{r_ϵ}. This surface is by construction a vertical graph over the annulus $\bar{B}_{2r_\epsilon} - B_{r_\epsilon}$ for some function U_K that can be expanded as

$$U_K(r, \theta) = \ln r + \mathcal{H}_\varphi(r, \theta) + V_K(r, \theta)$$

where $V_K(\epsilon, \varphi)$ depends nonlinearly on ϵ and φ. It is important to remark that the norm of the Dirichlet data satisfies $\| \varphi \|_{\mathcal{C}^{2,\alpha}(S^1)} \leqslant \kappa \epsilon$. The boundary of the surface corresponds to $r = r_\epsilon = \frac{1}{2\sqrt{\epsilon}}$. Furthermore the following estimates hold

$$\| V_K(\epsilon, \varphi)(r_\epsilon \, \cdot) \|_{\mathcal{C}^{2,\alpha}(\bar{B}_2 - B_1)} \leqslant \, c\, \epsilon$$

and

$$\| V_K(\epsilon, \varphi)(r_\epsilon \, \cdot) - V_K(\epsilon, \varphi')(r_\epsilon \, \cdot) \|_{\mathcal{C}^{2,\alpha}(\bar{B}_2 - B_1)} \leqslant \, c\, \epsilon^{\frac{1}{2}} \, \| \varphi - \varphi' \|_{\mathcal{C}^{2,\alpha}(S^1)} \tag{4.43}$$

where the constant $c > 0$ does not depend on ϵ.

4.9 The matching of Cauchy data

In the following we will need a new notation. Given an even function $f \in \mathcal{C}^{2,\alpha}(S^1)$ with the following Fourier expansion

$$f(\theta) = \sum_{n\in\mathbb{N}} a_n \cos(n\theta),$$

then we denote with $\pi''(f)$ the function

$$\sum_{n\geqslant 2} a_n \cos(n\theta)$$

and with $\pi'(f)$ the function

$$a_0 + a_1 \cos(\theta).$$

Along this section $r_\epsilon = \frac{1}{2\sqrt{\epsilon}}$ and φ is a function in $\pi''C^{2,\alpha}(S^1)$ such that $\|\varphi\|_{C^{2,\alpha}} \leqslant c\epsilon$.

In Section 4.7 we have shown the existence of a surface which is a graph on $\Sigma - B_{r_\epsilon}$ and is asymptotic to it. In other words we are able to solve the minimal surface equation, that we will write for short as follows

$$\begin{cases} L_p u^0 &= Q_\Sigma(u^0) \quad \text{on} \quad \Sigma - B_{r_\epsilon} \\ \pi'' u^0 &= \varphi \qquad\quad \text{on} \quad \partial B_{r_\epsilon} \end{cases}$$

where L_p denotes the linearized mean curvature operator about the plane.

So chosen a function φ, we can obtain a surface S^0, whose parametrization $X_{0,\varphi}$, in a neighbourhood of its boundary (a curve defined on ∂B_{r_ϵ}), satisfies

$$X_{0,\varphi} = \varphi + \pi'(u^0)$$

$$\partial_r X_{0,\varphi} = \partial_r u^0$$

In Section 4.6 we showed how to obtain a surface which is a graph on $\widetilde{M}^+_{\sigma,\alpha,0}$ (or $\widetilde{M}^+_{\sigma,0,\beta}$) and is asymptotic to it. We have solved a problem of the form

$$\begin{cases} \mathcal{L}_\sigma u^+ &= Q_\sigma(u^+) \quad \text{on} \quad \widetilde{M}^+_{\sigma,\alpha,0} \quad (\widetilde{M}^+_{\sigma,0,\beta}) \\ \pi'' u^+ &= \varphi - \pi'' g_t \quad \text{on} \quad \partial\widetilde{M}^+_{\sigma,\alpha,0} \quad (\partial\widetilde{M}^+_{\sigma,0,\beta}) \end{cases}$$

where $\mathcal{L}_\sigma$ denotes the Lamé operator and $g_t = \mathcal{O}(\epsilon)$.

So we can obtain two surfaces $S^+_{\alpha,0}$ and $S^+_{0,\beta}$, whose parametrizations $X^+_{\alpha,\varphi}$, $X^+_{\beta,\varphi}$ satisfy in a neighbourhood of its boundary

$$X^+_{\alpha,\varphi} = -(1+\kappa)\ln(2r) + 2\sin\tfrac{\alpha}{2} r\cos\theta + \varphi + \pi'(u^+ + g_t),$$
$$\partial_r X^+_{\alpha,\varphi} = -(1+\kappa)\tfrac{1}{r} + 2\sin\tfrac{\alpha}{2}\cos\theta + g_{t,d} + \partial_r u^+,$$
$$X^+_{\beta,\varphi} = -(1+\kappa)\ln(2r) + 2\sin\tfrac{\beta}{2} r\sin\theta + \varphi + \pi'(u^+ + g_t),$$
$$\partial_r X^+_{\beta,\varphi} = -(1+\kappa)\tfrac{1}{r} + 2\sin\tfrac{\beta}{2}\sin\theta + g_{t,d} + \partial_r u^+,$$

where $g_{t,d} = \mathcal{O}(\epsilon)$. In the following to simplify the notation we will write for short S^+ and $X_{+,\varphi}$.

Analogously, in Section 4.8 we showed the existence of a surface close to a Scherk type surface, whose boundary is a curve on ∂B_{r_ϵ}. In other words we are able to solve a problem of the form

$$\begin{cases} \mathcal{L}_M u^- &= Q(u^-) \quad \text{on} \quad \Sigma - B_{r_\epsilon} \\ \pi'' u^- &= \varphi - \pi'' g_b \quad \text{on} \quad \partial B_{r_\epsilon} \end{cases}$$

where $\mathcal{L}_M$ denotes the linearized mean curvature operator and $g_b = \mathcal{O}(\epsilon)$.

So we can obtain a surface S^- that has a parametrization $X_{-,\varphi}$, which in a neighbourhood of its boundary satisfies

$$X_{-,\varphi} = -\ln(2r) + \varphi + \pi'(u^- + g_b),$$
$$\partial_r X_{-,\varphi} = -\tfrac{1}{r} + g_{b,d} + \partial_r u^-$$

where $g_{b,d} = \mathcal{O}(\epsilon)$.

Finally, in Section 4.3 we have obtained the surface $M_k(\epsilon/2, \Phi)$, whose boundaries are curves about ∂B_{r_ϵ}. In particular we are able to solve, for $\Phi = (\varphi_t, \varphi_b, \varphi_m) \in (\pi'' C^{2,\alpha}(S^1))^3$ and $\|\varphi_i\|_{C^{2,\alpha}} \leqslant \epsilon$ the problem

$$\begin{cases} L_{M_k(\epsilon/2)} u_C &= Q(u_C) \quad \text{on} \quad M_k^T(\epsilon/2) \\ \pi'' u_C &= (\varphi_t - \pi'' f_t, \varphi_b - \pi'' f_b, \varphi_m - \pi'' f_m) \quad \text{on} \quad \partial M_k^T(\epsilon/2). \end{cases}$$

The functions f_t, f_b, f_m denote the higher order terms appearing in lemma 53. This result gives the parametrizations of part of the ends of $M_k(\epsilon/2)$ seen as graphs of appropriate functions over the $x_3 = 0$ plane. $f_{t,d}, f_{b,d}, f_{m,d}$ denote their derivatives. As for the catenoidal type ends these functions, that are defined on a neighbourhood of ∂B_{r_ϵ}, have the following expressions

$$U_t(r,\theta) = \sigma_t + \ln(2r) + \frac{\epsilon}{2} r\cos\theta + f_t(r,\theta), \tag{4.44}$$

$$\partial_r U_t(r,\theta) = \frac{1}{r} + \frac{\epsilon}{2}\cos\theta + f_{t,d}(r,\theta),$$
$$U_b(r,\theta) = -\sigma_b - \ln(2r) + \frac{\epsilon}{2} r\cos\theta + f_b(r,\theta), \tag{4.45}$$
$$\partial_r U_b(r,\theta) = -\frac{1}{r} + \frac{\epsilon}{2}\cos\theta + f_{b,d}(r,\theta),$$

where $f_i = \mathcal{O}(\epsilon)$, $f_{i,d} = \mathcal{O}(\epsilon^{3/2})$, $i = t, b$. As for the parametrization of the planar end, it satisfies

$$U_m(r,\theta) = f_m(r,\theta) = \mathcal{O}(r^{-k}), \tag{4.46}$$
$$\partial_r U_m(r,\theta) = f_{m,d}(r,\theta) = \mathcal{O}(r^{-k+1}),$$

in a neighbourhood of ∂B_{r_ϵ}.

Then we can obtain in particular a minimal graph over the planar end whose parametrization, $X_{m,\Phi}$ in a neighbourhood of its boundary, satisfies

$$X_{m,\Phi} = \varphi_m + \pi'(u_C + f_m)$$
$$\partial_r X_{m,\Phi} = f_{m,d} + \partial_r u_C.$$

Then we can obtain in particular a minimal graph over the top end whose parametrization, $X_{t,\Phi}$ in a neighbourhood of its boundary, satisfies

$$X_{t,\Phi} = -\ln(2r) + \tfrac{\epsilon}{2} r\cos\theta + \varphi_t + \pi'(u_C + f_t)$$
$$\partial_r X_{t,\Phi} = -\tfrac{1}{r} + \tfrac{\epsilon}{2}\cos\theta + f_{t,d} + \partial_r u_C.$$

Now we can define

$$\begin{array}{rcl} E_\epsilon : (\pi'' C^{2,\alpha}(S^1))^3 & \longrightarrow & (C^{2,\alpha}(S^1))^3 \times (C^{1,\alpha}(S^1))^3 \\ \Phi = (\phi_t, \phi_b, \phi_m) & \longrightarrow & [(X_{+,\phi_t}, X_{-,\phi_b}, X_{0,\phi_m}), (\partial_r X_{+,\phi_t}, \partial_r X_{-,\phi_b}, \partial_r X_{0,\phi_m})]_{|\partial B_{r_\epsilon}}. \end{array}$$

and

$$\begin{array}{rcl} F_\epsilon : \pi'' C^{2,\alpha}(S^1) & \longrightarrow & C^{2,\alpha}(S^1) \times C^{1,\alpha}(S^1) \\ \Phi = (\phi_t, \phi_b, \phi_m) & \longrightarrow & [(X_{t,\phi_t}, X_{b,\phi_b}, X_{m,\phi_m}), (\partial_r X_{t,\phi_t}, \partial_r X_{b,\phi_b}, \partial_r X_{m,\phi_m})]_{|\partial B_{r_\epsilon}}. \end{array}$$

We set $C_\epsilon := E_\epsilon - F_\epsilon$.

We want to prove that the surfaces S^+, S^-, S^0 and $M_k^T(\epsilon/2, \Phi)$ can be glued along their boundaries to obtain a C^∞-surface. Firstly we will show that these surface correspond in a C^1 way along the boundaries curves. This is true if it exists $\Psi = (\psi_1, \psi_2, \psi_3)$ such that $C_\epsilon(\Psi) = 0$. The existence of the appropriates boundary functions is proven in the following theorem. Finally, to show that the surface is C^∞, it is sufficient to apply the regularity theory.

Theorem 77. *For some ϵ_0 and every $0 < \epsilon < \epsilon_0$, there exists $\Psi = (\psi_1, \psi_2, \psi_3) \in [\pi'' C^{2,\alpha}(S^1)]^3$ which solves $C_\epsilon(\Psi) = 0$.*

Proof. We consider the harmonic extensions of ψ_i, $i = 1, 2, 3$, on the ends of $M_k(\xi, \epsilon)$, that is

1. $\bar{w}_t = \chi_+ \, H_{\psi_1}(s_\epsilon - s, \cdot)$, on the upper end;
2. $\bar{w}_b = \chi_- \, H_{\psi_2}(s - s_\epsilon, \cdot)$, on the lower end;
3. $\bar{w}_m = \chi_p \, \tilde{H}_{\rho_\epsilon, \psi_3}(1/r, \cdot)$ on the middle end

and its harmonic extensions

1. $w_t = \bar{H}_{v_\epsilon, \psi_1}$ on S^+;
2. $w_b = -\bar{H}_{-v_\epsilon, \psi_2}$ on S^-;
3. $w_m = \mathcal{H}_{\psi_3}$ on S^0

(see Section 4.3 for the definitions of the cut-off functions). We recall that the operators $\tilde{H}$, H, $\bar{\mathcal{H}}$ and $\mathcal{H}$ have been introduced respectively in Propositions 79, 80, 81 and 74. We consider the following maps

$$\begin{array}{rcl} E_0 : [\pi'' C^{2,\alpha}(S^1)]^3 & \longrightarrow & C^{2,\alpha}(S^1)^3 \times C^{1,\alpha}(S^1)^3 \\ \Psi & \longrightarrow & [(w_t, w_b, w_m), (\partial_r w_t, \partial_r w_b, \partial_r w_m)]_{|r_\epsilon} \end{array}$$

and

$$\begin{array}{rcl} F_0 : [\pi'' C^{2,\alpha}(S^1)]^3 & \longrightarrow & C^{2,\alpha}(S^1)^3 \times C^{1,\alpha}(S^1)^3 \\ \Psi & \longrightarrow & [(\bar{w}_t, \bar{w}_b, \bar{w}_m), (\partial_r \bar{w}_t, \partial_r \bar{w}_b, \partial_r \bar{w}_m)]_{|r_\epsilon}. \end{array}$$

Now using Fourier expansion of the function, we can see that $C_0 = E_0 - F_0$ has an inverse which is bounded independently of ϵ. In particular, the equation $C_0(\Psi) = 0$ has the unique solution $\Psi = (0, 0, 0)$. Now we consider $(C_\epsilon - C_0)(\Psi)$, whose expression is

$$\begin{gathered} \big(r \cos\theta(2b_1 - \epsilon/2) + \pi'(u^+ - u_C) + \pi'(g_t - f_t), \\ r\cos\theta(-\xi) + \pi'(u^- - u_C) + \pi'(g_b - f_b), \\ \pi'(u^0 - u_C) + \pi'(-f_m)\big)\,, \\ \big(\partial_r(u^+ - w_t) - \partial_r(u_C - \bar{w}_t) + g_{t,d} - f_{t,d} \\ \cos\theta(-\epsilon/2) + \partial_r(u^- - w_b) - \partial_r(u_C - \bar{w}_b) + g_{b,d} - f_{b,d} \\ \partial_r(u^0 - w_m) - \partial_r(u_C - \bar{w}_m) - f_{m,d}\big)\,. \end{gathered}$$

It is easy to prove that

$$||(C_\epsilon - C_0)(\Psi)||_{C^{2,\alpha}(S^1)^3 \times C^{1,\alpha}(S^1)^3} \leqslant c\epsilon.$$

In order to solve $C_\epsilon(\Psi) = 0$, we find a fixed point for the mapping

$$D_\epsilon(\Psi) := C_0^{-1}\left((C_\epsilon - C_0)(\Psi)\right).$$

□

4.10 Appendix A

Definition 78. *Given $\ell \in \mathbb{N}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, the space $\mathcal{C}^{\ell,\alpha}_{\nu}(B_{\rho_0}(0))$ is defined to be the space of functions in $\mathcal{C}^{\ell,\alpha}_{loc}(B_{\rho_0}(0))$ for which the following norm is finite*

$$\|\rho^{-\nu}\, w\|_{\mathcal{C}^{\ell,\alpha}(B_{\rho_0}(0))}.$$

Now we can state the following result.

Proposition 79. *There exists an operator*

$$\tilde{H} : C^{2,\alpha}(S^1) \longrightarrow C^{2,\alpha}_0(S^1 \times [\bar{\rho}, +\infty)),$$

such that for each even function $\varphi(\theta) \in C^{2,\alpha}(S^1)$, which is L^2-orthogonal to the constant function, then $w_\varphi = \tilde{H}_{\bar{\rho},\varphi}$ solves

$$\begin{cases} \triangle w_\varphi = 0 & on \quad S^1 \times [\bar{\rho}, +\infty) \\ w_\varphi = \varphi & on \quad S^1 \times \{\bar{\rho}\}. \end{cases}$$

Moreover,

$$||\tilde{H}_{\bar{\rho},\varphi}||_{C^{2,\alpha}_{-1}(S^1\times[\bar{\rho},+\infty))} \leqslant c\,||\varphi||_{C^{2,\alpha}(S^1)}, \tag{4.47}$$

for some constant $c > 0$.

Proof. We consider the decomposition of the function φ with respect to the basis $\{\cos(i\theta)\}$, that is

$$\varphi = \sum_{i=1}^{\infty} \varphi_i \cos(i\theta).$$

Then the solution w_φ is given by

$$w_\varphi(\rho,\theta) = \sum_{i=1}^{\infty} \left(\frac{\bar{\rho}}{\rho}\right)^i \varphi_i \cos(i\theta).$$

Since $\frac{\bar{\rho}}{\rho} \leqslant 1$, then $\left(\frac{\bar{\rho}}{\rho}\right)^i \leqslant \left(\frac{\bar{\rho}}{\rho}\right)$, we can conclude that $|w(r,\theta)| \leqslant c\rho^{-1}|\varphi(\theta)|$ and then $||w_\varphi||_{C^{2,\alpha}_{-1}} \leqslant c||\varphi||_{C^{2,\alpha}}$. □

Calcolo 1

$$\sup_{[s_0,t_\epsilon]\times S^1} e^{-\delta s}|w_\Phi|_{0,\alpha;[s,s+1]} \leqslant \sup_{[s_0,t_\epsilon]\times S^1} e^{-\delta s} e^{-\delta(t_\epsilon - s)}|\phi_t|_{2,\alpha} \leqslant$$

$$\sup_{[s_0,t_\epsilon]\times S^1} e^{-\delta t_\epsilon}|\phi_t|_{2,\alpha} \leqslant c_k \epsilon^{1+\delta/2}.$$

Calcolo 2

$$\sup_{s\in[s_0,t_\epsilon]\times S^1} e^{-\delta s}|w|^2_{0,\alpha;[s,s+1]} \leqslant \|w\|^2_{2,\alpha,\mu/2} \leqslant c|\varphi|^2_{2,\alpha} \sup_{[s_0,t_\epsilon]\times S^1} e^{-\delta s/2}e^{-\delta(t_\epsilon - s)}$$

$$\leqslant c|\varphi|^2_{2,\alpha} \sup_{[s_0,t_\epsilon]\times S^1} e^{-\delta(t_\epsilon - s/2)} \leqslant ce^{-\delta t_\epsilon}e^{\delta s_0/2}|\varphi|^2_{2,\alpha} \leqslant c\epsilon^{-\delta/2}e^{\delta s_0/2}|\varphi|^2_{2,\alpha} \leqslant c_k\epsilon^{2-\delta/2}.$$

Now we give the statement of an useful result whose proof is contained in [7].

Proposition 80. *There exists an operator*

$$H : \mathcal{C}^{2,\alpha}(S^1) \longrightarrow \mathcal{C}^{2,\alpha}_{-2}([0,+\infty)\times S^1),$$

such that for all $\varphi \in \mathcal{C}^{2,\alpha}(S^1)$*, even function and orthogonal to* 1 *and* $\cos\theta$*, in the* L^2*-sense, the function* $w = H_\varphi$ *solves*

$$\begin{cases} (\partial_s^2 + \partial_\theta^2)\, w = 0 & in \quad [0,+\infty)\times S^1 \\ w = \varphi & on \quad \{0\}\times S^1 \end{cases}$$

Moreover

$$\|H_\varphi\|_{\mathcal{C}^{2,\alpha}_{-2}} \leqslant c\,\|\varphi\|_{\mathcal{C}^{2,\alpha}},$$

for some constant $c > 0$.

Proposition 81. *There exists an operator*

$$\bar{\mathcal{H}}_{v_0} : C^{2,\alpha}(S^1) \longrightarrow C^{2,\alpha}_\mu(S^1\times[v_0,+\infty)),$$

$\mu \in (-2,-1)$*, such that for every function* $\varphi(v) \in C^{2,\alpha}(S^1)$*, which is* L^2*-orthogonal to* $e_{0,i}(u)$ *with* $i = 0,1$ *and even, the function* $w_\varphi = \bar{\mathcal{H}}_{v_0}(\varphi)$ *solves*

$$\begin{cases} \partial^2_{uu}w_\varphi + \partial^2_{vv}w_\varphi = 0 & on \quad S^1\times[v_0,+\infty) \\ w_\varphi = \varphi & on \quad S^1\times\{v_0\}. \end{cases}$$

Moreover,

$$||\bar{\mathcal{H}}_{v_0}(\varphi)||_{C^{2,\alpha}_\mu(S^1\times[v_0,+\infty])} \leqslant c\,||\varphi||_{C^{2,\alpha}(S^1)}, \tag{4.48}$$

for some constant $c > 0$.

Proof. We consider the decomposition of the function φ with respect to the basis $\{e_{0,i}(u)\}$, that is

$$\varphi = \sum_{i=2}^{\infty} \varphi_i e_{0,i}(u).$$

Then the solution w_φ is given by

$$w_\varphi(u,v) = \sum_{i=2}^{\infty} e^{-i(v-v_0)}\varphi_i e_{0,i}(u).$$

We recall that $\mu \in (-2,-1)$ so we have $-i \leqslant \mu$ from which it follows $|w_\varphi|_{2,\alpha;[v,v+1]} \leqslant e^{\mu(v-v_0)}|\varphi|_{2,\alpha}$ and

$$\|w_\varphi\|_{\mathcal{C}^{2,\alpha}_\mu} = \sup_{v\in[v_0,\infty]} e^{-\mu v}|w|_{2,\alpha;[v,v+1]} \leqslant \sup_{v\in[v_0,\infty]} e^{-\mu v}e^{\mu(v-v_0)}|\varphi|_{2,\alpha} \leqslant e^{-\mu v_0}|\varphi|_{2,\alpha}.$$

□

4.11 Appendix B

In section 4.5 we introduced the Jacobi operator about the surface $\widetilde{M}_{\sigma,\alpha,\beta}$. Its expression is $\mathcal{J} = \frac{-K}{T(x(u),y(v))}\mathcal{L}_\sigma$. Here we want to verify that the factor $\frac{-K}{T(x(u),y(v))}$ is bounded. It is well known that the Gauss curvature has the following expression in terms of the Weierstrass data g, dh:

$$K = -16\left(|g| + \frac{1}{|g|}\right)^{-4}|\frac{dg}{g}|^2|dh|^{-2}$$

We recall that $dh = \frac{\mu\, dz}{\sqrt{(z^2+\lambda^2)(z^2+\lambda^{-2})}}$. Since $|z^2+\lambda^2||z^2+\lambda^{-2}|$ and $T(x,y) = q^2\cos^2 x(u) + p^2\sin^2 y(v)$ have the same zeroes, that is the points D, D', D'', D''', then $-K/T$ is bounded.

We can give an estimate of the derivatives of K and $\sqrt{-K}$. We can write $\sqrt{-K} = \sqrt{T}\sqrt{\frac{-K}{T}}$. So it is sufficient to study the derivatives of T.

We recall that

$$l(x) = \sqrt{1-\sin^2\sigma\sin^2 x} \quad m(y) = \sqrt{1-\cos^2\sigma\cos^2 y}.$$

From the expression of T, using (4.12) it is easy to get:

$$\frac{\partial}{\partial u}\sqrt{T} = -\frac{\sin^2\sigma\sin 2x(u)}{2\sqrt{T}}l(x(u)),$$

$$\frac{\partial}{\partial v}\sqrt{T} = \frac{\cos^2\sigma\sin 2y(v)}{2\sqrt{T}}m(y(v)).$$

Then

$$\left|\frac{\partial}{\partial u}\sqrt{T}\right| = \frac{\sin^2\sigma|\sin 2x(u)|l(u)}{2\sqrt{\sin^2\sigma\cos^2 x(u) + \cos^2\sigma\sin^2 y(v)}} \leqslant \frac{\sin^2\sigma|\sin 2x(u)|}{2\sin\sigma|\cos x(u)|} \leqslant \sin\sigma,$$

$$\left|\frac{\partial}{\partial v}\sqrt{T}\right| = \frac{\cos^2\sigma|\sin 2y(v)|m(v)}{2\sqrt{\sin^2\sigma\cos^2 x(u)+\cos^2\sigma\sin^2 y(v)}} \leqslant \frac{\cos^2\sigma|\sin 2y(v)|}{2\cos\sigma|\sin y(v)|} \leqslant \cos\sigma.$$

We can conclude that the derivatives of $\sqrt{T}$ (and trivially also the derivatives of T) are bounded.

4.12 Appendix C

Proof of proposition 69. In section 4.4.2 we parametrized the surface $\widetilde{M}_{\sigma,\alpha,\beta}$ on the cylinder $\mathbb{S}^1\times\mathbb{R}$. We introduced the map $z(x,y):\mathbb{S}^1\times[0,\pi[\to\bar{\mathbb{C}}$ where x,y denote the sphero-conal coordinates. Let p,q be the real and the imaginary part of $z\in\bar{\mathbb{C}}$. If Z denotes the immersion of the surface in $\mathbb{R}^3$, N its normal vector, then it holds that

$$|Z_p|^2=|Z_q|^2=\Lambda,\quad |N_p|^2=|N_q|^2=-K\Lambda,$$

$$\langle N_p,N\rangle=\langle N_q,N\rangle=0,\quad \langle Z_p,Z_q\rangle=0,\quad \langle N_p,N_q\rangle=0,$$
$$\langle N_q,Z_q\rangle=-\langle N_p,Z_p\rangle,\quad \langle N_q,Z_p\rangle=\langle N_p,Z_q\rangle,$$

so

$$\langle N_p,Z_p\rangle=|N_p||Z_p|\cos\gamma_1=\sqrt{-K}\Lambda\cos\gamma_1,$$
$$\langle N_p,Z_q\rangle=|N_p||Z_q|\cos\gamma_2=\sqrt{-K}\Lambda\cos\gamma_2.$$

Here γ_1 is the angle between the vectors N_p and Z_p, γ_2 is the angle between the vectors N_p and Z_q.

The proof of proposition 69 is articulated in some lemmas. We recall that $Z_f=Z+fN$. We denote by E_f,F_f,G_f the coefficiens of the second fundamental form for Z_f. The following lemma gives the expression of the area energy functional.

Lemma 82.

$$A(f):=\int\sqrt{E_fG_f-F_f^2}\,dp\,dq,$$

with

$$E_fG_f-F_f^2=\Lambda^2+\Lambda(f_p^2+f_q^2)+2K\Lambda^2f^2+2f(f_q^2-f_p^2)\sqrt{-K}\Lambda\cos\gamma_1$$
$$-4ff_pf_q\sqrt{-K}\Lambda\cos\gamma_2-K\Lambda f^2(f_p^2+f_q^2)+f^4K^2\Lambda^2.$$

Proof. The coefficients of the second fundamental form are:

$$E_f = |\partial_p Z_f|^2 = |Z_p|^2 + f_p^2 + f^2|N_p|^2 + 2f\langle N_p, Z_p\rangle,$$
$$G_f = |\partial_q Z_f|^2 = |Z_q|^2 + f_q^2 + f^2|N_q|^2 + 2f\langle N_q, Z_q\rangle,$$
$$F_f = |\partial_p Z_f \cdot \partial_q Z_f| = f_p f_q + f(\langle Z_p, N_q\rangle + \langle Z_q, N_p\rangle).$$

Then

$$E_f G_f = |Z_p|^2|Z_q|^2 + f_p^2|Z_q|^2 + f_q^2|Z_p|^2 + f^2(|N_q|^2|Z_p|^2 + |N_p|^2|Z_q|^2)+$$
$$f^2(f_p^2|N_q|^2+f_q^2|N_p|^2)+f^4|N_p|^2|N_q|^2+4f^2(\langle N_p Z_p\rangle)(\langle N_q Z_q\rangle)+2f(f_p^2\langle N_q, Z_q\rangle+f_q^2\langle N_p, Z_p\rangle)+$$
$$f_p^2 f_q^2 + 2f(\langle N_q, Z_q\rangle|Z_p|^2 + \langle N_p, Z_p\rangle|Z_q|^2) + 2f^3(\langle N_q, Z_q\rangle|Z_p|^2 + \langle N_p, Z_p\rangle|Z_q|^2)$$

Since $\langle N_q, Z_q\rangle + \langle N_p, Z_p\rangle = 0$ and $|Z_p|^2 = |Z_q|^2$ we can conclude that the last two terms of the previous expression are zero. Since $\langle N_q, Z_p\rangle = \langle N_p, Z_q\rangle$ we have

$$F_f = f_p f_q + 2f\langle N_p, Z_q\rangle.$$

Then

$$F_f^2 = f_p^2 f_q^2 + 4f^2(\langle N_p, Z_q\rangle)^2 + 4f f_p f_q\langle N_p, Z_q\rangle.$$

So the expression of $E_f G_f - F_f^2$ is:

$$|Z_p|^2|Z_q|^2 + f_p^2|Z_q|^2 + f_q^2|Z_p|^2 + f^2(|N_q|^2|Z_p|^2 + |N_p|^2|Z_q|^2)+$$
$$f^2(f_p^2|N_q|^2+f_q^2|N_p|^2)+f^4|N_p|^2|N_q|^2+4f^2(\langle N_p, Z_p)(\langle N_q, Z_q\rangle)+2f(f_p^2\langle N_q, Z_q\rangle+f_q^2\langle N_p, Z_p\rangle)$$
$$-4f^2(\langle N_p, Z_q\rangle)^2 - 4f f_p f_q\langle N_p, Z_q\rangle.$$

Ordering the terms we get:

$$|Z_p|^2|Z_q|^2 + f_p^2|Z_q|^2 + f_q^2|Z_p|^2 + f^2(|N_q|^2|Z_p|^2 + |N_p|^2|Z_q|^2) - 4f^2(\langle N_p, Z_q\rangle)^2$$
$$+4f^2(\langle N_p, Z_p\rangle)(\langle N_q, Z_q\rangle) + 2f(f_p^2\langle N_q, Z_q\rangle + f_q^2\langle N_p, Z_p\rangle) - 4f f_p f_q\langle N_p, Z_q\rangle+$$
$$+f^2(f_p^2|N_q|^2 + f_q^2|N_p|^2) + f^4|N_p|^2|N_q|^2.$$

The expression of $E_f G_f - F_f^2$ becomes:

$$\Lambda^2 + \Lambda(f_p^2 + f_q^2) - 2K\Lambda^2 f^2 + 4f^2 K\Lambda^2\left(\cos^2\gamma_1 + \cos^2\gamma_2\right) +$$
$$+2f(f_q^2 - f_p^2)\sqrt{-K}\Lambda\cos\gamma_1 - 4f f_p f_q\sqrt{-K}\Lambda\cos\gamma_2 - K\Lambda f^2(f_p^2 + f_q^2) + f^4K^2\Lambda^2.$$

Using the relations $\langle N_q, Z_p\rangle = \langle N_p, Z_q\rangle$ and $\langle N_q, Z_q\rangle = -\langle N_p, Z_p\rangle$, it is possible to understand that the relative positions of these vectors are such that $\gamma_2 = \frac{\pi}{2} \pm \gamma_1$. So $\cos^2\gamma_2 = \cos^2(\frac{\pi}{2} \pm \gamma_1) = \sin^2\gamma_1$ and $\cos^2\gamma_1 + \cos^2\gamma_2 = 1$. Then we can write:

$$\Lambda^2 + \Lambda(f_p^2 + f_q^2) + 2K\Lambda^2 f^2 + 2f(f_q^2 - f_p^2)\sqrt{-K}\Lambda\cos\gamma_1$$
$$-4f f_p f_q\sqrt{-K}\Lambda\cos\gamma_2 - K\Lambda f^2(f_p^2 + f_q^2) + f^4K^2\Lambda^2.$$

□

The next lemma completes the proof of the proposition 69.

Lemma 83. *The surface whose immersion is given by* $Z + fN$*, is minimal if and only if* f *satisfies*

$$\mathcal{L}_\sigma f + Q_\sigma(f, f_u, f_v, f_{uu}, f_{uv}, f_{vv}) = 0.$$

Proof. The surface parameterized by $Z_f = Z + fN$ is minimal if and only the first variation of $A(f)$ is 0. That is

$$2\,DA_{|f}(g) = \int \frac{1}{\sqrt{(E_f G_f - F_f^2)}_{|f=0}} D_f(E_f G_f - F_f^2)\,(g)\,dp\,dq.$$

Thanks to the previous lemma it holds that

$$\frac{1}{\sqrt{(E_f G_f - F_f^2)}_{|f=0}} D_f(E_f G_f - F_f^2)(g) = \frac{1}{\Lambda}\big(2\Lambda(f_p g_p + f_q g_q) + 4K\Lambda^2 fg +$$

$$+2\sqrt{-K}\Lambda\cos\gamma_1\left[2ff_q g_q + gf_q^2 - 2ff_p g_p - gf_p^2\right] +$$

$$-4\sqrt{-K}\Lambda\cos\gamma_2\left[ff_q g_p + fg_q f_p + gf_p f_q\right] +$$

$$-2K\Lambda\left[fgf_p^2 + f_p g_p f^2 + fgf_q^2 + f_q g_q f^2\right] + 4K^2\Lambda^2 f^3 g\big)\,.$$

Reordering the summands, we have:

$$\frac{1}{\sqrt{(E_f G_f - F_f^2)}_{|f=0}} D_f(E_f G_f - F_f^2)(g) = 2\,(f_p g_p + f_q g_q + 2K\Lambda fg +$$

$$+\sqrt{-K}\cos\gamma_1\left[2f(f_q g_q - f_p g_p) + g(f_q^2 - f_p^2)\right] +$$

$$-2\sqrt{-K}\cos\gamma_2\left[f(f_q g_p + g_q f_p) + gf_p f_q\right] +$$

$$-K\left[fg(f_p^2 + f_q^2) + f^2(f_p g_p + f_q g_q)\right] + 2K^2\Lambda f^3 g\big)\,.$$

In the next computation we can skip the factor 2 in front of the last expression.

$$f_p g_p + f_q g_q + 2K\Lambda fg + Q_1(f, f_p, f_q)g - Q_2(f, f_p, f_q)g_p - Q_3(f, f_p, f_q)g_q = 0,$$

where

$$Q_1(f, f_p, f_q) = -(f_p^2 - f_q^2)\sqrt{-K}\cos\gamma_1 - 2f_p f_q\sqrt{-K}\cos\gamma_2 - Kf(f_p^2 + f_q^2) + 2K^2\Lambda f^3,$$

$$Q_2(f, f_p, f_q) = 2ff_p\sqrt{-K}\cos\gamma_1 + 2ff_q\sqrt{-K}\cos\gamma_2 + Kf^2 f_p,$$

$$Q_3(f, f_p, f_q) = -2ff_q\sqrt{-K}\cos\gamma_1 + 2ff_p\sqrt{-K}\cos\gamma_2 + Kf^2 f_q.$$

An integration by parts and a change of sign give us the equation:

$$(f_{pp} + f_{qq} - 2K\Lambda f - Q_1(f, f_p, f_q)+$$

$$+P_2(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq}) + P_3(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq}))\, g = 0,$$

where

$$P_2(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq}) = \partial_p Q_2(f, f_p, f_q)$$

and

$$P_3(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq}) = \partial_q Q_3(f, f_p, f_q).$$

That is

$$P_2(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq}) = 2(f_p^2 + ff_{pp})\sqrt{-K}\cos\gamma_1 + 2(f_pf_q + ff_{pq})\sqrt{-K}\cos\gamma_2+$$

$$+K(2ff_p^2 + f^2 f_{pp}) + 2f(f_p(\sqrt{-K}\cos\gamma_1)_p + f_q(\sqrt{-K}\cos\gamma_2)_p) + f^2 f_p K_p$$

and

$$P_3(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq}) = -2(f_q^2 + ff_{qq})\sqrt{-K}\cos\gamma_1 + 2(f_pf_q + ff_{pq})\sqrt{-K}\cos\gamma_2+$$

$$+K(2ff_q^2 + f^2 f_{qq}) + 2f(-f_q(\sqrt{-K}\cos\gamma_1)_q + f_p(\sqrt{-K}\cos\gamma_2)_q) + f^2 f_q K_q.$$

Now we change the variables passing from the variables (p, q) to the (u, v) variables. Then we want to understand how the minimal surfaces equation changes. We recall that p and q are the real and imaginary part of the variable z, the same that appears in the Weierstrass representation of the surface. It is known that the metric $\bar{g}$ induced on a surface whose immersion Z is given by the Weierstrass representation on a domain of the complex z-plane, can be expressed in terms of the metric $d\bar{s}^2 = dp^2 + dq^2$, by $\bar{g} = \Lambda(dp^2 + dq^2)$, where $\Lambda = |Z_p|^2 = |Z_q|^2$. It is well known that in this case then there exists a simple relation between the Laplace-Beltrami operators written with respect to the metrics $d\bar{s}^2$ and $\bar{g}$. As a fact they differ for a conformal factor:

$$\Delta_{d\bar{s}^2} = \frac{1}{\Lambda}\Delta_{\bar{g}}.$$

In section 69 we observed that the conformal factor related to the change of coordinates $(x, y) \to (u, v)$ is $-K/T$. So the conformal factor due to the change of coordinates $(p, q) \to (u, v)$ is obtained by multiplication of the conformal factors described above. Summarizing it holds that

$$f_{pp} + f_{qq} = \frac{-K\Lambda}{T}(f_{uu} + f_{vv}).$$

So we can write

$$\frac{-K\Lambda}{T}\,(f_{uu} + f_{vv}) + 2\,(-K\Lambda)\,f + R_1 + R_2 + R_3 = 0,$$

where

$$R_1(f, f_u, f_v) = -\frac{-K\Lambda}{T}\left[-(f_u^2 - f_v^2)\sqrt{-K}\cos\gamma_1 - 2f_uf_v\sqrt{-K}\cos\gamma_2 - Kf(f_u^2 + f_v^2)\right] - 2K^2\Lambda f^3$$

$$= \frac{-K\Lambda}{T}\left[(f_u^2 - f_v^2)\sqrt{-K}\cos\gamma_1 + 2f_uf_v\sqrt{-K}\cos\gamma_2 + Kf(f_u^2 + f_v^2) - 2KTf^3\right] =$$

$$\frac{-K\Lambda}{T}\bar{P}_1(f, f_u, f_v),$$

$$R_2(f, f_u, f_v, f_{uu}, f_{uv}, f_{vv}) = \frac{-K\Lambda}{T}P_2(f, f_u, f_v, f_{uu}, f_{uv}, f_{vv})$$

and

$$R_3(f, f_u, f_v, f_{uu}, f_{uv}, f_{vv}) = \frac{-K\Lambda}{T}P_3(f, f_u, f_v, f_{uu}, f_{uv}, f_{vv}).$$

We can write

$$\frac{-K\Lambda}{T}\left[f_{uu} + f_{vv} + 2T(u, v)f + \bar{P}_1(f) + P_2(f) + P_3(f)\right] = 0.$$

We can recognize the Lamé operator,

$$\mathcal{L}_\sigma f = f_{uu} + f_{vv} + 2(\sin^2\sigma\cos^2 x(u) + \cos^2\sigma\sin^2 y(v))f,$$

then, if we set $Q_\sigma = \bar{P}_1(f) + P_2(f) + P_3(f)$, the equation can be written

$$\mathcal{L}_\sigma f + Q_\sigma(f) = 0.$$

The estimate about Q_σ is an easy cnsequence of the fact that all its coefficients are bounded. That completes the proof. $\square$

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