Differentiable compactifications of symmetric spaces

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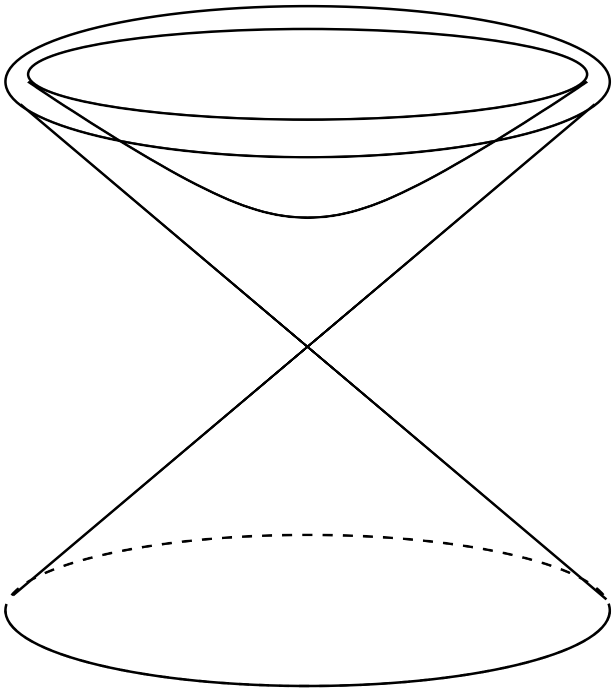
Construction of RH^n

$\mathsf{R}^n = \{(x_1, \ldots, x_n, y)\}$

$Q = \sum x_i^2 - y^2$

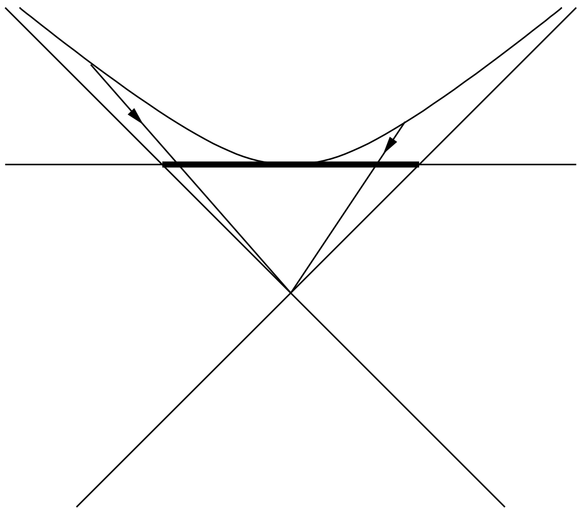
$\mathcal{H} = \{Q = -1 \text{ and } y > 0\}$

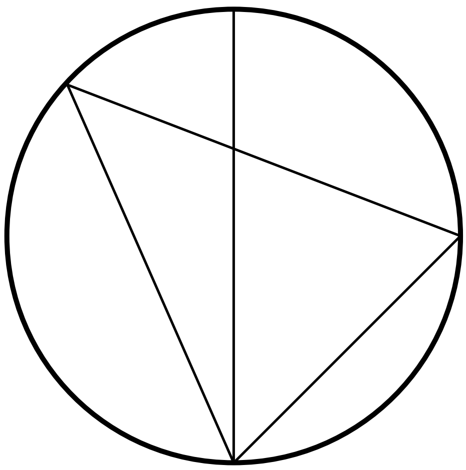


$\mathsf{RH}^n = (\mathcal{H}, Q_{|T\mathcal{H}}) = \mathsf{SO}_0(1, n)/\mathsf{SO}(n)$

The Klein ball model

Embedding $\mathsf{RH}^n \hookrightarrow \mathsf{RP}^n$:

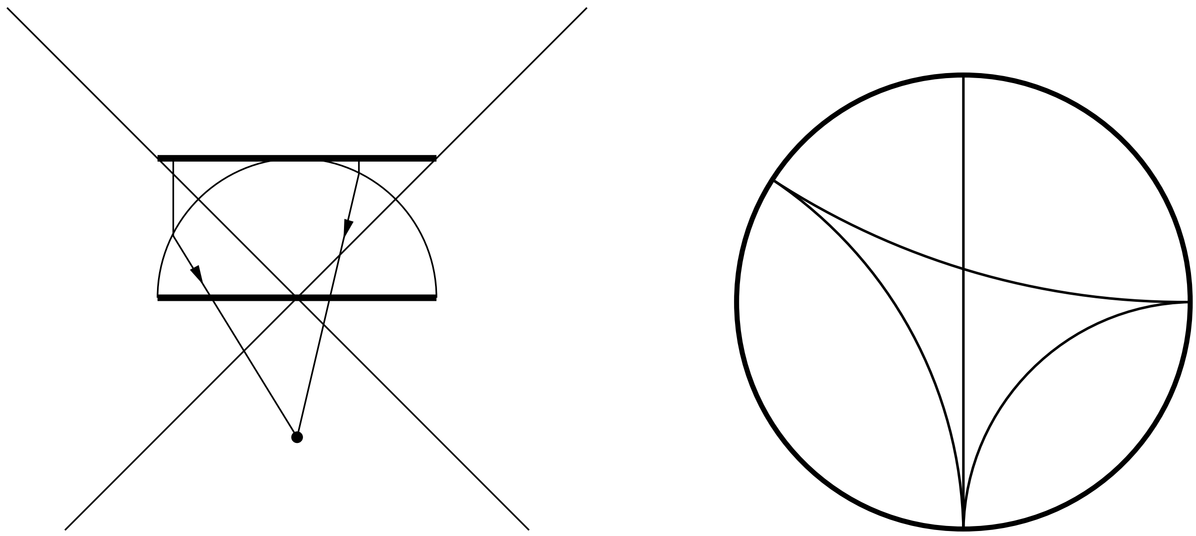




The action of the isometry group extends analytically.

The Poincaré ball model

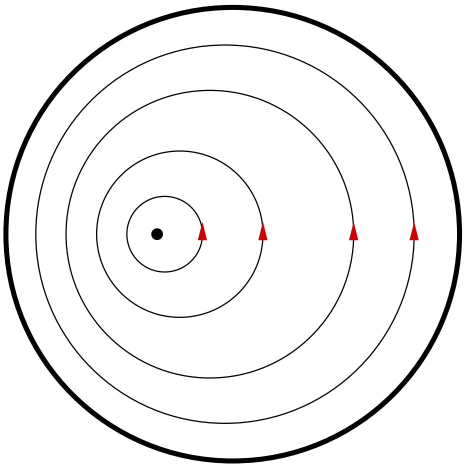
Composition of vertical and stereographic projections:



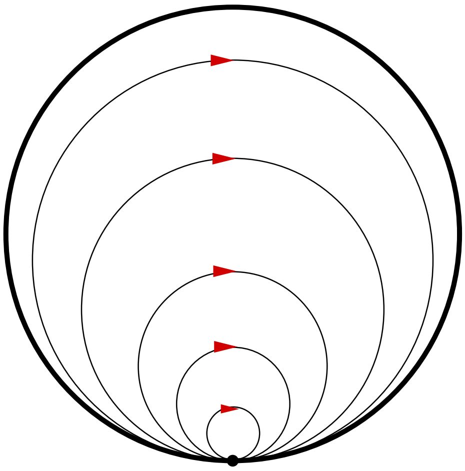
The action of the isometry group extends analytically.

Classification of isometries

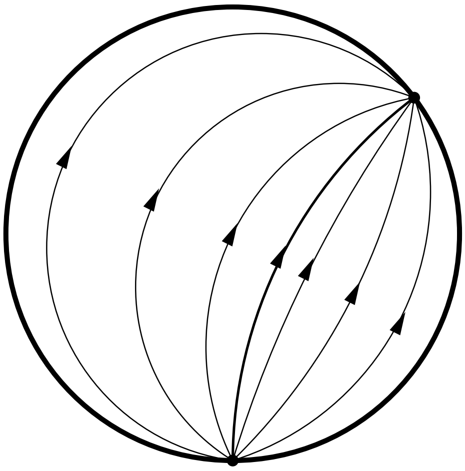
Elliptic



Parabolic



Hyperbolic



Differentiable compactifications

Given M a Riemanniann manifold,
$G = \text{Isom}_+(M)$, α the action of G on M,

a *differentiable compactification* of M is an embedding

$$\phi : M \hookrightarrow N$$

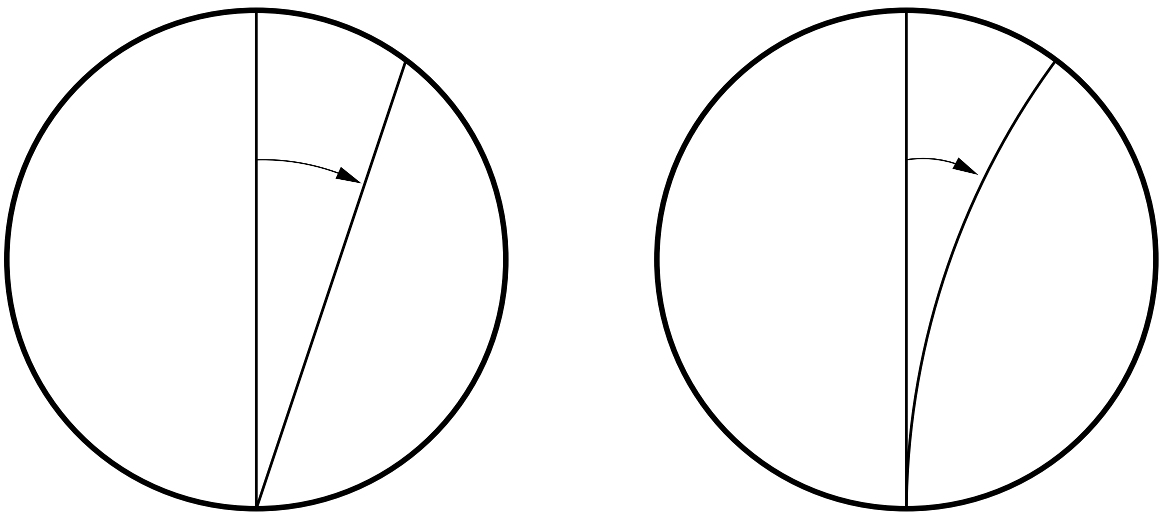
where

- N is a manifold with boundary,
- $\phi(M) = \text{int}(N)$,
- $\phi_*\alpha$ extends differentiably to N.

Compactifications of RH^n (I)

Proposition 1. *All differentiable compactifications of* RH^n *are topologically conjugate.*

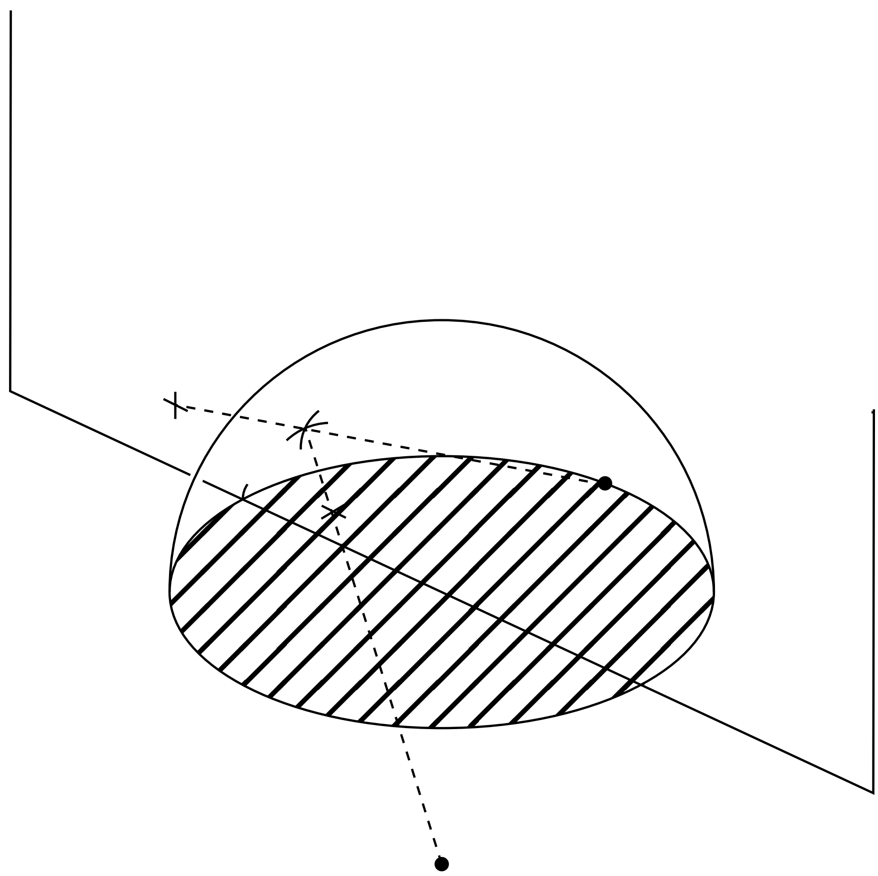
However, the conformal and projective compactifications are not C^1 conjugate:



The differential of a parabolic isometry at its fixed point is diagonalisable in the Poincaré ball, not in the Klein ball.

Half-space charts

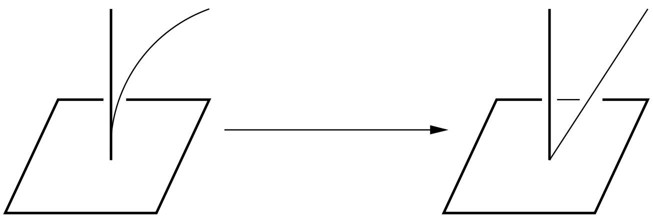
The Poincaré half-space:



Compactifications of RH^n (II)

The conformal compactification is the pullback of the projective one by the map

$$(x_1, \ldots, x_{n-1}, y) \mapsto (x_1, \ldots, x_{n-1}, y^2)$$



Theorem 1. *Every analytic compactification of* RH^n *is conjugate to the pullback of the projective one by the map*

$$(x_1, \ldots, x_{n-1}, y) \mapsto (x_1, \ldots, x_{n-1}, y^p)$$

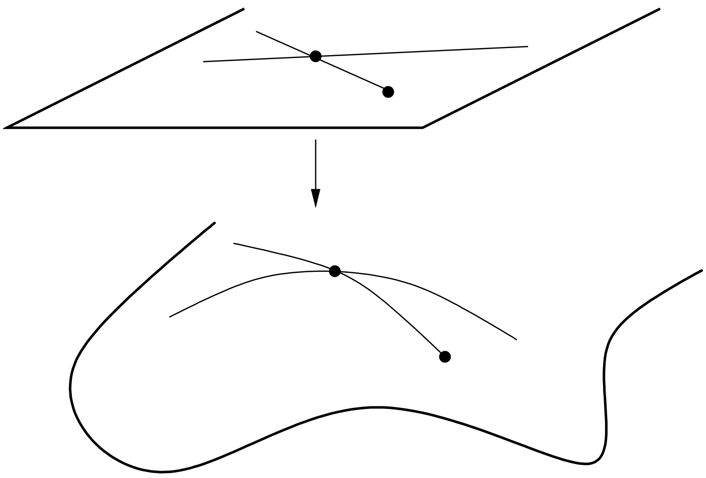
for some $p \in \mathsf{N}$.

Hadamard Manifolds

Let M be a Riemannian manifold. It is a *Hadamard manifold* if

- $\pi_1(M) = 0$ and
- its sectional curvature is non-positive.

Proposition 2. [Cartan-Hadamard] *In a Hamard manifold, the exponential maps are global diffeomorphisms.*

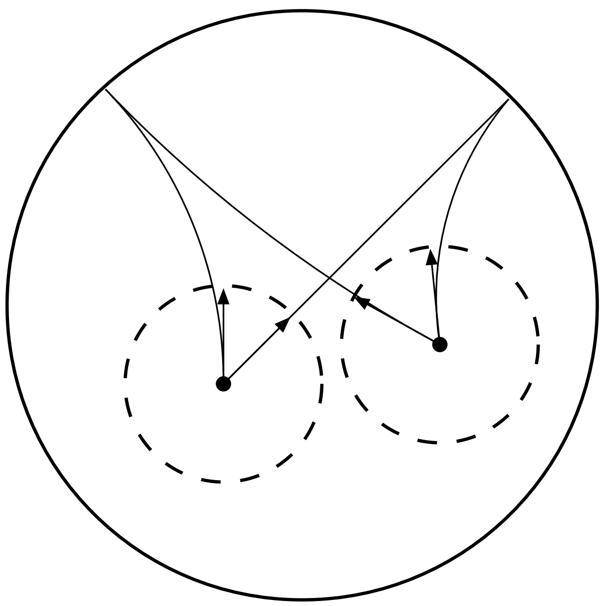


Geodesic boundary

$$M(\infty) = \{\text{unit speed geodesic ray}\}/\sim$$

Where $\sim$ stands for "staying at bounded distance one of another".

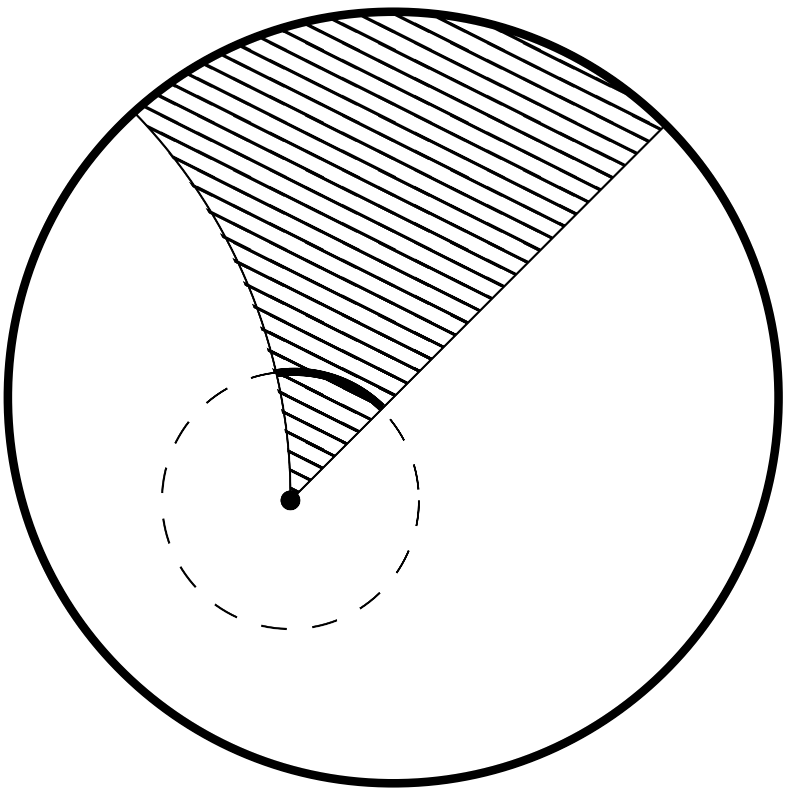
The visual projection $\pi_x : T_x^1 M \to M(\infty)$ identifies the boundary with each tangent unit sphere.



Hadamard compactification

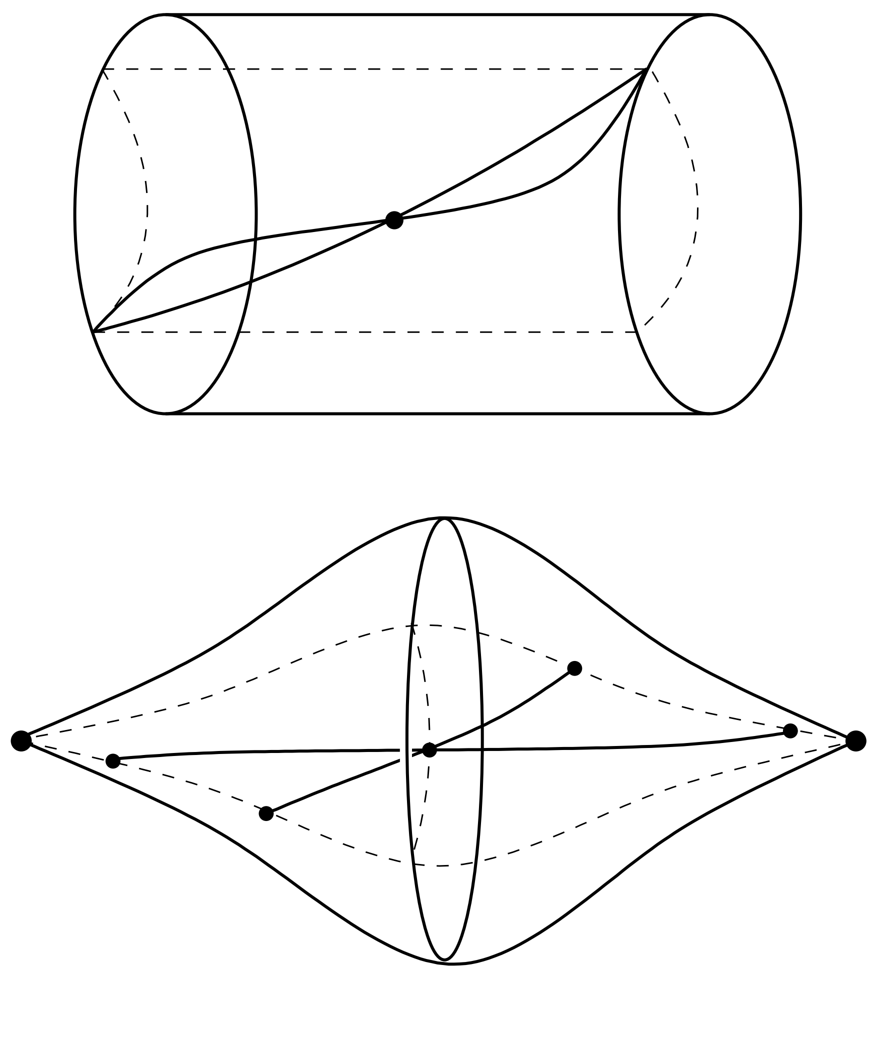
$\overline{M} = M \cup M(\infty)$ is endowed with the topology generated by the open cones:

$$\mathcal{C}(x, U) = \left\{\gamma_v(t); v \in U, t \in \mathsf{R}^+ \cup \{+\infty\}\right\}$$



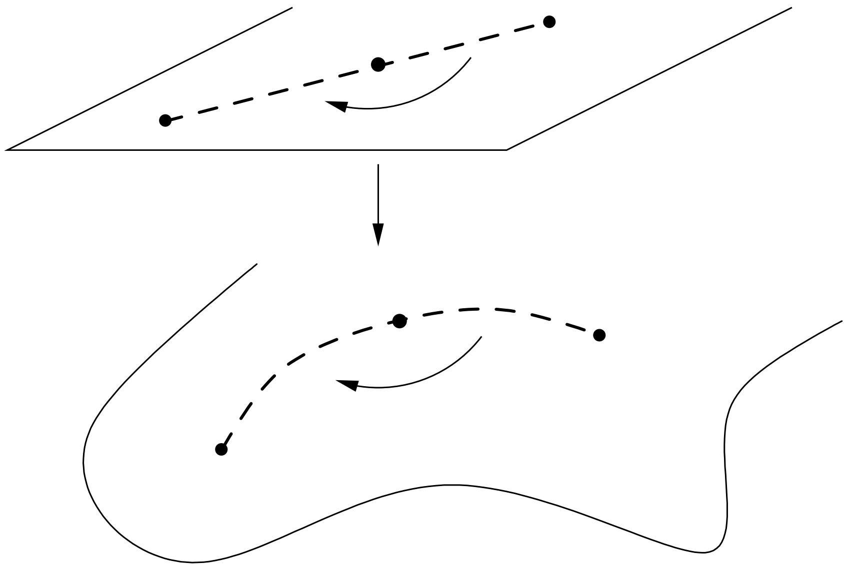
An example: $\mathbf{RH}^2 \times \mathbf{R}$

From the cylindrical representation of $\mathrm{RH}^2 \times \mathrm{R}$ we contract the caps and the side, and blow-up the two circular corners.



Geodesic symmetry

The geodesic symmetry based on $x \in M$ is the conjugate of $-\mathrm{Id}$ by the exponential map $\exp_x$.



Symmetric spaces

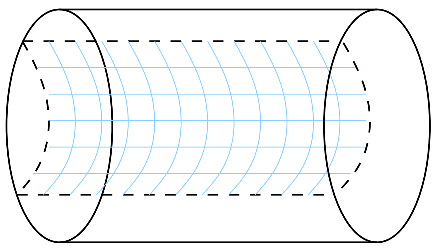
A Hadamard manifold M is a *symmetric space* if each geodesic symmetry is an isometry.

Examples :

- Euclidean space R^n,
- real hyperbolic space RH^n,
- $\mathsf{SL}(n;\mathsf{R})/\mathsf{SO}(n)$,
- product of symmetric spaces.

Rank

Let M be a symmetric space of non-positive curvature. The *rank* of M is the greatest r such that there is an embedding $\mathsf{R}^k \hookrightarrow M$ that is isometric and totally geodesic.



Examples :

- $\mathrm{rank}(\mathsf{R}^n) = n$,
- $\mathrm{rank}(\mathsf{RH}^n) = 1$,
- $\mathrm{rank}(\mathsf{SL}(n;\mathsf{R})/\mathsf{SO}(n)) = n-1$,
- $\mathrm{rank}(M_1 \times M_2) = \mathrm{rank}(M_1) + \mathrm{rank}(M_2)$.

The building at infinity

The image of an isometric, totally geodesic embedding $\mathrm{R}^k \hookrightarrow M$ is a *flat*.

Every geodesic is contained in a flat. A geodesic is *regular* if contained in only one flat, *singular* otherwise.

On the boundary at infinity, flats give *apartments*.

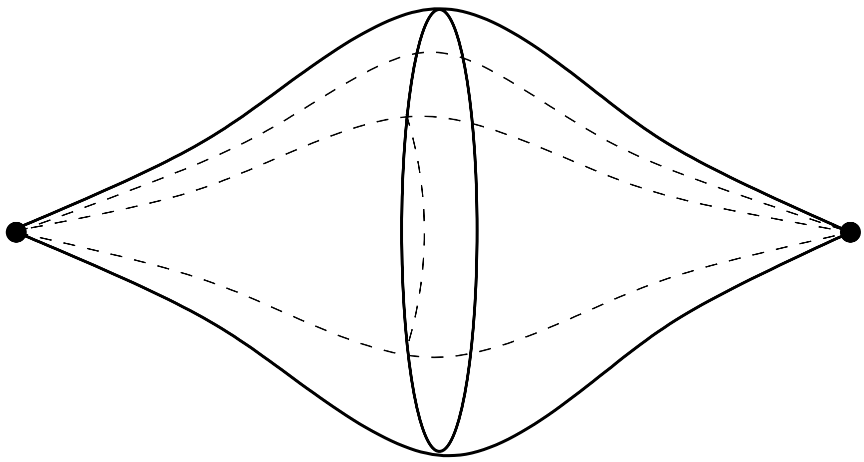
In an apartments, the connected components of the set of regular points are called *chambers*.

Proposition 3. *If M is non-Euclidean of higher rank, then the set of apartments is a thick building.*

An example: $\mathbf{RH}^2 \times \mathbf{R}$

The building at infinity of $\mathrm{RH}^2 \times \mathrm{R}$ is of type A_1.

The hyperplane system in T_xM is not essential, and the only panel of the building is made of the two singular points.



An impossibility theorem

Theorem 2. *Let M be a symmetric space of non-positive curvature. Assume that:*

- $M \neq \mathsf{R}^n$ *and*
- $\mathrm{rank}(M) \geqslant 2$.

Then M admits no differentiable compactification whose topology is that of the Hadamard compactification $\overline{M}$.

We give the proof in the simplest case: $\mathsf{RH}^2 \times \mathsf{R}$.

Proof of the main theorem (I)

Consider $M = \mathsf{RH}^2 \times \mathsf{R}$ and let:

- p be one of the singular points,
- $G = \mathsf{SO}_0(1,2) = \mathsf{Isom}_+(\mathsf{RH}^2)$.

Then G acts on M preserving p and $M(\infty)$.

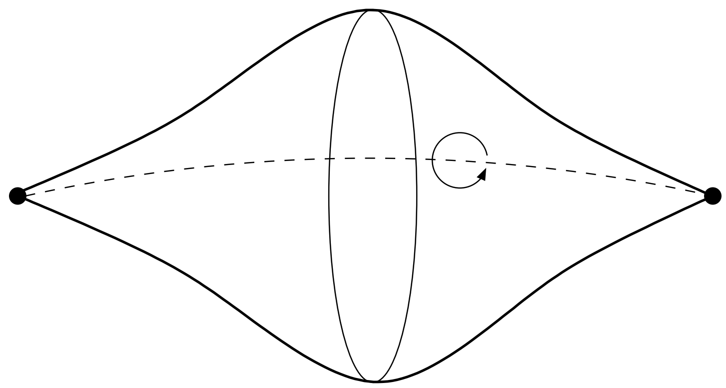
Assume we can find a differentiable structure on $\overline{M}$ such that $\mathsf{Isom}_+(M)$ acts differentiably.

Then we get a representation ρ of G on $T_pM(\infty)$:

$$\rho(g) = dg(p)|T_pM(\infty).$$

Proof of the main theorem (II)

Let $s \in G$ be a geodesic symmetry.



Then $\rho(s) = -\mathsf{Id}$.

Let s' be another geodesic symmetry. Then $\rho(s') = -\mathsf{Id}$ thus $\rho(ss') = \mathsf{Id}$.

But ss' is an hyperbolic element of G, thus ρ is neither trivial nor faithful. As $\mathsf{SO}_0(1,2)$ is a simple Lie group, we get a contradiction.