The Schoen–Webster Theorem: history and proofs

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General principle

Some geometries are *rigid* in the following sense: only few spaces admit many symmetries.

Let M be a space with some geometric structure, G be its symmetry group.

- a compact M has "many symmetries" if G is noncompact,
- a general M has "many symmetries" if G acts nonproperly,
- there are "few" such spaces if one can hope for a classification.

Not examples

Symplectic or contact geometry are not rigid:

If M is any symplectic or contact manifold, G is infinite-dimensional.

Examples

Riemannian geometry:

If M is a Riemannian manifold, then G acts properly.

Conformal geometry:

Theorem (Ferrand–Obata)

A Riemannian manifold whose *conformal* group acts non properly is conformally equivalent to the unit sphere, or to the Euclidean space.

Strictly pseudoconvex CR geometry: the Schoen–Webster Theorem.

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Complex operator

The complex structure of $\mathbb{C}^n$ is contained in J :

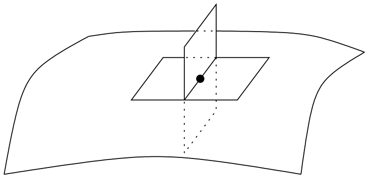
$$\begin{array}{ccc} T\mathbb{C}^n & \longrightarrow & T\mathbb{C}^n \\ (x,v) & \longmapsto & (x,iv) \end{array}$$

At every point, $J_x^2 = -\mathrm{Id}$. Moreover, J satisfies an integrability condition.

Hypersurfaces

A real hypersurface $H \subset \mathbb{C}^n$ inherits of some geometric structure:

- a hyperplane field $\xi = TH \cap J(TH)$
- a complex operator $J : \xi \to \xi$ (satisfying some integrability condition).



CR manifold

Definition

On a $(2n-1)$-manifold M, a *CR structure* is defined by the following data:

- a hyperplane field,
- a complex operator $J : \xi \to \xi$ (satisfying some integrability condition).

The Levi form

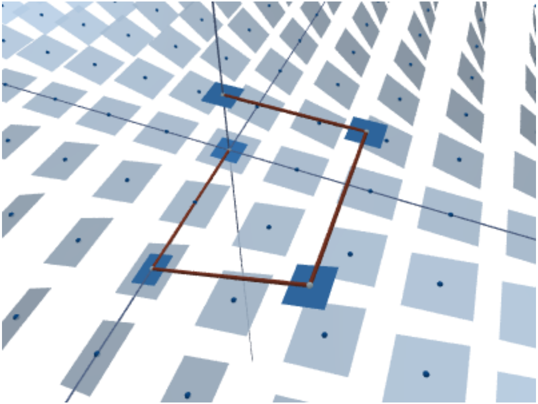
A *calibration* is the choice of a 1-form θ defining ξ.
Given a calibration, one defines a hermitian form on ξ:

Levi form associate to θ

$$\begin{aligned} L_\theta(v) &= d\theta(v, Jv) \qquad \forall v \in \xi \\ &= -\theta([V, JV]) \qquad \forall V \in \xi \text{ extending } v. \end{aligned}$$

The Levi form is homogeneous: $L_{f\theta} = fL_\theta$.

Representation of the Levi form



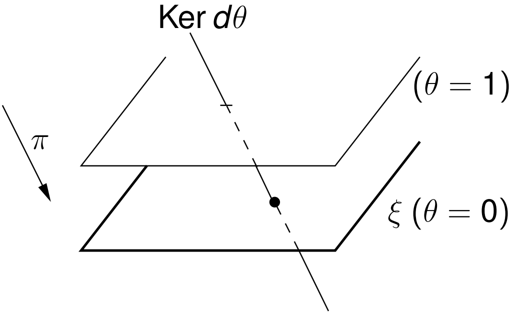
Different CR geometries

Signature of L	the manifold is	ξ induce
zero	Levi-flat	a foliation by complex manifolds
definite	strictly pseudoconvex	a contact structure

Webster metric

If (M, ξ, J) is a strictly pseudoconvex CR manifold and θ a calibration, then

- $\mathrm{Ker}(d\theta)$ is one-dimensional, transverse to ξ,
- one defines a Riemannian metric $W_\theta(v) = L_\theta(\pi v) + \theta(v)^2$.



Pseudoconformal geometry

A strictly pseudoconvex CR structure on a manifold M gives *a line bundle of Riemannian metrics* over M, but not a conformal structure.

The flat models

Standard sphere

The sphere $\mathcal{S}^{2n-1} = \{\sum |z_i|^2 = 1\} \subset \mathbb{C}^n$, endowed with the induced CR structure, is strictly pseudoconvex.

$\mathcal{S}^{2n-1} = \partial \mathbb{C}\mathrm{H}^n$, and $\mathrm{Aut}(\mathcal{S}^{2n-1}) = \mathrm{PU}(1, n)$ is the isometry group of $\mathbb{C}\mathrm{H}^n$.

Heisenberg group

The Heisenberg group $\mathcal{H}^{2n-1} = \mathcal{S} \setminus \{*\}$ is its noncompact analogous.

Their automorphism groups act nonproperly.
A CR manifold that is locally isomorphic to $\mathcal{S}$ is said to be *flat*.

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Complete statement

Theorem (Schoen–Webster)

A strictly pseudoconvex CR manifold whose automorphism group acts nonproperly is CR equivalent to either $\mathcal{S}$ or $\mathcal{H}$.

An action of G on M is proper if the map $G \times M \to M$ is proper, that is: for any compact $K \subset M$, the set

$$G_K = \{g \in G; gK \cap K \neq \varnothing\}$$

is compact.

Two steps

Local statement

A strictly pseudoconvex CR manifold whose automorphism group acts nonproperly is flat.

Local-to-global statement

A flat CR manifold whose automorphism group acts nonproperly is CR equivalent to either $\mathcal{S}$ or $\mathcal{H}$.

additional hypotheses

Compactness hypothesis

The manifold is assumed to be compact.

Connectedness hypothesis

The identity component of the automorphism group is assumed to act nonproperly.

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Chronology

1977: S. WEBSTER, local statement under both compactness and connectedness hypotheses,

1993: Y. KAMISHIMA, local-to-global statement under both hypotheses,

1995: R. SCHOEN, full statement without extra hypothesis,

1996: J. LEE, full statement under both hypotheses (relies on a partial result of Webster),

2007: C. FRANCES, full statement in a general setting (including the Ferrand–Obata Theorem).

Close works

Many author contributed to study *domains* or *complex manifold with boundary* having many automorphisms:

WONG, ROSAY, BURNS and SCHNIDER, PINCHUK, KLEMBECK, ...

Webster 1977

Let M be a strictly pseudoconvex CR manifold.

Theorem

If M is compact and $\mathrm{Aut}_0(M)$ is noncompact, then M is flat.

Geometric methods: relative invariants.

Theorem

If M is compact and $\mathrm{Aut}_0(M)$ admits a noncompact one-parameter Lie subgroup that has a fixed point, then M is globally equivalent to either $\mathcal{S}$ or $\mathcal{H}$.

Kamishima 1993

Theorem

If M is flat, compact and $\mathrm{Aut}_0(M)$ is noncompact then M is globally equivalent to the standard sphere.

Geometric methods, (G, X)-structures.

Lee 1996

Theorem

If M is compact and $\mathrm{Aut}_0(M)$ admit a closed noncompact one-parameter subgroup G_1, then G_1 has a fixed point.

Geometric methods, contact structures.

Schoen 1995

Theorem

If Aut(M) acts nonproperly, then M is CR equivalent to either $\mathcal{S}$ or $\mathcal{H}$.

Analytic methods: Yamabe problem.

Frances 2007

Let (M, B, ω) be a Cartan geometry modelled on $X = \partial\mathbb{K}\mathrm{H}^d$, with regular connection.

Theorem

If $\mathrm{Aut}(M, \omega)$ acts nonproperly on M, then M is isomorphic to either X or X with a point deleted.

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Scalar curvature and conformal laplacian

Let (M, g) be a n-dimensional compact Riemannian manifold and denote by R_g its scalar curvature.

Yamabe Problem

Prove that there exists a metric h conformal to g such that R_h is constant.

Change of metric in a conformal class

By a change of metric $h = u^{\frac{4}{n-2}} g$, we change the scalar curvature according to:

$$R_h = c_n^{-1} u^{-\frac{n+2}{n-2}} L_g u$$

where $c_n = \frac{n-2}{4(n-1)}$ and

$$L_g = \Delta_g + c_n R_g$$

is the *conformal laplacian*.

The Yamabe Theorem

Define the Yamabe invariant by:

$$Q(M,g) = \inf_{\|\phi\|_p=1} \int_M \phi L_g \phi$$

where $p = \frac{2n}{n-2}$.

Theorem

If $Q(M,g)$ is positive (resp. zero, negative) then there exists in $[g]$ a metric of constant scalar curvature equal to 1 (resp. 0, -1).

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When the Yamabe invariant is zero

If $Q(M,g)=0$, we can assume $R_g=0$. Then any conformal map F is an isometry:

- $F^*g = u^{\frac{4}{n-2}}g$ where $L_g u = \Delta_g u = 0$,
- thus u is constant, and F^*g has the same volume than g: $u=1$.

When the Yamabe invariant is negative

If $Q(M,g) < 0$, we can assume $R_g = -1$. Then any conformal map F is an isometry:

- $F^*g = u^{\frac{4}{n-2}} g$ where

$$\Delta_g u = c_n \left(u - u^{\frac{n+2}{n-2}} \right),$$

- Looking at maximum and minimum points, one sees that $u = 1$.

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Main Lemma

Let g be a metric on the unit ball $B \subset \mathbb{R}^n$ that is $\mathscr{C}^{2+\alpha}$-close to the euclidean metric.
Let h be a metric on M with $R_h = 0$ and $F : (B, g) \to (M, h)$ a conformal map, and define $\lambda = |dF_0|$.

Then there is a constant C that controls:

- the conformal dilation factor $C^{-1}\lambda \leqslant |dF| \leqslant C\lambda$,
- the eccentricity of the image of the g-ball of radius $1/2$

$$B_h(F(0), C^{-1}\lambda/2) \subset F(B_g(0, 1/2)) \subset B_h(F(0), C\lambda/2),$$

- the total curvature of h on $B_h(F(0), C^{-1}\lambda)$ is bounded by $C\lambda^{-2}$.

Proof of the Schoen–Webster Theorem

Assume M, g has positive Yamabe invariant and a noncompact conformal group.
There is a sequence of conformal diffeomorphisms F_i and a sequence of points x_i such that

$$|dF_i(x_i)| \to \infty.$$

For ε small enough, g is almost euclidean in all ε-balls.
Choose a sequence of points $y_i \notin F_i(B(x_i, \varepsilon))$.

Proof of the Schoen–Webster Theorem

- Since the Yamabe invariant is positive, L_g admit an inverse; normalising $L_g^{-1}\delta_{y_i}$ one constructs a metric g_i, singular at y_i with zero scalar curvature.
- When $i \to \infty$, $dF_i(x_i)$ explodes and the metrics g_i have a small total curvature on large balls (Main Lemma).
- At the end, there is a conformal diffeomorphism from $M \setminus \{*\}$ to $S \setminus \{*\}$, and one has to patch the missing point.

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Definition

Let (M, ξ, J) be a strictly pseudoconvex CR manifold.

Definition

A *relative invariant* is a family of functions $F_\theta : M \to \mathbb{R}$, indexed by the set of calibrations, where:

- F_θ is nonnegative, continuous, smooth outside its zero set,
- $F_{f\theta} = |f|^{-k} F_\theta$,
- F_θ vanishes on an open set U if and only if U is flat.

Existence

- In dimension $2n - 1 = 3$, there exists a degree -2 relative invariant (Cartan).
- In dimension $2n - 1 > 3$, there exists a degree -1 relative invariant:

the Chern-Moser curvature S is a tensor describing the nonsphericity of M.
The Levi form L_θ allows us to define a norm $\|S\|_\theta$, which is the needed relative invariant.

Canonical calibration

Let F_θ be a relative invariant (of degree -1).
Pick any calibration θ.
The *canonical calibration* defined by

$$\theta^* = F_\theta \theta,$$

- is a CR invariant,
- is continuous, smooth outside its zero set and vanishes only where M is flat,
- its Webster metric is a singular, invariant Riemannian metric.

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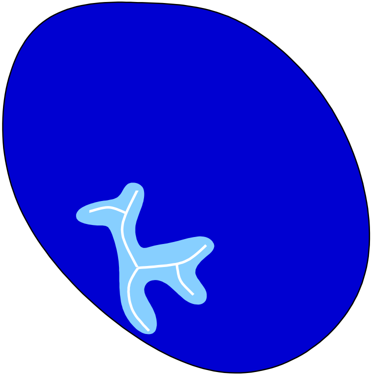
Canonical pseudometric

Assume M is compact.
If M is nonflat, let U be the set where θ^* does not vanish.
We define a pseudometric by:

$$d(x,y) = \inf_{\gamma} \quad \ell_{W_{\theta^*}}(\gamma)$$

Compacity

The set $U_\varepsilon = \{x \in U; d(x, M \setminus U) \geqslant \varepsilon\}$ is compact and has nonempty interior.
The action of $\mathrm{Aut}(M)$ preserves U_ε and W_{θ^*}, thus is compact.



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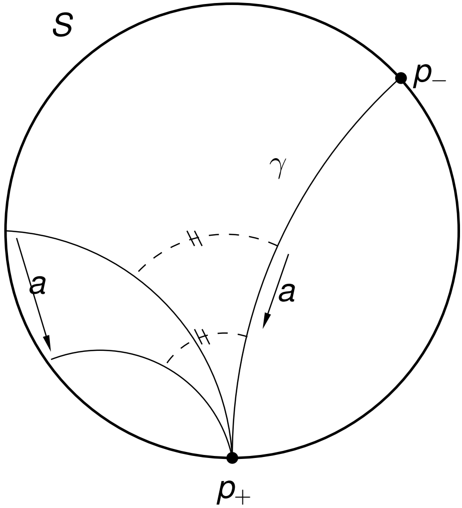
On the model

Proposition

Let $(\phi_k)_k$ be an unbounded sequence in $\mathrm{Aut}(\mathcal{S})$. There exists a subsequence, still denoted by $(\phi_k)_k$ and two points (which may be the same) p_+ and p_- on $\mathcal{S}$ such that:

$$\lim \phi_k(p) = p_+ \qquad \forall p \neq p_-$$
$$\lim \phi_k^{-1}(p) = p_- \qquad \forall p \neq p_+$$

From the complex hyperbolic space to its boundary



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Developing a flat CR manifold

Let M be a flat CR manifold. There exist a *developing map*:

$$D : \widetilde{M} \longrightarrow \mathcal{S}$$

that is,

- D is a local CR diffeomorphism and
- every $\tilde{f} \in \mathrm{Aut}(\widetilde{M})$ is semiconjugate to a $\phi \in \mathrm{Aut}(\mathcal{S})$:

$$D\tilde{f} = \phi D.$$

Follows from order two rigidity, and the high dimension of $\mathrm{Aut}(\mathcal{S})$.

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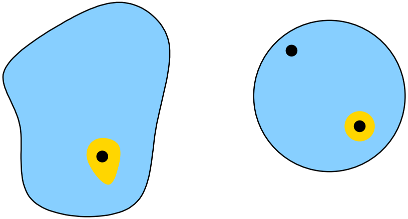
Dynamics of nonproper actions

- If $\mathrm{Aut}(M)$ acts nonproperly, then so do $\mathrm{Aut}(\widetilde{M})$.
- We can find an unbounded sequence $\tilde{f}_i$ mapping a convergent sequence $x_i \to x_\infty$ to another one $y_i \to y_\infty$.
- The semiconjugate sequence ϕ_i has a North-South dynamics with poles p_+ and p_-.
- We can assume that $D(y_\infty) = p_+$.

Extension of injectivity

The image of an injectivity domain includes the basin of attraction of p_+.



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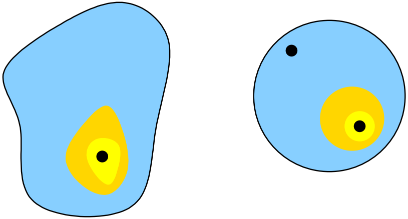

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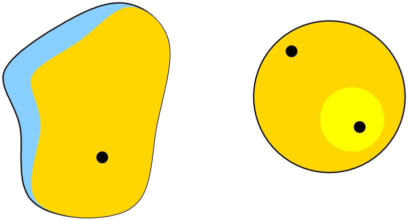

End of the proof

- If the injectivity domain is not all of $\widetilde{M}$, then only a point can be missing: $\widetilde{M}$ must be $\mathcal{S}$ of $\mathcal{H}$.
- The dynamics on $\widetilde{M}$ have only one attracting point, thus the covering map $\widetilde{M} \to M$ must be one sheeted.