A GEOMETRIC STUDY OF WASSERSTEIN SPACES: ISOMETRIC RIGIDITY IN NEGATIVE CURVATURE

by

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Abstract. — Given a metric space X, one defines its Wasserstein space $\mathscr{W}_2(X)$ as a set of sufficiently decaying probability measures on X endowed with a metric defined from optimal transportation. In this article, we continue the geometric study of $\mathscr{W}_2(X)$ when X is a simply connected, nonpositively curved metric spaces by considering its isometry group. When X is Euclidean, the second named author proved that this isometry group is larger than the isometry group of X. In contrast, we prove here a rigidity result: when X is negatively curved, any isometry of $\mathscr{W}_2(X)$ comes from an isometry of X.

1. Introduction

This article is part of a series where the Wasserstein space of a metric space is studied from an intrinsic, geometric point of vue. Given a Polish metric space X, the set of its Borel probability measures of finite second moment can be endowed *via* optimal transport with a natural distance; the resulting metric space is the *Wasserstein space* $\mathscr{W}_2(X)$ of X. We shall only give a minimal amount of background and we shall not discuss previous works, so as to avoid redundancy with the previous articles in the series. One can think of these Wasserstein spaces as geometric measure theory analogues of L^p spaces (here with $p = 2$), but they recall much more of the geometry of X. Our goal is to understand precisely how the geometric properties of X and $\mathscr{W}_2(X)$ are related.

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1.1. Main results. — Our setting here is at the intersection of the previous papers [**Klo08**] and [**BK12**]: as in the later we assume X is a Hadamard space, by which we mean a globally CAT(0), complete, locally compact space, and as in the former we consider the isometry group of $\mathscr{W}_2(X)$. Note that a weaker result concerning the isometric rigidity was included in a previous preprint version of [**BK12**], which has been divided due to length issues after a remark from a referee.

In [**Klo08**] it was proved that the isometry group of $\mathscr{W}_2(\mathbb{R}^n)$ is strictly larger than the isometry group of $\mathbb{R}^n$ itself, the case $n = 1$ being the most striking: some isometries of $\mathscr{W}_2(\mathbb{R})$ are exotic in the sense that they do not preserve the shape of measures. This property seems pretty uncommon, and our main result is the following.

Theorem 1.1. — *If X is a negatively curved geodesically complete Hadamard space then $\mathscr{W}_2(X)$ is isometrically rigid in the sense that its only isometries are those induced by the isometries of X itself.*

By "Negatively curved" we mean that X is not a line and the CAT(0) inequality is strict except for triangles all of whose vertices are aligned, see Section 2.2 for details.

As stated, our proof of Theorem 1.1 depends on strong, yet unpublished results of Lytchak and Nagano [**LN**]; but we shall treat many cases (manifolds, trees, some more general polyhedral complexes) without resorting to [**LN**].

In the process of proving Theorem 1.1, we also get the following result that seems interesting by itself.

Theorem 1.2. — *Let Y, Z be geodesic Polish spaces and assume that Y is geodesically complete and locally compact. Then $\mathscr{W}_2(Y)$ is isometric to $\mathscr{W}_2(Z)$ if and only if Y is isometric to Z.*

It is quite surprising that whether a fully general version of this result holds is an open question.

Note that the proof of Theorem 1.1 involves the inversion of some kind of Radon transform, that seems to be new (for trees it is in particular different from the horocycle Radon transform) and could be of interest for other problems.

1.2. Preliminaries and basic notions. — In this preliminary section, for the sake of self-containment, we briefly recall well-known general facts on Hadamard and Wasserstein spaces. One can refer to [**Bal95, BH99**] and [**Vil09, Vil03**] for proofs, further details and much more.

1.2.1. Wasserstein space. — Given a Polish (*i.e.* complete and separable metric) space X, one defines its (quadratic) Wasserstein space $\mathscr{W}_2(X)$ as the set of Borel probability measures μ on X that satisfy

$$\int_X d^2(x_0, x)\mu(dx) < +\infty$$

for some (hence all) point $x_0 \in X$, equipped by the distance W defined by:

$$\mathrm{W}^2(\mu_0, \mu_1) = \inf \int_{X\times X} d^2(x, y)\Pi(dxdy) \tag{1}$$

where the infimum is taken over all measures Π on $X \times X$ whose marginals are μ_0 and μ_1. Such a measure is called a transport plan and is said to be optimal if it achieves the infimum. In this setting, optimal plans always exist and W turns $\mathscr{W}_2(X)$ into a metric space.

1.2.2. Hadamard spaces. — Given any three points x, y, z in a geodesic Polish metric space X, there is up to congruence a unique comparison triangle x', y', z' in $\mathbb{R}^2$, that is a triangle that satisfies $d(x, y) = d(x', y')$, $d(y, z) = d(y', z')$, and $d(z, x) = d(z', x')$.

One says that X has (globally) *non-positive curvature* (in the sense of Alexandrov), or is CAT(0), if for all x, y, z the distances between two points on sides of this triangle is lesser than or equal to the distance between the corresponding points in the comparison triangle, see figure 1. Note that some authors call this property "globally CAT(0)".

Equivalently, X is CAT(0) if for any triangle x, y, z, any geodesic γ such that $\gamma_0 = x$ and $\gamma_1 = y$, and any $t \in [0, 1]$, the following inequality holds:

$$d^2(y, \gamma_t) \leqslant (1-t)d^2(y, \gamma_0) + td^2(y, \gamma_1) - t(1-t)\ell(\gamma)^2 \tag{2}$$

where $\ell(\gamma)$ denotes the length of γ (which is equal to $d(x, z)$).

We shall say that X is a Hadamard space if X is CAT(0), complete and locally compact. By the generalized Cartan-Hadamard theorem, this implies that X is simply connected. Our goal is now, as said above, to study the isometry group of $\mathscr{W}_2(X)$.

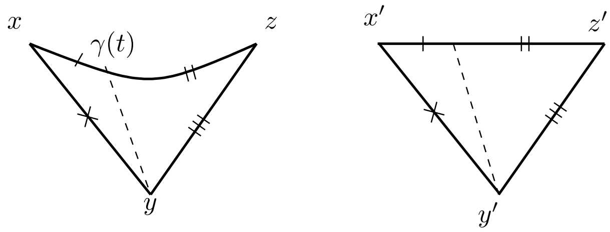

FIGURE 1. The CAT(0) inequality: the dashed segment is shorter in the triangle xyz than in the comparison triangle on the right.

1.3. Strategy of proof. — We have a natural morphism

$$\begin{aligned} \# : \operatorname{Isom} X &\to \operatorname{Isom} \mathscr{W}_2(X) \\ \varphi &\mapsto \varphi_\# \end{aligned}$$

where $\varphi_\#$ is the push-forward of measures:

$$\varphi_\#\mu(A) := \mu(\varphi^{-1}(A)).$$

An isometry Φ of $\mathscr{W}_2(X)$ is said to be *trivial* if it is in the image of $\#$, to *preserve shape* if for all $\mu \in \mathscr{W}_2(X)$ there is an isometry φ^μ of X such that $\Phi(\mu) = \varphi^\mu_\#\mu$, and to be *exotic* otherwise. When all isometries of $\mathscr{W}_2(X)$ are trivial, that is when $\#$ is an isomorphism, we say that $\mathscr{W}_2(X)$ is *isometrically rigid.*

When X is Euclidean, $\mathscr{W}_2(X)$ is not isometrically rigid as proved in [**Klo08**].

Our main result, Theorem 1.1 can now be phrased as follows: if X is a geodesically complete, negatively curved Hadamard space, then $\mathscr{W}_2(X)$ is isometrically rigid.

We first prove this assuming the set of regular points (namely those whose tangent cone is isometric to a Euclidean space plus an extra assumption if the dimension is one, see Section 4.1 for a precise definition) is a dense subset of X. The density of regular points holds in many examples such as manifolds, simplicial trees, polyhedral complexes whose faces are endowed with hyperbolic metrics such as $I_{p,q}$ buildings. Building on results of Lytchak and Nagano [**LN**], we shall prove that the density of regular points holds true for *any* geodesically complete, negatively curved Hadamard space, completing the proof of Theorem 1.1.

To prove isometric rigidity, we show first that an isometry must map Dirac masses to Dirac masses. The method is similar to that used in the

Euclidean case, and is valid for geodesically complete locally compact spaces without curvature assumption. It already yields Theorem 1.2, answering positively in a wide setting the following question: if $\mathscr{W}_2(Y)$ and $\mathscr{W}_2(Z)$ are isometric, can one conclude that Y and Z are isometric?

The second step is to prove that an isometry that fixes all Dirac masses must map a measure supported on a geodesic to a measure supported on the same geodesic. Here we use that X has negative curvature: this property is known to be false in the Euclidean case.

Then as in [**Klo08**] we are reduced to prove the injectivity of a specific Radon transform. The point is that this injectivity is known for $\mathbb{R}\mathrm{H}^n$ and is easy to prove for trees, but seems not to be known for other spaces. We give a simple argument for manifolds, then extend it to spaces with a dense set of regular points. Finally, we prove that all geodesically complete, negatively curved Hadamard spaces have a dense set of regular points.

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2. Reduction to a Radon transform

In this Section we reduce Theorem 1.1 to the injectivity of a kind of Radon transform, and also prove Theorem 1.2.

2.1. Characterization of Dirac masses.

— The characterization of Dirac masses follows from two lemmas, and can be carried out without any assumption on the curvature. Note that when we ask a space to be geodesically complete, we of course imply that it is geodesic. Note that all geodesics are assumed to be parameterized with constant speed and to be globally minimizing.

Lemma 2.1. — *Let Y be a locally compact, geodesically complete Polish space. Any geodesic segment in $\mathscr{W}_2(Y)$ issued from a Dirac mass can be extended to a complete geodesic ray.*

Proof. — Let us recall the measurable selection theorem (see for example [**Del75**], Corollary of Theorem 17): any surjective measurable map between Polish spaces admits a measurable right inverse provided its fibers are compact. Consider the restriction map

$$p^T : \mathscr{R}(Y) \to \mathscr{G}^{0,T}(Y)$$

where $\mathscr{G}^{0,T}(Y)$ is the set of geodesic segments parameterized on $[0,T]$ and $\mathscr{R}(Y)$ is the set of complete rays (geodesics are assumed here to be minimizing and to have constant, non-necessarily unitary, possibly zero speed and all sets of geodesics are endowed with the topology of uniform convergence on compact sets).

The fiber of a geodesic segment γ is closed in the set $\mathscr{R}_{\gamma_0,s}(Y)$ of geodesic rays of speed $s = s(\gamma)$ starting at γ_0. This set is compact by Arzela-Ascoli theorem (equicontinuity follows from the speed being fixed, while the pointwise relative compactness is a consequence of Y being locally compact and geodesic, hence proper).

Moreover the geodesic completeness implies the surjectivity of p^T. There is therefore a measurable right inverse q of p^T: it maps a geodesic segment γ to a complete ray whose restriction to $[0,T]$ is γ.

Let $(\mu_t)_{t\in[0,T]}$ be a geodesic segment of $\mathscr{W}_2(Y)$ of speed s with $\mu_0 = \delta_x$ for some point x.

For $t \in [0,T]$, let $e_t : \mathscr{G}^{0,T}(Y) \to Y$ be the evaluation map at time t. We know from the theory of optimal transport that there is a measure μ' on $\mathscr{G}^{0,T}(Y)$ such that $\mu_t = (e_t)_\#\mu'$, called the displacement interpolation of the geodesic segment.

Let $\mu = q_\#(\mu')$: it is a probability measure on $\mathscr{R}(Y)$ whose restriction $p^T_\#(\mu)$ is the displacement interpolation of (μ_t). For all $t > 0$ we therefore denote by μ_t the measure $(e_t)_\#\mu$ on X; for $t \leq T$, we retrieve the original mesure μ_t.

It is easy to see that $(\mu_t)_{t\geq 0}$ now defines a geodesic ray: since there is only one transport plan from a Dirac mass to any fixed measure, $\mathrm{W}^2(\mu_0,\mu_t) = \int s(\gamma)^2 t^2\, \mu(d\gamma) = s^2t^2$. Moreover the transport plan from μ_t to $\mu_{t'}$ deduced from μ gives $\mathrm{W}(\mu_t,\mu_{t'}) \leqslant s|t-t'|$. But the triangular inequality applied to μ_0, μ_t and $\mu_{t'}$ implies $\mathrm{W}(\mu_t,\mu_{t'}) \geqslant s|t-t'|$ and we are done. □

Lemma 2.2. — *If Y is a geodesic Polish space, given $\mu_0 \in \mathscr{W}_2(Y)$ not a Dirac mass and $y \in \operatorname{supp}\mu_0$, the geodesic segment from μ_0 to $\mu_1 := \delta_y$ cannot be geodesically extended for any time $t > 1$.*

Note that at least in the branching case the assumption $y \in \operatorname{supp}\mu_0$ is needed.

Proof. — Assume that there is a geodesic segment $(\mu_t)_{t\in[0,1+\varepsilon]}$ and let μ be one of its displacement interpolation. Since $y \in \operatorname{supp}\mu_0$, $\operatorname{supp}\mu$ contains a geodesic segment of X that is at y at times 0 and 1, and must therefore be constant.

Since μ_0 is not a Dirac mass, there is a $y' \neq y$ in $\operatorname{supp}\mu_0$. Let γ be a geodesic segment in $\operatorname{supp}\mu$ such that $\gamma_0 = y'$. Then $\gamma_1 = y$ lies between y' and $y'' := \gamma_{1+\varepsilon}$ on γ, and the transport plan Π from μ_0 to $\mu_{1+\varepsilon}$ defined by μ contains in its support the couples (y'', y') and (y, y).

But then cyclical monotonicity shows that Π is not optimal: it costs less to move y'' to y and y to y' by convexity of the cost (see figure 2). The dynamical transport μ cannot be optimal either, a contradiction. □

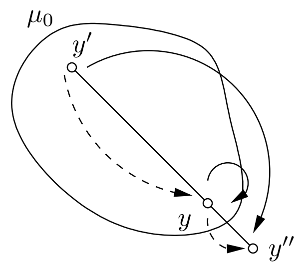

FIGURE 2. The transport shown with continuous arrows is less effective than the transport given by the dashed arrows.

We can now easily draw the consequences of these lemmas.

Proposition 2.3. — *Let Y, Z be geodesic Polish spaces and assume that Y is geodesically complete and locally compact. Any isometry from $\mathscr{W}_2(Y)$ to $\mathscr{W}_2(Z)$ must map all Dirac masses to Dirac masses.*

Except in the next corollary, we shall use this result with $Y = Z = X$. Note that we will need to require X being geodesically complete in addition to the Hadamard hypothesis.

Proof. — Denote by φ an isometry $\mathscr{W}_2(Y) \to \mathscr{W}_2(Z)$ and consider any $x \in Y$. If $\varphi(\delta_x)$ were not a Dirac mass, there would exist a geodesic segment $(\mu_t)_{t\in[0,1]}$ from $\varphi(\delta_x)$ to a Dirac mass (at a point $y \in \operatorname{supp}\varphi(\delta_x)$) that cannot be extended for times $t > 1$.

But $\varphi^{-1}(\mu_t)$ gives a geodesic segment issued from δ_x, that can therefore be extended. This is a contradiction since φ is an isometry. □

We can now prove Theorem 1.2, which we recall: let Y, Z be geodesic Polish spaces and assume that Y is geodesically complete and locally compact; then $\mathscr{W}_2(Y)$ is isometric to $\mathscr{W}_2(Z)$ if and only if Y is isometric to Z.

Proof. — Let φ be an isometry $\mathscr{W}_2(Y) \to \mathscr{W}_2(Z)$. Then φ maps Dirac masses to Dirac masses, and since the set of Dirac masses of a space is canonically isometric to the space, φ induces an isometry $Y \to Z$. the converse implication is obvious. □

Note that we do not know whether this result holds for general metric spaces. Also, we do not know if there is a space Y and an isometry of $\mathscr{W}_2(Y)$ that maps some Dirac mass to a measure that is not a Dirac mass.

2.2. Measures supported on a geodesic. — The characterization of measures supported on a geodesic relies on the following argument: when dilated from a point of the geodesic, such a measure has Euclidean expansion.

Lemma 2.4. — *Assume that X is negatively curved, and let γ be a maximal geodesic of X, μ be in $\mathscr{W}_2(X)$. Given a point $x \in \gamma$, denote by $(x^t \cdot \mu)_{t\in[0,1]}$ the geodesic segment from δ_x to μ.*

The measure μ is supported on γ if and only if for all $x, g \in \gamma$

$$\mathrm{W}(x^{\frac{1}{2}} \cdot \mu, x^{\frac{1}{2}} \cdot \delta_g) = \frac{1}{2}\,\mathrm{W}(\mu, \delta_g).$$

Proof. — If x, y are points of X, $(x+y)/2$ denotes the midpoint of x and y. The "only if" part is obvious since the transport problem on a convex subset of X only involves the induced metric on this subset, which here is isometric to an interval.

To prove the "if" part, first notice that the CAT(0) inequality in a triangle (x, y, g) yields the Thales inequality $d((x+y)/2, (x+g)/2) \leqslant \frac{1}{2}d(y, g)$, so that by direct integration

$$\mathrm{W}(x^{\frac{1}{2}} \cdot \mu, \delta_{(x+g)/2}) \leqslant \frac{1}{2}\,\mathrm{W}(\mu, \delta_g). \tag{3}$$

Note also that $x^{\frac{1}{2}} \cdot \delta_g = \delta_{(x+g)/2}$.

Assume now that μ is not supported on γ and let $y \in \operatorname{supp}\mu \setminus \gamma$, and $x, g \in \gamma$ such that x, y, g are not aligned. Since γ is maximal, this is possible, for example by taking $d(x, g) > d(x, y)$ in the branching case (any choice of $x \neq g$ would do in the non-branching case). Since X is negatively curved, we get that $d((x+y)/2,(x+g)/2) < \frac{1}{2}d(y,g)$ so that (2) is strict (see Figure 3). $\square$

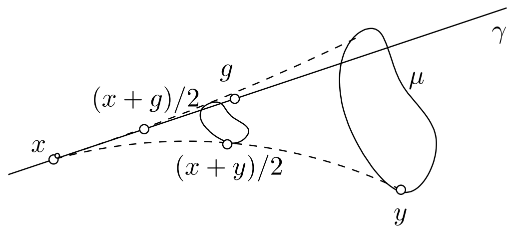

FIGURE 3. In negative curvature, midpoinds are closer than they would be in Euclidean space.

Corollary 2.5. — *If X is a negatively curved Hadamard space and γ is a maximal geodesic of X, any isometry of $\mathscr{W}_2(X)$ that fixes Dirac masses must preserve the subset $\mathscr{W}_2(\gamma)$ of measures supported on γ, and therefore induces an isometry on this set.*

Proof. — Let φ be an isometry of $\mathscr{W}_2(X)$ that fixes Dirac masses. Let γ be a maximal geodesic of X and $\mu \in \mathscr{W}_2(X)$ be supported on γ. If $\varphi(\mu)$ were not supported in γ, there would exist $x, g \in \gamma$ such that $\mathrm{W}(\varphi(\mu)_{1/2}, \delta_{(x+g)/2}) < \frac{1}{2}\mathrm{W}(\varphi(\mu), \delta_g)$. But φ is an isometry and $\varphi^{-1}(\delta_x) = \delta_x$, $\varphi^{-1}(\delta_g) = \delta_g$, $\varphi^{-1}(\delta_{(x+g)/2}) = \delta_{(x+g)/2}$ so that also $\varphi(\mu)_{1/2} = \varphi(\mu_{1/2})$. As a consequence, (2) would be a strict inequality too, a contradiction. $\square$

2.3. Isometry induced on a geodesic. — We want to deduce from the previous section that an isometry that fixes all Dirac masses must also fix every geodesically-supported measure. To this end, we have to show that the isometry induced on the measures supported in a given geodesic is trivial, in other terms to rule out the other possibilities exhibited in [**Klo08**], at which it is recommended to take a look before reading the proof below.

Proposition 2.6. — *Assume that X is a geodesically complete, negatively curved Hadamard space and let φ be an isometry of $\mathscr{W}_2(X)$ that fixes all Dirac masses. For all complete geodesics γ of X, the isometry induced by φ on $\mathscr{W}_2(\gamma)$ is the identity.*

We only address the geodesically complete case for simplicity, but the same result probably holds in more generality.

Proof. — Let x be a point not lying on γ. Such a point exists since X is negatively curved, hence not a line.

First assume that φ induces on $\mathscr{W}_2(\gamma)$ an exotic isometry. Let y, z be two points of γ and define $\mu_0 = \frac{1}{2}\delta_y + \frac{1}{2}\delta_z$. Then $\mu_n := \varphi^n(\mu_0)$ has the form $m_n\delta_{y_n} + (1 - m_n)\delta_{z_n}$ where $m_n \to 0$, $y_n \to \infty$, $z_n \to (y + z)/2 =: g$.

Let now γ' be a complete geodesic that contains $y' = (x + y)/2$ and $z' = (x+z)/2$ (see Figure 4). Since the midpoint of μ_0 and δ_x is supported on γ', so is the midpoint of μ_n and δ_x. This means that $(x + y_n)/2$ and $(x + z_n)/2$ lie on γ', and the former goes to infinity. This shows that φ also induces an exotic isometry on $\mathscr{W}_2(\gamma')$, thus that $(x + z_n)/2$ goes to the midpoint g' of y' and z'. But we already know that $(x + z_n)/2$ tends to the midpoint of x and g, which must therefore be g'.

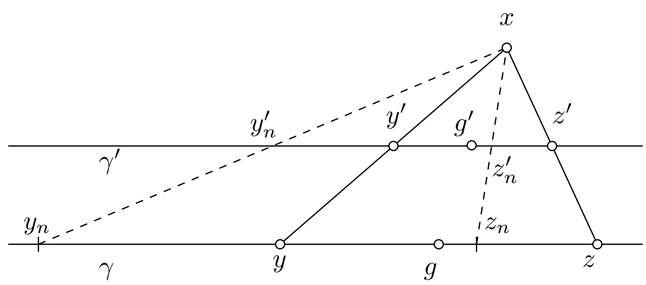

FIGURE 4. That an isometry acts exotically on a geodesic γ would imply that there is another geodesic γ' such that taking midpoints with x is an affine map. Then γ and γ' would be parallel, therefore they would bound a flat strip.

Since this holds for all choices of y and z, we see that the map $y \to (x + y)/2$ maps affinely γ to γ'. But the geodesic segment $[x\gamma_t]$ converges when $t \to \pm\infty$ to geodesic segments asymptotic to γ and $-\gamma$ respectively. It follows that γ' is parallel to γ, so that they must bound a flat strip (see [**Bal95**]). But this is forbidden by the negative curvature assumption.

A similar argument can be worked out in the case when φ induces an involution: given $y, z \in \gamma$ and their midpoint g, one can find measures μ_n supported on y and y_n, where $y_n \to g$ and μ_n has more mass on y_n than on y, such that $\varphi(\mu_n)$ is supported on z and a point z_n of γ, with more mass on z_n. It follows that the midpoints y', y'_n, z' with x are on a line, and we get the same contradiction as before.

The classification of isometries of $\mathscr{W}_2(\mathbb{R})$ shows that if φ fixes Dirac masses and is neither an exotic isometry nor an involution, then it is the identity. $\square$

Now we are able to link the isometric rigidity of $\mathscr{W}_2(X)$ to the injectivity of a Radon transform. The following definition relies on the following observation: since a geodesic is convex and X is Hadamard, given a point y and a geodesic γ there is a unique point $p_\gamma(y) \in \gamma$ closest to y, called the projection of y to γ.

Definition 2.7. — When X is geodesically complete, we define the *perpendicular Radon transform* $\mathscr{R}\,\mu$ of a measure $\mu \in \mathscr{W}_2(X)$ as the following map defined over complete geodesics γ of X:

$$\mathscr{R}\,\mu(\gamma) = (p_\gamma)_\#\mu.$$

In other words, this Radon transform recalls all the projections of a measure on geodesics.

The following result is now a direct consequence of Proposition 3.7.

Proposition 2.8. — *Assume that X is geodesically complete and negatively curved. If there is a dense subset $A \subset \mathscr{W}_2(X)$ such that for all $\mu \in \mathscr{W}_2(X)$ and all $\nu \in A$, we have*

$$\mathscr{R}\,\mu = \mathscr{R}\,\nu \Rightarrow \mu = \nu$$

then $\mathscr{W}_2(X)$ is isometrically rigid.

Proof. — Let φ be an isometry of $\mathscr{W}_2(X)$. Up to composing with an element of the image of #, we can assume that φ acts trivially on Dirac masses. Then it acts trivially on geodesically supported measures. For all $\nu \in A$, since $(p_\gamma)_\#\nu$ is the measure supported on γ closest to ν, one has $\mathscr{R}\,\varphi(\nu) = \mathscr{R}\,\nu$, hence $\varphi(\nu) = \nu$. We just proved that φ acts trivially on a dense set, so that it must be the identity. $\square$

3. Injectivity of the Radon transform

The proof of Theorem 1.1 shall be complete as soon as we get the injectivity required in Proposition 3.9. Note that in the case of the real hyperbolic space $\mathbb{R}\mathrm{H}^n$, we could use the usual Radon transform on the set of compactly supported measures with smooth density to get it (see [**Hel99**]).

3.1. The case of manifolds and their siblings. — Let us first give an argument that does the job for all manifolds. We shall give a more general, but somewhat more involved argument afterwards.

Proposition 3.1. — *Assume that X is a Hadamard smooth manifold . Let A be the set of finitely supported measures. For all $\mu \in \mathscr{W}_2(X)$ and all $\nu \in A$, if $\mathscr{R}\,\mu = \mathscr{R}\,\nu$ then $\mu = \nu$.*

Note that A is dense in $\mathscr{W}_2(X)$ so that this proposition ends the proof of Theorem 1.1 in the case of manifolds.

Proof. — Write $\nu = \sum m_i \delta_{x_i}$ where $\sum m_i = 1$. Note that since X is a *negatively* curved manifold, it has dimension at least 2.

First, we prove under the assumption $\mathscr{R}\,\mu = \mathscr{R}\,\nu$ that μ must be supported on the x_i. Let x be any other point, and consider a geodesic γ such that $\gamma_0 = x$ and $\dot\gamma_0$ is not orthogonal to any of the geodesics (xx_i). Then for all i, $p_\gamma(x_i) \neq x$ and there is an $\varepsilon > 0$ such that the neighborhood of size ε around x on γ does not contain any of these projections. It follows that $\mathscr{R}\,\nu(\gamma)$ is supported outside this neighborhood, and so does $\mathscr{R}\,\mu(\gamma)$. But the projection on γ is 1-Lipschitz, so that μ must be supported outside the ε neighborhood of x in X. In particular, $x \notin \operatorname{supp}\mu$.

Now, if γ is a geodesic containing x_i, then $\mathscr{R}\,\nu(\gamma)$ is finitely supported with a mass at least m_i at x_i. For a generic γ, the mass at x_i is exactly m_i. It follows immediately, since μ is supported on the x_i, that its mass at x_i is m_i. □

The above proof mainly uses the fact that given a point x and a finite number of other points x_i, there is a geodesic $\gamma \ni x$ such that $p_\gamma(x_i) \neq x$ for all i. It follows that the proof can be adjusted to get the general case. To this end, we need to introduce some extra definitions.

3.2. Regular subset of a Hadamard space. — Assume that X is a Hadamard geodesically complete space and let $p \in X$ be a point. We set Σ'_p the set of all nontrivial geodesics starting at p. The angle $\angle$ is a pseudo-metric on Σ'_p. The space of directions Σ_p at p is the completion with respect to $\angle$ of the quotient metric space obtained from Σ'_p by the relation $\angle = 0$. Under these assumptions, the space of directions at p is a compact CAT(1) space whose diameter is smaller or equal to π (see [**BBI01**] for a proof). We shall also use the geometric dimension of a CAT space introduced by Kleiner in [**Kle99**].

***Definition 3.2* (Geometric dimension).** — Let U be an open subset of a locally compact CAT(1) space. Then the dimension of U is 0 if U is discrete. Otherwise, it is defined as

$$\dim U = 1 + \sup_{p \in U} \dim \Sigma_p.$$

Given a geodesically complete Hadamard space X, one defines its *regular set* $\mathrm{Reg}(X)$ as the subset of X made of points p whose space of directions Σ_p is isometric to a standard sphere $\mathbb{S}^k$ where k is any integer (or equivalently, whose tangent cone at p is isometric to the Euclidean space of dimension $k+1$). When $k=0$, we further require the existence of an open neighborhood of p which is isometric to an open segment in $\mathbb{R}$.

Proposition 3.3. — *Assume that X is a geodesically complete Hadamard space. Let A be the set of measures supported on finite set of points all located in the regular set of X. For all $\mu \in \mathscr{W}_2(X)$ and all $\nu \in A$, if $\mathscr{R}\mu = \mathscr{R}\nu$ then $\mu = \nu$.*

Proof. — Write $\nu = \sum m_i \delta_{x_i}$ where $\sum m_i = 1$. Consider one of the x_i, and let $m_i' := \mu(\{x_i\})$. Let us first assume that Σ_{x_i} is isometric to $\mathbb{S}^{n_i}$ with $n_i \geq 1$.

Let γ be a geodesic and x be a point that does not belong to the image of γ. Then, the angle at $y = p_\gamma(x)$ between the geodesic σ from y to x and γ satisfies $\angle_y(\sigma', \gamma') \geq \pi/2$ according to the first variation formula. The same argument yields $\angle_y(\sigma', -\gamma') \geq \pi/2$. Thus, if we further assume that $\Sigma_y = \mathbb{S}^k$ with $k \geq 1$, we finally get

$$\angle_y(\sigma', \gamma') = \angle_y(\sigma', -\gamma') = \pi/2.$$

Using this property, it is now easy to find a geodesic γ with $x_i \in \gamma$ and $p_\gamma(x_j) \neq x_i$ for all $j \neq i$. From this and $\mathscr{R}\mu = \mathscr{R}\nu$, we get $m_i' \leq m_i$. It also follows that for some $\varepsilon > 0$, the measure $\mathscr{R}\mu(\gamma) = \mathscr{R}\nu(\gamma)$ is concentrated outside $B(x_i, \varepsilon) \setminus x_i$. Since p_γ is one Lipschitz (see [**BH99**, Proposition 2.4]), it also follows that μ is concentrated outside $B(x_i, \varepsilon) \setminus x_i$.

Let $\gamma^\perp$ denote the set of points x such that the geodesic segment $[xx_i]$ is orthogonal to γ at x_i. By definition of the Radon transform, $\mu(\gamma^\perp \setminus x_i) = m_i - m_i'$ and for all $x \in \operatorname{supp}\mu \setminus \gamma^\perp$, we have $p_\gamma(x) \notin B(x_i, \varepsilon)$.

Choose a second geodesic $\gamma_2 \ni x_i$, close enough to γ to ensure that for all $x \in \operatorname{supp}\mu \setminus \gamma^\perp$, we still have $p_{\gamma_2}(x) \notin B(x_i, \varepsilon)$ (up to schrinking

ε a bit if necessary). We can moreover assume that γ_2 enjoys the same properties we asked to γ, so that

$$\forall x \in \operatorname{supp}\mu \setminus \gamma^\perp \cap \gamma_2^\perp, \quad p_{\gamma_2}(x) \notin B(x_i,\varepsilon).$$

Let $n_i - 1$ be the dimension of Σ_{x_i}. We can construct inductively a family $\gamma_1,\dots,\gamma_{n_i}$ of geodesics chosen as above, and such that their velocity vectors at x_i span its tangent space $T_{x_i}X$ which is isometric to $\mathbb{R}^{n_i}$. For all point x in

$$\operatorname{supp}\mu \setminus \cap_\alpha \gamma_\alpha^\perp,$$

we get $p_{\gamma_n}(x) \notin B(x_i,\varepsilon)$. But $\cap_\alpha \gamma_\alpha^\perp = \{x_i\}$, and considering $\mathscr{R}\,\mu(\gamma_n) = \mathscr{R}\,\nu(\gamma_n)$ we get $m_i' = m_i$.

It remains to treat the case when Σ_{x_i} is isometric to $\mathbb{S}^0$. Recall that in that case, we further assume the existence of an open neighborhood V of x_i isometric to a short open segment. It is then clear that choosing any geodesic γ going through x_i, $p_\gamma(x) \neq x_i$ for any $x \neq x_i$.

Since μ is a probability measure and $\sum m_i = 1$, we deduce $\mu = \nu$. $\square$

To get our main result, it remains to prove that $\operatorname{Reg}(X)$ is a dense subset of X. This is a consequence of deep results of Lytchak and Nagano [**LN**] on the structure of spaces with an upper curvature bound.

3.3. Fine properties of Hadamard spaces. — In this section, we report the results we need from the paper [**LN**], some of these results are also contained in [**OT**]. We then use them below to complete the proof of Theorem 1.1 in the general case. For simplicity, we give the statements for a geodesically complete Hadamard space X only, see [**LN**] for the full results.

Given U a relatively compact open subset of X, the authors introduce the set of *n-regular* points $R_n(U)$ defined by

$$R_n(U) = \{x \in U; \Sigma_x \text{ is isometric to } \mathbb{S}^{n-1} * Z\}$$

where (Z,d) is a metric space and $\mathbb{S}^{n-1} * Z$ is the spherical join of $\mathbb{S}^{n-1}$ and Z. We shall use the following results they prove:

Theorem 3.4 **(Lytchak-Nagano)**. — *Let U be a locally compact open subset of X. Then, the Hausdorff dimension of $U \setminus R_n(U)$ satisfies*

$$\dim_{\mathcal{H}}(U \setminus R_n(U)) \leq n-1.$$

(This is [**LN**, Theorem 1.1].)

***Lemma 3.5* (Lytchak-Nagano).** — *Let U be a relatively compact open subset of X and assume that* $\dim U = 1$. *Then, for each $x \in R_1(U)$, there exists a bilipschitz embedding of a small open neighborhood of x into $\mathbb{R}$.*

(Obtained by combining [**LN**, Lemma 9.6] (see also the discussion above the statement) and [**LN**, Lemma 11.5])

***Theorem 3.6* (Lytchak-Nagano).** — *Let U be a relatively compact open subset of X. Then, there exists an integer $n \in \mathbb{N}$ such that $\mathcal{H}^n(U) \in (0, +\infty)$. Moreover, the geometric dimension of U coincides with its Hausdorff dimension, namely* $\dim U = n$.

(This is [**LN**, Theorem 13.1] which is an improvement of earlier result in [**OT**]).

3.4. Density of the regular set. — In this section, we use the tools described above to prove the following statement, completing the proof of Theorem 1.1.

***Theorem 3.7*.** — *Let X be a geodesically complete Hadamard space. Then* $\mathrm{Reg}(X)$ *is a dense subset of X.*

Proof. — Given any $x \in X$, we have to prove that x is in the closure of $\mathrm{Reg}(X)$.

We first introduce a definition: we call the minimal dimension of a neighborhood of x the *local dimension* at x and denote it by n_x; moreover there is an open, relatively compact neighborhood U of x such that $\dim U = n_x$. To see this, recall that X must have relatively compact balls, and apply Theorem 4.6 to $(\dim B(x, 1/k))_{k \geq 1}$: it is a decreasing sequence with positive integer value, thus has a limit which is reached for some k. Moreover, we get that n_x is also the Hausdorff dimension of U.

The theorem follows from two claims: first $R_{n_x}(U) \subset \mathrm{Reg}(X)$, second x is in the closure of $R_{n_x}(U)$.

To prove the first claim, observe that if Z is any non-empty space, the compactness of Σ_x then yields that $\mathbb{S}^{n_x - 1} * Z$ has dimension at least n_x; since $\dim U = n_x$, this shows that all points of $R_{n_x}(U)$ have their space of direction isometric to $\mathbb{S}^{n_x - 1}$. When $n_x = 1$, Lemma 4.5 further ensures that points in $R_{n_x}(U)$ have a neighborhood isometric to an interval, proving the claim.

We prove the second claim by contradiction. If x were not in the closure of $R_{n_x}(U)$, by Theorem 4.4 it would lie in an open set $U' \subset U$ of Hausdorff dimension less than n_x. By Theorem 4.6, U' has dimension less than n_x, contradicting the definition of local dimension. $\square$

4. Appendix: the Radon transform on trees

In this appendix we prove a side result, not needed in the proof of Theorem 1.1, showing that in the case of simplicial trees one can explicitely inverse the Radon transform introduced above.

Let X be a locally finite tree that is not a line. We describe X by a couple (V,E) where V is the set of vertices; E is the set of edges, each endowed with one or two endpoints in V and a length.

For all $x \in V$, let $k(x)$ be the valency of x, that is the number of edges incident to x. We assume that no vertex has valency 2, since otherwise we could describe the same metric space by a simpler graph. Note also that edges with only one endpoint have infinite length since X is assumed to be complete.

In this setting, X is geodesically complete if and only if it has no leaf (vertex of valency 1). Assume this, and let γ be a complete geodesic. If x is a vertex lying on γ and $C_1, \dots, C_{k(x)}$ are the connected components of $X \setminus \{x\}$, let $\perp^x(\gamma)$ be the union of x and of the C_i not meeting γ (see Figure **??**). It inherits a tree structure from X. In fact, $\perp^x(\gamma)$ depends only upon the two edges e, f of γ that are incident to x. We therefore let $\perp^x(ef) = \perp^x(\gamma)$.

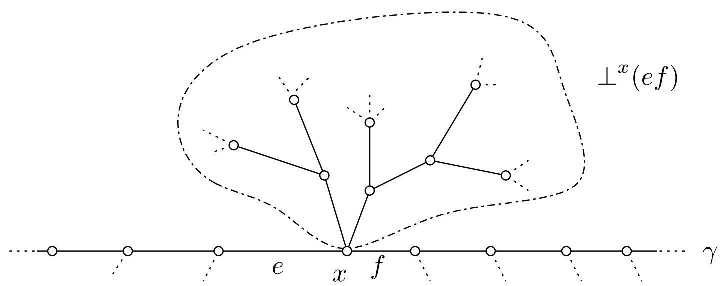

FIGURE 5. A perpendicular to a geodesic.

The levels of p_γ are called the *perpendiculars* of γ, they also are the bissectors of its points. They are exactly:

– the sets $\perp^x(\gamma)$ where x is a vertex of γ.
– the sets $\{x\}$ where x is a point interior to an edge of γ.

A measure $\mu \in \mathscr{W}_2(X)$ can be decomposed into a part supported outside vertices, which is obviously determined by the projections of μ on the various geodesics of X, and an atomic part supported on vertices. Therefore, we are reduced to study the perpendicular Radon transform reformulated as follows for functions defined on V instead of measures on X.

***Definition 4.1* (combinatorial Radon transform)**
A *flag* of X is defined as a triple (x, ef) where x is a vertex, $e \neq f$ are edges incident to x and ef denotes an unordered pair. Let us denote the set of flags by F; we write $x \in ef$ to say that $(x, ef) \in F$.

Given a summable function h defined on the vertices of X, we define its *combinatorial perpendicular Radon transform* as the map

$$\begin{array}{rccc} \mathscr{R}\,h: & F & \to & \mathbb{R} \\ & (x, ef) & \mapsto & \displaystyle\sum_{y \in \perp^x(ef)} h(y) \end{array}$$

where the sum is on vertices of $\perp^x(ef)$.

It seems that this Radon transform has not been studied before, contrary to the transforms defined using geodesics [**?**], horocycles [**?, ?**] and circles [**?**].

***Theorem 4.2* (Inversion formula).** — *Two maps $h, l : V \to \mathbb{R}$ such that $\sum h = \sum l$ and $\mathscr{R}\,h = \mathscr{R}\,l$ are equal. More precisely, we can recover h from $\mathscr{R}\,h$ by the following inversion formula:*

$$h(x) = \frac{1}{k(x)-1} \sum_{ef \ni x} \mathscr{R}\,h(x, ef) - \frac{k(x)-2}{2} \sum_{y \in V} h(y)$$

where the first sum is over the set of pairs of edges incident to x.

Proof. — The formula relies on a simple double counting argument:

$$\begin{aligned} \sum_{ef \ni x} \mathscr{R}\,h(x, ef) &= \sum_{ef \ni x} \sum_{y \in \perp^x(ef)} h(y) \\ &= \sum_{y \in V} h(y) n_x(y) \end{aligned}$$

where $n_x(y)$ is the number of flags (x, ef) such that $y \in \perp^x(ef)$. If $y \neq x$, let e_y be the edge incident to x starting the geodesic segment from x to y. Then $y \in \perp^x(ef)$ if and only if $e, f \neq e_y$. Therefore, $n_x(y) = \binom{k(x)-1}{2}$. But $n_x(x) = \binom{k(x)}{2}$, so that

$$\sum_{ef \ni x} \mathscr{R}\, h(x, ef) = \binom{k(x)-1}{2} \sum_{y \in V} h(y) + (k(x)-1)h(x).$$

□

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