

# SHARP QUANTITATIVE ISOPERIMETRIC INEQUALITIES IN THE $L^1$ MINKOWSKI PLANE

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ABSTRACT. An isoperimetric inequality bounds from below the perimeter of a domain in terms of its area. A *quantitative* isoperimetric inequality is a stability result: it bounds from above the distance to an isoperimetric minimizer in terms of the isoperimetric deficit. In other words, it measures how close to a minimizer an almost optimal set must be.

The euclidean quantitative isoperimetric inequality has been thoroughly studied, in particular in [Hal92] and [FMP08], but the  $L^1$  case has drawn much less attention.

In this note we prove two quantitative isoperimetric inequality in the  $L^1$  Minkowski plane with sharp constants and determine the extremal domains for one of them. It is usually (but not here) difficult to determine the extremal domains in a quantitative isoperimetric inequality: the only such known result is for the Euclidean plane, due to Alvino, Ferone and Nitsch [AFC09], see also [Nit08].

## 1. STATEMENT OF THE RESULTS

We consider the plane  $\mathbb{R}^2$  endowed with the  $L^1$  metric:

$$|(x_1, x_2) - (y_1, y_2)| = |x_1 - y_1| + |x_2 - y_2|.$$

We denote the boundary of a set using  $\partial$ . The notation  $|\cdot|$  shall be used to denote the size of an object, whatever its nature. If  $A$  is an measurable plane set then  $|A|$  is its Lebesgue measure, also called its area ; if  $v$  is a vector  $|v|$  is its  $L^1$  norm ; if  $\gamma$  is a curve,  $|\gamma|$  is its  $L^1$  length, namely

$$|\gamma| = \sup \sum_{i=1}^{n-1} |\gamma(t_i) - \gamma(t_{i+1})|$$

where the supremum is on all increasing sequences  $(t_1, t_2, \dots, t_n)$  taking values in the definition interval of  $\gamma$ . A curve is said to be ( $L^1$ -)rectifiable if its length is finite. For example, a curve defined on a segment whose coordinates function are both motononic is rectifiable and its length is the distance between its endpoints. See e.g. [AT04] for more information on the length of curves in a metric space.

By a *domain* of the plane, we mean the closure of the bounded component of a Jordan curve. In particular, domains are compact and connected. All rectangles and squares considered are assumed to have their sides parallel to the coordinate axes. The square centered at 0 with side length  $2\lambda$  is denoted by  $B_\infty(\lambda)$ : it is the  $\lambda$  ball of the  $L^\infty$  metric. Squares are known to minimize  $L^1$  perimeter among plane domains of given area; in particular, since the  $L^1$  perimeter of a unit area square is 4, the following isoperimetric inequality holds for all domains:  $|\partial A| \geq 4|A|^{1/2}$ .

Note that this kind of isoperimetric inequality can be recast as the problem of minimizing a boundary energy given by the integral over the boundary of the domain of a function defined on the set of possible tangent direction. The minimizing shape can be explicitly constructed and is known as the Wulff shape (or Wulff crystal, since it is often a polyhedra) of the integrand, see e.g. [Tay78, BM94].

The measure of the distance between compact plane sets  $A, B$  we use in our main result is the  $L^\infty$  Hausdorff metric :

$$d_\infty(A, B) = \inf\{\lambda \geq 0 \mid A \subset B + B_\infty(\lambda) \text{ and } B \subset A + B_\infty(\lambda)\}.$$

Let us explain why this metric is natural here. One way to prove that almost isoperimetric domains are close to minimizers is to prove that they contain a minimizer of radius  $r$  and are included in another of radius  $R$ , with small radii difference  $R - r$  and same center. In the euclidean space, such inclusions imply that the considered domain is at Hausdorff distance at most  $(R - r)/2$  from some ball; the Bonnesen inequality gives a bound on the possible  $R - r$  in terms of the isoperimetric deficit, and implies results similar to ours in the Euclidean case; see [Ber05, Bon29, Oss78]. However, balls and minimizers are different in the  $L^1$  plane, so that if  $A$  is between concentric squares of radii  $R$  and  $r$ , one can only say that it is at  $L^1$  Hausdorff distance  $R - r$  from some square, while the  $L^\infty$  Hausdorff distance bound is the expected  $(R - r)/2$ .

It would certainly be possible to use the  $L^1$  Hausdorff metric, and we expect that arguments of the same kind that those we use to prove Theorem 1, but more involved, would give a constant better than the  $1/16$  obtained using the inequality  $d_1 \leq 2d_\infty$  and Theorem 1.

**Theorem 1.** *Let  $A$  be a domain of the  $L^1$  Minkowski plane whose boundary is a rectifiable curve, and assume that*

$$(1) \quad |\partial A|^2 \leq (16 + \varepsilon)|A|.$$

*Then there is a square  $S$  such that*

$$(2) \quad d_\infty(A, S)^2 \leq \frac{\varepsilon|A|}{64}.$$

We shall also see that Theorem 1 is sharp and show that up to  $L^1$  isometry and homothety the domains that achieve the bound are the rectangles and the squares with one square deleted at a corner.

A second possible measure of the distance between domains of the same area, which presents the advantage to be suitable to higher dimension as well, is simply the gap between their area and that of their intersection. In this respect we prove the following.

**Proposition 1.** *Let  $A$  be a domain of the  $L^1$  Minkowski plane whose boundary is a rectifiable curve, and assume that (1) holds with  $\varepsilon$  sufficiently small. Then there is a square  $S$  such that  $|S| = |A|$  and:*

$$(3) \quad |S \cap A| \geq \left(1 - \frac{\sqrt{\varepsilon}}{4} + O(\varepsilon)\right) |A|.$$

*In terms of Fraenkel asymmetry, this reads:*

$$(4) \quad \frac{|S \Delta A|}{|A|} \leq \frac{\sqrt{\varepsilon}}{2} + O(\varepsilon)$$

We shall see that the  $1/2$  constant in (4) is sharp.

Surprisingly enough, it seems that these results are new, although similar ones can be deduced from the much more general [FMP09] (but with a non-optimal constant) and the Bonnesen-like inequalities of [PWZ93] (but only when  $A$  is convex).

## 2. PROOF OF THE INEQUALITIES

Assume  $A$  satisfies (1) for some  $\varepsilon$  and let  $R$  be the smallest rectangle containing  $A$ . This rectangle plays the role of a convex hull.

**Lemma 1.** *We have  $|\partial A| \geq |\partial R|$ .*

*Proof.* Since  $R$  is minimal, each of its sides contains a point of the boundary of  $A$ . Denote  $r_1, r_2, r_3, r_4$  such points so that  $r_i$  and  $r_{i+1}$  lie on two adjacent sides of  $R$  for all  $i$  (modulo 4). It is possible that some  $r_i = r_{i+1}$ , but this does not affect what follows.

There are four curves  $\gamma_i$  in  $\partial A$  that connect  $r_i$  to  $r_{i+1}$  and meet only at their endpoints (see figure 1). Similarly, the boundary of  $R$  is made of four curves  $\eta_i$  connecting  $r_i$  to  $r_{i+1}$ . Since  $R$  is a rectangle, the  $\eta_i$  are  $L^1$  geodesics. The length of  $\gamma_i$  is at least  $|r_i - r_{i+1}| = |\eta_i|$ , so that

$$|\partial A| = |\gamma_1| + |\gamma_2| + |\gamma_3| + |\gamma_4| \geq |\eta_1| + |\eta_2| + |\eta_3| + |\eta_4| = |\partial R|.$$

□

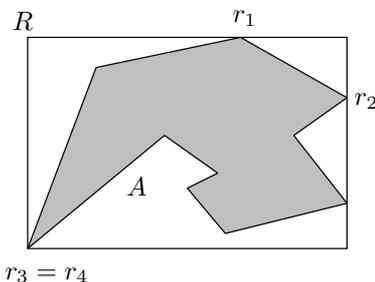


FIGURE 1. The  $L^1$  perimeter of  $A$  is at least that of  $R$

Let  $\ell$  and  $\alpha$  be such that  $\ell - 2\alpha$  and  $\ell + 2\alpha$  are the side lengths of  $R$ . In other words, if  $\ell_1$  and  $\ell_2$  are respectively the small and great side length of  $R$ , then  $\ell = (\ell_1 + \ell_2)/2$  and  $\alpha = (\ell_2 - \ell_1)/4$ .

**Lemma 2.** *We have*

$$(5) \quad |A| \leq \ell^2 \leq \frac{16 + \varepsilon}{16} |A|$$

and

$$(6) \quad \alpha^2 \leq \frac{\varepsilon |A|}{64}$$

*Proof.* From previous lemma we have  $|\partial A| \geq 4\ell$ , so that using (1) we get  $16\ell^2 \leq (16 + \varepsilon)|A|$ . Since  $A \subset R$  we have  $|A| \leq |R| = \ell^2 - 4\alpha^2$  and (5) follows.

Next we have

$$\begin{aligned} 16\ell^2 &\leq (16 + \varepsilon)(\ell^2 - 4\alpha^2) \\ 0 &\leq \varepsilon\ell^2 - 4(16 + \varepsilon)\alpha^2 \\ \alpha^2 &\leq \frac{\varepsilon\ell^2}{4(16 + \varepsilon)} \\ \alpha^2 &\leq \frac{\varepsilon}{64}|A| \end{aligned}$$

and we are done.  $\square$

Note that this lemma is sufficient to deduce the  $L^1$  isoperimetric inequality and its equality case: if  $\varepsilon = 0$ , then  $\alpha = 0$  and  $|A| = \ell^2$ .

**2.1. Proof of Theorem 1.** We have  $\min_S d_\infty(A, S) \geq \alpha$  (where  $S$  runs over all squares, see figure 2) and if there is equality, Lemma 2 is sufficient to conclude. We therefore assume  $\delta := \min_S d_\infty(A, S) > \alpha$ .

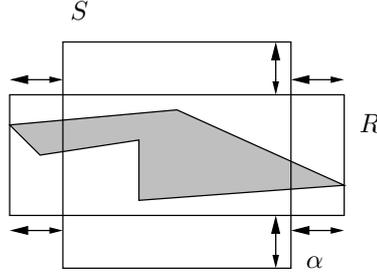


FIGURE 2. The closest square to  $R$ .

The following is the main step of the proof; it improves the inequality  $|A| \leq \ell^2 - 4\alpha^2$  obtained in Lemma 2 by a term that is function of  $\delta$ .

**Lemma 3.** *We have either*

$$|A| \leq \ell^2 - 4\alpha^2 - 8\delta(\delta - \alpha)$$

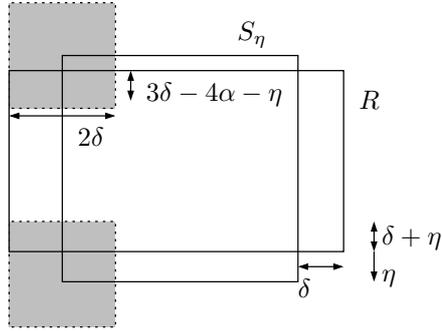
or

$$|A| \leq \ell^2 - 4\alpha^2 - 4\delta^2.$$

*Proof.* Choose the origin so that  $R$  has its bottom side at height 0. Let  $S_\eta$  be the square that is at distance  $\delta$  from each short side of  $R$  (in particular the centers of  $R$  and  $S_\eta$  have the same horizontal projection) and whose bottom side is at height  $\eta$ .

We consider negative values of  $\eta$ , precisely  $\eta \in [-\delta, 3\delta - 4\alpha]$ . In this way,  $R$  (and thus,  $A$ ) is contained in a  $L^\infty$  neighborhood of size  $\delta$  around  $S_\eta$ , thus there is some point  $p_\eta \in S_\eta$  that is at  $L^\infty$  distance at least  $\delta$  from  $A$ .

This excludes  $A$  from intersecting a square centered at  $p_\eta$ . Having in mind to bound  $A$  from above we can directly assume to be in the worst case that, for each  $\eta$ ,  $p_\eta$  is a corner of  $S_\eta$ , either always on its left side or always on its right side. Up


 FIGURE 3. The domain  $A$  avoids one of the grey squares.

to an horizontal reflection we can directly assume that  $p_\eta$  is either the upper or the lower left corner of  $S_\eta$ .

If  $p_\eta$  is the lower left corner of  $S_\eta$ , then there is a  $2\delta \times (\delta + \eta)$  sub-rectangle of  $R$  excluded. Otherwise, there is a  $2\delta \times (3\delta - 4\alpha - \eta)$  sub-rectangle of  $R$  excluded. In computing these values we have assumed that  $\eta < \delta$  and  $3\delta - 4\alpha - \eta < \delta$  respectively, otherwise there is simply an excluded square of area  $4\delta^2$ .

Let  $x$  be the supremum of the  $\eta$  such that  $p_\eta$  is a lower corner. There is an excluded sub-rectangle of area  $2\delta \times (\delta + x)$  and for all  $\eta > x$  the point  $p_\eta$  must be a higher corner, so that there is another excluded sub-rectangle of area  $2\delta \times (3\delta - 4\alpha - x)$ .

Summing up, either there are excluded sub-rectangles of total area at least  $2\delta \times 4(\delta - \alpha)$ , or there is an excluded sub-rectangle of area at least  $4\delta^2$ , and we get the desired bounds on  $|A|$ .  $\square$

We can now conclude the proof of Theorem 1. First, if  $|A| \leq (\ell^2 - 4\alpha^2) - 8\delta(\delta - \alpha)$  then we have

$$\begin{aligned} |\partial A|^2 &\leq (16 + \varepsilon)|A| \\ (4\ell)^2 &\leq (16 + \varepsilon)(\ell^2 - 4\alpha^2 - 8\delta(\delta - \alpha)) \\ 0 &\leq \varepsilon\ell^2 - 4(16 + \varepsilon)(\alpha^2 + 2\delta(\delta - \alpha)) \\ \alpha^2 + 2\delta(\delta - \alpha) &\leq \frac{\varepsilon\ell^2}{4(16 + \varepsilon)} \\ 2\delta^2 - 2\alpha\delta + \alpha^2 &\leq \frac{\varepsilon}{64}|A| \end{aligned}$$

the last inequality coming from (5).

Since the function  $x \mapsto 2\delta^2 - 2x\delta + x^2$  is minimal when  $x = \delta$ , we have  $2\delta^2 - 2\alpha\delta + \alpha^2 \geq 2\delta^2 - 2\delta^2 + \delta^2 = \delta^2$ , so that  $\delta^2 \leq \frac{\varepsilon}{64}|A|$ .

In the case when  $|A| \leq \ell^2 - 4\alpha^2 - 4\delta^2$ , we get:

$$\begin{aligned} |\partial A|^2 &\leq (16 + \varepsilon)(\ell^2 - 4\alpha^2 - 4\delta^2) \\ 16\ell^2 &\leq 16\ell^2 + \varepsilon\ell^2 - 4(16 + \varepsilon)(\alpha^2 + \delta^2) \\ \alpha^2 + \delta^2 &\leq \frac{\varepsilon\ell^2}{4(16 + \varepsilon)} \\ \delta^2 &\leq \frac{\varepsilon}{64}|A| \end{aligned}$$

**2.2. Proof of Proposition 1.** Let  $\mu = \max_S |S \cap A|/|A|$  where  $S$  runs over the squares having same area than  $A$ .

**Lemma 4.** *One of the following holds:*

$$\begin{aligned} \mu &\geq 2 - \frac{\ell^2 - 4\alpha^2}{|A|} \\ \mu &\geq 2 - \frac{\ell + 2\alpha}{\sqrt{|A|}}. \end{aligned}$$

*Proof.* Define  $S_0$  to be a square that shares a corner of  $R$  and intersects its interior, and that have the same area than  $A$  (see figure 4). The definition of  $\mu$  implies that  $|A \cap S_0| \leq \mu|A|$ .

If  $\sqrt{|A|} \leq \ell - 2\alpha$  we have:

$$\begin{aligned} |A| &\leq |A \cap S_0| + |R \setminus S_0| \\ &\leq \mu|A| + (\ell + 2\alpha)(\ell - 2\alpha - \sqrt{|A|}) + \sqrt{|A|}(\ell + 2\alpha - \sqrt{|A|}) \\ &\leq \mu|A| + \ell^2 - 4\alpha^2 - \sqrt{|A|}(\ell + 2\alpha) + \sqrt{|A|}(\ell + 2\alpha) - |A| \\ &\leq \ell^2 - 4\alpha^2 + (\mu - 1)|A| \end{aligned}$$

and then  $\mu|A| \geq 2|A| - (\ell^2 - 4\alpha^2)$ .

Otherwise, we get

$$\begin{aligned} |A| &\leq \mu|A| + (\ell - 2\alpha)(\ell + 2\alpha - \sqrt{|A|}) \\ &\leq \mu|A| + \sqrt{|A|}(\ell + 2\alpha - \sqrt{|A|}) \end{aligned}$$

thus  $\mu|A| \geq 2|A| - (\ell + 2\alpha)\sqrt{|A|}$ . □

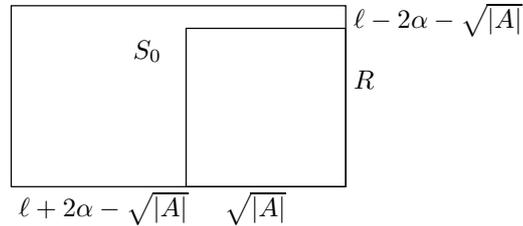


FIGURE 4. The domain  $A$  is included in  $R$  and cannot meet a too large proportion of  $S_0$

We now prove Proposition 1. If the first conclusion in Lemma 4 holds, using Lemma 2 it comes

$$\begin{aligned}\mu &\geq 2 - \frac{(\ell^2 - 4\alpha^2)(16 + \varepsilon)}{16\ell^2} \\ &\geq 2 - \frac{16 + \varepsilon}{16} = 1 - \frac{\varepsilon}{16}\end{aligned}$$

If the second conclusion holds, using Lemma 2 we get

$$\begin{aligned}\mu &\geq 2 - \frac{\ell}{\sqrt{|A|}} - \frac{2\alpha}{\sqrt{|A|}} \\ &\geq 2 - \sqrt{1 + \frac{\varepsilon}{16}} - \frac{\sqrt{\varepsilon}}{4} \\ &\geq 1 - \frac{\sqrt{\varepsilon}}{4} + O(\varepsilon)\end{aligned}$$

But for all sufficiently small  $\varepsilon$ , this second expression is smaller than  $1 - \varepsilon/16$ , and Proposition 1 is proved.

### 3. SHARPNESS

Two examples showing sharpness of Theorem 1 steam out from its proof.

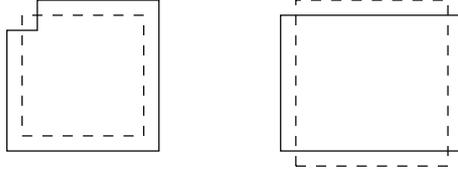


FIGURE 5. Two domain that are almost isoperimetric and as far as possible from squares: a square with a small corner deleted and a rectangle with sides of almost the same lengths.

The first one is the domain  $S'_\delta$  obtained from the unit square by deleting a  $2\delta \times 2\delta$  square at one corner ( $\delta < 1/2$ ). We have  $|S'_\delta| = 1 - 4\delta^2$  and  $|\partial S'_\delta| = 4$ , so that (1) holds with

$$\varepsilon = \frac{64\delta^2}{1 - 4\delta^2}$$

and  $\inf_S d_\infty(S, S'_\delta) = \delta$  so that equality holds in (2).

The second one is the rectangle  $R_\alpha$  whose side length are  $1 - 2\alpha$  and  $1 + 2\alpha$  (where  $\alpha < 1/2$ ). We have  $|R_\alpha| = 1 - 4\alpha^2$ ,  $|\partial R_\alpha| = 4$  and  $\inf_S d_\infty(S, R_\alpha) = \alpha$  so that (2) is an equality once again.

Let us show that  $S'_\delta$  and  $R_\alpha$  are the only possible (up to homothety and  $L^1$  isometry) exemple realizing equality in both (1) and (2) for the same  $\varepsilon$ . In the first case of Lemma 3, for  $2\delta^2 - 2\alpha\delta + \alpha^2 \leq \delta^2$  to be an equality it is necessary that  $\alpha = \delta$ , so that  $A$  must be equal to  $R$  (otherwise  $R$  would have smaller isoperimetric deficit and same distance to squares). In the second case of the lemma, one is lead to  $\alpha = 0$  in the last lines of the proof of Theorem 1, so that  $R$  is a square and according to the proof of Lemma 3,  $A$  is contained in a  $S'_\delta$  having the same isoperimetric deficit and the same minimal rectangle. They must therefore be equal.

At last,  $R_\alpha$  shows asymptotic sharpness of Proposition 1:

$$\sup_{|S|=|R_\alpha|} |S \cap R_\alpha| = (1 - 2\alpha)\sqrt{1 - 4\alpha^2} = 1 - 2\alpha + o(\alpha)$$

and

$$1 - \frac{1}{4}\sqrt{\varepsilon} = 1 - 2\alpha + o(\alpha)$$

when  $\varepsilon$  takes the extremal value  $64\alpha^2/(1 - 4\alpha^2)$ .

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#### REFERENCES

- [AFC09] A. Alvino, V. Ferone, and Nitsch C. A sharp isoperimetric inequality in the plane. preprint, 2009.
- [AT04] Luigi Ambrosio and Paolo Tilli. *Topics on analysis in metric spaces*, volume 25 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2004.
- [Ber05] Felix Bernstein. Über die isoperimetrische Eigenschaft des Kreises auf der Kugeloberfläche und in der Ebene. *Math. Ann.*, 60(1):117–136, 1905.
- [BM94] John E. Brothers and Frank Morgan. The isoperimetric theorem for general integrands. *Michigan Math. J.*, 41(3):419–431, 1994.
- [Bon29] T. Bonnesen. *Les problèmes des isopérimètres et des isépiphanes*. Gauthier-Villars, Paris, 1929.
- [FMP08] N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative isoperimetric inequality. *Ann. of Math. (2)*, 168(3):941–980, 2008.
- [FMP09] A. Figalli, F. Maggi, and A. Pratelli. A mass transportation approach to quantitative isoperimetric inequalities. preprint, 2009.
- [Hal92] R. R. Hall. A quantitative isoperimetric inequality in  $n$ -dimensional space. *J. Reine Angew. Math.*, 428:161–176, 1992.
- [Nit08] Carlo Nitsch. The quantitative isoperimetric inequality for planar convex domains. *Boll. Unione Mat. Ital. (9)*, 1(3):573–589, 2008.
- [Oss78] Robert Osserman. The isoperimetric inequality. *Bull. Amer. Math. Soc.*, 84(6):1182–1238, 1978.
- [PWZ93] Carla Peri, Jörg M. Wills, and Andreana Zucco. On Blaschke’s extension of Bonnesen’s inequality. *Geom. Dedicata*, 48(3):349–357, 1993.
- [Tay78] Jean E. Taylor. Crystalline variational problems. *Bull. Amer. Math. Soc.*, 84(4):568–588, 1978.