The linear request problem

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This article propose an alternative approach to a problem introduced by Galatolo and Pollicott, consisting in perturbing a dynamical system in order for its absolutely continuous invariant measure ("acim", assumed to exist) to change in a prescribed way. Instead of using transfer operators, we rely on the continuity equation to produce a perturbation of the system which is an infinitesimal conjugacy. This allows us to work in any dimension and dispense from any dynamical hypothesis. In particular, we don't need to assume the dynamical system is hyperbolic; but if it is, we obtain our acim-prescribed perturbations within this class.

1 Introduction

When one studies a deformation of a dynamical system with a particularly nice invariant measure (e.g. absolutely invariant measures, or more generally SRB or physical measures), a natural question is how this particular measure, if it continues to exist and being unique, is deformed in the process. The linear response theory seeks conditions under which a first-order perturbation of a dynamical system implies a first-order perturbation of that particular measure, depending linearly on the system perturbation; see for example [Rue09, BS12].

A recent article by Galatolo and Pollicott [GP16] starts studying the opposite direction: which perturbation of a dynamical system should be requested in order to achieve a target response for the particular measure? We shall call this the *linear request* problem, in antonymy with the classical linear response theory.

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Our goal here is to propose a simple point of view on these matters, which easily gives solutions to the linear request problem when the system acts on a compact manifold and admits a smooth enough invariant density (and which could probably be pushed to further generality).

Our starting point is to look at diffeomorphisms, which act on dynamical systems by conjugacy and on measures by push-forward. Let us give a first, easy example of this point of view.

Proposition 1.1. *Let M be a compact Riemannian manifold with volume form ω and $T : M \to M$ a smooth map having an absolutely continuous invariant probability with smooth positive density, $f\omega$, and assume $g\omega$ is any absolutely continuous probability with smooth positive density.*

Then there is a map $R : M \to M$ which is smoothly conjugated to T and which preserves $g\omega$.

Proof. A classical theorem of Moser [Mos65] ensures that there is a smooth diffeomorphism $\Phi : M \to M$ such that $\Phi_*(f\omega) = g\omega$. Then $R = \Phi \circ T \circ \Phi^{-1}$ preserves $g\omega$. □

In particular, as long as their exist *some* smooth measure with positive density on M which is invariant under a smooth hyperbolic map, then *every* smooth measure with positive density on M is also invariant under some smooth hyperbolic map. If say M is the n-torus, then every smooth measure with positive density is invariant under some hyperbolic map, and under some minimal smooth map, and under some k-periodic maps for all $k \in \mathbb{N}$, etc.

The above result does not *a priori* ensures that R is close to T whenever g is close to f; however it is possible to achieve this; more precisely, we will see that one can induce this idea on first-order perturbations, replacing diffeomorphisms with vector fields. The action on systems translates into a first-order perturbation of the system, and the action on measures translates into a first-order perturbation of the (density of the) measure. The main task to get a linear request formula is then to find a solution to a simple, well-understood elliptic PDE. Our result is the following.

Theorem 1.2. *Let $k \geq 0, \alpha \in (0,1)$ and $T : M \to M$ be a $C^{k+2,\alpha}$ map acting on a compact smooth Riemannian manifold, preserving a $C^{k+1,\alpha}$ volume form $f\omega$. Let $g : M \to \mathbb{R}$ be a $C^{k,\alpha}$ function such that $\int_M g\omega = 0$.*

If the dimension of M is $n \geq 2$, then there is an infinite-dimensional space of $C^{k+1,\alpha}$ vector fields w, each of which admits a family (T_t) of maps, all $C^{k+1,\alpha}$ conjugated to T, such that $T_0 = T$ and

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T_t(x) = w(x) \qquad \forall x \in M \tag{1}$$

and such that T_t preserves a $C^{k,\alpha}$ volume form $f_t\omega$ with

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f_t(x) = g(x) \qquad \forall x \in M \tag{2}$$

If $n = 1$, then the same result holds, but with a one-dimensional family of solutions.

Moreover in both cases there is a particular w that can be singled out by either an optimization property, or equivalently by a particular form requirement.

It could be expected that instead of pointwise derivative, one could differentiate T_t and f_t "in $C^{k,\alpha}$ norm", but this would at the very least complicate the proofs. Obtaining derivatives in C^k norm would not be as big a deal, and should be achievable with our proofs by tracking the $o(t)$ and their norms. However for the sake of clarity and simplicity, we stick to the weaker point-wise framework.

The fact that we formulate the regularity in the $C^{k,\alpha}$ spaces is partially motivated by the theory of elliptic PDE, where this kind of assumptions allows loosing as little regularity as possible, see Section 3.4 and references therein.

2 From measure perturbation to map perturbation through vector fields

We fix the manifold M, assumed to be compact in order to avoid any completeness issue, and endowed with a Riemannian metric (whose role will be mostly of normalization).

Let $C^{k,\alpha}(M)$ be the space of $C^{k,\alpha}$ functions $M \to \mathbb{R}$ (i.e. functions having all partial derivatives of order k well defined and α-Hölder continuous) and $\Gamma^{k,\alpha}(M)$ the space of $C^{k,\alpha}$ vector fields on M.

2.1 Action of vector fields on maps

Elements of $\Gamma^{k+1,\alpha}(M)$ can act on dynamical systems (i.e. maps $T : M \to M$) in two natural manners.

First, as a direct perturbation of T: given a vector field w, one can consider the family of maps defined as

$$x \mapsto T(x) + tw(x).$$

where we use the notation $x + u := \exp_x(u)$ whenever $x \in M$ and $u \in T_xM$ (see Section 3.1).

More generally, we say that a family $(T_t)_t$ with $T_0 = T$ has w as tangent vector field if for all $x \in M$ and small enough t:

$$T_t(x) = T(x) + tw(x) + o(t).$$

This is only a reformulation of (1), and despite the use of the exponential map, the metric does not in fact matter (different metrics will only yield different $o(t)$). Note that all our remainder terms $o(t)$ will be uniform in x.

Second, each $v \in \Gamma^{k+1,\alpha}(M)$ defines a flow $(\Phi_v^t)_{t\in\mathbb{R}}$, where Φ_v^t is a $C^{k+1,\alpha}$ diffeomorphism of M for all t and

$$\Phi_v^t(x) = x + tv(x) + o(t).$$

We can now conjugate T by Φ_v^t and let t go to zero: as soon as T and v are C^1, for all $x \in M$ and small t we get

$$\begin{aligned}
\Phi_v^t \circ T \circ \Phi_v^{-t}(x) &= \Phi_v^t \circ T\big(x - tv(x) + o(t)\big) \\
&= \Phi_v^t\Big(T(x) - tDT(x)\cdot v(x) + o(t)\Big) \\
&= T(x) - tDT(x)\cdot v(x) + \\
&\qquad tv\Big(T(x) - tDT(x)\cdot v(x) + o(t)\Big) + o(t) \\
&= T(x) - tDT(x)\cdot v(x) + tv(T(x)) + o(t)
\end{aligned}$$

and the family $(\Phi_v^t \circ T \circ \Phi_v^{-t})$ has $DT \cdot v + v \circ T$ as tangent vector field. If T is $C^{k+2,\alpha}$ and v is $C^{k+1,\alpha}$, this vector field lie in $\Gamma^{k+1,\alpha}(M)$, see Section 3.2.

Definition 2.1. Assuming T is at least $C^{k+2,\alpha}$ with $k \geq 0$, we define the *conjugacy operator*

$$\begin{aligned}
\mathcal{S}_T : \Gamma^{k+1,\alpha}(M) &\to \Gamma^{k+1,\alpha}(M) \\
v &\mapsto DT \cdot v + v \circ T
\end{aligned}$$

In other words, $\mathcal{S}_T(v)$ is the tangent vector field of the perturbation of T by conjugacy by the flow of v. The above operator would also make sense with $k+1=0$, but its interpretation would fall down since a $C^{0,\alpha}$ vector field may not define a flow.

2.2 Action of vector fields on measures

When a flow (Φ_v^t) acts on a measure $f\omega$, the *continuity equation* states that the density f_t of $(\Phi_v^t)_*(f\omega)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f_t + \nabla\cdot(fv) = 0$$

where $\nabla\cdot$ is the Riemannian divergence operator (see Lemma 3.2 below).

Remark 2.2. The continuity equation holds true in the distribution sense even at very low regularity: if μ is *any* probability measure on M, v is a vector field with $\|v\| \in L^2(\mu)$, (Φ^t) is a family of Borel measurable maps with $\Phi^t(x) = x + tv(x) + o(t)$, and $\mu_t := \Phi_*^t\mu$, then for all smooth test function $\varphi : M \to \mathbb{R}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int \varphi\,\mathrm{d}\mu_t - \int \nabla\varphi\cdot v\,\mathrm{d}\mu = 0$$

where $\cdot$ is the inner product induced on vectors by the Riemannian metric.

Definition 2.3. As soon as $f \in C^{k+1,\alpha}(M)$, we are thus lead to define the *continuity operator*

$$\begin{aligned}
\mathcal{M}_f : \Gamma^{k+1,\alpha}(M) &\to C^{k,\alpha}(M) \\
v &\mapsto -\nabla\cdot(fv).
\end{aligned}$$

In other words, $\mathcal{M}_f(v)$ is the derivative at $t=0$ of the density of $(\Phi_v^t)_*(f\omega)$.

2.3 Proof of the main Theorem

Now, we shall try to find inverse images under $\mathcal{M}_f$ to solve the linear request problem. Luckily enough, this is a well-known result as it involves a linear, uniformly elliptic differential operator.

Proposition 2.4. *Assume $n \geq 2$. Given $f \in C^{k+1,\alpha}(M)$ positive, for all $g \in C^{k,\alpha}(M)$ such that $\int g\,\omega = 0$ there exist an infinite-dimensional affine space of solutions $v \in \Gamma^{k+1,\alpha}(M)$ to the equation*

$$\mathcal{M}_f(v) = g. \tag{3}$$

Among them, precisely one minimizes $\int \|v\|^2 f\omega$. This minimizing solution is also the only solution which is the gradient of a function.

Proof. First, let us prove that there is a solution to (3) which is a gradient: this amounts to find a function u such that

$$\nabla \cdot (f \nabla u) = -g$$

which is a linear, uniformly elliptic equation with $C^{k,\alpha}$ coefficients, well-known to have a solution $u \in C^{k+2,\alpha}$, see Section 3.4. The vector field $v_0 = \nabla u$ is thus a gradient solution of (3).

To find other solutions, one only has to find divergence-free vector fields $\bar{v}$: then $v = v_0 + \frac{1}{f}\bar{v}$ is solution of $\mathcal{M}_f(v) = g$. It is well-known (see Section 3.5) that there is an infinite-dimensional space of such vector fields as soon as $n \geq 2$.

Conversely, any solution has the form $v_0 + \frac{1}{f}\bar{v}$ for some divergence-free vector field $\bar{v}$, and we have

$$\begin{aligned}\int \left\|v_0 + \frac{1}{f}\bar{v}\right\|^2 f\omega &= \int \|v_0\|^2 f\omega + 2\int (\nabla u)\cdot \bar{v}\,\omega + \int \left\|\frac{1}{f}\bar{v}\right\|^2 f\omega \\ &= \int \|v_0\|^2 f\omega - 2\int u(\nabla \cdot \bar{v})\,\omega + \int \left\|\frac{1}{f}\bar{v}\right\|^2 f\omega \\ &= \int \|v_0\|^2 f\omega + \int \left\|\frac{1}{f}\bar{v}\right\|^2 f\omega\end{aligned}$$

so that v_0 is indeed uniquely minimizing among solutions. □

In dimension 1, the problem can easily be solved but we have less room to maneuver.

Proposition 2.5. *Assume $M = \mathbb{R}/\tau\mathbb{Z}$, the Riemannian circle of length τ. Given $f \in C^{k+1,\alpha}(M)$ positive, for all $g \in C^{k,\alpha}(M)$ such that $\int g\,\omega = 0$ there exist a one dimensional affine space of solutions $v \in \Gamma^{k+1,\alpha}(M)$ to the equation*

$$\mathcal{M}_f(v) = g. \tag{4}$$

Among them, precisely one minimizes $\int \|v\|^2 f\omega$. This minimizing solution is also the only solution which is the gradient of a function.

Proof. Equation (4) can be rewritten as $-(fv)' = g$. Since g has vanishing average, its τ-periodic lift to $\mathbb{R}$ has a τ-periodic antiderivative G, and (4) is easily solved by setting $v = (c - G)/f$ where c is any constant. There is precisely one value of c which gives such a v vanishing average, i.e. makes it a gradient. The minimization property follows as before. $\square$

Proof of Theorem 1.2. Let $T : M \to M$ be a $C^{k+2,\alpha}$ map (with $k \geq 0$, $\alpha \in (0,1)$) assumed to have an absolutely continuous invariant probability measure $f\omega$ where $f \in C^{k+1,\alpha}(M)$ is positive and let $g \in C^{k,\alpha}(M)$.

From Proposition 2.4 (or Proposition 2.5 if $n = 1$) we know that there is an infinite-dimensional (or 1-dimensional) affine space V of vector fields $v \in \Gamma^{k+1,\alpha}(M)$ such that $\mathcal{M}_f(v) = -g$.

Section 2.1 shows that, given any $v \in V$, the vector field $w = \mathcal{S}_T(v) \in \Gamma^{k+1,\alpha}(M)$ is tangent to the perturbation $T_t = \Phi_v^t \circ T \circ \Phi_v^{-t}$ of T.

But by Section 2.1, T_t preserves the absolutely continuous measure

$$\Phi_{v*}^t f\omega = \Big(f + t\mathcal{M}_f(v) + o(t)\Big)\omega = \Big(f + tg + o(t)\Big)\omega.$$

A particular solution can be singled out: take $w_0 = \mathcal{S}_T(v_0)$ where v_0 is the unique gradient solution of $\mathcal{M}_f(v_0) = -g$. $\square$

Remark 2.6. Since the maps T_t are $C^{k+1,\alpha}$ conjugated to T, the perturbation we obtain share many properties with T. It has the same topological entropy, is hyperbolic if T was hyperbolic, is minimal if T was minimal, etc.

Remark 2.7. One could hope to use the same strategy in the other direction to obtain linear response formulas. However, contrary to $\mathcal{M}_f$ the operator $\mathcal{S}_T$ is far from being onto, so this approach seems hopeless. This relates to the fact that, even if expanding maps T are structurally stable, i.e. are conjugated to any of their small perturbations, the conjugacy is only topological: the spectrum of the derivative at a fixed point is indeed a differentiable conjugacy invariant and arbitrarily close maps can have different spectra. Now, if one pushes forward an absolutely continuous measure by a mere homeomorphism, the resulting measure may not be absolutely continuous. So, even if one can realize a first-order perturbation of T by a topological conjugacy, the absolutely continuous invariant measure of the perturbed map may exist without being equal to the corresponding pushed forward measure.

2.4 A model case

Let us now spell out what happens in the model case when $M = \mathbb{R}/\mathbb{Z}$ and $T(x) = 2x \mod 1$; the acim of T is then the Lebesgue measure, which coincides with the Riemannian volume ω, and we thus have $f \equiv 1$.

For all $v \in \Gamma^{k+1,\alpha}(\mathbb{R}/\mathbb{Z})$ we have

$$\mathcal{S}_T(v) : x \mapsto 2v(x) + v(2x)$$
$$\mathcal{M}_f(v) : x \mapsto -v'(x)$$

Let $g \in C^{k,\alpha}(M)$ be a function with vanishing average, which will be identified with its 1-periodic lift to $\mathbb{R}$. The gradient solution to $\mathcal{M}_f(v) = g$ is given by

$$v_0(x) = -\int_0^x g(t)\,\mathrm{d}t + \int_0^1 \int_0^y g(t)\,\mathrm{d}t\,\mathrm{d}y$$

(the only primitive of g which has vanishing average).

To compare with [GP16], let us consider the case $g(x) = \sin(2\pi x)$. Then

$$v_0(x) = \frac{1}{2\pi}\cos(2\pi x)$$

and the corresponding perturbation of T is

$$w_0 = \mathcal{S}_T(v_0) : x \mapsto \frac{1}{\pi}\cos(2\pi x) + \frac{1}{2\pi}\cos(4\pi x).$$

Meanwhile, the $L^2(f\omega)$-norm minimizing perturbation found in [GP16] is $w_1(x) = \frac{1}{2\pi}\cos(4\pi x)$; this begs for the question: is the difference in the minimized energy, or in the restriction to the image of $\mathcal{S}_T$?

This is easily answered, as any solution to $\mathcal{M}_f(v) = -g$ has the form $v = v_0 + c$ where c is a constant, and then $\mathcal{S}_T(v_0 + c) = w_0 + 3c$ has $L^2(f\omega)$ norm equal to the norm of w_0 plus $3|c|$ by Parseval's identity. Hence, not only does v_0 minimize the $L^2(f\omega)$ norm among solutions of $\mathcal{M}_f(v) = g$, but w_0 also minimizes the $L^2(f\omega)$ norm among "infinitesimal conjugacies", i.e. vector fields solving the request problem and lying in the image of $\mathcal{S}_T$.

Note that w_1 does writes as $DT \cdot v_1 + v_1 \circ T$ for some unique continuous vector field v_1, which is α-Hölder for all $\alpha < 1$ but not C^1; this vector field can be computed explicitly as

$$v_1(x) = \sum_{k\geq 0} \frac{(-1)^k}{2^{k+1}} \sin(2\pi \cdot 2^k \cdot x);$$

more generally we observe:

Proposition 2.8. *For all $w \in \Gamma^{0,\alpha}(\mathbb{R}/\mathbb{Z})$, there is exactly one $v \in \Gamma^{0,\alpha}(\mathbb{R}/\mathbb{Z})$ such that $2v(x) + v(2x) = w(x)$ for all x.*

Proof. A solution to $2v + v(2\cdot) = w$ is a fixed point of the map

$$\begin{aligned} F \qquad \Gamma^{0,\alpha}(\mathbb{R}/\mathbb{Z}) &\to \Gamma^{0,\alpha}(\mathbb{R}/\mathbb{Z}) \\ v &\mapsto \frac{1}{2}\big(w - v(2\cdot)\big) \end{aligned}$$

which is a $2^{\alpha-1}$-contraction in the norm $\|\cdot\|_\alpha$, and thus has a unique fixed point. □

3 Toolbox

In this last section we collect all classical statements we needed above, and provide proofs or references for them.

3.1 Additive notation for the exponential map

We used for convenience the additive notation $x+u := \exp_x(u)$, which needs an infinitesimal associativity to reach its potential: the *a priori* ambiguous notation $x+u_1+u_2$, which we define as $x+(u_1+u_2)$, can be identified with $(x+u_1)+u_2$ up to a small error (note that these expressions do not both make sense unless u_1 and u_2 are vector *fields*).

Lemma 3.1. *Let u_1, u_2 be C^1 vector fields on M. Then we have*

$$(x+tu_1(x)) + tu_2\big(x+tu_1(x)\big) = x + t(u_1(x)+u_2(x)) + o(t).$$

Proof. Fix $x \in M$; for all $v \in T_xM$, let $\bar{u}_2(x+v) \in T_{x+v}M$ be the parallel translate of $u_2(x)$ along the geodesic defined by $\gamma_t = \exp_x(tv)$. This a priori depends on v, not only on $x+v$, but restricting to small enough v it defines a vector fields $\bar{u}_2$ on a neighborhood of x, which coincides with u_2 at x and such that $\bar{u}_2(x+v) = u_2(x+v) + O(v)$.

We then have

$$\begin{aligned}(x+tu_1(x)) + tu_2(x+tu_1(x)) &= \exp_{x+tu_1(x)}(tu_2(x+tu_1(x))) \\ &= \exp_{x+tu_1(x)}(t\bar{u}_2(x+tu_1(x))) + O(t^2) \\ &= \exp_x(tu_1(x)+tu_2(x)) + O(t^2)\end{aligned}$$

where the last line follows from $D(\exp_x)_x = \mathrm{Id}_{T_xM}$, and the last remainder term involves the curvature of M which can be uniformly bounded. □

3.2 Hölder regularity

Let us start by gathering a few useful observations about the Hölder regularity. Since a change of metric shall only change the involved norms in a multiplicatively controlled way, and M is assumed to be compact, we can restrict to objects defined on a chart and use any coordinate system there. We shall thus consider functions defined on an open neighborhood U of a compact domain K of $\mathbb{R}^n$ and denote directional derivatives by ∂_i (where $i \in \{1,\dots,n\}$), and higher order derivatives by $\partial^\ell_{\underline{i}}$ (where $\underline{i} = (i_1,\dots,i_\ell)$ is an unordered ℓ-tuple, and $\ell =: |\underline{i}|$). The case of any more complicated object (vector field or map) can be recovered from its coordinate functions.

A function $f : K \to \mathbb{R}$ is said to be $C^{k,\alpha}$ if it extends C^k to U, and all its kth order derivatives are α-Hölder on K. Equivalently, f is $C^{k,\alpha}$ if its $C^{k,\alpha}$ norm

$$\|f\|_{C^{k,\alpha}} := \|f\|_\infty + \sum_{\substack{1\le \ell\le k\\ |\underline{i}|=\ell}} \|\partial^\ell_{\underline{i}} f\|_\infty + \sum_{|\underline{i}|=k} \sup_{x\neq y\in K} \frac{|\partial^k_{\underline{i}} f(x) - \partial^k_{\underline{i}} f(y)|}{|x-y|^\alpha}$$

is finite, where $|\cdot|$ denotes absolute value or Euclidean norm in $\mathbb{R}^n$ and $\|\cdot\|_\infty$ is the uniform norm of K.

This norm makes the space of $C^{k,\alpha}$ functions a Banach space. We also have a good behavior with respect to products: for all $f, g : K \to \mathbb{R}$, we have

$$\|fg\|_{C^{k,\alpha}} \lesssim \|f\|_{C^{k,\alpha}} \|g\|_{C^{k,\alpha}}$$

where $A \lesssim B$ means that there is a constant c depending only on K (or M), k and α such that $A \leq cB$.

Observe now that as soon as $k \geq 0$,

$$\|f\|_{C^{k+1,\alpha}} = \|f\|_\infty + \sum_{i=1}^{n} \|\partial_i f\|_{C^{k,\alpha}}$$

which makes it possible to prove some inequalities by induction instead of managing higher-order derivatives.

Notably, this shows that as soon as $k \geq 1$ the $C^{k,\alpha}$ class is stable by composition: if F, G are $C^{k,\alpha}$ maps defined from compact domains of Euclidean spaces such that $F \circ G$ makes sense, then it is $C^{k,\alpha}$. This is a direct consequence of the 1-dimensional case, which can be proven by induction on k since when $f, g : \mathbb{R} \to \mathbb{R}$ we have

$$\begin{aligned}
\|f \circ g\|_{C^{k+1,\alpha}} &\lesssim \|f \circ g\|_\infty + \|(f \circ g)'\|_{C^{k,\alpha}} \\
&\lesssim \|f\|_\infty + \|(f' \circ g)g\|_{C^{k,\alpha}} \\
&\lesssim \|f\|_\infty + \|(f' \circ g)\|_{C^{k,\alpha}} \|g\|_{C^{k,\alpha}}
\end{aligned}$$

and

$$\begin{aligned}
\|f \circ g\|_{C^{1,\alpha}} &\lesssim \|f\|_\infty + \|f' \circ g\|_{C^{0,\alpha}} \|g\|_{C^{0,\alpha}} \\
&\lesssim \|f\|_\infty + \|f'\|_{C^{0,\alpha}} \|g\|_{C^1}^\alpha \|g\|_{C^{0,\alpha}}
\end{aligned}$$

Note that we do need $k \geq 1$, as the composition of two α-Hölder maps is only α^2-Hölder in general.

3.3 The continuity equation

Let v be a $C^{k+1,\alpha}$ vector field and $f\omega$ be a measure with positive, $C^{k,\alpha}$ density. The following pointwise version of the continuity equation can obviously not claim any originality, but it seems simpler to provide a proof rather than a reference.

Lemma 3.2 (continuity equation)**.** *The n-forms $(\Phi_v^t)_*(f\omega) =: f_t\omega$ satisfy*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f_t + \nabla \cdot (fv) = 0$$

where $\nabla \cdot$ is the Riemannian divergence operator.

Proof. Let $\varphi \in C^\infty(M)$ be any test function. Then we have for small t:

$$\begin{aligned}
\int \varphi\, f_t\omega &= \int \varphi\, \big((\Phi_v^t)_*(f\omega)\big) \\
&= \int \varphi \circ \Phi_v^t\, f\omega \\
&= \int \varphi \circ (\operatorname{Id} + tv + o(t))\, f\omega \\
&= \int \big(\varphi + t\nabla\varphi \cdot v + o(t)\big) f\omega \\
&= \int \varphi\, f\omega + t \int \nabla\varphi \cdot (fv)\,\omega + o(t) \\
&= \int \varphi\, f\omega - t \int \varphi\, \nabla \cdot (fv)\,\omega + o(t).
\end{aligned}$$

Note that above, the $o(t)$ can be taken uniform in x, hence its integral is indeed a $o(t)$. From this computation follows $f_t = f - t\nabla \cdot (fv) + o(t)$. □

3.4 The modified Poisson equation

Let us consider the equation

$$\nabla \cdot (f\nabla u) = -g \tag{5}$$

where $f \in C^{k+1,\alpha}$, $g \in C^{k,\alpha}$ and $u : M \to \mathbb{R}$ is the unknown. When $f \equiv 1$, this is called the *Poisson equation*: $\Delta u = g$. Many books treat in some detail the resolution of the Poisson equation on a compact manifold, see e.g. [Aub98], but we did not find spelled out explicitly a resolution of (5). However it can be handled in the very same way as the Poisson equation, and for the convenience of the non-PDEist reader we provide a combination of proofs and references.

One could try to perform a change of conformal factor to reduce to the Poisson equation; unfortunately, this only works in dimension different than 2. The approach we take is through the Lax-Milgram theorem: first one observes that by integration by parts, (5) is equivalent to

$$\int \varphi\nabla \cdot (f\nabla u)\,\omega = -\int \varphi g\,\omega \qquad \forall \varphi \in C^\infty(M)$$

$$\int \nabla\varphi \cdot \nabla u\, f\omega = \int \varphi g\,\omega \qquad \forall \varphi \in C^\infty(M) \tag{6}$$

and we are thus asking for u solving

$$Q(u, \varphi) = L(\varphi) \qquad \forall \varphi \in C^\infty(M)$$

where Q and L are the bilinear form, respectively the linear form, defined by each side of (6), on the domain $H_1(M)$ of Sobolev functions. Recall that $H_1(M)$ can be defined as the set of $L^2(\omega)$ functions whose gradient in the distribution sense is $L^2(\omega)$, or as the completion of $C^\infty(M)$ with respect to the norm

$$\|\varphi\|_{H_1} = \int \varphi^2 \, \omega + \int \|\nabla\varphi\|^2 \, \omega$$

where $\|\cdot\|$ is the norm in each tangent space induced by the Riemannian metric.

The Lax-Milgram theorem would precisely give us a weak solution $u \in H_1$ if we proved that Q and L are continuous in the H_1 norm (which follows from the Cauchy-Schwarz inequality), and that Q is coercive (i.e. the seminorm it induces on $H_1(M)$ is equivalent to the H_1 norm). That last statement does not hold as constants are in the kernel of Q; we thus decompose

$$H_1(M) = H_1^\perp \overset{\perp}{\oplus} \{\text{constants}\}$$

where $H_1^\perp$ is the subspace of functions of vanishing ω-average. Now, on a compact manifold one has :

Proposition 3.3 (Poincaré inequality)**.** *There exist a constant C depending only on M and its metric such that for all $\varphi \in H_1(M)$, denoting by $\bar\varphi = \frac{1}{\mathrm{vol}(M)} \int \varphi \, \omega$ the average of φ, it holds*

$$\int |\varphi - \bar\varphi|^2 \, \omega \leq C \int \|\nabla\varphi\|^2 \, \omega$$

This inequality is very classical, but hard to give a precise reference to. Looking at the Rayleigh quotient, one sees it is equivalent to $\lambda_1(M) > 0$ where λ_1 is the first eigenvalue of $-\Delta$ (in $H_1^\perp$). This follows for example from Cheeger's bound $\lambda_1 \geq h_C^2/4$ where Cheeger's constant $h_C(M)$ can be bounded below in terms of the diameter of M and a (possibly negative) lower bound on its Ricci curvature, see for example Chapter IV of [Bér86].

On $H_1^\perp$, the Poincaré inequality yields

$$\begin{aligned} Q(\varphi, \varphi) &\geq \min(f) \int \|\nabla\varphi\|^2 \, \omega \\ &\geq \min(f) \Big(\frac{1}{2} \int \|\nabla\varphi\|^2 \, \omega + \frac{1}{2C} \int \varphi^2 \, \omega \Big) \\ &\geq C' \|\varphi\|_{H_1} \end{aligned}$$

which is precisely the coercivity of Q restricted to $H_1^\perp$. It follows from the Lax-Milgram theorem that there is a $u \in H_1^\perp$, such that:

$$Q(u, \varphi) = L(\varphi) \qquad \forall \varphi \in H_1^\perp$$

To get rid of the restriction that φ must have vanishing average, observe that given $\varphi \in H_1(M)$, its centered version $\varphi - \bar\varphi$ is in $H_1^\perp$; on the one hand, $\nabla\varphi = \nabla(\varphi - \bar\varphi)$ so that $Q(u, \varphi) = Q(u, \varphi - \bar\varphi)$ and on the other hand,

$$L(\varphi - \bar\varphi) = \int \varphi g \omega - \bar\varphi \int g \omega = L(\varphi)$$

since g has vanishing average. It follows that $Q(u,\varphi) = L(\varphi)$ for all $\varphi \in H_1(M)$, in particular for all smooth φ: $u \in H_1(M)$ is a weak solution of (5).

The last step we need is to improve this into a strong solution. This is well-known but subtle, and is purely local: the manifold case is handled just as in the $\mathbb{R}^n$ case, using charts. We refer for example to [Aub98] (Theorem 3.55 page 85) for a suitable statement, which itself refers to [LU68], where it is proved that $u \in C^{k+2,\alpha}(M)$ whenever the coefficients of (5) are $C^{k,\alpha}$. This happens as soon as $f \in C^{k+1,\alpha}(M)$ and $g \in C^{k,\alpha}(M)$.

3.5 Divergence-free vector fields

Let us prove that, as soon as $n \geq 2$, M carries an infinite-dimensional space of divergence-free vector fields. Recall that one can define the divergence of a vector field v by

$$d(i_v\omega) =: (\nabla \cdot v)\omega.$$

Assume $n \geq 2$ and let β be any smooth $(n-2)$ differential form on M. Then $d\beta$ is a closed $(n-1)$-form, and since ω is non-singular there exist a unique vector field v_β such that $i_{v_\beta}\omega = d\beta$. Then $d(i_{v_\beta}\omega) = d^2\beta = 0$ so that v_β is divergence free.

If γ is another $(n-2)$-form and $v_\gamma = v_\beta$, then $d\beta = d\gamma$ so that $\beta - \gamma$ is closed. The space of closed $(n-2)$ forms is infinite codimensional, so that we obtain an infinite-dimensional space of divergence-free vector fields.

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