Dirichlet uniformly well-approximated numbers

Lingmin LIAO
(joint work with Dong Han Kim)

Université Paris-Est Créteil (University Paris 12)

Hyperbolicity and Dimension
CIRM, Luminy
December 3rd 2013
Outline

1. The question
2. Metric Diophantine approximation
3. Dirichlet uniform approximation
4. Ideas and proofs
I. Dirichlet

Denote by $\| \cdot \|$ the distance to the nearest integer.

**Uniform Dirichlet Theorem (1842):**
Let $\theta$, $Q$ be real numbers with $Q \geq 1$. There exists an integer $n$ with $1 \leq n \leq Q$, such that

$$\|n\theta\| < Q^{-1}.$$ 

In other words,

$$\{ \theta : \forall Q > 1, \|n\theta\| < Q^{-1} \text{ has a solution } 1 \leq n \leq Q \} = \mathbb{R}.$$

**Asymptotic Dirichlet Theorem:**
For any real $\theta$, there exist infinitely many integers $n$ such that

$$\|n\theta\| < n^{-1}.$$ 

In other words,

$$\{ \theta : \|n\theta\| < n^{-1} \text{ for infinitely many } n \} = \mathbb{R}.$$
II. Our question

Corresponding to Uniform Dirichlet Theorem, we study a question of Bugeaud and Laurent 2005: for fixed $\theta$, what is the size (Hausdorff dimension) of the following set

$$U_\beta[\theta] := \left\{ y : \forall Q \gg 1, \|n\theta - y\| < Q^{-1/\beta} \text{ has a solution } 1 \leq n \leq Q \right\}$$

$$= \lim \inf \bigcup_{n=1}^{Q} B\left(n\theta, Q^{-1/\beta}\right).$$

Corresponding to Asymptotic Dirichlet Theorem, there is a result of Bugeaud 2003, Troubetzkoy-Schmeling 2003: For all $\theta \in \mathbb{R} \setminus \mathbb{Q}$, $\beta \leq 1$, define

$$L_\beta[\theta] := \{ y : \|n\theta - y\| < n^{-1/\beta} \text{ for infinitely many } n \}$$

$$= \lim \sup B\left(n\theta, n^{-1/\beta}\right).$$

Then

$$\dim_H(L_\beta[\theta]) = \beta.$$
Metric Diophantine approximation
I. General context

$(X, d)$ a complete metric space.

- $\{x_n\}_{n \geq 1}$ a sequence of points in $X$.
- $\{r_n\}$ a sequence of real numbers.

Define (i.o. = infinitely often)

$$\mathcal{L}(\{x_n\}, \{r_n\}) := \limsup B(x_n, r_n) = \{y : d(y, x_n) < r_n \text{ i.o.}\}$$

**Question**: What is the size of $\mathcal{L}(\{x_n\}, \{r_n\})$?
- Its measure (if we have a measure $\mu$ on $X$)?
- Its Hausdorff dimension?
II. Three examples

- $X = \mathbb{R}$, $\{x_n\} = \mathbb{Q}$:
  
  \[
  \left\{ y : |y - \frac{p}{q}| < \frac{1}{q^\beta} \text{ i.o.} \right\}.
  \]

- $X$ is a probability space and $\{x_n\}$ is a random sequence.
  
  Example: $x_n$ are i.i.d.
  
  \[
  \left\{ y : |y - x_n| < \frac{1}{n^\beta} \text{ i.o.} \right\}.
  \]

- $(X, T)$ is a dynamical system and $x_n = T^n x$.
  
  Example: $X = \mathbb{R}/ \mathbb{Z}$, $Tx = 2x \mod 1$.
  
  \[
  \left\{ y : |y - T^n x| < \frac{1}{n^\beta} \text{ i.o.} \right\}.
  \]
### III. I.i.d. sequences

- $X = \mathbb{T} = \mathbb{R}/\mathbb{Z}$.
- $\omega = \{\omega_n\}_{n \geq 1}$ is an i.i.d. sequence uniformly distributed on $\mathbb{T}$.
- $r_n$ a sequence of positive numbers decreasing to 0.

We are interested in the following set:

$$L[\omega] := \left\{ y \in \mathbb{T} : |\omega_n - y| < \frac{r_n}{2} \text{ i.o.} \right\} = \limsup_{n \to \infty} (\omega_n - \frac{r_n}{2}, \omega_n + \frac{r_n}{2})$$

→ points covered by infinitely many random intervals $\omega_n + (-\frac{r_n}{2}, \frac{r_n}{2})$.

**Question (Dvoretzky 1956):** When almost surely $L[\omega] = \mathbb{T}$?

**Case $r_n = c/n$**:
- **Kahane 1959**: $L[\omega] = \mathbb{T}$ a.s. if $c > 1$.
- **Erdős conjecture** : $L[\omega] = \mathbb{T}$ a.s. if and only if $c \geq 1$.
- **Billard 1965**: For $c < 1$ : No.
- **Mandelbrot 1972, Orey (independently)** : $L[\omega] = \mathbb{T}$ a.s. if $c = 1$. 
IV. I.i.d. sequences (Continued)

General case:

**Shepp 1972**: \( \mathcal{L}[\omega] = \mathbb{T} \) a.s. if and only if

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} e^{r_1 + \cdots + r_n} = \infty.
\]

A dimension result:

**Fan-Wu 2004**: If \( r_n = c/n^\kappa, c > 0 \) and \( \kappa > 1 \), then almost surely

\[
\dim_H \mathcal{L}[\omega] = 1/\kappa.
\]

Other related works: **Barral, Fan, Kahane, Jaffard, Jonasson-Steif ...**
V. Dynamical sequences

- $(X, T)$ a dynamical system, $x_n = T^n x$.

We are interested in the sets:

\[ \mathcal{L}[x] := \{ y \in X : d(T^n x, y) < r_n \text{ i.o.} \}. \]

**Known results**: Fan-Schmeling-Troubetzkoy 2013 for the doubling map and L-Seuret 2012 for Markov piecewise interval maps.

**Remarque**: Different to the Shrinking targets problems.

\[ S(y) := \{ x \in X : d(T^n x, y) < r_n \text{ i.o.} \}. \]

For the measure of $S(y)$, there are “Dynamical Borel-Cantelli Lemma” studied by: Boshernitzan, Chernov-Kleinbock, Chazottes, Fayad, Galatalo-Rousseau-Saussol, Haydn-Nicol-Persson-Vaienti, Kim....

VI. Asymptotic Approximation-Lebsgue measure

**Khintchine**: Let $\Psi : \mathbb{N} \to \mathbb{R}^+$ be a decreasing function. Consider:

$$L_\Psi := \{ \theta : \|n\theta\| < \Psi(n) \text{ i.o.} \}$$

- $L_\Psi$ is of Lebesgue measure zero if $\sum \Psi(n) < \infty$;
- $L_\Psi$ is of Lebesgue measure full if $\sum \Psi(n) = \infty$.

**Duffin-Schaefer 1941** Conjecture: If $\Psi$ is not decreasing, then $L_\Psi$ is of Lebesgue measure full if $\sum \phi(n)\Psi(n)/n = \infty$,

where $\phi$ is the Euler function.

**Haynes-Pollington-Velani 2012**:

Yes, if $\sum \phi(n)(\Psi(n)/n)^{1+\epsilon} = \infty$, with $\epsilon > 0$ small.

**Beresnevich-Harman-Haynes-Velani 2013**:

Yes, if $\sum \phi(n)\frac{\Psi(n)}{n \exp(c \log \log n)(\log \log \log n)} = \infty$, with $c > 0$. 
VII. Asymptotic Approximation-Hausdorff Dimension

Define

\[ \mathcal{L}_\beta(y) := \{ \theta : ||n\theta - y|| < n^{-1/\beta} \text{ i.o.} \} \]

**Jarník 1929, Besicovith 1934** : For \( \beta \leq 1 \),

\[ \dim_H(\mathcal{L}_\beta(0)) = \frac{2\beta}{\beta + 1}. \]

**Levesley 1998** : For any \( y \in \mathbb{R} \), and \( \beta \leq 1 \),

\[ \dim_H(\mathcal{L}_\beta(y)) = \frac{2\beta}{\beta + 1}. \]

Define

\[ \mathcal{L}_\beta[\theta] := \{ y : ||n\theta - y|| < n^{-1/\beta} \text{ i.o.} \} \]

**Bernik-Dodson 1999** : For all \( \theta \in \mathbb{R} \setminus \mathbb{Q} \), and \( \beta \leq 1 \),

\[ \omega \cdot \beta \leq \dim_H(\mathcal{L}_\beta[\theta]) \leq \beta \]

where \( \omega \leq 1 \) is a real number such that \( ||n\theta|| \geq n^{-1/\omega} \text{ ev.} \).

**Bugeaud 2003, Troubetzkoy-Schmeling 2003** : For all \( \theta \in \mathbb{R} \setminus \mathbb{Q} \), \( \beta \leq 1 \):

\[ \dim_H(\mathcal{L}_\beta[\theta]) = \beta. \]
Dirichlet uniform approximation
I. Uniformly approximated points

In general, we consider the sizes of the following sets:

\[ U_{\beta}(0) := \{ \theta : \forall Q \gg 1, \exists 1 \leq n \leq Q, \|n\theta\| < Q^{-1/\beta} \}, \]

\[ U_{\beta}(y) := \{ \theta : \forall Q \gg 1, \exists 1 \leq n \leq Q, \|n\theta - y\| < Q^{-1/\beta} \}. \]

and for fixed \( \theta \),

\[ U_{\beta}[\theta] := \{ y : \forall Q \gg 1, \exists 1 \leq n \leq Q, \|n\theta - y\| < Q^{-1/\beta} \}. \]

Remark:

\[ U_{\beta}(0) = \begin{cases} \mathbb{Q} & \text{if } \beta < 1, \\ \mathbb{R} & \text{if } \beta \geq 1. \end{cases} \]
II. A known result in higher dimensional cases

**Theorem (Y. Cheung (Ann. Math 2011))**

The set

\[
\{(\theta_1, \theta_2) : \forall \delta > 0, \forall Q \gg 1, \ 1 \leq \exists n \leq Q, \ \max \{\|n\theta_1\|, \|n\theta_2\|\} < \frac{\delta}{Q^{1/2}}\}.
\]

has Hausdorff dimension 4/3.

**Remark**: The set is called **singular points set**. It has some important geometric meaning. Let \( G = \text{SL}_3 \mathbb{R}, \Gamma = \text{SL}_3 \mathbb{Z} \). For \( t > 0, x = (x_1, x_2) \), let

\[
g_t = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}, \quad h_x = \begin{pmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -x_2 \\ 0 & 0 & 1 \end{pmatrix}
\]

The vector \( x \) is singular iff \( (g_t h_x \Gamma)_{t \geq 0} \) is a divergent trajectory of homogeneous flow on \( G/\Gamma \) induced by \( g_t \).
III. Our result

For an irrational $\theta$, define $w(\theta) := \sup\{\beta > 0 : \liminf_{j \to \infty} j^\beta \|j\theta\| = 0\}$.

**Theorem (L, D.H. Kim)**

Let $\theta$ be an irrational with $w(\theta) \geq 1$. Then $\dim_H (U_\beta[\theta])$ equals to

$$\begin{cases} 
1, & \beta > w(\theta), \\
\lim_{k \to \infty} \frac{\log(n_1^n \|n_1\theta\|n_2^n \|n_2\theta\| \cdots n_{k-1}^n \|n_{k-1}\theta\| \cdot n_k^{\beta+1})}{\log(n_k \|n_k\theta\|^{-1})}, & 1 < \beta < w(\theta), \\
\lim_{k \to \infty} \frac{-\log(n_1^n \|n_1\theta\|^\beta n_2^n \|n_2\theta\|^\beta \cdots n_{k-1}^n \|n_{k-1}\theta\|^\beta)}{\log(n_k \|n_k\theta\|^{-1})}, & \frac{1}{w(\theta)} < \beta < 1, \\
0, & \beta < \frac{1}{w(\theta)}. 
\end{cases}$$

where $n_k$ is the (maximal) subsequence of $(q_k)$ such that

$$\begin{cases} 
n_k^\beta \|n_k\theta\| < 1, & \text{if } 1 < \beta < w(\theta), \\
n_k^{1/\beta} \|n_k\theta\| < 1, & \text{if } 1/w(\theta) < \beta < 1.
\end{cases}$$
IV. Our result - Continued

**Theorem (L-D.H. Kim)**

For any irrational \( \theta \) with \( w(\theta) = w > 1 \) we have

\[
\dim_H (U_\beta[\theta]) = 1, \quad \beta \geq w,
\]

\[
\frac{w\beta - 1}{w^2 - 1} \leq \dim_H (U_\beta[\theta]) \leq \frac{\beta + 1}{w + 1}, \quad 1 \leq \beta < w,
\]

\[
0 \leq \dim_H (U_\beta[\theta]) \leq \frac{w\beta - 1}{w^2 - 1}, \quad \frac{1}{w} < \beta < 1,
\]

\[
\dim_H (U_\beta[\theta]) = 0, \quad \beta \leq \frac{1}{w}.
\]

If \( w(\theta) = 1 \), then we have

\[
\dim_H (U_\beta[\theta]) = 1, \quad \beta > 1,
\]

\[
\frac{1}{2} \leq \dim_H (U_\beta[\theta]) \leq 1, \quad \beta = 1,
\]

\[
\dim_H (U_\beta[\theta]) = 0, \quad \beta < 1.
\]

If \( w(\theta) = \infty \), then \( \dim_H (U_\beta[\theta]) = 0 \).
V. Graphs and comparing with the asymptotic case

\begin{align*}
\text{case } w(\theta) > 1 & \\
\text{case } w(\theta) = 1 & \\
\end{align*}
VI. Remarks

Remark 1 : The results depend on $w(\theta)$.

Remark 2 : For the case $\beta < 1$, optimize the upper bound w.r.t. $w(\theta)$:

$$\dim_H (\mathcal{U}_\beta[\theta]) \leq \frac{\beta^2}{2(1 + \sqrt{1 - \beta^2})}.$$ 

Since $1 + \sqrt{1 - \beta^2} > 1$, we have $\dim_H (\mathcal{U}_\beta[\theta]) < \frac{\beta}{2}$.

Recall that for $\beta < 1$,

- $\mathcal{U}_\beta[\theta] \subset \mathcal{L}_\beta[\theta]$ except for a countable set of points.
- $\dim_H (\mathcal{L}_\beta[\theta]) = \beta$.

Remark 3 : No mass transference principle for uniform approximations.

Remark 4 : Our results give an answer for the case of dimension one of a problem of Bugeaud and Laurent 2005.
VII. Examples

(i) Let $\theta$ be an irrational with $w(\theta) = w > 1$ and $q_{k+1} > q_k^w$ for all $k$. Then for each $1/w < \beta < w$ we have

$$\dim_H (U_\beta[\theta]) = \frac{w\beta - 1}{w^2 - 1}.$$

(ii) Assume that $\theta$ is an irrational with $w(\theta) = w > 1$ and with the subsequence $\{k_i\}$ of $q_{k+1} > q_k^w$ satisfying that $a_{n+1} = 1$ for $n \neq k_i$ and $q_{k+1} > (q_k)^{2^i}$. Then we have

$$\dim_H (U_\beta[\theta]) = \begin{cases} 
\frac{\beta + 1}{w + 1}, & \text{for } 1 \leq \beta < w, \\
0, & \text{for } \beta < 1.
\end{cases}$$
VIII. Examples-continued

(iii) Let $\theta = \frac{\sqrt{5} - 1}{2}$, of which partial quotients $a_k = 1$ for all $k$. Note that $w(\theta) = 1$. Then $\mathcal{U}_\beta[\theta] = \mathbb{T}$ for $\beta = 1$. Thus, we have

$$\dim_H (\mathcal{U}_\beta[\theta]) = \begin{cases} 
1, & \beta \geq 1, \\
0, & \beta < 1.
\end{cases}$$

(iv) Let $\theta$ be the irrational with partial quotient $a_k = k$ for all $k$. Then $w(\theta) = 1$ and

$$\dim_H (\mathcal{U}_\beta[\theta]) = \begin{cases} 
1, & \beta > 1, \\
\frac{1}{2}, & \beta = 1, \\
0, & \beta < 1.
\end{cases}$$
Ideas and proofs
I. Relation with the hitting time

Let $T_\theta$ be the rotation on $\mathbb{R}/\mathbb{Z}$.

Define

$$\tau^\theta_r(x, y) = \inf\{n : T^n_\theta x \in B(y, r)\}.$$ 

Define the hitting time rates:

$$R^\theta(x, y) := \liminf_{r \to 0} \frac{\log \tau^\theta_r(x, y)}{-\log r}, \quad \overline{R}^\theta(x, y) := \limsup_{r \to 0} \frac{\log \tau^\theta_r(x, y)}{-\log r}.$$ 

We have

$$\mathcal{L}_\beta(y) \approx \{\theta : R^\theta(0, y) \leq \beta\}, \quad \mathcal{U}_\beta(\theta) \approx \{\theta : \overline{R}^\theta(0, y) \leq \beta\},$$

and

$$\mathcal{L}_\beta[\theta] \approx \{y : R^\theta(0, y) \leq \beta\}, \quad \mathcal{U}_\beta[\theta] \approx \{y : \overline{R}^\theta(0, y) \leq \beta\}.$$
II. A result on hitting time

For an irrational $\theta$, let

$$w = w(\theta) := \sup\{\beta > 0 : \lim_{j \to \infty} j^\beta \|j\theta\| = 0\}.$$  

D. H. Kim and B. K. Seo 2003: for every $x$ and $y$

$$\limsup_{r \to 0} \frac{\log \tau_r^\theta(x, y)}{-\log r} \leq w, \quad \liminf_{r \to 0} \frac{\log \tau_r(x, y)}{-\log r} \leq 1.$$  

and for almost every $x$ and $y$

$$\limsup_{r \to 0} \frac{\log \tau_r^\theta(x, y)}{-\log r} = w, \quad \liminf_{r \to 0} \frac{\log \tau_r(x, y)}{-\log r} = 1.$$
III. Cantor structure

Let

\[ G_n = \bigcup_{i=1}^{n} B \left( i\theta, \left( \frac{1}{n} \right)^{1/\beta} \right) \quad \text{and} \quad F_k = \bigcap_{n=q_k}^{q_{k+1}-1} G_n. \]

Then we have

\[ U_\beta[\theta] = \bigcup_{\ell=1}^{\infty} \bigcap_{n=\ell}^{\infty} G_n = \bigcup_{\ell=1}^{\infty} \bigcap_{k=\ell}^{\infty} F_k. \]

We will show that for all \( \ell \geq 1 \), the Hausdorff dimensions of \( \bigcap_{k=\ell}^{\infty} F_k \) are the same.

Take

\[ E_n := \bigcap_{k=1}^{n} F_k. \]

Then \( E_n \) is the union of the intervals at level \( n \).
IV. Calculation tools

For each $k$, $E_k$ is a union of finite intervals. Suppose $E_0 \supset E_1 \supset E_2 \supset \ldots$. Let $F = \bigcap_{n=0}^{\infty} E_n$.

**Fact (Falconer’s book p.64)**

Suppose each interval of $E_{k-1}$ contains at least $m_k$ intervals of $E_k$ ($k = 1, 2, \ldots$) which are separated by gaps of at least $\varepsilon_k$, where $0 < \varepsilon_{k+1} < \varepsilon_k$ for each $k$. Then

$$\dim_H(F) \geq \lim_{k \to \infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \varepsilon_k)}.$$ 

**Fact (Falconer’s book p.59)**

Suppose $F$ can be covered by $\ell_k$ sets of diameter at most $\delta_k$ with $\delta_k \to 0$ as $k \to \infty$. Then

$$\dim_H(F) \leq \lim_{k \to \infty} \frac{\log \ell_k}{-\log \delta_k}.$$
V. Three Distances Theorem

Given an irrational number $\theta$ and a positive integer $N$, if one arranges the points $\{\theta\}, \{2\theta\}, \ldots, \{N\theta\}$ in ascending order, the distance between consecutive points can have at most three different lengths, and if there are three, one will be the sum of the other two. More precisely,

Theorem (V. T. Sós 1958)

If $q_{k-1} + q_k \leq N < q_k + q_{k+1}$, let $N = q_{k-1} + q_k + nq_k + j$, with $0 \leq n \leq a_{k+1} - 1$ and $0 \leq j \leq q_k - 1$. Then there are $q_{k-1} + q_k + nq_k$ intervals of length $\|q_k\theta\|$, $q_k - j$ intervals of length $\|q_{k-1}\theta\| - n\|q_k\theta\|$ and $k$ intervals of length $\|q_{k-1}\theta\| - (n + 1)\|q_k\theta\|$.

Corollary

If $N = q_k$, for some $k \geq 0$, then there are $q_k - q_{k-1}$ intervals of length $\|q_{k-1}\theta\|$ and $q_{k-1}$ intervals of length $\|q_{k-1}\theta\| + \|q_k\theta\|$.
VI. Structure of $n$-th level intervals-(i)

Lemma

If

$$2 \left( \frac{1}{q_{k+1}} \right)^{1/\beta} \geq \|q_{k-1}\theta\| + \|q_k\theta\|, \quad (1)$$

then we have $F_k = \mathbb{T}$.

Facts :

- If $1 < \beta < w(\theta)$ and $q_k^\beta \|q_k\theta\| \geq 1$, then (1) works, thus $F_k = \mathbb{T}$.
- If $\beta = 1$ and $a_{k+1} = 1$, then (1) works, thus $F_k = \mathbb{T}$. 
VII. Structure of $n$-th level intervals-(ii)

Lemma

For any $\beta \geq 1$, we have

$$F_k = \bigcup_{i=1}^{q_k} \left( i \theta - q_k^{-1/\beta}, i \theta + r_k(i) \right)$$

where $r_k(i)$ equals to

$$\begin{cases} 
\min \left( a_{k+1} q_k \theta + q_k^{-1/\beta}, \min_{1 \leq c \leq a_{k+1}} \left( (c - 1) q_k \theta + (cq_k + i - 1)^{-1/\beta} \right) \right), & i \leq q_k - 1, \\
\min \left( (a_{k+1} - 1) q_k \theta + q_k^{-1/\beta}, \min_{1 \leq c < a_{k+1}} \left( (c - 1) q_k \theta + (cq_k + i - 1)^{-1/\beta} \right) \right), & i > q_k - 1.
\end{cases}$$
VIII. Structure of $n$-th level intervals-(iii)

Lemma

For any $\beta \geq 1$, we have

$$\bigcup_{i=1}^{q_k} \left( i\theta - q_{k+1}^{-1/\beta} \bigcup i\theta + C_{\beta} \left( \frac{\|q_k\theta\|}{q_k} \right)^{\frac{1}{1+\beta}} - 2\|q_k\theta\| \right)$$

$$\subset F_k \subset \bigcup_{i=1}^{q_k} \left( i\theta - q_{k+1}^{-1/\beta} \bigcup i\theta + C_{\beta} \left( \frac{\|q_k\theta\|}{q_k} \right)^{\frac{1}{1+\beta}} \right),$$

where $C_{\beta} = \beta^{-\frac{\beta}{1+\beta}} + \beta^{\frac{1}{1+\beta}}$. 
IX. Lower bound for $1 < \beta < w(\theta)$

Suppose $1 < \beta < w(\theta)$. Take $(k_i)_{i \geq 1}$ those that $q_{k_i}^\beta \|q_{k_i}\| < 1$. Then

$$\bigcap_{k=1}^{\infty} F_k = \bigcap_{i=1}^{\infty} F_{k_i}.$$ 

Define

$$E_i = \bigcap_{j=1}^{i} F_{k_j}, \quad \text{and} \quad F = \bigcap_{i=1}^{\infty} E_i.$$ 

Let $m_{i+1}$ be the number of intervals of $E_{i+1}$ contained in $E_i$. Then

$$m_{i+1} \geq \frac{q_{k_{i+1}}}{2} \left( \frac{\|q_{k_i} \theta\|}{q_{k_i}} \right)^{\frac{1}{1+\beta}}.$$ 

Let $\varepsilon_i$ be the smallest gap between the intervals in $E_i$. Then

$$\varepsilon_i \geq \frac{1}{2} \|q_{k_i} - \theta\|.$$ 

Using the formula in Falconer’s book (page 64), we get the lower bound.
X. Upper bound for $1 < \beta < w(\theta)$

The set $E_i$ can be covered by $\ell_i$ sets of diameter at most $\delta_i$, with

$$\ell_i \leq \frac{1}{\|q_{k_1-1}\theta\|} \cdot \frac{C_\beta + 2}{\|q_{k_2-1}\theta\|} \left(\frac{\|q_{k_1}\theta\|}{q_{k_1}}\right)^{\frac{1}{\beta+1}} \cdots \frac{C_\beta + 2}{\|q_{k_i-1}\theta\|} \left(\frac{\|q_{k_i-1}\theta\|}{q_{k_i-1}}\right)^{\frac{1}{\beta+1}},$$

$$\delta_i = (C_\beta + 2) \left(\frac{\|q_{k_i}\theta\|}{q_{k_i}}\right)^{\frac{1}{\beta+1}}.$$

Using the formula in Falconer’s book (page 59), we get the upper bound.
XI. Some words for the other cases

Lemma

If $\beta < 1$, then for large $q_k$,

$$\max(c_k, 1) \cdot q_k \bigcup_{i=1}^{(c_k+2)q_k} B\left(i\theta, q_k^{-1/\beta}\right) \subset F_k \subset \bigcup_{i=1}^{(c_k+2)q_k} B\left(i\theta, q_k^{-1/\beta}\right).$$

where $c_k = \left\lfloor \left(\|q_k\theta\|^{\beta q_k}\right)^{-1/(1+\beta)} \right\rfloor$. All intervals are disjoint.

More efforts are needed for $\beta = 1$. For example, the following lemma.

Lemma

If $\frac{1}{(b+1)(b+2)} \leq q_k \|q_k\theta\| < \frac{1}{b(b+1)}$, $b \geq 1$, then we have

$$\bigcup_{1 \leq i \leq q_k} \left(i\theta - \frac{1}{q_k+1}, i\theta + (b - 1)q_k\theta + \frac{1}{(b + 1)q_k}\right) \subset F_k.$$