On minimal decomposition of $p$-adic homographic dynamical systems

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Géométrie et systèmes dynamiques archimédiens et non-archimédiens

Université d’Orléans
March 12th 2013
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3. Ideas and methods
Affine polynomial dynamical systems on $\mathbb{Q}_p$
I. The $p$-adic numbers

- $p \geq 2$ a prime number.
  \[
  \forall n \in \mathbb{N}, \quad n = \sum_{i=0}^{N} a_i p^i \quad (a_i = 0, 1, \ldots, p - 1)
  \]

- Ring $\mathbb{Z}_p$ of $p$-adic integers:
  \[
  \mathbb{Z}_p \ni x = \sum_{i=0}^{\infty} a_i p^i.
  \]

- Field $\mathbb{Q}_p$ of $p$-adic numbers: fraction field of $\mathbb{Z}_p$:
  \[
  \mathbb{Q}_p \ni x = \sum_{i=v(x)}^{\infty} a_i p^i, \quad (\exists v(x) \in \mathbb{Z}).
  \]

Absolute value: $|x|_p = p^{-v(x)}$, metric: $d(x, y) = |x - y|_p$. 

\[
\begin{align*}
3\mathbb{Z}_3 & \\
2 + 3\mathbb{Z}_3 & \\
1 + 3\mathbb{Z}_3 & 
\end{align*}
\]
Arithmetic in $\mathbb{Q}_p$:

Addition and multiplication: similar to the decimal way. "Carrying" from left to right.

Example: $x = (p - 1) + (p - 1) \times p + (p - 1) \times p^2 + \cdots$, then

- $x + 1 = 0$. So,
  
  $$-1 = (p - 1) + (p - 1) \times p + (p - 1) \times p^2 + \cdots.$$ 

- $2x = (p - 2) + (p - 1) \times p + (p - 1) \times p^2 + \cdots$.

We also have substraction and division.

Then we can define polynomials and rational maps.
II. Equicontinuous dynamics

We say $T : X \to X$ is equicontinuous if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s. t. } d(T^n x, T^n y) < \epsilon \ (\forall n \geq 1, \forall d(x, y) < \delta).$$

**Theorem**

Let $X$ be a compact metric space and $T : X \to X$ be an equicontinuous transformation. Then the following statements are equivalent:

1. $T$ is minimal.
2. $T$ is uniquely ergodic.
3. $T$ is ergodic for any/some invariant measure with $X$ as its support.

- **Fact**: 1-Lipschitz transformation is equicontinuous.
- **Fact**: Polynomial $f \in \mathbb{Z}_p[x] : \mathbb{Z}_p \to \mathbb{Z}_p$ is equicontinuous.
III. Polynomial dynamical systems on $\mathbb{Z}_p$

- Let $f \in \mathbb{Z}_p[x]$ be a polynomial with coefficients in $\mathbb{Z}_p$.
- Polynomial dynamical systems: $f : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$, noted as $(\mathbb{Z}_p, f)$.

**Theorem (Ai-Hua Fan, L 2011) minimal decomposition**

Let $f \in \mathbb{Z}_p[x]$ with $\deg f \geq 2$. The space $\mathbb{Z}_p$ can be decomposed into three parts:

$$\mathbb{Z}_p = A \sqcup B \sqcup C,$$

where

- $A$ is the finite set consisting of all periodic orbits;
- $B := \sqcup_{i \in I} B_i$ ($I$ finite or countable)
  - $B_i :$ finite union of balls,
  - $f : B_i \rightarrow B_i$ is minimal;
- $C$ is attracted into $A \sqcup B$. 
IV. Conjugate classes

Given a positive integer sequence \((p_s)_{s \geq 0}\) such that \(p_s | p_{s+1}\).

Profinite groupe : \(\mathbb{Z}(p_s) := \lim\leftarrow \mathbb{Z}/p_s\mathbb{Z}\).

Odometer : The transformation \(\tau : x \mapsto x + 1\) on \(\mathbb{Z}(p_s)\).


Let \(E\) be a compact set in \(\mathbb{Z}_p\) and \(f : E \to E\) a 1-lipschitzian transformation. If the dynamical system \((E, f)\) is minimal, then

- \((E, f)\) is conjugate to the odometer \((\mathbb{Z}(p_s), \tau)\) where \((p_s)\) is determined by the structure of \(E\).

Theorem (Fan-L 2011 : Minimal components of polynomials)

Let \(f \in \mathbb{Z}_p[X]\) be a polynomial and \(O \subset \mathbb{Z}_p\) a clopen set, \(f(O) \subset O\). Suppose \(f : O \to O\) is minimal.

- If \(p \geq 3\), then \((O, f|_O)\) is conjugate to the odometer \((\mathbb{Z}(p_s), \tau)\) where \((p_s)_{s \geq 0} = (k, kd, kdp, kdp^2, \ldots)\) \((1 \leq k \leq p, d|(p-1))\).
- If \(p = 2\), then \((O, f|_O)\) is conjugate to \((\mathbb{Z}_2, x + 1)\).
V. Affine polynomials on $\mathbb{Z}_p$

Let $T_{a,b}x = ax + b \ (a, b \in \mathbb{Z}_p)$. Denote

$$U = \{z \in \mathbb{Z}_p : |z| = 1\}, \quad V = \{z \in U : \exists m \geq 1, \text{s.t. } z^m = 1\}.$$ 

**Easy cases:**

1. $a \in \mathbb{Z}_p \setminus U$ $\Rightarrow$ one attracting fixed point $b/(1 - a)$.
2. $a = 1, b = 0$ $\Rightarrow$ every point is fixed.
3. $a \in V \setminus \{1\}$ $\Rightarrow$ every point is on a $\ell$-periodic orbit, with $\ell$ the smallest integer $\geq 1$ such that $a^\ell = 1$.

**Theorem (AH. Fan, MT. Li, JY. Yao, D. Zhou 2007)** **Case $p \geq 3$**:

4. $a \in (U \setminus V) \cup \{1\}, \ v_p(b) < v_p(1 - a) \Rightarrow p^{v_p(b)}$ minimal parts.
5. $a \in U \setminus V, \ v_p(b) \geq v_p(1 - a) \Rightarrow (\mathbb{Z}_p, T_{a,b})$ is conjugate to $(\mathbb{Z}_p, ax)$.

Decomposition: $\mathbb{Z}_p = \{0\} \sqcup \sqcup_{n \geq 1} p^n U$.

(1) One fixed point $\{0\}$.

(2) All $(p^n U, ax) (n \geq 0)$ are conjugate to $(U, ax)$.

For $(U, T_{a,0}) : p^{v_p(a^{\ell-1})(p - 1)/\ell}$ minimal parts, with $\ell$ the smallest integer $\geq 1$ such that $a^\ell \equiv 1 (\text{mod } p)$. 

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Theorem (Fan-Li-Yao-Zhou 2007) **Case** $p = 2$:

1. $a \in (\mathbb{U} \setminus \mathbb{V}) \cup \{1\}$, $v_p(b) < v_p(1 - a)$.
   - $v_p(b) = 0 \Rightarrow p^{v_p(a+1)-1}$ minimal parts.
   - $v_p(b) > 0 \Rightarrow p^{v_p(b)}$ minimal parts.

2. $a \in \mathbb{U} \setminus \mathbb{V}$, $v_p(b) \geq v_p(1 - a)$
   
   $\Rightarrow (\mathbb{Z}_p, T_{a,b})$ is conjugate to $(\mathbb{Z}_p, ax)$.

Decomposition: $\mathbb{Z}_p = \{0\} \sqcup \sqcup_{n \geq 1} p^n \mathbb{U}$.

1. One fixed point $\{0\}$.

2. All $(p^n \mathbb{U}, ax)(n \geq 0)$ are conjugate to $(\mathbb{U}, ax)$.

For $(\mathbb{U}, T_{a,0})$ : $2^{v_2(a^2-1)-2}$ minimal parts.

**Remark** : For the case $p = 2$, all minimal parts (except for the periodic orbits) are conjugate to $(\mathbb{Z}_2, x + 1)$. 
VI. Affine polynomials on $\mathbb{Q}_p$

Let $\varphi$ be an affine map defined by

$$\varphi(x) = ax + b \ (a, b \in \mathbb{Q}_p, a \neq 0, (a, b) \neq (1, 0)).$$

If $|a| \neq 1$: easy! For $|a| = 1$, we have the following conjugacy:

- $a \neq 1$:

\[
\begin{align*}
\mathbb{Q}_p & \xrightarrow{ax + b} \mathbb{Q}_p \\
\mathbb{Q}_p & \xrightarrow{x - \frac{b}{1-a}} \mathbb{Q}_p \\
\mathbb{Q}_p & \xrightarrow{ax} \mathbb{Q}_p
\end{align*}
\]

- $a = 1$:

\[
\begin{align*}
\mathbb{Q}_p & \xrightarrow{x + b} \mathbb{Q}_p \\
\mathbb{Q}_p & \xrightarrow{x - \frac{b}{1-a}} \mathbb{Q}_p \\
\mathbb{Q}_p & \xrightarrow{x + 1} \mathbb{Q}_p
\end{align*}
\]
VII. Affine polynomials on $\mathbb{Q}_p$ and local fields-continued

Theorem (AH. Fan, Y. Fares 2011)

If $K = \mathbb{Q}_p$, then

1. $\varphi(x) = x + 1 : \mathbb{Q}_p = \mathbb{Z}_p \cup \bigcup_{n=1}^{\infty} p^n \mathbb{U}$.  
   - $\mathbb{Z}_p$ is minimal.
   - $p^n \mathbb{U}$ contains $p^{n-1}(p - 1)$ minimal balls with radius 1.

2. $\varphi(x) = ax$ ($a$ is not a root of unity) : $\mathbb{Q}_p = \{0\} \cup \bigcup_{n \in \mathbb{Z}} p^n \mathbb{U}$.  
   - 0 is fixed.
   - All subsystems on $p^n \mathbb{U}$ are conjugate to $(\mathbb{U}, \varphi)$.

For $(\mathbb{U}, \varphi)$:

(1) Case $p \geq 3$ : $p^{v_p(a^\ell - 1)}(p - 1)/\ell$ minimal balls of same radius, with $\ell$ the smallest integer $\geq 1$ such that $a^\ell \equiv 1 (\text{mod } p)$.

(2) Case $p = 2$ : $2^{v_2(a^2 - 1) - 2}$ minimal balls of same radius.
VIII. Conjugacy and Classification

Let $\varphi_1(x) = a_1 x + b_1$ and $\varphi_2(x) = a_2 x + b_2$ be two affine maps with $a_1 \neq 1$, $a_2 \neq 1$.

Theorem (Fan-Fares 2011)

1. $p \geq 3 : (\mathbb{Q}_p, \varphi_1)$ and $(\mathbb{Q}_p, \varphi_2)$ are conjugate (topological) if and only if the orders of $a_1$ and $a_2$ in the multiplicative group $(\mathbb{Z}_p/p\mathbb{Z}_p)^\times$ are equal.

2. $p = 2 : (\mathbb{Q}_p, \varphi_1)$ and $(\mathbb{Q}_p, \varphi_2)$ are always conjugate (topological).
$p$-adic homographic dynamical systems
I. Projective line over $\mathbb{Q}_p$

For $(x_1, y_1), (x_2, y_2) \in \mathbb{Q}_p^2 \setminus \{(0, 0)\}$, we say that $(x_1, y_1) \sim (x_2, y_2)$ if $\exists \lambda \in \mathbb{Q}_p^* \ s.t.
\begin{align*}
x_1 = \lambda x_2 \quad \text{and} \quad y_1 = \lambda y_2.
\end{align*}

Projective line over $\mathbb{Q}_p$:

$\mathbb{P}^1(\mathbb{Q}_p) := (\mathbb{Q}_p^2 \setminus \{(0, 0)\})/\sim$

Spherical metric: Let $P = [x_1, y_1], Q = [x_2, y_2] \in \mathbb{P}^1(\mathbb{Q}_p)$, define

$\rho(P, Q) = \frac{|x_1 y_2 - x_2 y_1|_p}{\max\{|x_1|_p, |y_1|_p\} \max\{|x_2|_p, |y_2|_p\}}$

Viewing $\mathbb{P}^1(\mathbb{Q}_p)$ as $K \cup \{\infty\}$, for $z_1, z_2 \in \mathbb{Q}_p \cup \{\infty\}$ we define

$\rho(z_1, z_2) = \frac{|z_1 - z_2|_p}{\max\{|z_1|_p, 1\} \max\{|z_2|_p, 1\}}$ \quad if $z_1, z_2 \in \mathbb{Q}_p,$

and

$\rho(z, \infty) = \begin{cases} 1, & \text{if } |z|_p \leq 1; \\ 1/|z|_p, & \text{if } |z|_p > 1. \end{cases}$
II. Homographic maps

Let

\[ \phi(x) = \frac{ax + b}{cx + d} \quad \text{with} \quad a, b, c, d \in \mathbb{Q}_p, \quad ad - bc \neq 0, \]

which induces an 1-to-1 map \( \phi : \mathbb{P}^1(\mathbb{Q}_p) \rightarrow \mathbb{P}^1(\mathbb{Q}_p) \).

- \( \phi(-d/c) = \infty \), and \( \phi(\infty) = a/c \).
- \( \phi \) is a composition of \( \phi_1(x) = \alpha x, \phi_2(x) = x + \beta, \phi_3(x) = x \mapsto 1/x \).
- if \( \mathbb{D}(a, r) \) is a disk in \( \mathbb{P}^1(\mathbb{Q}_p) \), then \( \phi(\mathbb{D}(a, r)) \) is also a disk.
  (a disk in \( \mathbb{P}^1(\mathbb{Q}_p) \) is a disk in \( \mathbb{Q}_p \) or a complement of a disk in \( \mathbb{Q}_p \).)

1. \( \phi_1(\mathbb{D}(a, r)) = \mathbb{D}(\alpha a, r|\alpha|_p) \).
2. \( \phi_2(\mathbb{D}(a, r)) = \mathbb{D}(a + \beta, r) \).
3. if \( 0 \in \mathbb{D}(a, r) \), \( \phi_3(\mathbb{D}(a, r)) = \mathbb{P}^1(K) \setminus \overline{\mathbb{D}}(0, 1/r) \).
4. if \( 0 \notin \mathbb{D}(a, r) \), \( \phi_3(\mathbb{D}(a, r)) = \mathbb{D}(a^{-1}, r|a|_p^{-2}) \).
III. Fixed points and dynamics
The dynamics of $\phi$ depends on its fixed points which are the solution of

$$\frac{ax + b}{cx + d} = x \iff cx^2 + (d - a)x - b = 0.$$ 

Discriminant : $\Delta = (d - a)^2 + 4bc$.

- If $\Delta = 0$, then $\phi$ has only one fixed point $x_0$ in $\mathbb{Q}_p$ and $\phi(x)$ is conjugate to a translation $\psi(x) = x + \alpha$ for some $\alpha \in \mathbb{Q}_p$ by $g(x) = \frac{1}{x-x_0}$.

- If $\Delta \neq 0$ and $\sqrt{\Delta} \in \mathbb{Q}_p$, then $\phi$ has two fixed points $x_1, x_2 \in \mathbb{Q}_p$ and $\phi$ is conjugate to a multiplication $x \mapsto \beta x$ for some $\beta \in \mathbb{Q}_p$ by $g(x) = \frac{x-x_2}{x-x_1}$.

- If $\Delta \neq 0$ and $\sqrt{\Delta} \notin \mathbb{Q}_p$, then $\phi$ has no fixed point in $\mathbb{Q}_p$. But $\phi$ has two fixed points $x_1, x_2 \in \mathbb{Q}_p(\sqrt{\Delta})$. So we will study the dynamics of $\phi$ on $\mathbb{P}^1(\mathbb{Q}_p(\sqrt{\Delta}))$ then its restriction on $\mathbb{P}^1(\mathbb{Q}_p)$. 

IV. Notations

- $K$ is a finite extension of $\mathbb{Q}_p$.
- Still denote by $|\cdot|_p$ the extended absolute value of $K$.
- Degree: $d = [K : \mathbb{Q}_p]$. Ramification index: $e$
- Valuation function: $v_p(x) := -\log_p(|x|_p)$. $\text{Im}(v_p) = \frac{1}{e}\mathbb{Z}$.
- $\mathcal{O}_K := \{x \in K : |x|_p \leq 1\}$: the local ring of $K$,
- $\mathcal{P}_K := \{x \in K : |x|_p < 1\}$: its maximal ideal.
- Residual field: $\overline{K} = \mathcal{O}_K/\mathcal{P}_K$. Then $\overline{K} = \mathbb{F}_{p^f}$, with $f = d/e$.

**Quadratic extensions**:

- 7 quadratic extensions of $\mathbb{Q}_2$:
  - $\mathbb{Q}_2(\sqrt{-1})$, $\mathbb{Q}_2(\sqrt{\pm 2})$, $\mathbb{Q}_2(\sqrt{\pm 3})$, $\mathbb{Q}_2(\sqrt{\pm 6})$.
- 3 quadratic extensions of $\mathbb{Q}_p (p \geq 3)$:
  - $\mathbb{Q}_p(\sqrt{p})$, $\mathbb{Q}_p(\sqrt{N_p})$, $\mathbb{Q}_p(\sqrt{pN_p})$,
  where $N_p$ is the smallest quadratic non-residue module $p$. 
V. Uniformizer and representation

An element $\pi \in K$ is a uniformizer if $v_p(\pi) = 1/e$.

Define $v_\pi(x) := e \cdot v_p(x)$ for $x \in K$. Then $\text{Im}(v_\pi) = \mathbb{Z}$, and $v_\pi(\pi) = 1$.

Let $C = \{c_0, c_1, \ldots, c_{p^f-1}\}$ be a fixed complete set of representatives of the cosets of $P_K$ in $O_K$. Then every $x \in K$ has a unique $\pi$-adic expansion of the form

$$x = \sum_{i=i_0}^{\infty} a_i \pi^i,$$

where $i_0 \in \mathbb{Z}$ and $a_i \in C$ for all $i \geq i_0$.

Example: For $\mathbb{Q}_p(\sqrt{p})$ ($p \geq 3$), take $\pi = \sqrt{p}$, and

$$x = a_0 + a_1 \sqrt{p} + a_2 p + a_3 p^{3/2} + a_4 p^2 + \cdots.$$
VI. Minimal decomposition ($\phi$ admits no fixed point)

**Theorem (AH. Fan, SL. Fan, L, YF. Wang (preprint))**

Suppose that $\phi$ has no fixed points in $\mathbb{P}^1(\mathbb{Q}_p)$ and $\phi^n \neq id$ for each integer $n > 0$. Then

1. the system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is decomposed as a finite number of minimal subsystems;
2. these minimal subsystems are topologically conjugate to each other;
3. the number of minimal subsystems is determined by the number

$$\lambda := \frac{(a + d) + \sqrt{\Delta}}{(a + d) - \sqrt{\Delta}}.$$

Denote

- $K = \mathbb{Q}_p(\sqrt{\Delta})$ be the quadratic extension of $\mathbb{Q}_p$ generated by $\sqrt{\Delta}$.
- $\pi$ be an uniformizer of $K$.
- $\mathbb{K}$ be the residue field of $K$.
- $\ell$ be the order in the group $\mathbb{K}^*$ of $\lambda$. 
VII. The case $p \geq 3$

Theorem (Fan-Fan-L-Wang, $K = \mathbb{Q}_p(\sqrt{N_p})$ is unramified)

The dynamics $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is decomposed as $((p + 1)p^{v_p(\lambda^\ell - 1) - 1})/\ell$ minimal subsystems. Each subsystem is topologically conjugate to the adding machine on an odometer $\mathbb{Z}_{(p_s)}$ with $(p_s) = (\ell, \ell p, \ell p^2, \cdots)$.

Theorem (Fan-Fan-L-Wang, $K = \mathbb{Q}_p(\sqrt{p}), \mathbb{Q}_p(\sqrt{pN_p})$ is ramified)

(1) If $|a + d|_p > |\sqrt{\Delta}|_p$, then $\lambda = 1 \pmod{\pi}$. The dynamics $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is decomposed as $2p^{(v_{\pi}(\lambda^p - 1) - 3)/2}$ minimal subsystems. Moreover, each minimal subsystem is conjugate to the adding machine on the odometer $\mathbb{Z}_{(p_s)}$ with $(p_s) = (1, p, p^2, \cdots)$.

(2) If $|a + d|_p < |\sqrt{\Delta}|_p$, then $\lambda = -1 \pmod{\pi}$. The dynamics $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is decomposed as $p^{(v_{\pi}(\lambda^p + 1) - 3)/2}$ minimal subsystems. Moreover, each minimal subsystem is conjugate to the adding machine on the odometer $\mathbb{Z}_{(p_s)}$ with $(p_s) = (2, 2p, 2p^2, \cdots)$. 
VIII. The case $p = 2$

**Theorem (FFLW, $K = \mathbb{Q}_2(\sqrt{-3})$ is unramified)**

The dynamical system $(\mathbb{P}^1(\mathbb{Q}_2), \phi)$ is decomposed as $3 \cdot 2^{v_2(\lambda^2 - 1) - 2}/\ell$ minimal subsystems. Moreover, each minimal system is conjugate to the adding machine on the odometer $\mathbb{Z}_{(p_s)}$ with $(p_s) = (\ell, \ell 2, \ell 2^2, \cdots)$.

**Theorem (FFLW, $K = \mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{-6}), \mathbb{Q}_2(\sqrt{6})$ ramified)**

1. If $|a + d|_2 > |\sqrt{\Delta}|_2$, then $v_\pi(\lambda - 1) \geq 3$ is odd and the dynamical system $(\mathbb{P}^1(\mathbb{Q}_2), \phi)$ is decomposed as $2^{v_\pi(\lambda - 1) - 1}/2$ minimal subsystems.

2. If $|a + d|_2 < |\sqrt{\Delta}|_2$, then $v_\pi(\lambda + 1) \geq 3$ is odd and the dynamical system $(\mathbb{P}^1(\mathbb{Q}_2), \phi)$ is decomposed as $2^{v_\pi(\lambda + 1) - 1}/2$ minimal subsystems. Moreover, each minimal system is conjugate to the adding machine on the odometer $\mathbb{Z}_{(p_s)}$ with $(p_s) = (1, 2, 2^2, \cdots)$. 
IX. The case $p = 2$ (continued)

Theorem (FFLW, $K = \mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{3})$ is ramified)

(1) If $|a + d|_2 = |\sqrt{\Delta}|_2$, $v_\pi(\lambda^2 + 1) \geq 2$ is even and the system $(\mathbb{P}^1(\mathbb{Q}_2), \phi)$ is decomposed as $2^{(v_\pi(\lambda^2 + 1) - 2)/2}$ minimal subsystems.

(2) If $|a + d|_2 > |\sqrt{\Delta}|_2$, then $v_\pi(\lambda - 1) \geq 4$ is even and the system $(\mathbb{P}^1(\mathbb{Q}_2), \phi)$ is decomposed as $2^{v_\pi(\lambda - 1)/2}$ minimal subsystems.

(3) If $|a + d|_2 < |\sqrt{\Delta}|_2$, $v_\pi(\lambda + 1) \geq 4$ is even and the system $(\mathbb{P}^1(\mathbb{Q}_2), \phi)$ is decomposed as $2^{v_\pi(\lambda + 1)/2}$ minimal subsystems.

Moreover, each minimal system is conjugate to the adding machine on the odometer $\mathbb{Z}_{(p_s)}$ with $(p_s) = (1, 2, 2^2, \cdots)$. 
X. Minimal conditions

Corollary (FFLW, case $p \geq 3$)

*The system $(\mathbb{P}^1(\mathbb{Q}_p), \phi)$ is minimal if and only if one of the following conditions satisfied*

1. $K = \mathbb{Q}_p(\sqrt{\Delta})$ is unramified, $\ell = p + 1$ and $v_p(\lambda^\ell - 1) = 1$,
2. $K = \mathbb{Q}_p(\sqrt{\Delta})$ is ramified and $v_\pi(\lambda^p + 1) = 3$.

Corollary (FFLW, case $p = 2$)

*The system $(\mathbb{P}^1(\mathbb{Q}_2), \phi)$ is minimal if and only if one of the following conditions satisfied*

1. $K = \mathbb{Q}_2(\sqrt{\Delta}) = \mathbb{Q}_2(\sqrt{-3})$, $\ell = 3$ and $v_2(\lambda^{2\ell} - 1) = 2$,
2. $K = \mathbb{Q}_2(\sqrt{\Delta}) = \mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{3})$, $|a + b|_2 = |\sqrt{\Delta}|_2$ and $v_\pi(\lambda^2 + 1) = 2$. 
Ideas and methods
I. Conjugacy and restriction

Let $x_1, x_2$ be the two fixed points in $K \setminus \mathbb{Q}_p$. Let $g(x) = \frac{x-x_2}{x-x_1}$. Denote $\hat{K} = \mathbb{P}^1(K)$. Remark that $\hat{\mathbb{Q}}_p = \mathbb{P}^1(\mathbb{Q}_p)$ is invariant under $\phi$.

Step I : Do minimal decomposition of $(K, \lambda x)$.
Step II : Find $g(\hat{\mathbb{Q}}_p)$ and determine the restriction $(g(\hat{\mathbb{Q}}_p), \lambda x)$.
Step III : Go back to $\hat{\mathbb{Q}}_p$.
II. Methods for minimal decomposition of $\mathbb{Z}_p, \mathbb{Q}_p, K$.

Fan, Li, Yao, Zhou : Fourier analysis.

Our méthodes :

Theorem (Anashin 1994, Chabert, Fan and Fares 2009)

Let $X \subset \mathcal{O}_K$ be a compact set.
$\varphi : X \to X$ is minimal $\iff$
$\varphi_k : X/\pi^k \mathcal{O}_K \to X/\pi^k \mathcal{O}_K$ is minimal for all $k \geq 1$.

Predicting the behavior of $\varphi_{k+1}$ by the structure of $\varphi_k$.

- Consider the cycle $(x_1, \ldots, x_k)$ in $\mathcal{O}_K/\pi^n \mathcal{O}_K$,
- Each $x_i$ is lift to be $p^f$ points $\{x_i + t\pi^n : 0 \leq t < p^f\}$ in $\mathcal{O}_K/\pi^{n+1} \mathcal{O}_K$.

Linearization : $g := \varphi^k$,

$$g(x_1 + t\pi^n) \equiv x_1 + (a_n t + b_n)\pi^n \pmod{\pi^{n+1}}$$

with $a_n = g'(x_1)$, $b_n = \frac{g(x_1) - x_1}{\pi^n}$.

Linear maps $\Phi : \Phi(t) = a_n t + b_n$. 
III. Ideas and methods (continued)

Lifts of the cycle \((x_1, \ldots, x_k)\):

Let \(X_{n+1} = \{x_i + t\pi^n : 0 \leq t < p^f\}\)

- \(a_n \equiv 1, b_n \not\equiv 0 \mod \pi : \varphi_{n+1}\big|_{X_{n+1}}\) has \(p^f-1\) cycles of length \(pk\).
  We say \(\sigma\) grows.

- \(a_n \equiv 1, b_n \equiv 0 \mod \pi : \varphi_{n+1}\big|_{X_{n+1}}\) has \(pf\) cycles of length \(k\).
  We say \(\sigma\) splits.

- \(a_n \equiv 0 \mod \pi : \varphi_{n+1}\big|_{X_{n+1}}\) has a single cycle of length \(k\) and the remaining points of \(X\) are mapped into this cycle by \(\varphi^k\).
  We say \(\sigma\) grows tails.

- \(a_n \not\equiv 0, 1 \mod \pi : \varphi_{n+1}\big|_{X_{n+1}}\) has a single cycle of length \(k\) and \((p^f - 1)/\ell\) cycles of length \(k\ell\).
  We say \(\sigma\) partially splits.
Behavior of $\varphi_{n+1}$

Case 1

Case 2

Case 3

Case 4
IV. Subsystems and types
Let $\vec{E} = (E_1, E_2, \cdots)$ be a vector with $E_i \in \mathbb{N}^*$. A compact

$$X = \bigsqcup_{i=1}^{k} (x_i + \pi^n \mathcal{O}_K)$$

is called of type $(k, \vec{E})$ if:

- It is a $k$-cycle growing at level $n$ and all the lifts of this $k$-cycle split $E_1 - 1$ times.
- Then, all $E_1$-th generations of descendants grow and then all the lifts split $E_2 - 1$ times.
- Further, all the lifts of these descendants at level $E_1 + E_2$ split $E_3 - 1$ times, ....

If $X$ is of type $(k, \vec{E})$, then $(X, f)$ is decomposed into

- countable if the extension degree $d = 1$,
- uncountable (cardinality of $\mathbb{R}$) many, if $d > 1$,

minimal subsystems, each is conjugate to the odometer $(\mathbb{Z}_{(p_s)}, \tau)$ with

$$(p_s) = (k, \underbrace{k p, \cdots, k p}_{E_1}, \underbrace{k p^2, \cdots, k p^2}_{E_2}, \underbrace{k p^3, \cdots, k p^3}_{E_3}, \cdots).$$

If $\vec{E} = (e, e, e, \ldots)$, we call simply that $X$ is of type $(k, e)$. 

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V. Minimal decomposition for $\alpha x + \beta$ on $K$

We need only to treat $\varphi(x) = x + 1$ and $\varphi(x) = \alpha x$.

Let $U = \{x \in K : |x|_p = 1\}$.

**Theorem (Shi-Lei Fan and L, preprint)**

1. $\varphi(x) = x + 1 : K = \mathcal{O}_K \cup \bigcup_{n=1}^{\infty} \pi^n U$.
   - $\mathcal{O}_K$ is of type $(1, e)$.
   - $\pi^n U$ contains $p^{(n-1)f}(p^f - 1)$ balls with radius 1, each is of type $(1, e)$.

2. $\varphi(x) = \alpha x$ ($\alpha$ is not a root of unity) : $K = \{0\} \cup \bigcup_{n \in \mathbb{Z}} \pi^n U$.
   - $0$ is fixed and all subsystems on $\pi^n U$ are conjugate to $(U, \varphi)$.
   - For $(U, \varphi) : (1)$ Case $p \geq 3$ : Denote by $\ell$ the smallest integer $\geq 1$ such that $\alpha^\ell \equiv 1 \pmod{\pi}$. The subsystem $U$ is decomposed into

$$ (p^f - 1) \cdot p^{v_\pi(\alpha^\ell - 1)f - f}/\ell $$

balls of same radius and each is of type $(\ell, \vec{E})$ where

$$ \vec{E} = (v_\pi(\alpha^{\ell p-1}/\alpha-1), v_\pi(\alpha^{\ell p^2-1}/\alpha-1), \ldots, v_\pi(\alpha^{\ell p^N-1}/\alpha-1), e, e, \ldots), $$

$N$ is the largest integer such that $v_\pi((\alpha^{\ell p^N+1} - 1)/((\alpha^{\ell p^N} - 1)) \neq e$. 
Theorem (Shi-Lei Fan and L, preprint)

For $(\mathcal{U}, \varphi)$:

(2) Case $p = 2$:

Denote by $\ell$ the smallest integer $\geq 1$ such that $\alpha^\ell \equiv 1 \pmod{\pi}$.

The subsystem $\mathcal{U}$ is decomposed into

\[(p^f - 1) \cdot p^{v_{\pi}(\alpha^\ell - 1)f - f} / \ell\]

compact sets and each compact set is of type $(\ell, \vec{E})$ with

\[\vec{E} = \left( v_{\pi}(\alpha^\ell + 1), v_{\pi}(\alpha^{\ell p} + 1), \ldots, v_{\pi}(\alpha^{\ell p^N + 1}), e, e, \ldots \right), \]

where $N$ the biggest integer such that $v_{\pi}(\alpha^{\ell p^N + 1}) \neq e$. 