Dynamical properties of the negative beta transformation

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Outline

1. The $(-\beta)$-transformation and $(-\beta)$-shift

2. Dynamical properties of $(-\beta)$-transformation

3. $(-\beta)$-number VS $\beta$-number

4. Questions
The $(-\beta)$-transformation and $(-\beta)$-shift
I. The $\beta$-transformation and the $(-\beta)$-transformation

\[ \beta = \frac{1 + \sqrt{5}}{2}. \]

\textbf{Fig.}: $\beta$-transformation (left) and $(-\beta)$-transformation (right), $\beta = \frac{1 + \sqrt{5}}{2}$. 
II. The $(-\beta)$-transformation

Define $T_{-\beta} : (0, 1] \to (0, 1]$ by

$$T_{-\beta}(x) := -\beta x + \lfloor \beta x \rfloor + 1.$$ 

Let

$$d_{-\beta,1}(x) = \lfloor \beta x \rfloor + 1, \quad d_{-\beta,n}(x) = d_{-\beta,1}(T_{-\beta}^{n-1}(x)) \quad \text{for } n \geq 1.$$ 

Then

$$x = \frac{-d_{-\beta,1}}{-\beta} + \frac{-d_{-\beta,2}}{(-\beta)^2} + \frac{-d_{-\beta,3}}{(-\beta)^3} + \frac{-d_{-\beta,4}}{(-\beta)^4} + \cdots.$$ 

Sequence $d_{-\beta}(x) = d_{-\beta,1}(x)d_{-\beta,2}(x) \cdots \longrightarrow (-\beta)$-expansion of $x$.

Example: $\beta = \frac{1+\sqrt{5}}{2}$,

$$1 = \frac{-2}{-\beta} + \frac{-1}{(-\beta)^2} + \frac{-1}{(-\beta)^3} + \frac{-1}{(-\beta)^4} + \cdots.$$ 

$\longrightarrow$ expansion of 1 $\longrightarrow$ $2\bar{1} = 21111\ldots$. 
III. Remarks about the definition

- **Ito and Sadahiro 2009**: On the interval \([-\frac{\beta}{\beta+1}, \frac{1}{\beta+1})\):

  \[ x \mapsto -\beta x - \left( -\beta x + \frac{\beta}{\beta+1} \right). \]

→ conjugate to our \(T_{-\beta}\) through the conjugacy function

\(\phi(x) = \frac{1}{\beta+1} - x\). So all results can be translated to our case.

- Our definition is one case of generalized \(\beta\)-transformations studied by **Góra 2007** and **Faller 2008** *(Ph.D Thesis)*.

IV. Other works on \((-\beta)\)-transformations

- P. Ambrož, D. Dombek, Z. Masáková, and E. Pelantová
- A. Bertrand-Mathis
- K. Dajani and C. Kalle
- C. Frougny and A. Lai
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V. Admissible sequence and \((-\beta)-shift\)

A sequence \(a_1a_2\cdots\) is said admissible if \(\exists x \in (0, 1], d_{-\beta}(x) = a_1a_2\cdots\).

Alternate order: \(a_1a_2\cdots \prec b_1b_2\cdots\) if and only if

\[\exists k \geq 1, \ a_i = b_i \text{ for } i < k \ \text{ and } \ (-1)^k(b_k - a_k) < 0.\]

Denote \(a_1a_2\cdots \preceq b_1b_2\cdots\), if \(a_1a_2\cdots \prec b_1b_2\cdots\) or \(a_1a_2\cdots = b_1b_2\cdots\).

The \((-\beta)-shift\) \(S_{-\beta}\) on the alphabet \(\{1, \ldots, \lfloor \beta \rfloor + 1\}\) is the closure of the set of admissible sequences.

Define

\[
d^*_\beta(0) := \lim_{x \to 0^+} d_{-\beta}(x)
= \begin{cases} 
1b_1b_2\cdots b_{q-1}(b_q - 1), & \text{if } d_{-\beta}(1) = \overline{b_1\cdots b_{q-1}b_q} \\
1d_{-\beta}(1) & \text{otherwise.}
\end{cases}
\]
### VI. Admissible sequence and \((-\beta)\)-shift (continued)

#### Theorem (Ito-Sadahiro 2009)

A sequence \(a_1 a_2 \cdots\) is admissible if and only if for each \(n \geq 1\)
\[
d^*_\beta(0) < a_n a_{n+1} \cdots \leq d_-\beta(1).
\]

A sequence \(a_1 a_2 \cdots\) is in \(S_-\beta\) if and only if for each \(n \geq 1\)
\[
d^*_\beta(0) \leq x_n x_{n+1} \cdots \leq d_-\beta(1).
\]

#### Theorem (Frougny-Lai 2009)

The \((-\beta)\)-shift is of finite type if and only if \(d_-\beta(1)\) is purely periodic.

#### Theorem (Ito-Sadahiro 2009)

The \((-\beta)\)-shift is sofic if and only if \(d_-\beta(1)\) is eventually periodic.

#### Theorem (Frougny-Lai 2009)

If \(\beta\) is a Pisot number, then the \((-\beta)\)-shift is sofic.
VI. Admissible sequence and $(-\beta)$-shift (continued)

**Theorem (Ito-Sadahiro 2009)**

A sequence $a_1a_2\cdots$ is admissible if and only if for each $n \geq 1$

$$d^*_{-\beta}(0) < a_na_{n+1}\cdots \leq d_{-\beta}(1).$$

A sequence $a_1a_2\cdots$ is in $S_{-\beta}$ if and only if for each $n \geq 1$

$$d^*_{-\beta}(0) \leq x_nx_{n+1}\cdots \leq d_{-\beta}(1).$$

**Theorem (Frougny-Lai 2009)**

The $(-\beta)$-shift is of **finite type** if and only if $d_{-\beta}(1)$ is purely periodic.

**Theorem (Ito-Sadahiro 2009)**

The $(-\beta)$-shift is **sofic** if and only if $d_{-\beta}(1)$ is eventually periodic.

**Theorem (Frougny-Lai 2009)**

If $\beta$ is a Pisot number, then the $(-\beta)$-shift is sofic.
VI. Admissible sequence and \((-\beta)\)-shift (continued)

Theorem (Ito-Sadahiro 2009)

A sequence \(a_1a_2\cdots\) is admissible if and only if for each \(n \geq 1\)

\[ d^*_\beta(0) < a_na_{n+1}\cdots \leq d_\beta(1). \]

A sequence \(a_1a_2\cdots\) is in \(S_\beta\) if and only if for each \(n \geq 1\)

\[ d^*_\beta(0) \leq x_nx_{n+1}\cdots \leq d_\beta(1). \]

Theorem (Frougny-Lai 2009)

The \((-\beta)\)-shift is of finite type if and only if \(d_\beta(1)\) is purely periodic.

Theorem (Ito-Sadahiro 2009)

The \((-\beta)\)-shift is sofic if and only if \(d_\beta(1)\) is eventually periodic.

Theorem (Frougny-Lai 2009)

If \(\beta\) is a Pisot number, then the \((-\beta)\)-shift is sofic.
Dynamical properties of $(−\beta)$-transformation
I. Some notions of dynamical systems

Suppose $T : X \to X$ be a dynamical system.

- **locally eventually onto**: if for every nonempty open subset $U \subset X$, there exists a positive integer $n_0$ such that for every $f^{n_0}(U) = X$.

- **exactness**: $T$ acting on $(X, \mathcal{B}, \mu)$ is called exact if
  \[
  \bigcap_{n=0}^{\infty} T^{-n} \mathcal{B} = \{X, \emptyset\}
  \]
  or equivalently, for any positive measure set $A$ with $T^n(A) \in \mathcal{B}$ ($n \geq 0$),
  \[
  \mu(T^n(A)) \to 1 \ (n \to \infty).
  \]

- **maximal entropy measure**: the measure attaining the maximum of
  \[
  \sup\{h_\mu : \mu \text{ invariant}\}.
  \]

- **intrinsic ergodicity**: the maximal entropy measure is unique.
II. General piecewise monotone transformation

\[ T : [0, 1] \rightarrow [0, 1]. \]
- finite partition of \([0, 1] : P = \{P_1, \ldots, P_N\}.\]
- on each \(P_i\), \(T\) is monotonic, Lipschitz continuous and \(|T'| \geq \rho > 1.\)

Lasota-Yorke 1974: There is an invariant measure \(d\mu = h d\lambda\), where \(d\lambda\) is the Lebesgue measure and \(h\) is a density of bounded variation.

Keller 1978: The set \(\{h \neq 0\}\) is a finite union of intervals.

Wagner 1979: We can decompose \([0, 1] = \bigcup_{i=1}^{s} A_i \cup B\), such that
- on each \(A_i\) there is an invariant measure which is equivalent to the Lebesgue measure restricted to \(A_i\)
- each \(A_i\) can be decomposed as \(A_i = \bigcup_{j=1}^{m_i} A_{ij}\) and \(T^{m_i}\) is exact on each \(A_{ij}\).
- the set \(B\) satisfies \(T^{-1}B \subset B\) and \(\lim_{n \to \infty} \lambda(T^{-n}B) = 0\).

Hofbauer 1981: The number of maximal entropy measures is finite. If \(T\) is topological transitive, then it is intrinsic ergodic.
III. Dynamical properties - the \(( -\beta )\) case

Ito-Sadahiro 2009 : Let \( h_{-\beta} \) be a real-valued function defined on \((0, 1]\) by

\[
h_{-\beta}(x) = \sum_{n \geq 1, \ T_{-\beta}^n(1) \geq x} \frac{1}{(-\beta)^n}.
\]

Then the measure \( h_{-\beta}(x)\,d\lambda \) is an invariant measure of \( T_{-\beta} \).

Remark : The density may be zero on some intervals. So the invariant measure is not equivalent to the Lebesgue measure. (Different to the \( \beta \) case).

Góra 2007 : for \( \beta > \gamma_1 = 1.3247... \) (the smallest Pisot number), the transformation \( T_{-\beta} \) is exact and he conjectured that this would hold for all \( \beta > 1 \).

Faller 2008 : \( \beta > \sqrt[3]{2} \), \( T_{-\beta} \) admits a unique maximal measure.
IV. How many gaps?

A question:
For a given $\beta$, how many intervals (gaps) on which the density $h_{-\beta}$ equals to 0?
When $\beta$ decreases, the numbers should be like

0, 1, 2, 5, 10, 21,

What is the next?
V. Our results - Notations

For each $n \geq 0$, let $\gamma_n$ be the positive real number defined by

$$\gamma_{n}^{g_{n}+1} = \gamma_{n} + 1, \quad \text{with} \quad g_{n} = \lfloor 2^{n+2}/3 \rfloor.$$

Then

$$2 > \gamma_0 > \gamma_1 > \gamma_2 > \cdots > 1.$$

Note that $\gamma_0$ is the golden ratio and that $\gamma_1$ is the smallest Pisot number.

For each $n \geq 0$ and $1 < \beta < \gamma_n$, set

$$G_n(\beta) = \left\{ G_{m,k}(\beta) \mid 0 \leq m \leq n, 0 \leq k < \frac{2^{m+1}+(-1)^{m}}{3} \right\},$$

with open intervals

$$G_{m,k}(\beta) = \begin{cases} (T_{-\beta}^{2^{m+1}+k}(1), T_{-\beta}^{(2^{m+2}-(-1)^{m})/3+k}(1)) & \text{if } k \text{ is even}, \\ (T_{-\beta}^{(2^{m+2}-(-1)^{m})/3+k}(1), T_{-\beta}^{2^{m+1}+k}(1)) & \text{if } k \text{ is odd}. \end{cases}$$
VI. Our results-Theorems

We call an interval a gap if the density of the invariant measure is zero on it.

**Theorem (L-Steiner arXiv 2011)**

If $\beta \geq \gamma_0$, then there is no gap. If $\gamma_{n+1} \leq \beta < \gamma_n$, $n \geq 0$, then the set of gaps is $G_n(\beta)$ which consists of $g_n = \left\lfloor \frac{2^{n+2}}{3} \right\rfloor$ disjoint non-empty intervals.

Define

$$G(\beta) = \begin{cases} \emptyset & \text{if } \beta \geq \gamma_0, \\ \bigcup_{I \in G_n(\beta)} I & \text{if } \gamma_{n+1} \leq \beta < \gamma_n, \ n \geq 0. \end{cases}$$

**Theorem (L-Steiner arXiv 2011)**

The transformation $T_{-\beta}$ is locally eventually onto on $(0, 1] \setminus G(\beta)$,

$$T_{-\beta}^{-1}(G(\beta)) \subset G(\beta) \quad \text{and} \quad \lim_{n \to \infty} \lambda(T_{-\beta}^{-n}(G(\beta))) = 0.$$
VII. Our results-Theorems (continued)

Define a morphism on the symbolic space \( \{1, 2\}^\mathbb{N} \) by

\[
\varphi : 1 \mapsto 2, \quad 2 \mapsto 211.
\]

**Theorem (L-Steiner arXiv 2011)**

(1) For every \( n \geq 0 \), we have

\[
d_{-\beta n}(1) = \varphi^n(21^\omega).
\]

Hence

\[
\lim_{\beta \to 1} d_{-\beta}(1) = \lim_{n \to \infty} \varphi^n(2) = 2112221121121122 \cdots.
\]

(2) When \( \beta \) tends to 1, the \((-\beta)\) shift \( S_{-\beta} \) tends to the substitution dynamical system determined by 2112221121121122 \( \cdots \).

**Remarks**: 

(1) Thue-Morse sequence: 0110 1001 0110 \( \ldots \) \( \rightarrow \emptyset \) 110 1001 0110 \( \ldots \).

Then count the numbers of consecutive ones and zeros:

\[
\begin{align*}
&11 \quad 0 \quad 1 \quad 00 \quad 1 \quad 0110 \ldots, \\
&\underline{2} \quad \underline{1} \quad \underline{1} \quad \underline{2}
\end{align*}
\]

(2) \(|\varphi^m(2)| = |\varphi^{m+1}(1)| = g_n + \frac{1 - (-1)^n}{2} \) and \(|\varphi^m(21)| = 2^{m+1} \).
VIII. Our results-Corollaries

Corollary

For any $\beta > 1$, the transformation $T_{-\beta}$ is exact.

Corollary

For any $\beta > 1$, the transformation $T_{-\beta}$ has a unique maximal measure, and hence is intrinsic ergodic.

Corollary

The set of periodic points is dense in $(0, 1] \setminus G(\beta)$. 

IX. Our results-Proofs
For every word $a_1 \cdots a_n \in \{1, 2\}^n$, $n \geq 0$, define the polynomial

$$P_{a_1 \cdots a_n} = (-X)^n + \sum_{k=1}^{n} a_k (-X)^{n-k} \in \mathbb{Z}[X]$$

**Lemma**

For $1 \leq m < n$, we have

$$P_{a_1 \cdots a_n} = (-X)^{n-m} (P_{a_1 \cdots a_m} - 1) + P_{a_{m+1} \cdots a_n}.$$ 

For $n \geq 0$ we have the identities:

- $X^{\frac{1+(-1)^n}{2}} P_{\varphi^n(2)} + X^{\frac{1-(-1)^n}{2}} P_{\varphi^n(11)} = X + 1 = X^{\frac{1+(-1)^n}{2}} + X^{\frac{1-(-1)^n}{2}}$

- $1 - P_{\varphi^n(1)} = X^{\frac{1+(-1)^n}{2}} \prod_{k=0}^{n-1} (X^{\varphi^k(1)} - 1)$

- $P_{\varphi^n(21)} - P_{\varphi^n(2)} = (X^{g_n+1} - X - 1) \prod_{k=0}^{n-1} (X^{\varphi^k(1)} - 1)$
X. Our results-Proofs-continued
Let $1 < \beta < \gamma_n, n \geq 0$. Then the elements of $F_n(\beta)$ and $G_n(\beta)$ are intervals of positive length which form a partition of $(0, 1]$. Moreover, we have

1. $d_{-\beta}(1)$ starts with $\varphi^{n+1}(2)$, $T|\varphi^{n+1}(2)|(1) \in F_{n,0}$,

2. $1/\beta$ is an interior point of $F_{n,g_n-1}$,

3. $F_{n,k} = T^k(F_{n,0})$ for all $0 \leq k < g_n$,

4. $T^{g_n}(F_{n,0}) = F_{n,g_n} \cup F_{n,0}$, $T(F_{n,g_n}) = F_{n+1,0}$, if $n$ is even,

5. $T^{g_n}(F_{n,0}) = F_{n,g_n} \cup F_{n+1,0}$, $T(F_{n,g_n}) = F_{n,0}$, if $n$ is odd,
XI. Hofbauer’s recent work

**Fig.**: Piecewise linear transformation with two different negative slopes.

**Theorem (Hofbauer preprint 2011)**

*We can explicitly construct the non-wandering set which is a union of periodic orbits and some closed intervals.*
$(−\beta)$-number VS $\beta$-number
I. Definitions

Extend the definition of $T_\beta$ to 1 by $T_\beta(1) := \beta - \lfloor \beta \rfloor$

- **\(\beta\)-number (Parry Number)**: number $\beta > 1$ such that the orbit of 1 under $T_\beta$ is eventually periodic.

- **\((-\beta)\)-number**: number $\beta > 1$ such that the orbit of 1 under $T_{-\beta}$ is eventually periodic.

- **Pisot Number**: algebraic integer number $\beta > 1$, whose conjugates all have modulus $< 1$.

- **Perron Number**: algebraic integer number $\beta > 1$, whose conjugates all have modulus $< \beta$. 
II. Results

**Schmidt 1980, Bertrand 1977**: All Pisot numbers are $\beta$-numbers.

**Frougny-Lai 2009**: All Pisot numbers are $(−\beta)$-numbers.

**Lind 1984, Denker-Grillenberger-Sigmund 1976**: All $\beta$-numbers are Perron numbers.

**Solomyak 1994**: All conjugates of a $\beta$-number have modulus less than golden number.

**Masáková-Pelantová arXiv 2010**: All conjugates of a $(−\beta)$-number have modulus less than 2, so all $(−\beta)$-numbers with modului $\geq 2$ are Perron numbers.

**Theorem (L-Steiner, arXiv 2011)**

*All $(−\beta)$-numbers are Perron numbers.*
III. Results—Conitnued

Lemma

Let $\beta > 1$ such that $\beta^4 = \beta + 1$, i.e., $\beta \approx 1.2207$. Then $T_{-\beta}^{10}(1) = T_{-\beta}^{5}(1)$, and $(T_{-\beta}^{n}(1))_{n \geq 0}$ is aperiodic.

Lemma

Let $\beta > 1$ such that $\beta^7 = \beta^6 + 1$, i.e., $\beta \approx 1.2254$. Then $T_{-\beta}^{7}(1) = 0$, and $(T_{-\beta}^{n}(1))_{n \geq 0}$ is aperiodic.

Theorem (L-Steiner, arXiv 2011)

The set $(-\beta)$-numbers and the set of $\beta$-numbers do not include each other.
Questions
I. About the dynamics

- Are the periodic points for the \((-\beta)\)-shift uniformly distributed with respect to the unique measure of maximal entropy?
- Are the invariant measures concentrated on periodic orbits dense in the set of all invariant measures?
- Characterization of the \(\beta\) such that the corresponding \((-\beta)\)-shift satisfies the specification property.
- Characterization of the \(\beta\) such that the corresponding \((-\beta)\)-shift is synchronizing.
II. Classification and size
Classical Rényi $\beta$ case (Blanchard 1989, Schmeling 1997) :

- Class C1. simple Parry numbers ($S_\beta$ is a subshift of finite type) → dense.
- Class C2. Parry numbers ($S_\beta$ is a sofic.) → at most countable.
- Class C3. ($S_\beta$ satisfies the specification property) → Lebesgue measure 0, Hausdorff dimension 1.
- Class C4. ($S_\beta$ is synchronizing) → Lebesgue measure 0, Hausdorff dimension 1.
- Class C5. (the rest) → Lebesgue measure 1.

Question : What is about the $-\beta$ case?
III. Univoque set and size

Rényi’s $\beta$ case:
Let $J_\beta := [0, (\lceil \beta \rceil - 1)/(\beta - 1)]$. We are interested in the following set

$$\mathbb{U} := \{(x, \beta) : \beta > 1, x \in J_\beta, x \text{ has exactly one expansion in base } \beta\},$$

and the one dimensional sections:

$$\mathbb{U}_\beta := \{x \in J_\beta : (x, \beta) \in \mathbb{U}\}, \quad \mathbb{U} := \{\beta > 1 : (1, \beta) \in \mathbb{U}\}.$$

- $\mathbb{U}$ : $\text{Leb} = 0$, $HD = 2$ (de Vries-Komornik 2010).
- $\mathbb{U}_\beta$ : (Glendinning-Sidorov 2001)
  1. $1 < \beta \leq (1 + \sqrt{5})/2$ : two elements;
  2. $(1 + \sqrt{5})/2 < \beta < \beta_{KL}$ : countably infinite;
  3. $\beta_{KL} < \beta \leq 2$ : positive Hausdorff dimension. (tends 1 when $\beta \to 2$ : detailed proof in Jordan-Shmerkin-Solomyak 2010)

Here $\beta_{KL} \approx 1.787$ is the Komornik-Loreti constant.
- $\mathbb{U}$ : continuum many (Erdős-Horváth-Joó 1991), $\text{Leb} = 0$, $HD = 1$ (Daróczy-Kátai 1995).

Question : What is about the $-\beta$ case?
IV. Schmidt conjecture

Salem Number: algebraic integer number $\beta > 1$, whose conjugates all have modulus $\leq 1$ and at least one $= 1$.

Denote $\text{Per}(\beta)$, $\text{Per}(-\beta)$ the sets of eventually periodic points for $T_\beta$ and $T_{-\beta}$ respectively.

Schmidt 1980: If $\mathbb{Q} \cap [0, 1) \subset \text{Per}(\beta)$, then $\beta$ is either a Pisot number or a Salem number.

Masáková-Pelantová arXiv 2010: If $\mathbb{Q} \cap (0, 1] \subset \text{Per}(-\beta)$, then $\beta$ is either a Pisot number or a Salem number.

Conversely,

Bertrand 1977: If $\beta$ is a Pisot number, then $\mathbb{Q} \cap (0, 1] \subset \text{Per}(\beta)$.

Frougny-Lai 2009: If $\beta$ is a Pisot number, then $\mathbb{Q} \cap (0, 1] \subset \text{Per}(-\beta)$.

Schmidt conjecture, 1980

If $\beta$ is a Salem number, then $\mathbb{Q} \cap (0, 1] \subset \text{Per}(\beta)$.

Question: Schmidt conjecture for $-\beta$ case?
V. Schmidt conjecture-progress (β case)

Fact: The degree of a Salem number is even and $\geq 4$.

Boyd 1989: If $\beta$ is a Salem number of degree 4, then the orbit of 1 under $T_{\beta}$ is eventually periodic.

"There are also some very large orbits which have been shown to be finite: an example is given for which the preperiod length is 39420662 and the period length is 93218808".