On a class of stochastic semilinear PDE’s

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ABSTRACT

We consider stochastic semilinear partial differential equations with Lipschitz nonlinear terms. We prove existence and uniqueness of an invariant measure and the existence of a solution for the corresponding Kolmogorov equation in the space $L^2(H;\nu)$, where $\nu$ is the invariant measure. We also prove the closability of the derivative operator and an integration by parts formula. Finally, under boundness conditions on the nonlinear term, we prove a Poincaré inequality, a logarithmic Sobolev inequality and the iperc-contractivity of the transition semigroup.

Key words: Differential stochastic equation; invariant measure; Kolmogorov equation; log-Sobolev inequality; spectral gap

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1 Introduction and setting of the problem

We are concerned with the following semilinear equation perturbed by noise in the Hilbert space $H$ of all $2\pi$-periodic real functions

\[
\begin{aligned}
  dX &= (D^2\xi X - X + D\xi F(X))dt + dW, \\
  X(0)(\xi) &= x(\xi), \quad \xi \in [0,2\pi], 
\end{aligned}
\]

where $x \in H$, $F \in C^1(H;H)$ with $DF \in C_b(H;\mathcal{L}(H))$ and $W$ is a cylindrical Wiener process defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with values in $H$. We shall denote by $\langle \cdot, \cdot \rangle$ the inner product in $H$, defined by

\[
\langle x, y \rangle = \int_0^{2\pi} x(\xi)y(\xi)d\xi, \quad x, y \in H
\]
and by $| \cdot |_2$ the corresponding norm. We shall prove that (1) admits a unique mild solution in the space $C_W([0,T]; H)$, consisting of all stochastic processes $X(\cdot, x) \in C([0,T]; L^2(\Omega; H))$ which are adapted to $W(t)$. We recall that a treatment of the Cauchy problem for an extensive class of Burgers-type equations can be found in [11]. We shall also prove the differentiability of $X(t, x)$ with respect to $x$ and some approximation theorems both for $X(t, x)$ and its derivative. Through the mild solution $X(t, x)$ of (1) we shall define the transition semigroup $\{P_t\}_{t \geq 0}$ as

$$ P_t \varphi(x) = \mathbb{E}[\varphi(X(t, x))], $$

where $\varphi : H \in \mathbb{R}$ is Borel and bounded. We shall prove strong Feller and irreducibility properties of the transition semigroup $\{P_t\}_{t \geq 0}$ in order to ensure, thanks to the Doob theorem, the uniqueness of an invariant measure for the transition semigroup. We recall that a Borel probability measure $\nu$ is invariant for the semigroup $P_t$ if we have

$$ \int_H P_t \varphi d\nu = \int_H \varphi d\nu $$

for all $\varphi : H \rightarrow \mathbb{R}$ continuous and bounded. Then we shall present some sufficient conditions on $F$ that imply the existence (and consequently, by the Doob theorem, the uniqueness) of an invariant measure. The existence of an invariant measure $\nu$ allow us to extend uniquely $P_t$ to a strongly continuous semigroup (still denoted by $P_t$) in $L^2(H; \nu)$. We shall denote by $K_2$ its infinitesimal generator. Then we shall show that $K_2$ is the closure of the following differential operator

$$ K_0 \varphi(x) = \frac{1}{2} Tr[D^2 \varphi(x)] + \langle (D_x^2 - I)x + D_x F(x), D\varphi(x) \rangle $$

where $Tr$ denote the trace, $D$ denote the derivative with respect to $x$ and $\varphi$ belong to a suitable subspace of $L^2(H; \nu)$ that will be rigorously defined in the following. This kind of result was proved for a Burgers equation with coloured noise (see [3]). In the present situation (Lipschitz nonlinearities and a white noise perturbation) the result seems to be new. An extensive survey on second order partial differential operators in Hilbert spaces can be found in the monographs [1], [2], [6]. A second new result of this paper is the closability of the operator $D$ in $L^2(H; \nu)$ and that $D(K_2)$ is included in the Sobolev space $W^{1, 2}(H; \nu)$. This implies the integration by parts formula

$$ \int_H \varphi K_2 \varphi d\nu = -\frac{1}{2} \int_H |D\varphi|^2 d\nu, \quad \varphi \in D(K_2). $$

(3)
Moreover (but only in the case $\|DF\|_0 < 2$) we shall show that by (3) it follows a Poincaré-type inequality, i.e.

$$\int_H |\varphi - \overline{\varphi}|^2 d\nu \leq \frac{1}{2(1 - \frac{\|DF\|_0^2}{4})} \int_H |D\varphi|^2 d\nu, \quad \varphi \in W^{1,2}(H, \nu). \quad (4)$$

As consequence of (4) we shall derive that the spectrum of $K_2$ in the space $L^2_0(H; \nu) = \{\varphi \in L^2(H; \nu) : \int_H \varphi d\nu = 0\}$ is contained in the half space $\{\lambda \in \mathbb{C} : \Re \lambda < -(1 - \|DF\|_0^2/4)\}$. Moreover, we shall prove a logarithmic Sobolev inequality and consequently the hypercontractivity of $P_t$.

This paper is organized as follows: in the next section we introduce some notations and some functional spaces that will be used in what follows. Section 3 is devoted in proving existence and uniqueness of a mild solution $X(t, x)$ of problem (1) and to its differentiability with respect to $x$, and in section 4 we prove some approximation theorems. In section 5 we introduce the transition semigroup $P_t$, and in sections 6, 7 we discuss the strong Feller and irreducibility properties respectively. In section 8 we prove the existence of an invariant measure. In section 9 we study the infinitesimal generator $K_2$ of the semigroup $P_t$ in $L^2(H; \nu)$, where $\nu$ is an invariant measure for $P_t$. Section 10 is devoted to the integration by parts formula, and section 11 to the Sobolev space $W^{1,2}(H; \nu)$, i.e. the domain of the closure of $D$ in $L^2(H; \nu)$. Finally, the Poincaré inequality, the spectral gap and the logarithmic Sobolev inequality are discussed in section 12.

2 Preliminaries

Let us write problem (1) in an abstract form. For this it is convenient to consider

the complete orthogonal system $\{e_k\}_{k \in \mathbb{Z}}$ in $H$ given by

$$e_k(\xi) = \begin{cases} \frac{1}{\sqrt{2\pi}} \cos(k\xi), & k \geq 0, \; \xi \in [0, 2\pi], \\ \frac{1}{\sqrt{2\pi}} \sin(k\xi), & k < 0, \; \xi \in [0, 2\pi]. \end{cases}$$

We represent any element $x \in H$ by its Fourier series

$$x = \sum_{k \in \mathbb{Z}} x_k e_k, \quad x_k = \langle x, e_k \rangle,$$

and for any $\sigma \geq 0$ we define the set

$$H^2_\sigma = \{x \in H : |x|_{2,\sigma} < \infty\},$$

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where 

\[ |x|_{2,\sigma} = \left( \sum_{k \in \mathbb{Z}} (1 + k^2)^{\sigma/2} |x_k|^2 \right)^{1/2}. \]

Now, we define a linear operator \( A : D(A) \to H \) by

\[ Ax(\xi) = D_\xi^2 x(\xi) - x(\xi), \quad \xi \in [0, 2\pi], \quad D(A) = H^2_\#(0, 2\pi). \]

The linear operator \( A \) is selfadjoint and \( Ae_k = -(1 + k^2)e_k, \quad k \in \mathbb{Z} \). Clearly we have that \( |(-A)^{\sigma/2} x|_2 = |x|_{2,\sigma} \) and \( |(-A + I)^{1/2} x|_2 = |D_\xi x|_2 \). The cylindrical Wiener process \( W(t) \) is formally defined by

\[ W(t) = \sum_{z \in \mathbb{Z}} \beta_k(t)e_k, \quad t \geq 0, \]

where \( \{\beta_k\} \) is a sequence of mutually independent real Brownian process in a given probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Finally (1) can be written as

\[
\begin{cases}
  dX(t) = (AX(t) + D_\xi F(X(t)))dt + dW(t), \\
  X(0) = x \in H
\end{cases}
\]

(5)

In the following we will denote by \( \| \cdot \|_0 \) the supremum norm in the space \( C(H; L^2(H)) \). Clearly the conditions on \( F \) implies \( \|DF\|_0 < \infty \). We write (5) in the following mild form

\[ X(t) = e^{tA}x + \int_0^t D_\xi e^{(t-s)A}F(X(t))ds + W_A(t) \]

(6)

where \( W_A(t) \) is the stochastic convolution

\[ W_A(t) = \int_0^t e^{(t-s)A}dW(s) = \sum_{k \in \mathbb{Z}} \int_0^t e^{-(t-s)k^2}e_kd\beta_k(s). \]

Notice that for any \( \sigma \in [0, 1/2) \) we have that \( W_A(t) \in L^2(\Omega; H^\sigma_\#) \), since

\[ \|W_A(t)\|_{L^2(\Omega; H^\sigma_\#)}^2 = \mathbb{E}|W_A(t)|_{2,\sigma}^2 \leq \sum_{k \in \mathbb{Z}} \frac{(1 + k^2)^{\sigma}}{2(1 + k^2)} < \infty. \]

In order to give a precise meaning to equation (6), we introduce, for any \( t > 0 \), the linear mapping

\[ K(t) : H \to H, \quad x \mapsto K(t)x, \quad K(t)x = D_\xi e^{tA}x. \]

We have
Lemma 1. $K(t)$ is a linear bounded mapping from $H$ into itself. Moreover there exists $\kappa > 0$ such that

$$|K(t)x|_2 \leq \kappa e^{-t}t^{-1/2}|x|_2, \quad x \in H \quad (7)$$

Proof. For any $t > 0$ we have

$$D_\xi e^{tA}x = \sum_{k \in \mathbb{Z}} k:e^{-(1+k^2)t}x_k e^{-k}.$$

Then

$$|D_\xi e^{tA}x|_2^2 = \sum_{k \in \mathbb{Z}} k^2 e^{-2(1+k^2)t}|x_k|^2 \leq \sup_{k \in \mathbb{Z}} k^2 e^{-(1+k^2)t}|x|_2^2.$$ 

Since, as it can be easily seen,

$$\sup_{k \in \mathbb{Z}} k^2 e^{-2(1+k^2)t} \leq \frac{1}{4\sqrt{e}}t^{-1}e^{-2t},$$

the conclusion follows.

In the following will be useful the next Lemma 2.

Lemma 2. Suppose $b \geq 0$, $\beta > 0$ and that $a(t)$ is a nonnegative function locally integrable on $0 \leq t < T$ fulfilling

$$u(t) \leq a(t) + b \int_0^t (t-s)^{\beta-1}u(s)ds, \quad t \in [0,T].$$

Then we have

$$u(t) \leq a(t) + \theta \int_0^t E'_\beta(\theta(t-s))a(s)ds, \quad 0 \leq t < T$$

where

$$\theta = (b\Gamma(\beta))^{1/\beta}, \quad E_\beta(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\beta+1)}, \quad E'_\beta(z) = \frac{d}{dz}E_\beta(z).$$

Moreover

$$E'_\beta(z) \sim \frac{z^{\beta-1}}{\Gamma(\beta)} \text{ as } z \to 0^+, \quad E'_\beta(z) \sim E_\beta(z) \sim \frac{e^z}{\beta} \text{ as } z \to +\infty,$$

and if $a(t) = a$, constant, then $u(t) \leq aE_\beta(\theta t)$.

Proof. See e.g. Lemma 7.1.1 on [12].
3 The mild solution $X(t, x)$ and its differentiability

**Theorem 1.** For any $x \in H$ and $T > 0$ there exists a unique mild solution $X \in C_W([0, T]; H)$ of equation (5).

*Proof.* Existence and uniqueness of a solution of equation (5) follows easily by the fixed point method in the space $C_W([0, T]; H)$.

We prove here that the mild solution $X(t, x)$ of (6) is differentiable with respect to $x$ and that for any $h \in H$ it holds

$$DX(t, x) \cdot h = \eta^h(t, x),$$

where $\eta^h(t, x)$ is the mild solution of the equation

$$\begin{align*}
\frac{d}{dt}\eta^h(t, x) &= A\eta^h(t, x) + D\xi(DF(X(t, x) \cdot \eta^h(t, x))) \\
\eta^h(0, x) &= h
\end{align*}$$ (8)

This means that $\eta^h(t, x)$ is the solution of the integral equation

$$\eta^h(t, x) = e^{tA}h + \int_0^t K(t-s)DF(X(s, x)) \cdot \eta^h(s, x)ds, \quad t \geq 0. \quad (9)$$

**Theorem 2.** Assume that $X(t, x)$ is the solution of equation (6). Then it is differentiable with respect to $x$ $\mathbb{P}$-a.s., and for any $h \in H$ we have

$$DX(t, x) \cdot h = \eta^h(t, x), \quad \mathbb{P} \text{- a.s.} \quad (10)$$

and

$$|\eta^h(t, x)|_2 \leq e^{\left(\frac{\|DF\|_2^2}{4}\right)t}|h|_2, \quad t \geq 0 \quad (11)$$

*Proof.* Arguing as in the proof of Theorem 1, we notice that (8) has a unique mild solution $\eta^h(t, x)$ in $C_W([0, T]; H)$. Let us prove (11). By multiplying both sides of (8) by $\eta^h(t, x)$ and integrating on $[0, 2\pi]$ we have

$$\frac{1}{2}\frac{d}{dt}|\eta^h(t, x)|_2^2 = \langle A\eta^h(t, x), \eta^h(t, x) \rangle + \langle D\xi(DF(X(t, x)) \cdot \eta^h(t, x)), \eta^h(t, x) \rangle.$$ 

Integrating by parts and applying the H"older inequality we find

$$\frac{1}{2}\frac{d}{dt}|\eta^h(t, x)|_2^2 \leq \langle A\eta^h(t, x), \eta^h(t, x) \rangle + \frac{\|DF\|_2^2}{4}|\eta^h(t, x)|_2^2 + |D\xi\eta^h(t, x)|_2^2 = $$

$$= \left(\frac{\|DF\|_2^2}{4} - 1\right)|\eta^h(t, x)|_2^2.$$
Then (11) follows by Gronwall’s lemma.

Now we prove that \( \eta^h(t, x) \) fulfills (10). For this fix \( T > 0, x, h \in H \) such that \( |h|_2 \leq 1 \). We claim that there exist a constant \( C_T > 0 \) and a function \( \sigma_T(\cdot): H \to \mathbb{R}^+ \), with \( \sigma_T(h) \to 0 \) as \( h \to 0 \), such that

\[
|X(t, x + h) - X(t, x) - \eta^h(t, x)|_2 \leq C_T \sigma_T(h)|h|_2, \quad \mathbb{P} - \text{a.s.}
\]

Setting \( r_h(t, x) = X(t, x + h) - X(t, x) - \eta^h(t, x) \), \( r_h(t, x) \) satisfies the equation

\[
r_h(t, x) = \int_0^t K(t - s) [F(X(s, x + h)) - F(X(s, x))] ds + \int_0^t K(t - s) DF(X(s, x)) \cdot \eta^h(s, x) ds.
\]

Consequently we have that

\[
r_h(t, x) = \int_0^t K(t - s) \int_0^1 DF(\rho(\zeta, s)) \cdot \eta^h(s, x) ds + \int_0^t K(t - s) DF(X(s, x)) \cdot \eta^h(s, x) ds = \\
= \int_0^t K(t - s) \int_0^1 DF(\rho(\zeta, s)) \cdot \eta^h(s, x) ds + \\
+ \int_0^t K(t - s) \int_0^1 (DF(\rho(\zeta, s)) - DF(X(s, x))) \cdot \eta^h(s, x) ds.
\]

where \( \rho(\zeta, s) = \zeta X(s, x + h) + (1 - \zeta) X(s, x) \). Notice that since \( F \in C_b^1(H) \) and \( X(t, x) \) is continuous with respect to \( x \) uniformly in \([0, T]\), there exists a function \( \sigma_T: H \to \mathbb{R}^+ \) such that \( \sigma_T \to 0 \) as \( h \to 0 \) and

\[
|DF(\rho(\zeta, s)) - DF(X(s, x))|_2 \leq \sigma_T(h).
\]  

(12)

Setting

\[
\gamma_T = \sup_{t \in [0,T]} e^{(\|DF\|_0^2/4-1)t},
\]

and taking into account (11),(12), we find

\[
| \int_0^t K(t - s) \int_0^1 (DF(\rho(\zeta, s)) - DF(X(s, x))) \cdot \eta^h(s, x) ds |_2 \leq \\
\leq \kappa \int_0^t e^{-(t-s)(t-s)-1/2} ds \sigma_T(h)|h|_2 \leq \kappa \Gamma(1/4) \gamma_T \sigma_T(h)|h|_2.
\]
It follows that
\[ |r_h(t, x)|_2 \leq \|DF\|_0 \int_0^t e^{-(t-s)}(t-s)^{-1/2}|r_h(s, x)|_2 ds + \kappa \Gamma(\frac{1}{4}) \gamma_T \sigma_T(h) |h|_2, \]
and thus by Lemma 2 we have \( |r_h(t, x)|_2 \leq \kappa \Gamma(1/2) \gamma_T^{1/2} (\theta T) \sigma_T(h), |h|_2 \), where \( \theta = (\|DF\|_0 \Gamma(1/2))^2 \). This implies (10).

4 Approximation of \( X(t, x) \) and \( \eta^h(t, x) \)

In this section we consider the approximated problem
\[
\begin{align*}
\frac{dX_n(t)}{dt} &= (AX_n(t) + D_{\xi,n}(F(X_n(t))))dt + dW(t), \\
X_n(0) &= x \in H, 
\end{align*}
\]
where \( D_{\xi,n} \in L(H) \) is defined by \( D_{\xi,n} = D_{\xi} \circ P_n \) and \( P_n \) is the projection of \( H \) into the linear span of \( \{e_{-n}, \ldots, e_n\} \). We also consider problem (13) in its mild form, i.e.
\[
X_n(t) = e^{tA}x + \int_0^t K_n(t-s)F(X_n(s))ds + W_A(t),
\]
where \( K_n(t) = D_{\xi,n}e^{tA} \). Notice that \( D_{\xi,n} \circ F : H \to H \) is a nonlinear Lipschitz continuous function, and so, as it is well known (see, for example, [7]), problem (13) admits a mild solution in \( C(W([0, T]; H)) \). Moreover, for any \( n \in \mathbb{N}, t \geq 0 \) we have that \( K_n(t) \in L(H) \) and it holds
\[
\|K_n(t)\|_{L(H)} < \|K(t)\|_{L(H)}, \\
K_n(\cdot) \to K(\cdot) \text{ in } C([t_0, T]; L(H)), \quad 0 < t_0 < T.
\]
We have

**Theorem 3.** If \( X_n(t, x) \) and \( X(t, x) \) are the solutions of problem (14) and (6) respectively, then
\[
\lim_{n \to \infty} X_n(\cdot, x) = X(\cdot, x), \quad \text{in } C_W([0, T]; H).
\]

**Proof.** Let us fix \( \varepsilon > 0 \). Taking into account (6), (14) we have
\[
X(t, x) - X_n(t, x) = \int_0^t (K(t-s) - K_n(t-s))F(X(s))ds + \\
+ \int_0^t K_n(t-s)(F(X(s)) - F(X_n(s)))ds
\]

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Moreover, taking into account (15), for all $0 < t_0 < t \leq T$ it holds

$$\left| \int_0^t (K(t-s) - K_n(t-s))F(X(s))ds \right|_2 \leq$$

$$\leq \int_0^t \|K_n(s) - K(s)\|_{\mathcal{L}(\mathbb{H})}ds \|DF\|_0 \sup_{0 \leq t \leq T} |X(t)|_2 \leq$$

$$\leq (2 \int_0^{t_0} \|K(s)\|_{\mathcal{L}(\mathbb{H})} + \int_{t_0}^t \|K(s) - K_n(s)\|_{\mathcal{L}(\mathbb{H})}ds) \|DF\|_0 \sup_{0 \leq t \leq T} |X(t)|_2 \leq$$

$$\leq (4\kappa \sqrt{t_0} + T \sup_{t_0 \leq t \leq T} \|K(t) - K_n(t)\|_{\mathcal{L}(\mathbb{H})}) \|DF\|_0 \sup_{0 \leq t \leq T} |X(t)|_2$$

and

$$\left| \int_0^t K_n(t-s)(F(X(s)) - F(X_n(s)))ds \right|_2 \leq$$

$$\leq \kappa \|DF\|_0 \int_0^t (t-s)^{-1/2}e^{-(t-s)}|X(s) - X_n(s)|_2 ds.$$
Denote with $\eta^h_n(t,x)$ the mild solution of problem
\[
\begin{cases}
\frac{d}{dt}\eta^h_n(t,x) = A\eta^h_n(t,x) + D\xi_n(DF(X_n(t,x) \cdot \eta^h_n(t,x)), \\
\eta^h_n(0,x) = h.
\end{cases}
\] (17)

It is well known that the solution $X_n(t,x)$ of problem (13) it is differentiable with respect to $x$ P–a.s. (see, for example, [7]), and that
\[
\langle DX_n(t,x), h \rangle = \eta^h_n(t,x), \quad h \in H, t \geq 0.
\]
Moreover it is easy to see that (11) still holds for $\eta^h_n(t,x)$. We have also the next

**Theorem 4.** If $\eta^h(t,x)$ and $\eta^h_n(t,x)$ are the solutions of problems (8), (17) respectively, then for all $h \in H$
\[
\lim_{n \to \infty} \eta^h_n(t,x) = \eta^h(t,x)
\] (18)
in $C_W([0,T];H)$

**Proof.** The proof is similar to that of Theorem 3

5 The transition semigroup

The transition semigroup corresponding to the mild solution $X(t,x)$ of (6) is defined by
\[
P_t \varphi(x) = \mathbb{E}[\varphi(X(t,x))], \quad \varphi \in B_b(H), \ t \geq 0, \ x \in H.
\] (19)
Let us also consider the approximating semigroup
\[
P^n_t \varphi(x) = \mathbb{E}[\varphi(X_n(t,x))], \quad \varphi \in B_b(H), \ t \geq 0, \ x \in H,
\] (20)
for all $n \in \mathbb{N}$, where $X_n(t,x)$ is the solution of (14). We have obviously
\[
\|P_t \varphi\|_0 \leq \|\varphi\|_0, \quad \varphi \in B_b(H),
\]
and by the dominated convergence theorem it follows that
\[
\lim_{n \to \infty} P^n_t \varphi(x) = P_t \varphi(x), \quad \varphi \in C_b(H), \ x \in H.
\]
If $F \in C^1_b(H;H)$, by Theorem 2 we have that, for all $\varphi \in C^1_b(H)$, $P_t \varphi(x)$ and $P^n_t \varphi(x)$ are differentiable with respect to $x$ and it holds
\[
\langle DP_t \varphi(x), h \rangle = \mathbb{E}\langle D\varphi(X(t,x)), \eta^h(t,x) \rangle, \quad h \in H,
\]
\[ \langle DP^n_t \varphi(x), h \rangle = \mathbb{E} \langle D \varphi(X_n(t,x)), \eta^n_h(t,x) \rangle, \quad h \in H. \]

Moreover, by Theorem 3 and (18) it follows that for all \( \varphi \in C^1_0(H) \), \( h \in H \),
\[
\lim_{n \to \infty} \langle DP^n_t \varphi(x), h \rangle = \langle DP_t \varphi(x), h \rangle
\]
in \( C([0,T]; \mathbb{R}) \).

6 Strong Feller property

In order to prove the strong Feller property of the transition semigroup \( P_t \), i.e for all \( \varphi \in B_b(H), \ t > 0 \), it follows that \( P_t \varphi \in C_b(H) \), we shall use the Bismut-Elworthy formula (see [4]). Since \( D\xi F \) is not Lipschitz continuous, we will apply the Bismut-Elworthy formula to the approximated transition semigroup \( P^n_t \), defined in (20), and then let \( n \to \infty \).

**Lemma 3.** If \( \varphi \in C^2_b(H) \) and \( t > 0 \) we have, for all \( n \in \mathbb{N} \), \( P^n_t \varphi \in C^1_b(H) \) and, for any \( h \in H \),
\[
\langle DP^n_t \varphi(x), h \rangle = \frac{1}{t} \mathbb{E} \left[ \varphi(X_n(t,x)) \int_0^t \eta^n_h(s,x) \, dW(s) \right]. \tag{21}
\]

**Proof.** See [4]. \qed

Formula (21) remains true also for \( \varphi \in C_b(H) \), since we can pointwise approximate a \( C^2_b(H) \)-function by a sequence of \( C^2_b(H) \)-functions.

**Theorem 5.** The transition semigroup \( P_t \) defined in (19) is strong Feller.

**Proof.** Step 1. If \( \varphi \in C^2_b(H) \), for all \( t > 0 \) we have
\[
|DP_t \varphi(x)|_2 \leq t^{-1} \frac{\sqrt{2}}{\|DF\|_0} (e^{\frac{\|DF\|_0^2}{4}t} - 1)^{1/2} \|\varphi\|_0.
\]

In fact by (21), using the Hölder inequality and recalling (11), for all \( n \in \mathbb{N} \) we have
\[
|\langle DP^n_t \varphi(x), h \rangle|^2 \leq t^{-2} \|\varphi\|_0^2 \int_0^t \|\eta^n_h(s,x)\|_2^2 ds \leq t^{-2} \|\varphi\|_0^2 \int_0^t e^{\frac{\|DF\|_0^2}{4}s} |h|^2 ds = t^{-2} \|\varphi\|_0^2 \|DF\|_0^2 (e^{\frac{\|DF\|_0^2}{4}t} - 1) |h|^2.
\]

Now, letting \( n \to \infty \), the conclusion holds for the arbitrariness of \( h \).

Step 2. For any \( \varphi \in B_b(H), \ t > 0 \) and \( x, y \in H \) it holds
\[
|P_t \varphi(x) - P_t \varphi(y)| \leq t^{-1} \frac{\sqrt{2}}{\|DF\|_0} (e^{\frac{\|DF\|_0^2}{4}t} - 1)^{1/2} \|\varphi\|_0 |x - y|_2 \tag{22}
\]

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In order to prove the step we need to approximate \( \varphi \) by a sequence of \( \mathcal{C}^2_b(H) \)-functions. Since \( \mathcal{C}^2_b(H) \) is not dense in \( B_b(H) \), we will use a suitable pointwise approximation. Fix \( t > 0 \) and \( x, y \in H \). Let us define a signed measure \( \zeta \) setting

\[
\zeta = \lambda_{t,x} - \lambda_{t,y},
\]

where \( \lambda_{t,x} \), \( \lambda_{t,y} \) are the law of \( X(t,x) \) and \( X(t,y) \) respectively, and consider a sequence \( \{\varphi_n\} \) of \( \mathcal{C}^2_b(H) \)-functions such that

\[
\lim_{n \to \infty} \varphi_n(x) = \varphi(x) \quad \zeta\text{-a.s.}, \quad \|\varphi_n\|_0 \leq \|\varphi\|_0 \quad \forall n \in \mathbb{N}.
\]

By step 1 we have

\[
|P_t\varphi_n(x) - P_t\varphi_n(y)| \leq \sup_{0 \leq \theta \leq 1} \|D P_t \varphi_n(\theta x + (1-\theta)y)\|_{\mathcal{L}(H)} |x - y|_2 \leq \frac{\sqrt{2}}{\|DF\|_0} (e^{\|DF\|_2 t} - 1)^{1/2} \|\varphi_n\|_0 |x - y|_2.
\]

By the dominate convergence theorem, it follows that (22) holds and so \( P_t \varphi \in C_b(H) \) as claimed. Theorem 5 is proved.

7 Irreducibility

A basic tool for proving irreducibility of \( P_t \) is the approximate controllability of the following controlled system

\[
\begin{cases}
y'(t) = A y(t) + D \xi F(y(t)) + u(t) \\
y(0) = x
\end{cases}
\]

where \( u \in L^2([0,T];H) \). Let us denote by \( y(\cdot, x; u) \) the mild solution of (23), that is the solution of the integral equation

\[
y(t) = e^{tA} x + \int_0^t K(t-s)F(y(s))ds + \sigma_u(t),
\]

where

\[
\sigma_u(t) = \int_0^t e^{(t-s)A} u(s)ds.
\]

We say that the sistem (23) is approximatively controllable if for any \( \varepsilon > 0, T > 0, x, z \in H \), there exists \( u \in L^2([0,T];H) \) such that

\[
|y(T, x; u) - z| \leq \varepsilon.
\]

We have
Lemma 4. The system (23) is approximatively controllable.

Proof. Let be $\varepsilon > 0$, $T > 0$, $x$, $z \in H$. we have to show that there exists $u \in L^2([0, T]; H)$ such that (25) holds.

Step 1. The mapping

$$\sigma : L^2([0, T]; H) \rightarrow C_0([0, T]; H) \quad u \mapsto \sigma_u,$$

where

$$C_0([0, T]; H) = \{x \in C([0, T]; H) : x(0) = 0\}$$

has dense range. In fact is easy to check that the set

$$D_0 = \{\varphi \in C^1([0, T]; D(A)) : x(0) = 0\}$$

is dense in $C_0([0, T]; H)$. Now let $\varphi \in D_0$ and set

$$u(t) = \varphi(t) - A\varphi(t) - D\xi\varphi(t).$$

It is clear that $\sigma_u = \varphi$, so the range of $\sigma$ is dense as claimed.

Step 2. Conclusion.

Choose $\psi \in C([0, T]; H)$ such that $\psi(0) = 0$, $\psi(T) = z$, for instance

$$\psi(t) = \frac{T-t}{T}x + \frac{t}{T}z, \quad t \in [0, T],$$

and set

$$g(t) = \psi(t) - e^{tA}x - \int_0^t K(t-s)F(\psi(s))ds, \quad t \in [0, T].$$

Now, given $\varepsilon > 0$, by Step 1 there exists $u \in L^2([0, T]; H)$ such that

$$|\sigma_u(t) - g(t)| \leq C, \quad t \in [0, T],$$

where the constant $C$ will be choosen later. Let us show that (25) holds. In fact, let $y(\cdot; x; u)$ be the solution of (23). By (7) we have

$$|y(t) - \psi(t)|_2 \leq \int_0^t |K(t-s)(F(y(s)) - F(\psi(s)))|_2ds + |\sigma_u(t) - g(t)|_2 \leq$$

$$\leq \kappa\|DF\|_0 \int_0^t e^{-(t-s)}(t-s)^{-1/2}|y(s) - \psi(s)|_2ds + |\sigma_u(t) - g(t)|_2.$$

Then by Lemma 2 it follows that

$$|y(t) - \psi(t)|_2 \leq CE_{1/2}(\theta t)$$

and consequently

$$|y(T) - z|_2 \leq CE_{1/2}(\theta T).$$

Now it is enough to choose $C < E_{1/4}(\theta T)^{-1}\varepsilon$. □ □
Theorem 6. The transition semigroup $P_t$ defined in (19) is irreducible.

Proof. Let be $\varepsilon, T > 0, x, z \in H$. We have to show that

$$P_t X_{B^c(z,\varepsilon)}(x) = \mathbb{P}(|X(t, x) - z|_2 > \varepsilon) < 1,$$

(26)

where $X(t, x)$ is the solution of (6). For this purpose we choose a control $u \in L^2([0, T]; H)$ such that $|y(T, x; u) - z|_2 \leq \varepsilon/2$, where $y$ is the solution of (24). Since

$$|X(T, x) - z|_2 \leq |X(T, x) - y(T, x)|_2 + \frac{\varepsilon}{2},$$

we have

$$\mathbb{P}(|X(T, x) - z|_2 > 1) \leq \mathbb{P}(|X(T, x) - y(T, x)|_2 > \frac{\varepsilon}{2}).$$

(27)

But by (7) it holds

$$|X(t, x) - y(t)|_2 \leq \int_0^t |K(t - s)(F(X(s, x)) - F(y(s, x)))|_2 ds + |W_A(t) - \sigma_u(t)|_2 \leq$$

$$\leq \kappa |DF|_0 \int_0^t e^{-(t-s)}(t-s)^{-1/2}|X(s, x) - y(s, x)|_2 ds + |W_A(t) - \sigma_u(t)|_2.$$ 

consequently, by Lemma 2, it follows that

$$|X(t, x) - y(t)|_2 \leq |W_A(t) - \sigma_u(t)|_2 + \theta \int_0^t E_0^{1/2} \theta(t - s)|W_A(s) - \sigma_u(s)|_2.$$

Moreover, since $W_A(\cdot)$ is a nondegenerate continuous Gaussian random variable, we have that $\mathbb{P}(\sup_{t \in [0, T]} |W_A(t) - \sigma(t)|_2 > \varepsilon) < 1$. This implies that

$$\mathbb{P}(|X(T, x) - y(T)|_2 > \varepsilon) \leq \mathbb{P}(|W_A(t) - \sigma_u(t)|_2 + \theta \int_0^t E_0^{1/2} \theta(t - s)|W_A(s) - \sigma_u(s)|_2 > \varepsilon) < 1,$$

and therefore (26) is proved.

8 Existence and uniqueness of an invariant measure

In this section we shall assume that
Hypotesis 6.1.

\[
\| F \|_0 < \infty \quad \text{(28)}
\]

or

\[
\| DF \|_0 < 2 \quad \text{(29)}
\]

or

\[
F \in C^1(\mathbb{R}; \mathbb{R}) \text{ and } \| F' \|_0 < \infty \quad \text{(30)}
\]

In order to prove the existence of an invariant measure we set

\[
Y(t) = X(t, x) - W_A(t),
\]

where \( X(t, x) \) is the solution of problem (5). Since \( Y(t) \) is the solution of the integral equation

\[
Y(t) = e^{tA}x + \int_0^t K(t - s)F(Y(s) + W_A(s))ds,
\]

it follows easily that \( Y(t) \) is the strong solution of

\[
\begin{cases}
\frac{d}{dt} Y(t) = AY(t) + D\xi F(Y(t) + W_A(t)), \\
Y(0) = x.
\end{cases}
\quad \text{(31)}
\]

Multiplying both sides of (31) by \( Y(t) \) and integrating over \([0, 2\pi]\) we find

\[
\frac{1}{2} \frac{d}{dt} |Y(t)|_2^2 = \langle AY(t), Y(t) \rangle + \langle D\xi F(Y(t) + W_A(t)), Y(t) \rangle. \quad \text{(32)}
\]

We have the next

**Lemma 5.** Assume that (28) holds. Then for all \( 0 \leq \varepsilon \leq 1 \) it holds

\[
|Y(t)|_2^2 + \int_0^t e^{-2(1-\varepsilon)(t-s)}|D\xi Y(s)|_2^2 ds \leq |x|_2 e^{-2(1-\varepsilon)t} + \| F \|_0^2 \int_0^t e^{-2(1-\varepsilon)(t-s)} ds
\]

\[
\text{(33)}
\]

**Proof.** Fix \( 0 \leq \varepsilon \leq 1 \). By (28) and (32) it holds

\[
\frac{1}{2} \frac{d}{dt} |Y(t)|_2^2 \leq \langle AY(t), Y(t) \rangle + \| F \|_0^2 + |D\xi Y(t)|_2^2 =
\]

\[
= -|Y(t)|_2^2 + \| F \|_0^2 \leq -(1-\varepsilon)|Y(t)|_2^2 - \| F \|_0^2
\]

Now (33) follows by the Gronwall lemma.
Lemma 6. Assume that (29) holds. Then for all $\|DF\|_0^2/4 < \varepsilon \leq 1$ it holds

$$|Y(t)|^2 + (1 - \frac{\|DF\|_0^2}{4\varepsilon}) \int_0^t e^{-(1-\varepsilon)(t-s)}|D_\xi Y(s)|^2 ds \leq$$

$$\leq e^{-(1-\varepsilon)t}|x|^2 + \frac{\|DF\|_0^2}{4\varepsilon} \left( \int_0^t e^{-(1-\varepsilon)(t-s)}\|W_A(s)\|^2 ds \right) \tag{34}$$

Proof. Fix $\|DF\|_0^2/4 < \varepsilon \leq 1$. Integrating by parts and applying the Young inequality we find, for all $M > 0$,

$$||D_\xi F(Y(t) + W_A(t)), Y(t)|| \leq \|F(Y(t) + W_A(t)) \|_2 |D_\xi Y(t)|_2 \leq$$

$$\leq \|DF\|_0 |Y(t)|_2 |D_\xi Y(t)|_2 + \|DF\|_0 W_A(t)|_2 |D_\xi Y(t)|_2 \leq$$

$$\leq \varepsilon |Y(t)|_2 + \frac{\|DF\|_0^2}{4\varepsilon} |D_\xi Y(t)|_2 + \frac{\|DF\|_0^2}{2M} |W_A(t)|_2 + \frac{M}{2} |D_\xi Y(t)|_2.$$  

Then by (32) we have

$$\frac{1}{2} \frac{d}{dt} |Y(t)|^2 \leq - (1-\varepsilon)|Y(t)|^2 + \frac{M}{2} + \frac{\|DF\|_0^2}{4\varepsilon} - 1)|D_\xi Y(t)|_2 + \frac{\|DF\|_0^2}{2M} |W_A(t)|_2$$

Since $\|DF\|_0 < 2$ we can set $M = 1 - \frac{\|DF\|_0^2}{4\varepsilon}$ and so we find

$$\frac{1}{2} \frac{d}{dt} |Y(t)|^2 \leq - (1-\varepsilon)|Y(t)|^2 - \frac{1}{2} (1 - \frac{\|DF\|_0^2}{4\varepsilon}) |D_\xi Y(t)|_2 + \frac{\|DF\|_0^2}{1 - \frac{\|DF\|_0^2}{4\varepsilon}} |W_A(t)|_2.$$  

Now applying the Gronwall lemma we find (34). \hfill \Box

Lemma 7. If (30) holds, then $DF = F'$ and for all $0 \leq \varepsilon \leq 1$ it holds

$$|Y(t)|^2 + \int_0^t e^{2(1-\varepsilon)(t-s)}|D_\xi Y(s)|^2 ds \leq$$

$$\leq e^{-2(1-\varepsilon)t}|x|^2 + \|DF\|_0^2 \int_0^t e^{2(1-\varepsilon)(t-s)}\|W_A(s)\|^2 ds \tag{35}$$

Proof. Since $F \in C^1(\mathbb{R};\mathbb{R})$, it is easy to see that for all $x \in H$, $\xi \in [0, 2\pi]$, we have $F(x)(\xi) = F(x(\xi))$ and therefore $DF = F'$. We have also that for all $x \in H^1$ it holds

$$\langle D_\xi F(x), x \rangle = 0. \tag{36}$$

In fact we have

$$\langle D_\xi F(x), x \rangle = -\langle F(x), D_\xi x \rangle =$$

$$= \int_0^{2\pi} F(x(\xi)) D_\xi x(\xi) d\xi = \int_0^{2\pi} D_\xi (\int_0^{x(\xi)} F(\xi') d\xi') d\xi = 0.$$  

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This implies that for all $x \in H^1$, $y \in H$ it holds
\[
|\langle D_\xi F(x + y), x \rangle| \leq \|DF\|_0 |y|_2 |D_\xi x|_2
\] (37)

In fact, taking into account (36) we have
\[
|\langle D_\xi F(x + y), x \rangle| = |\langle F(x + y), D_\xi x \rangle| = |\langle F(x + y) - F(x), D_\xi x \rangle|
\]
that implies (37). Now fix $\varepsilon \geq 0$. Then, by (32), it follows
\[
\frac{1}{2} \frac{d}{dt} |Y(t)|_2^2 \leq \langle AY(t), Y(t) \rangle + \|DF\|_0 |W_A(t)|_2 |D_\xi Y(t)|_2 =
\]
\[
= -|Y(t)|_2^2 - \frac{|D_\xi Y(t)|_2^2}{2} + \frac{\|DF\|_0}{2} |W_A(t)|_2^2 \leq
\]
\[
\leq -(1 - \varepsilon)|Y(t)|_2^2 - \frac{|D_\xi Y(t)|_2^2}{2} + \frac{\|DF\|_0}{2} |W_A(t)|_2^2,
\]
and so applying the Gronwall lemma yields (35).

Now we are able to prove the main result of this section.

**Theorem 7.** Let $X(t, x)$ be the mild solution of problem (5). If hypothesis 5.1 holds then there exists a unique invariant measure for the transition semigroup $P_t$ defined in (19).

**Proof.** Since for Theorem 5 and Theorem 6 the transition semigroup $P_t$ is strong Feller and irreducible, it is sufficient to prove the existence of an invariant measure (see [5]). So, fix $x \in H$ and denote with $\lambda_{t,x}$ the law of $X(t, x)$ and by $\mu_T$ the measure
\[
\mu_T = \frac{1}{T} \int_0^T \lambda_{t,x} dt
\]
Now we shall prove that the family of measure $\left\{ \mu_T \right\}_{T \geq 0}$ is tight. So, denote with $B_R$ the set $B_R = \{ x \in H^{1/4} : |x|_{2,1/4}^2 \leq R \}$. Notice that since $H^{1/4} \subset H$ with compact embedding, the set $B_R$ is compact in $H$. Moreover we have
\[
|X(t, x)|_{1/4}^2 - 2|W_A(t)|_{1/4}^2 \leq 2|Y(t, x)|_{1/4}^2 \leq
\]
\[
\leq 2|2(-A + I)^{1/4}Y(t, x)|_{1/4}^2 \leq 8|D_\xi Y(t, x)|_2^2,
\]
where $Y(t, x)$ is a strong solution of (31). Setting $\varepsilon = 1$ in (33), (34), (35), it is clear that there exists a constant $C(x)$, depending by $x$, such that
\[
\int_0^T |D_\xi Y(t, x)|_2^2 dt \leq C(x)(1 + T).
\]
Then

\[
\mu(B_R^c) = \frac{1}{T} \int_0^T \lambda_{t,x}(B_R^c) dt = \frac{1}{T} \int_0^T \mathbb{P}(|X(t,x)|^2_{1/4} > R) dt \leq \frac{1}{TR} \int_0^T \mathbb{P}(|X(t,x)|^2_{1/2} > R) dt + \frac{2}{TR} \int_0^T \mathbb{E}|W_A(t)|^2_{1/2} dt \leq \frac{8}{R} \mathbb{E}|D_\xi Y(t,x)|^2_{1/2} dt + \frac{2}{R} \sup_{t>0} \mathbb{E}|W_A(t)|^2_{1/2}.
\]

So, it follows that \( \{\mu_T\}_{t \geq 0} \) is tight. Now, by the Krylov-Bogoliubov theorem it follows that there exists an invariant measure for the transition semigroup \( P_t \). The theorem is proved.

\[\square\]

**Lemma 8.** Assume that hypothesis 5.1 holds. Then for any \( n \in \mathbb{N} \) we have

\[
\int_H |x|^{2n} \nu(dx) < +\infty \quad (38)
\]

**Proof.** Fix \( n \in \mathbb{N} \). For any \( t > 0 \) we have

\[
|X(t,x)|^{2n}_{1/2} \leq 2n|Y(t,x)|^{2n}_{1/2} + 2n|W_A(t)|^{2n}_{1/2},
\]

where \( Y(t,x) \) is the solution of problem (31). Setting \( \varepsilon = (1 + \|DF\|^2_{0}/4)/2 \) in (33), (34), (35) it is clear that there exist \( \gamma_n, c_n > 0 \) such that

\[
\mathbb{E}|Y(t,x)|^{2n}_{1/2} \leq c_n(1 + e^{-\gamma_n t}|x|^{2n}_{1/2}).
\]

Then, since \( W_A(t) \) is a gaussian random variable, it follows that for some \( c'_n > 0 \) it holds

\[
\mathbb{E}|X(t,x)|^{2n}_{1/2} \leq c'_n(1 + e^{-\gamma_n t}|x|^{2n}_{1/2}).
\]

Now denote with \( \lambda_{t,x} \) the law of \( X(t,x) \). For any \( \alpha > 0 \) it holds

\[
\int_H \frac{|y|^{2n}_{1/2}}{1 + \alpha|y|^{2n}_{1/2}} \lambda_{t,x}(dy) \leq \int_H |y|^{2n}_{1/2} \lambda_{t,x}(dy) = \mathbb{E}|X(t,x)|^{2n}_{1/2} \leq c'_n(1 + e^{-\gamma_n t}|x|^{2n}_{1/2}).
\]

Since \( \lambda_{t,x} \) converges weakly to \( \nu \), it follows that

\[
\int_H \frac{|y|^{2n}_{1/2}}{1 + \alpha|y|^{2n}_{1/2}} \lambda_{t,x}(dy) \leq c'_n.
\]

Letting \( \alpha \to 0 \) yields (38). \(\square\)
We are concerned with the semigroup $P_t$ in $L^2(H, \nu)$, where $\nu$ is the unique invariant measure for $P_t$. In the following we only assume that $\int_H |x|^2 d\nu < \infty$. We denote by $K_2$ the infinitesimal generator of $P_t$ in $L^2(H, \nu)$ and by $E_A(H)$ is linear span of the set of the functions

$$x \mapsto \cos(\langle x, h \rangle), x \mapsto \sin(\langle x, h \rangle), x \in H, h \in D(A').$$

Let us consider the Kolmogorov operator

$$K_0\varphi = L\varphi - \langle F(x), D\xi D\varphi \rangle, \quad \varphi \in E_A(H)$$

where

$$L\varphi(\cdot) = \frac{1}{2} \text{Tr}[D^2 \varphi(\cdot)] + \langle \cdot, AD\varphi(\cdot) \rangle$$

is the Ornstein-Uhlenbek generator (see [7]). Notice that $E_A(H) \subset L^2(H, \nu)$ and $E_A(H)$ is dense in $L^2(H, \nu)$, since $D(A)$ is dense in $H$. Our aim is to prove that $K_2 = K_0$ in $L^2(H, \nu)$. First we have

**Lemma 9.** For any $\varphi \in E_A(H)$ we have $\varphi \in D(K_2)$ and $K_2\varphi = K_0\varphi$.

**Proof.** By Itô’s formula it follows that for all $\varphi \in E_A(H)$

$$\lim_{t \searrow 0} \frac{1}{t} \left( P_t \varphi(x) - \varphi(x) \right) = K_0\varphi(x), \quad x \in H$$

pointwise. Now it is sufficient to show that $\frac{1}{t}(P_t \varphi(x) - \varphi(x))$, $t \in (0, 1]$ is equibounded in $L^2(H, \nu)$. For all $\varphi \in E_A(H)$ and $x \in H$ we have

$$|P_t \varphi(x) - \varphi(x)|_2^2 \leq 2t \int_0^t \mathbb{E}|L\varphi(X(s, x))|_2^2 ds + 2t \int_0^t \mathbb{E}(|D\xi D\varphi(X(s, x)), F(X(s, x))|_2^2 ds.$$ 

It is clear that there exist two positive constants $a, b$ (depending on $\varphi$) such that for all $x \in H$ it holds

$$|L\varphi(x)|_2 \leq a + b|x|_2, \quad |\langle x, D\xi D\varphi(x) \rangle| \leq b|x|_2.$$

Then we have

$$|P_t \varphi_h(x) - \varphi_h(x)|_2^2 \leq 2t \int_0^t \mathbb{E}(a + b|x|_2) |X(s, x)|_2^2 ds + 2bt \int_0^t |F(X(s, x))|_2^2 ds.$$
Integrating with respect to $\nu$ and taking into account the invariance of $P_t$ with respect to $\nu$ yields

$$\|P_t\varphi_h - \varphi_h\|_{L^2(H,\nu)}^2 \leq 2t^2 \left( \int_H (a + b|x|_2)^2 \nu(dx) + b \int_H |F(x)|^2 d\nu(dx) \right).$$

Since $\int_H |x|^2 d\nu < \infty$ by assumption, the equiboundness of $t^{-1}(P_t\varphi_h - \varphi_h)$ follows easily.

Before concluding that $K_2 = K_0$ we need two lemmas

**Lemma 10.** There exists a constant $c_1 > 0$ such that for all $h \in H$ we have

$$|\eta^{Dh}(t, x)|_2 \leq c_1 \left( t^{-1/2} + e^{\theta t} \right) |h|_2, \quad (39)$$

where $\theta = (\kappa \|DF\|_0 \Gamma(1/2))^2$ and $\eta^z(t, x)$ is defined as in (8).

**Proof.** Notice that by the density of $H^1$ in $H$ it is sufficient to prove (39) for $h \in H^1$. So, if $h \in H^1$, $\eta^{Dh}(t, x)$ is the solution of

$$\eta^{Dh}(t, x) = K(t)h + \int_0^t K(t - s)\langle DF(X(s, x)), \eta^{Dh}(s, x) \rangle ds.$$

By (7) it follows that

$$|\eta^{Dh}(t, x)|_2 \leq \kappa t^{-1/2} |h|_2 + \kappa \|DF\|_0 \int_0^t (t - s)^{-1/2} |\eta^{Dh}(s, x)|_2 ds$$

and by Lemma 2 that

$$|\eta^{Dh}(t, x)|_2 \leq \kappa t^{-1/2} |h|_2 + \kappa \theta \int_0^t E_{1/2}'(\theta(t - s)) s^{-1/2} ds |h|_2. \quad (40)$$

Since $E_{1/2}(\cdot) : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function with

$$E_{1/2}'(z) \sim 2z^{-1/2} \text{ as } z \to 0^+, \quad E_{1/2}'(z) \sim 2e^z \text{ as } z \to +\infty,$$

it is clear that there exists a constant $C > 0$ such that

$$E_{1/2}'(z) \leq C(z^{-1/2} + e^z) \quad z > 0.$$

Taking into account that

$$\int_0^t (t - s)^{-1/2} s^{-1/2} ds = \int_0^1 (1 - s)^{-1/2} s^{-1/2} ds = \beta(1/2, 1/2),$$

where $\beta(\cdot, \cdot)$ is the Euler beta function, by an easy computation we find (39)
Lemma 11. Assume that \( f \in C^1_b(H) \). Then there exists \( c_2 > 0 \) such that for all \( \lambda > \theta \), \( h \in H \) we have

\[
|\langle D_\xi D\varphi(x), h \rangle| \leq c_2(\lambda^{-1/2} + \frac{1}{\lambda-\theta})\|f\|_1\|h\|_2
\]

(41)

where

\[ \theta = \left( \kappa \|DF\|_0 \Gamma(1/2) \right)^2 \]

and

\[ \varphi(x) = \int_0^\infty e^{-\lambda t} P_t f(x) dt. \]

Proof. Notice that since \( H^1 \) is dense in \( H \) it is sufficient to prove (41) for \( h \in H^1 \). We have

\[
\langle D_\xi D\varphi(x), h \rangle = -\langle D\varphi(x), D_\xi h \rangle =
\]

\[ = -\int_0^\infty e^{-\lambda t} \mathbb{E}\left[ \langle Df(X(t,x)), \langle X_x(t,x), D_\xi h \rangle \rangle \right] dt =
\]

\[ = -\int_0^\infty e^{-\lambda t} \mathbb{E}\left[ \langle Df(X(t,x)), \eta D_\xi h(t,x) \rangle \right] dt. \]

So, taking into account (39),

\[
|\langle D_\xi D\varphi(x), h \rangle|_2 \leq \|f\|_1 \int_0^\infty e^{-\lambda t} |\eta D_\xi h(t,x)|_2 dt \leq
\]

\[ \leq \|f\|_1 c_1 \int_0^\infty e^{-\lambda t} (t^{-1/2} + e^{\theta t}) dt |h|_2
\]

that implies (41). \( \square \)

Now we are able to prove the main result of this section.

Theorem 8. \( K_2 \) is the closure of \( K_0 \) in \( L^2(H,\nu) \).

Proof. By Lemma 9 we know that \( K_2 \) extends \( K_0 \). Since \( K_2 \) is dissipative, so is \( K_0 \). Consequently, \( K_0 \) is closable. Let us denote by \( \overline{K_0} \) its closure. We have to show that \( K_2 = \overline{K_0} \). For this purpose, we will show that the range of \( \lambda - \overline{K_0} \) is dense in \( L^2(H,\nu) \) for some \( \lambda > 0 \). In fact by the Lumer-Philipps theorem this implies that \( \overline{K_0} \) is \( m \)-dissipative and it is the generator of a semigroup of contraction. Therefore, since \( K_2 \) extends \( K_0 \), it must coincide with \( \overline{K_0} \). So, if \( f \in \mathcal{E}_A(H) \), we are interested in solving the problem

\[ \lambda \varphi - \overline{K_0} \varphi = f. \]
Setting
\[ \varphi(x) = R(\lambda, K_2)f(x) = \int_0^\infty e^{-\lambda t}P_tf(x)dt, \]
we will show in the next two steps that \( \varphi \in D(K_0) \), that implies \( \lambda \varphi - K_0 \varphi = f \), since \( K_2 \) extends \( K_0 \). Let us denote with \( C_{b,1}(H) \) the Banach space of all continuous function \( \psi : H \to \mathbb{R} \) such that \( \psi(x)/(1 + |x|^2) \in C_b(H) \).

**Step 1.** \( \varphi \in D(L, C_{b,1}(H)) \)

Notice that by Theorem 5 it follows that \( \varphi \in C^1_b(H) \). We have to compute the derivative
\[ \frac{d}{dt} R_t \varphi|_{t=0}, \]
where
\[ R_t \varphi(x) = \mathbb{E}[\varphi(Z(t,x))], \quad Z(t,x) = e^{tA}x + W_A(t). \]

We have
\[ R_t \varphi(x) = \mathbb{E}[\varphi(Z(t,x))] = \mathbb{E}[\varphi(X(t,x) - \int_0^t K(t-s)F(X(s,x))ds)] = \]
\[ = P_t \varphi(x) - \mathbb{E}[\langle D\varphi(X(t,x)), \int_0^t K(t-s)F(X(s,x))ds \rangle] + o(t), \] \( (42) \)
where \( \lim_{t \to 0} \frac{o(t)}{t} = 0 \). Now, since \( \forall x \in H \) we have that \( \mathbb{P}-a.s. \)
\[ \lim_{t \to 0^+} \frac{1}{t} \langle D\varphi(X(t,x)), \int_0^t K(t-s)F(X(s,x))ds \rangle = -\langle D_\xi D\varphi(x), F(x) \rangle \] \( (43) \)
Taking into account \( (41) \) with \( \lambda > \theta \) we find
\[ \frac{1}{t} \langle D\varphi(X(t,x)), \int_0^t K(t-s)F(X(s,x))ds \rangle = \]
\[ = e^{\lambda t} \frac{1}{t} \langle D\varphi(x), \int_0^t K(t-s)F(X(s,x))ds \rangle = \]
\[ = e^{\lambda t} \frac{1}{t} \langle D_\xi D\varphi(x), \int_0^t e^{(t-s)\lambda}F(X(s,x))ds \rangle | \leq \]
\[ \leq e^{\lambda t} c_2(\lambda^{1/2} + \frac{1}{\lambda - \theta})\|DF\|_0 \frac{1}{t} \sup_{0 \leq s \leq T} |X(s,x)| \] \( (44) \)
So, since \( (43), (44) \) hold, we can apply the dominated convergence theorem in \( (42) \) and obtain
\[ \lim_{t \to 0} \frac{R_t \varphi(x) - \varphi(x)}{t} = \lambda \varphi(x) - f(x) + \langle D_\xi D\varphi(x), F(x) \rangle \]
for all $x \in H$. Now we have to prove that $t^{-1}(R_t \varphi(x) - \varphi(x))$ is equibounded in $C_{b,1}(H)$ for $t \in (0, 1]$. To see this we need to observe that by (6) it follows easily that

$$
\sup_{t \in [0, 1]} \mathbb{E}|X(t, x)|_2 \leq |x|_2 C,
$$

where $C = \sup_{t \geq 0} \mathbb{E}|W(t)|_2 E_{1/2}(\theta)$. Now set

$$
C' = c_2(\lambda^{-1/2} + \frac{1}{\lambda - \theta})\|DF\|_0 C.
$$

By (42) and (44) we have

$$
\frac{|R_t \varphi(x) - \varphi(x)|}{t(1 + |x|_2)} \leq \frac{|P_t \varphi(x) - \varphi(x)|}{t(1 + |x|_2)} + \frac{1}{t(1 + |x|_2)} \mathbb{E}[\langle D\varphi(X(t, x)), \int_0^t K(t-s)F(X(s, x))ds \rangle] + \frac{o(t)}{t} \leq \frac{\lambda t - 1}{t} \|\varphi\|_{b,1} + C' \frac{\lambda^t (1 - e^{-t})|x|_2}{t(1 + |x|_2)} + \frac{o(t)}{t}
$$

which is equibounded in $(0, 1]$. 

**Step 2.** $D(L, C_{b,1}(H)) \subset D(K_0)$. 

By Proposition 2.6 of [8] there exists a 4-index sequence $(\varphi_{n_1, n_2, n_3, n_4}) \subset \mathcal{E}_A(H)$ such that for all $x \in H$

$$
\lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \lim_{n_3 \to \infty} \lim_{n_4 \to \infty} \varphi_{n_1, n_2, n_3, n_4}(x) = \varphi(x)
$$

$$
\lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \lim_{n_3 \to \infty} \lim_{n_4 \to \infty} L\varphi_{n_1, n_2, n_3, n_4}(x) = L\varphi(x)
$$

$$
\lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \lim_{n_3 \to \infty} \lim_{n_4 \to \infty} D\varphi_{n_1, n_2, n_3, n_4}(x) = D\varphi(x)
$$

and

$$
\sup_{n_1, n_2, n_3, n_4} \{\|\varphi_{n_1, n_2, n_3, n_4}\|_{b,2} + \|D\varphi_{n_1, n_2, n_3, n_4}\|_{b,2} + \|L\varphi_{n_1, n_2, n_3, n_4}\|_{b,2}\} < \infty.
$$

So, by the dominated convergence theorem it follows that

$$
\lim_{n \to \infty} K_0 \varphi_n = \lim_{n \to \infty} (L\varphi_n - \langle D\xi D\varphi_n, F \rangle) = L\varphi - \langle D\xi D\varphi, F \rangle
$$

in $L^2(H, \nu)$, that implies $\varphi \in D(K_0)$. The theorem is proved. \end{proof}
10 Integration by parts formula

Let us denote by $\nu$ the invariant measure for the transition semigroup $P_t$ and by $K_2$ its infinitesimal generator in $L^2(H, \nu)$.

**Proposition 9.** The operator $D : \mathcal{E}_A(H) \to C_b(H, H)$, $\varphi \mapsto D\varphi$, is uniquely extendible to a linear bounded operator $D : D(K_2) \to L^2(H, \nu; H)$. Moreover, (3) holds.

**Proof.** Let $\varphi \in \mathcal{E}_A(H)$. Taking into account Lemma 9 and that $\varphi^2 \in \mathcal{E}_A(H)$ by a simple computation we see that $K_2(\varphi^2) = 2\varphi K_2\varphi + |D\varphi|^2_2$. Integrating both sides over $H$ with respect to $\nu$ and taking into account that

\[ \int_H K_2(\varphi^2) d\nu = 0 \]

by the invariance of $P_t$ with respect to $\nu$, it follows (3). Now we will prove that (3) holds for all $\varphi \in D(K_2)$. Let us fix $\varphi \in D(K_2)$. Since $\mathcal{E}_A(H)$ is a core for $P_t$, there exists a sequence $\{\varphi_n\} \subset \mathcal{E}_A(H)$ such that $\varphi_n \to \varphi$, $K_2 \varphi_n \to K_2 \varphi$ in $L^2(H, \nu)$, and consequently

\[ \int_H |D(\varphi_n - \varphi_m)|^2 d\nu \leq 2 \int_H |\varphi_n - \varphi_m|^2 |K_2(\varphi_n - \varphi_m)|^2 d\nu \leq \|\varphi_n - \varphi_m\|^2_{L^2(H, \nu)} + \|K_2(\varphi_n - \varphi_m)\|^2_{L^2(H, \nu)} \]

Therefore the sequence $\{D\varphi_n\}$ is Cauchy in $L^2(H, \nu; H)$, and the conclusion follows.

11 The Sobolev space $W^{1,2}(H, \nu)$

We want to show that the mapping

\[ D : \mathcal{E}_A(H) \subset L^2(H, \nu) \to L^2(H, \nu; H), \varphi \mapsto D\varphi \]

is closable.

**Theorem 10.** $D$ is closable. Moreover, if $\varphi$ belongs to the domain $\overline{D}$ of the closure of $D$ and $\overline{D}\varphi = 0$ we have that $\overline{D}P_t \varphi = 0$ for any $t > 0$.

**Proof.** Since $\|R(\lambda, L)\|_{L^2(H, \nu)} \leq \sqrt{\lambda/t}$, $\lambda > 0$ and (8) holds, we can apply Proposition 3.5 of [9]. The theorem is proved.
We define by $W^{1,2}(H,\nu)$ the domain of $D$ in $L^2(H,\nu)$. By (3) it follows that $D(K_2) \subset W^{1,2}(H,\nu)$. We have the next

**Proposition 11.** Let $\varphi \in L^2(H,\nu)$ and $t \geq 0$. Set $u(t, x) = P_t \varphi(x)$. Then, for any $T > 0$, we have $u \in L^2(0, T; W^{1,2}(H,\nu))$ and the following identity holds

$$\int_H (P_t \varphi)^2 d\nu + \int_0^t ds \int_H |D P_s \varphi|^2 d\nu = \int_H \varphi^2 d\nu. \quad (45)$$

**Proof.** Let first $\varphi \in D(K_2)$. We have that $P_t \varphi \in D(K_2)$ and for all $t \geq 0$

$$\frac{d}{dt} P_t \varphi(x) = K_2 P_t \varphi(x).$$

Multiplying both sides of this identity by $P_t \varphi(x)$ and integrating with respect to $x$ over $H$, by (3) yields

$$\frac{1}{2} \frac{d}{dt} \int_H (P_t \varphi)^2 d\nu = \int_H P_t \varphi K_2 P_t \varphi d\nu - \frac{1}{2} \int_H |D P_t \varphi|^2 d\nu.$$

Integrating with respect to $t$ it yields (45). Now the conclusion follows by the density of $D(K_2)$ in $L^2(H,\nu)$. \qed

Letting $t \to \infty$ in (45) we have

**Proposition 12.** For any $\varphi \in L^2(H,\nu)$, we have

$$\int_H |\varphi - \bar{\varphi}|^2 d\nu = \int_0^\infty dt \int_H |D P_t \varphi|^2 d\nu, \quad (46)$$

where $\bar{\varphi} = \int_H \varphi d\nu$.

12 Poincaré and log-Sobolev inequality, spectral gap

As in (46) we will use the notation $\bar{\varphi} = \int_H \varphi d\nu$. Let us prove the Poincaré inequality.

**Theorem 13** (Poincaré inequality). Let us assume $\|DF\|_0 < 2$. Then for any $\varphi \in W^{1,2}(H,\nu)$ inequality (4) holds.
Proof. Let first $\varphi \in E(H)$. Then for any $h \in H$ and $t \geq 0$ we have
\[ \langle DP_t \varphi(x), h \rangle = \mathbb{E}[\langle D\varphi(X(t,x)), \eta^h(t,x) \rangle] \]
Taking into account (10) it follows that
\[ |\langle DP_t \varphi(x), h \rangle|^2 \leq \mathbb{E}[|D\varphi(X(t,x))|^2]|h|^2 \]
\[ \leq e^{2|\frac{\|DF\|_0^2}{4} - 1|t}\mathbb{E}[|D\varphi(X(t,x))|^2]|h|^2 = e^{2|\frac{\|DF\|_0^2}{4} - 1|t}P_t(|D\varphi|^2_2)(x)|h|^2. \]
By the arbitrariness of $h$ it yields
\[ |DP_t \varphi(x)|^2 \leq e^{2|\frac{\|DF\|_0^2}{4} - 1|t}P_t(|D\varphi|^2_2)(x) \]
for all $x \in H$, $s \geq 0$. Taking into account (46) and the invariance of $\nu$, we obtain
\[ \int_H |\varphi - \overline{\varphi}|^2 d\nu \leq \int_0^\infty dt \int_H e^{2|\frac{\|DF\|_0^2}{4} - 1|t}P_t(|D\varphi|^2_2)dv = \]
\[ = \frac{1}{2(1 - \frac{\|DF\|_0^2}{4})} \int_H |D\varphi|^2_2 d\nu \]
and the conclusion follows. If $\varphi \in W^{1,2}(H,\nu)$ we proceed by density.

Now we show that if (4) holds then there is a gap in the spectrum of $K_2$ and that the convergence to the equilibrium point is exponential. We have

**Theorem 14** (Spectral gap). *Let us assume $\|DF\|_0 < 2$. Then we have*
\[ \sigma(K_2) \setminus \{0\} \subset \{\lambda \in \mathbb{C} : \mathbb{R}{\lambda} \leq -(1 - \frac{\|DF\|_0^2}{4})\}. \]  
(47)
*Moreover*
\[ \int_H |P_t \varphi - \overline{\varphi}|^2 d\nu \leq e^{2(1 - \frac{\|DF\|_0^2}{4})t} \int_H |\varphi|^2 dv. \]  
(48)
*Proof.* The proof is an easy consequence of (3) and (4) (see [9] and [10, Prop. 2.3]).

**Theorem 15** (Log-Sobolev inequality). *Let us assume $\|DF\|_0 < 2$. Then or any $\varphi \in W^{1,2}(H,\nu)$ we have*
\[ \|\varphi^2 \log(\varphi^2)\|_{L^2(H,\nu)} \leq \frac{1}{1 - \frac{\|DF\|_0^2}{4}} \|D\varphi\|_{L^2(H,\nu)}^2 + \|\varphi^2\|_{L^2(H,\nu)} \log(\|\varphi^2\|_{L^2(H,\nu)}). \]  
(49)
*Moreover, the transition semigroup $\{P_t\}_{t \geq 0}$ is hypercontractive.*

Proof. Let us take $\varphi \in \mathcal{E}_A(H)$, with $\varphi \neq 0$. We have
\[
\frac{d}{dt} \int_H P_t(\varphi^2) \log(P_t(\varphi^2)) d\nu = \int_H K_2 P_t(\varphi^2) \log(P_t(\varphi^2)) d\nu + \int_H K_2 P_t(\varphi^2) d\nu,
\]
where the last term vanishes due to the invariance of $\nu$. Moreover, it holds the identity
\[
\int_H g'(\varphi) K_2 \varphi d\nu = -\frac{1}{2} \int_H g''(\varphi) |D\varphi|^2 d\nu.
\]
(50)

Since
\[
DP_t(\varphi^2) = 2E[\varphi(X(t,x)) D\varphi(X(t,x)) \cdot X_x(t,x)]
\]
it follows, from the Hölder inequality and (11) that
\[
|DP_t(\varphi^2)|^2 \leq 4E[|\varphi|^2(X(t,x))] \mathbb{E}[|D\varphi|^2(X(t,x))] e^{-2(1 - \|DF\|_0^2/4) t} = 4e^{-2(1 - \|DF\|_0^2/4) t} P_t(|\varphi|^2)(x) P_t(|D\varphi|^2)(x).
\]

Therefore, by (50) it yields
\[
\frac{d}{dt} \int_H P_t(\varphi^2) \log(P_t(\varphi^2)) d\nu \geq -2e^{-2(1 - \|DF\|_0^2/4) t} \int_H P_t(|D\varphi|^2)(x) d\nu = -2e^{-2(1 - \|DF\|_0^2/4) t} \int_H |D\varphi|^2(x) d\nu
\]
due to the invariance of $\nu$. Integrating with respect to $t$ gives
\[
\int_H P_t(\varphi^2) \log(P_t(\varphi^2)) d\nu \geq \int_H \varphi^2 \log(\varphi^2) d\nu - \frac{1 - e^{-2(1 - \|DF\|_0^2/4) t}}{2(1 - \|DF\|_0^2/4)} \int_H |D\varphi|^2(x) d\nu.
\]
Then the conclusion follows letting $t \to \infty$. Finally, as shown in [10], a logarithmic Sobolev inequality implies that the transition semigroup $\{P_t\}_{t \geq 0}$ is hypercontractive.

\[
\square
\]

References


