Fokker-Planck equation for Kolmogorov operators
with unbounded coefficients

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Abstract

Given a real and separable Hilbert space $H$ we consider the Fokker-Planck equation

$$\int_{H} \varphi(x) \mu(dx) - \int_{H} \varphi(x) \mu(dx) = \int_{0}^{t} \left( \int_{H} K_{0} \varphi(x) \mu_{s}(dx) \right) ds,$$

where $K_{0}$ is the Kolmogorov differential operator

$$K_{0} \varphi(x) = \frac{1}{2} \text{Trace} [BB^{*}D^{2} \varphi(x)] + \langle x, A^{*}D\varphi(x) \rangle + \langle D\varphi(x), F(x) \rangle,$$

$x \in H$, $\varphi : H \to \mathbb{R}$ is a suitable smooth function, $A : D(A) \subset H \to H$ is linear, $F : H \to H$ is a globally Lipschitz function and $B : H \to H$ is linear and continuous. In order prove existence and uniqueness of a solution for the above equation, we show that $K_{0}$ can be extended, in a suitable way, of the infinitesimal generator associated to the solution of a certain stochastic differential equation in $H$.

Key words: Reaction-diffusion equation, Kolmogorov operator, transition semigroup, Fokker-Planck equation.

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1 Introduction

Let $H$ be a separable real Hilbert space (with norm $|\cdot|$ and inner product $\langle \cdot, \cdot \rangle$), and let $B(H)$ be its Borel $\sigma$-algebra. $\mathcal{L}(H)$ denotes the usual Banach space of all linear and continuous operators in $H$, endowed with the supremum norm $\| \cdot \|_{\mathcal{L}(H)}$. We consider the stochastic differential equation in $H$

$$\begin{cases} dX(t) = (AX(t) + F(X(t)))dt + BdW(t), & t \geq 0 \\ X(0) = x \in H, \end{cases}$$

where

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**Hypothesis 1.1.**

(i) \( A : D(A) \subseteq H \to H \) is the infinitesimal generator of a strongly continuous semigroup \( e^{tA} \) of type \( G(M, \omega) \), i.e. there exist \( M \geq 0 \) and \( \omega \in \mathbb{R} \) such that \( \| e^{tA} \|_{L(H)} \leq Me^{\omega t}, \ t \geq 0 \);

(ii) \( B \in \mathcal{L}(H) \) and for any \( t > 0 \) the linear operator \( Q_t \), defined by

\[
Q_t x = \int_0^t e^{sA} B B^* e^{sA^*} x \, ds, \quad x \in H, \ t \geq 0
\]

has finite trace;

(iii) \( F : H \to H \) is a Lipschitz continuous map with Lipschitz constant equal to \( \kappa > 0 \);

(iv) \( (W(t))_{t \geq 0} \) is a cylindrical Wiener process, defined on a stochastic basis \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) and with values in \( H \).

It is well known that under hypothesis (1.1) problem (1) has a unique mild solution \((X(t, x))_{t \geq 0, x \in H}\) (see, for instance, [7]), that is for any \( x \in H \) the process \((X(t, x))_{t \geq 0}\) is adapted to the filtration \((\mathcal{F}_t)_{t \geq 0}\), it is continuous in mean square and it fulfills the integral equation

\[
X(t, x) = e^{tA} x + \int_0^t e^{(t-s)A} B dW(s) + \int_0^t e^{(t-s)A} F(X(s, x)) ds
\]

for any \( t \geq 0 \). Moreover, a straightforward computation shows that for any \( T > 0 \) there exists \( c > 0 \) such that

\[
\sup_{t \in [0, T]} |X(t, x) - X(t, y)| \leq c|x - y|, \quad \forall x, y \in H,
\]

and

\[
\sup_{t \in [0, T]} \mathbb{E}[|X(t, x)|] \leq c(1 + |x|), \quad x \in H,
\]

where the expectation is taken with respect to \( \mathbb{P} \) (see, for instance, [6, Proposition 3.3]). Now denote by \( C_b(H) \) the Banach space of all uniformly continuous and bounded functions \( \varphi : H \to \mathbb{R} \), endowed with the supremum norm \( \| \varphi \|_0 = \sup_{x \in H} |\varphi(x)|, \varphi \in C_b(H) \). Moreover, for any \( k > 0 \), let \( C_{b,k}(H) \) be the space of all functions \( \varphi : H \to \mathbb{R} \) such that the function \( H \to \mathbb{R}, x \mapsto (1 + |x|^k)^{-1} \varphi(x) \) belongs to \( C_b(H) \). The space \( C_{b,k}(H) \) is a Banach space, endowed with the norm \( \| \varphi \|_{b,k} = \| (1 + |\cdot|^k)^{-1} \varphi \|_0 \). In the following, we shall denote by \( (C_{b,k}(H))^* \) the topological dual space of \( C_{b,k}(H) \). As we shall see in Proposition 2.2, estimates (3), (4) allow us to define the transition operator associated to equation (2) in the space \( C_{b,1}(H) \), by the formula

\[
P_t \varphi(x) = \mathbb{E}[\varphi(X(t, x))], \quad \varphi \in C_{b,1}(H), \ t \geq 0, \ x \in H.
\]

The family of operators \( P_t, t \geq 0 \) maps \( C_{b,1}(H) \) into \( C_{b,1}(H) \) and enjoys the semigroup property, but it is not a strongly continuous semigroup (cf Proposition 2.2). However, we can define the infinitesimal generator of \( P_t, t \geq 0 \) in
We stress the fact that the operator $K$ is an extension of the operator $K_0$. Under Hypothesis (1.1), the operator $(K, D(K))$ is an extension of $K_0$, and for any $\varphi \in \mathcal{E}_A(H)$ we have $\varphi \in D(K)$ and $K \varphi = K_0 \varphi$. Finally, $\mathcal{E}_A(H)$ is a $\pi$-core for $(K, D(K))$.

As a consequence we have the third main result
Theorem 1.4. For any \( \mu \in \mathcal{M}_1(H) \) there exists an unique family of measures \( \{\mu_t, t \geq 0\} \subset \mathcal{M}_1(H) \) fulfilling (7) and the Fokker-Planck equation

\[
\int_H \varphi(x)\mu_t(dx) - \int_H \varphi(x)\mu(dx) = \int_0^t \left( \int_H K_0\varphi(x)\mu_s(dx) \right) ds, \tag{10}
\]

\( t \geq 0, \varphi \in \mathcal{E}_A(H), \) and the solution is given by \( \mu^*_t \mu t \geq 0. \)

In [9] a similar problem when \( F : H \to H \) is Lipschitz continuous and bounded has been investigated. Due to the fact that the nonlinearity is bounded, all the results are stated in the space \( C_0(H) \). In our paper we deal with unbounded nonlinearities and we need to develop a notion of semigroup and associated infinitesimal generator in the weighted space \( C_{0,1}(H) \).

The motivation of this work is to have a better understanding on the relationships between the stochastic differential equation (1) and the Kolmogorov equations for measures in finite dimension have been the object of several papers. We recall that in the papers [3] have been stated sufficient conditions in order to ensure existence of a weak solution for parabolic differential operators of the form

\[
Lu(t, x) = a^{ij}(t, x)\partial_{x_i}\partial_{x_j}u(t, x) + b^i(t, x)\partial_{x_i}u(t, x), \quad (t, x) \in (0, 1) \times \mathbb{R}^d
\]

where \( u \in C_0^\infty((0, 1) \times \mathbb{R}^d) \) and \( a^{ij}, b^i : (0, 1) \times \mathbb{R}^d \to \mathbb{R}, 1 \leq i, j \leq d \) are suitable locally integrable functions. With similar techniques, in [2] the problem is studied for parabolic differential operators of the form \( Lu(t, x) = u_t(t, x) + Hu(t, x), u \in C_0^\infty((0, 1) \times \mathbb{R}^d) \). The infinite dimensional framework has been investigated in [4], where it is considered an equation for measures of the form

\[
\int_X L_{A,B}\psi(x)\mu(dx) = 0, \quad \forall \psi \in \mathcal{K},
\]

where \( X \) is a locally convex space, \( \mathcal{K} \) is a suitable set of cylindrical functions and \( L_{A,B} \) is formally given by

\[
L_{A,B}\psi(x) = \sum_{i,j=1}^\infty A_{i,j}\partial_{e_i}\partial_{e_j}\psi(x) + \sum_{i=1}^\infty B_i\partial_{e_i}\psi(x),
\]

with \( \mu \)-measurable functions \( A_{i,j}, B_i \) and vectors \( e_i \in X \). Under some integrability assumptions on \( A_{i,j}, B_i \), the authors prove existence of a measure \( \mu \), possibly infinite, satisfying the above equation.

It is worthwhile to mention the papers [10] and [12], where the problem of extending a second order differential operator related to a Burgers stochastic differential equation is considered.
We stress that in our paper we prove uniqueness results.

In the recent papers [1], it has been proved uniqueness of the Fokker-Planck equation for Kolmogorov operators like $K_0$ with a time dependent quasi-dissipative nonlinear term. Due to the time dependent drift, the solution of the SPDE defines a semigroup of operator in the space $C_T([0,T]; C_b(H))$ (that is the space of all the continuous functions $f : [0,T] \to C_b(H)$ such that $f(T) = 0$). On the other side, the linear operators $A, Q$ have to enjoy stronger properties in order to have a.s. path continuity for the solution of the SPDE.

The paper is organized as follows: in the next section we introduce the notions of $\pi$-convergence and we prove some results about the transition semigroup (5) in the space $C_{b,1}(H)$. Sections 3, 4, 5 concern proofs of Theorems 1.2, 1.3, 1.4, respectively.

2 Notations and preliminary results

If $E, E'$ are, respectively, a topological space and a Banach space with norm $| \cdot |_{E'}$, we denote by $C_b(E, E')$ the Banach space of the bounded continuous function $\varphi : E \to E'$ endowed with the supremum norm

$$\| \varphi \|_{C_b(E, E')} := \sup_{x \in E} |\varphi(x)|_{E'}.$$ 

When $E' = \mathbb{R}$ we write $C_b(E, \mathbb{R})$. If $E = H$, we simply denote by $\| \cdot \|_0$ the supremum norm of $C_b(H)$. We also denote by $C_{b,1}(H)$ the Banach space of all continuous functions $f : H \to \mathbb{R}$ such that

$$\| f \|_{0,1} := \|(1 + | \cdot |)^{-1} f\|_0 < \infty.$$ 

$C_{b,1}(H)$ denotes the space of all the functions $f \in C_b(H)$ which are Fréchet differentiable with uniformly continuous and bounded differential $DF \in C_b(H; L(H; E))$.

We deal with semigroups of operators which are not, in general, strongly continuous. For this reason, we introduce the notion of $\pi$-convergence in the space $C_b(E)$ (see [9], [11]).

Definition 2.1. A sequence $\{ \varphi_n \}_{n \in \mathbb{N}} \subset C_b(E)$ is said to be $\pi$-convergent to a function $\varphi \in C_b(E)$ if

$$\lim_{n \to \infty} \varphi_n(x) = \varphi(x), \quad \forall x \in E$$

and

$$\sup_{n \in \mathbb{N}} \| \varphi_n \|_{C_b(E)} < \infty.$$ 

Similarly, the $m$-indexed sequence $\{ \varphi_{n_1, \ldots, n_m} \}_{n_1 \in \mathbb{N}, \ldots, n_m \in \mathbb{N}} \subset C_b(E)$ is said to be $\pi$-convergent to $\varphi \in C_b(E)$ if for any $i \in \{1, \ldots, m-1\}$ there exists an $i$-indexed sequence $\{ \varphi_{n_1, \ldots, n_i} \}_{n_1 \in \mathbb{N}, \ldots, n_i \in \mathbb{N}} \subset C_b(E)$ such that

$$\lim_{n_{i+1} \to \infty} \varphi_{n_1, \ldots, n_{i+1}} \equiv \varphi_{n_1, \ldots, n_i}, \quad i \in \{1, \ldots, m-1\}$$

and

$$\lim_{n_1 \to \infty} \varphi_{n_1} \equiv \varphi.$$
We shall write
\[
\lim_{n_1 \to \infty} \cdots \lim_{n_m \to \infty} \varphi_{n_1, \ldots, n_m} \xrightarrow{\pi} \varphi
\]
or \(\varphi_n \xrightarrow{\pi} \varphi\) as \(n \to \infty\), when the sequence has one index.

Note that since the convergence is pointwise we can not take a diagonal sequence. However, in order to avoid heavy notations, we shall often assume that the sequence has one index.

As easily seen the \(\pi\)-convergence implies the convergence in \(L^p(E; \mu)\), for any \(p \in [1, \infty)\) and any Borel finite measure \(\mu\) on \(E\).

Let \(k > 0\). We shall often use the fact that if for a sequence \((\varphi_n)_{n \in \mathbb{N}} \subset C_{b,k}(H)\) we have that \((1 + |x|^k)^{-1} \varphi_n \xrightarrow{\pi} \varphi \in C_{b,k}(H)\) as \(n \to \infty\), then the sequence converges to \(\varphi\) in \(L^p(H; \mu)\), for any \(\mu \in \mathcal{M}_b(H)\), \(p \in [1, \infty)\). This argument may be viewed as an extension of the \(\pi\)-convergence to the spaces \(C_{b,k}(H)\).

In Theorem 1.3 we claim that \(\mathcal{E}_A(H)\) is a \(\pi\)-core for \((K, D(K))\). This means that if \(\varphi \in \mathcal{E}_A(H)\) we have \(\varphi \in D(K)\) and \(K \varphi = K_0 \varphi\). In addition, if \(\varphi \in D(K)\), there exist \(m \in \mathbb{N}\) and an \(m\)-indexed sequence \((\varphi_{n_1, \ldots, n_m})_{n_1, \ldots, n_m \in \mathbb{N}} \subset \mathcal{E}_A(H)\) such that

\[
\lim_{n_1 \to \infty} \cdots \lim_{n_m \to \infty} \varphi_{n_1, \ldots, n_m} \xrightarrow{\pi} \varphi_{\cdot, \ldots, \cdot}, \quad \lim_{n_1 \to \infty} \cdots \lim_{n_m \to \infty} K_0 \varphi_{n_1, \ldots, n_m} \xrightarrow{\pi} K \varphi_{\cdot, \ldots, \cdot}.
\]

The construction of such a sequence is detailed in section 4.

2.1 The transition semigroup in \(C_{b,1}(H)\)

This section is devoted to study the semigroup \(P_t, t \geq 0\) in the space \(C_{b,1}(H)\).

**Proposition 2.2.** Formula (5) defines a semigroup of operators \(P_t, t \geq 0\) in \(C_{b,1}(H)\), and there exist a family of probability measures \(\{\pi_t(x, \cdot), t \geq 0, x \in H\} \subset \mathcal{M}_1(H)\) and two constants \(c_0 > 0, \omega_0 \in \mathbb{R}\) such that

(i) \(P_t \in \mathcal{L}(C_{b,1}(H))\) and \(\|P_t\|_{\mathcal{L}(C_{b,1}(H))} \leq c_0 e^{\omega_0 t}\);

(ii) \(P_t \varphi(x) = \int_H \varphi(y) \pi_t(x, dy), \) for any \(t \geq 0, \varphi \in C_{b,1}(H), x \in H\);

(iii) for any \(\varphi \in C_{b,1}(H), x \in H, \) the function \(\mathbb{R}^+ \to \mathbb{R}, t \mapsto P_t \varphi(x)\) is continuous.

(iv) \(P_t P_s = P_{t+s}, \) for any \(t, s \geq 0\) and \(P_0 = I;\)

(v) for any \(\varphi \in C_{b,1}(H)\) and any sequence \((\varphi_n)_{n \in \mathbb{N}} \subset C_{b,1}(H)\) such that

\[
\lim_{n \to \infty} \frac{\varphi_n}{1 + |\cdot|} \xrightarrow{\pi} \frac{\varphi}{1 + |\cdot|}
\]

we have, for any \(t \geq 0,\)

\[
\lim_{n \to \infty} \frac{P_t \varphi_n}{1 + |\cdot|} \xrightarrow{\pi} \frac{P_t \varphi}{1 + |\cdot|}.
\]
Proof. (i). Take $\varphi \in C_{0,1}(H)$, $t \geq 0$. We have to show that $P_t \varphi \in C_{0,1}(H)$, that is the function $x \mapsto (1 + |x|)^{-1}P_t \varphi(x)$ is uniformly continuous and bounded. Take $\varepsilon > 0$ and let $\theta_{\varphi} : \mathbb{R}^+ \to \mathbb{R}$ be the modulus of continuity of $(1 + |\cdot|)^{-1} \varphi$. We have
\[
\frac{P_t \varphi(x)}{1 + |x|} - \frac{P_t \varphi(y)}{1 + |y|} = I_1(t, x, y) + I_2(t, x, y) + I_3(t, x, y),
\]
where
\[
I_1(t, x, y) = \mathbb{E} \left[ \frac{\varphi(X(t, x))}{1 + |X(t, x)|} - \frac{\varphi(X(t, y))}{1 + |X(t, y)|} \right],
\]
\[
I_2(t, x, y) = \mathbb{E} \left[ \frac{\varphi(X(t, y))}{1 + |X(t, y)|} \left( \frac{|X(t, x)| - |X(t, y)|}{1 + |x|} \right) \right],
\]
\[
I_3(t, x, y) = \mathbb{E} \left[ \frac{\varphi(X(t, y))}{1 + |X(t, y)|} \left( \frac{1}{1 + |x|} - \frac{1}{1 + |y|} \right) \right].
\]
For $I_1(t, x, y)$ we have, by taking into account (3), (4), that there exists $c > 0$ such that
\[
|I_1(t, x, y)| \leq \mathbb{E} \left[ \theta_{\varphi}(X(t, x) - X(t, y)) \frac{1 + |X(t, x)|}{1 + |x|} \right] \leq \theta_{\varphi}(c|x - y|) \mathbb{E} \left[ \frac{1 + |X(t, x)|}{1 + |x|} \right] \leq c \theta_{\varphi}(c|x - y|).
\]
Then, there exists $\delta_1 > 0$ such that $|I_1(t, x, y)| \leq \varepsilon/3$, for any $x, y \in H$ such that $|x - y| \leq \delta_1$. For $I_2(t, x, y)$ we have, by elementary inequalities,
\[
|I_2(t, x, y)| \leq \frac{||\varphi||_{0,1}}{1 + |x|} \mathbb{E} \left[ |X(t, x) - X(t, y)| \right]
\leq \frac{||\varphi||_{0,1}}{1 + |x|} \mathbb{E} \left[ |X(t, x)| - |X(t, y)| \right] \leq ||\varphi||_{0,1} c|x - y|.
\]
Then there exists $\delta_2 > 0$ such that $|I_2(t, x, y)| \leq \varepsilon/3$, for any $x, y \in H$ such that $|x - y| \leq \delta_2$. Similarly, for $I_3(t, x, y)$ we have
\[
|I_3(t, x, y)| \leq ||\varphi||_{0,1} \frac{1 + \mathbb{E}[|X(t, x)|]}{1 + |x|} \frac{|x| - |y|}{1 + |y|}
\leq c ||\varphi||_{0,1} (1 + c)|x - y|.
\]
for some $c > 0$. Then, there exists $\delta_3 > 0$ such that $|I_3(t, x, y)| \leq \varepsilon/3$, for any $x, y \in H$ such that $|x - y| \leq \delta_3$. Finally, for any $x, y \in H$ with $|x - y| \leq \min\{\delta_1, \delta_2, \delta_3\}$ we find that
\[
\frac{P_t \varphi(x)}{1 + |x|} - \frac{P_t \varphi(y)}{1 + |y|} < \varepsilon
\]
as claimed. Now, by taking into account (4), there exists $c > 0$ such that
\[
|\frac{P_t \varphi(x)}{1 + |x|}| \leq ||\varphi||_{0,1} \frac{1 + \mathbb{E}[|X(t, x)|]}{1 + |x|} \leq c ||\varphi||_{0,1}
\]
Then $P_t \varphi \in C_{0,1}(H)$. Note that by (4) it follows that the operators $P_t$ are bounded in a neighborhood of 0. Hence, the existence of the two constants $c_0 > 0$, $\omega_0 \in \mathbb{R}$ follows by (iv) and by a standard argument. Notice that by the same argument follows\(^1\) (v).
\(^1\)Of course, to prove (iv)-(v) we do not use this statement of (i)
(ii). Take \( \varphi \in C_{b,1}(H) \), and consider a sequence \( (\varphi_n)_{n \in \mathbb{N}} \subset C_b(H) \) such that
\[
\lim_{n \to \infty} \frac{\varphi_n}{1 + |\cdot|} = \frac{\varphi}{1 + |\cdot|}.
\]
(11)

Since \( \pi_t(t,\cdot) \) is the image measure of \( X(t,x) \) in \( H \), the representation (ii) holds for any \( \varphi_n \), that is
\[
P_t\varphi_n(x) = \mathbb{E}[\varphi_n(X(t,x))] = \int_H \varphi_n(y)\pi_t(dy).
\]

Since (4) holds we have \( \pi(x,\cdot) \in \mathcal{M}_1(H) \), and by (11) there exists \( c > 0 \) such that \( |\varphi_n(x)| \leq c(1 + |x|) \), for any \( n \in \mathbb{N} \), \( x \in H \). Finally, the result follows by the dominated convergence theorem.

(iii). For any \( \varphi \in C_{b,1}(H) \), \( x \in H \), \( t, s \geq 0 \) we have
\[
P_t\varphi(x) - P_s\varphi(x) = \mathbb{E}\left[ \frac{\varphi(X(t,x))}{1 + |X(t,x)|} - \frac{\varphi(X(s,x))}{1 + |X(s,x)|} (1 + |X(t,x)|) \right]
+ \mathbb{E}\left[ \frac{\varphi(X(s,x))}{1 + |X(s,x)|} \left( |X(t,x)| - |X(s,x)| \right) \right].
\]

Then
\[
|P_t\varphi(x) - P_s\varphi(x)| \leq \mathbb{E}[\theta_{\varphi}(|X(t,x) - X(s,x)|)] (1 + |X(t,x)|) + \|\varphi\|_{0.1}\mathbb{E}[|X(t,x) - X(s,x)|],
\]
(12)

where \( \theta_{\varphi} : \mathbb{R}^+ \to \mathbb{R} \) is the modulus of continuity of \( (1 + |\cdot|)^{-1}\varphi \). Note also that since for any \( x \in H \) the process \( (X(t,x))_{t \geq 0} \) is continuous in mean square, we have
\[
\lim_{t \to s} |X(t,x) - X(s,x)| = 0 \quad \mathbb{P}\text{-a.s.}
\]

Hence, by taking into account that \( \theta_{\varphi} : \mathbb{R}^+ \to \mathbb{R} \) is bounded and that (4) holds, we can apply the dominated convergence theorem to show that the first term in the right-hand side of (12) vanishes as \( t \to s \). Finally, the fact that the second term in the right-hand side of (12) vanishes as \( t \to s \) may be proved by the same argument.

(iv). Take \( \varphi \in C_{b,1}(H) \), and consider a sequence \( (\varphi_n)_{n \in \mathbb{N}} \subset C_b(H) \) such that
\[
(1 + |\cdot|)^{-1}\varphi_n \overset{p}{\to} (1 + |\cdot|)^{-1}\varphi \quad \text{as} \quad n \to \infty.
\]

By the markovianity of the process \( X(t,x) \) it follows that (iv) holds true for any \( \varphi_n \). Then, by (iii) it follows that \( (1 + |\cdot|)^{-1}P_t\varphi_n \overset{p}{\to} (1 + |\cdot|)^{-1}P_t\varphi \) as \( n \to \infty \), still by (iii) we find
\[
\frac{P_t\varphi_n}{1 + |\cdot|} = \lim_{n \to \infty} \frac{P_t\varphi_n}{1 + |\cdot|} = \lim_{n \to \infty} \frac{P_tP_s\varphi_n}{1 + |\cdot|} = \frac{P_tP_s\varphi}{1 + |\cdot|}.
\]

This concludes the proof. \( \square \)

**Remark 2.3.** We recall that for any \( k > 0 \), \( T > 0 \) there exists \( c_k > 0 \) such that
\[
\sup_{t \in [0,T]} \mathbb{E}[|X(t,x)|^k] \leq c_k(1 + |x|^k),
\]

that implies \( \{\pi_t(x,\cdot), t \geq 0, x \in H\} \subset \bigcap_{k \geq 0} \mathcal{M}_k(H) \). Consequently, all the results of this section are true with \( C_{b,k}(H) \) replacing \( C_{b,1}(H) \).
Here we collect some useful properties of the infinitesimal generator \((K, D(K))\).

**Proposition 2.4.** Let \(X(t, x)\) be the mild solution of problem (1) and let \(P_t, t \geq 0\) be the associated transition semigroups in the space \(C_{b,1}(H)\) defined by (5). Let also \((K, D(K))\) be the associated infinitesimal generators, defined by (6). Then

(i) for any \(\varphi \in D(K)\), we have \(P_t\varphi \in D(K)\) and \(KP_t\varphi = P_tK\varphi, t \geq 0;\)

(ii) for any \(\varphi \in D(K)\), \(x \in H\), the map \([0, \infty) \to \mathbb{R}, t \mapsto P_t\varphi(x)\) is continuously differentiable and \((d/dt)P_t\varphi(x) = P_tK\varphi(x);\)

(iii) given \(c_0 > 0\) and \(\omega_0\) as in Proposition 2.2, for any \(\lambda > \omega_0\) the linear operator \(R(\lambda, K)\) on \(C_{b,1}(H)\) done by

\[
R(\lambda, K)f(x) = \int_0^\infty e^{-\lambda t}P_t f(x)dt, \quad f \in C_{b,1}(H), x \in H
\]

satisfies, for any \(f \in C_{b,1}(H)\)

\[
R(\lambda, K) \in \mathcal{L}(C_{b,1}(H)), \quad \|R(\lambda, K)\|_{\mathcal{L}(C_{b,1}(H))} \leq \frac{c_0}{\lambda - \omega_0}
\]

\[
R(\lambda, K)f \in D(K), \quad (\lambda I - K)R(\lambda, K)f = f.
\]

We call \(R(\lambda, K)\) the resolvent of \(K\) at \(\lambda\).

**Proof.** (i). It is proved by taking into account (6) and (iii) of Proposition 2.2.

(ii). This follows easily by (i) and by (iii) of Proposition 2.2.

(iii). By (i) of Proposition 2.2 we have

\[
\left\|\int_0^\infty e^{-\lambda t}P_t f dt\right\|_{0,1} \leq c_0 \int_0^\infty e^{-\lambda t} dt \|f\|_{0,1} = \frac{c_0 \|f\|_{0,1}}{\lambda - \omega_0}.
\]

Finally, the fact that \(R(\lambda, K)f \in D(K)\) and \((\lambda I - K)R(\lambda, K)f = f\) hold can be proved in a standard way (see, for instance, [5], [11]).

### 3 Proof of Theorem 1.2

In order to prove this theorem, we need some results about the transition semigroup \(P_t, t \geq 0\) in the space \(C_b(H)\). Denote by \(\pi_t(x, \cdot)\) the image measure of the mild solution \(X(t, x)\) of problem (1). Since for any \(\varphi \in C_b(H)\) the representation

\[
P_t\varphi(x) = \int_\mathbb{R} \varphi(y)\pi_t(x, dy), \quad x \in H, t \geq 0
\]

holds (cf (ii) of Proposition 2.2) and \(X(t, x)\) is continuous in mean square, it follows easily that \(P_t, t \geq 0\) is a semigroup of contraction operators in the space \(C_b(H)\). Moreover, we have that for any \(x \in H, \varphi \in C_b(H)\) the function \(\mathbb{R}^+ \to \mathbb{R}, t \mapsto P_t\varphi(x)\) is continuous (cf (iii) of Proposition 2.2). This means that \(P_t, t \geq 0\) is stochastically continuous Markov semigroup, in the sense introduced in [9].
We denote by \( (K, D(K, C_b(H))) \) the infinitesimal generator of \( P_t \) is the space \( C_b(H) \), defined by

\[
\left\{ \begin{array}{l}
D(K, C_b(H)) = \left\{ \varphi \in C_b(H) : \exists g \in C_b(H), \lim_{t \to 0^+} \frac{P_t \varphi(x) - \varphi(x)}{t} = g(x), \right. \\
x \in H, \sup_{t \in (0, 1)} \left\| \frac{P_t \varphi - \varphi}{t} \right\|_0 < \infty \left. \right\}\right. \\
K \varphi(x) = \lim_{t \to 0^+} \frac{P_t \varphi(x) - \varphi(x)}{t}, \quad \varphi \in D(K, C_b(H)), x \in H. \tag{13}
\]

It is clear that \( D(K, C_b(H)) \subset D(K) \). The key result we use to prove the Theorem is the following, proved in [9]

**Theorem 3.1.** For any \( \mu \in \mathcal{M}(H) \) there exists a unique family of measures \( \{\mu_t, t \geq 0\} \subset \mathcal{M}(H) \) such that

\[
\int_0^T |\mu_t|(H)dt < \infty, \quad \forall T > 0 \tag{14}
\]

and

\[
\int_H \varphi(x)\mu_t(dx) - \int_H \varphi(x)\mu(dx) = \int_0^t \left( \int_H K\varphi(x)\mu_s(dx) \right) ds \tag{15}
\]

holds for any \( t \geq 0, \varphi \in D(K, C_b(H)) \).

We split the proof into two lemmata.

**Lemma 3.2.** The formula

\[
\langle \varphi, P^*_t F \rangle_{\sigma(C_{b,1}(H), (C_{b,1}(H))^*)} = \langle P_t \varphi, F \rangle_{\sigma(C_{b,1}(H), (C_{b,1}(H))^*)} \tag{16}
\]

defines a semigroup of linear operators in \( (C_{b,1}(H))^* \). Finally, \( P^*_t : \mathcal{M}_1(H) \to \mathcal{M}_1(H) \) and it maps positive measures into positive measures.

**Proof.** This lemma can be proved by the same argument of Lemma 3.1 in [9]. The fact that \( P^*_t : \mathcal{M}_1(H) \to \mathcal{M}_1(H) \) follows by Proposition 2.2 and by using the dominated convergence theorem. \( \square \)

**Lemma 3.3.** For any \( \mu \in \mathcal{M}_1(H) \) there exists a unique family of finite measures \( \{\mu_t, t \geq 0\} \subset \mathcal{M}_1(H) \) fulfilling (7), (8), and this family is given by \( P^*_t \mu, t \geq 0 \).

**Proof.** We first check that \( P^*_t \mu, t \geq 0 \) satisfies (7), (8). By Proposition 3.2, for any \( \mu \in \mathcal{M}_1(H) \), formula (16) defines a family \( P^*_t \mu, t \geq 0 \) of measures on \( H \). Moreover, by (i) of Proposition 2.2 it follows that for any \( T > 0 \) it holds

\[
\sup_{t \in [0,T]} \|P^*_t \mu\|_{(C_{b,1}(H))^*} = \sup_{t \in [0,T]} \int_H (1 + |x|)\|P^*_t \mu\|(dx) < \infty.
\]

Hence, (7) holds. We now show (8). By (i), (ii), (iv) of Proposition 2.2 and by the dominated convergence theorem it follows easily that for any \( \varphi \in C_{b,1}(H) \) the function

\[
\mathbb{R}^+ \to \mathbb{R}, \quad t \mapsto \int_H \varphi(x) P^*_t \mu(dx) \tag{17}
\]
is continuous. Clearly, $P_0^t \mu = \mu$. Now we show that if $\varphi \in D(K)$ then the function (17) is differentiable. Indeed, by taking into account (6) and (i) of Proposition 2.4, for any $\varphi \in D(K)$ we can apply the dominated convergence theorem to obtain

$$
\frac{d}{dt} \int_H \varphi(x) P_t^* \mu(dx) =
\lim_{h \to 0} \frac{1}{h} \left( \int_H P_{t+h} \varphi(x) \mu(dx) - \int_H P_t \varphi(x) \mu(dx) \right)
= \lim_{h \to 0} \int_H \left( \frac{P_{t+h} \varphi(x) - P_t \varphi(x)}{h} \right) \mu(dx)
= \lim_{h \to 0} \int_H \left( \frac{P_t \varphi(x) - \varphi(x)}{h} \right) \mu(dx)
= \int_H \lim_{h \to 0} \left( \frac{P_t \varphi(x) - \varphi(x)}{h} \right) P_t^* \mu(dx) = \int_H K \varphi(x) P_t^* \mu(dx).
$$

Then, by arguing as above, it follows that the differential of the mapping defined by (17) is continuous. This clearly implies that $P_t^* \mu$, $t \geq 0$ satisfies (8). In order to show uniqueness of such a solution, by the linearity of the problem it is sufficient to show that if $\mu = 0$ and $\{ \mu_t, t \geq 0 \} \subset M_1(H)$ is a solution of equation (8), then $\mu_t = 0$, for any $t \geq 0$. Note that equation (8) holds in particular for $\varphi \in D(K, D(K))$ (cf (13)) and consequently (15) holds, for any $\varphi \in D(K, D(K))$. Note also that by (7) follows that $\{ \mu_t, t \geq 0 \}$ fulfils (14). Hence, by Theorem 3.1, it follows that $\mu_t = 0$, $\forall t \geq 0$. This concludes the proof.

## 4 Proof of Theorem 1.3

We split the proof in several steps. We start by studying the Ornstein-Uhlenbeck operator in $C_{b,1}(H)$ that is, roughly speaking, the case $F = 0$ in (9). In Proposition 4.3 we shall prove Theorem 1.4 in the case $F = 0$. Then, Corollary 4.4 will show that $(K, D(K_0))$ is an extension of $K_0$ and $K \varphi = K_0 \varphi$ for any $\varphi \in \mathcal{E}_A(H)$. In order to complete the proof of the theorem, we shall present several approximation results. Finally, Lemma 4.6 will complete the proof.

### 4.1 The Ornstein-Uhlenbeck semigroup in $C_{b,1}(H)$

An important role in what follows it is played by the Ornstein-Uhlenbeck semigroup $R_t$, $t \geq 0$ in the space $C_{b,1}(H)$, defined by the formula

$$
R_t \varphi(x) = \begin{cases} 
\varphi(x), & t = 0, \\
\int_H \varphi(e^{tA}x + y) N_Q_1(dy), & t > 0
\end{cases}
$$

where $\varphi \in C_{b,1}(H)$, $x \in H$ and $N_Q_1$ is the gaussian measure of zero mean and covariance operator $Q_1$ (cf Hypothesis 1.1). It is well known that the representation

$$
R_t \varphi(x) = \mathbb{E} \left[ \varphi \left( e^{tA}x + \int_0^t e^{(t-s)A} B dW(s) \right) \right]
$$

(18)
Proof. Let \( x \in C_b,1(H), t \geq 0 \). Hence, the Ornstein-Uhlenbeck semigroup \( R_t, t \geq 0 \) is the transition semigroup (5) in the case \( F = 0 \). Consequently, \( R_t, t \geq 0 \) satisfies statements (i)–(v) of Proposition 2.2. It is well known the following identity

\[
R_t(e^{i\langle \cdot, h \rangle})(x) = e^{i\langle e^{tA}x, h \rangle - \frac{1}{2}\langle Q_t h, h \rangle},
\]  

which implies \( R_t : \mathcal{E}_A(H) \to \mathcal{E}_A(H) \), for any \( t \geq 0 \). We define the infinitesimal generator \( L : D(L) \to C_b,1(H) \) of \( R_t, t \geq 0 \) in \( C_b,1(H) \) as in (6), with \( L \) replacing \( K \) and \( R_t \) replacing \( P_t \).

**Theorem 4.1.** Let \( P_t, t \geq 0 \) be the semigroup (5) and let \( R_t, t \geq 0 \) be the Ornstein-Uhlenbeck semigroup (18). We denote by \( (K, D(K)), (L, D(L)) \) the corresponding infinitesimal generators in \( C_b,1(H) \). Then we have \( D(L) \cap C_b^1(H) = D(K) \cap C_b^1(H) \) and \( K\varphi = L\varphi + (D\varphi, F) \), for any \( \varphi \in D(L) \cap C_b^1(H) \).

**Proof.** Let \( X(t, x) \) be the mild solution of equation (1) and set

\[
Z_A(t, x) = e^{tA}x + \int_0^t e^{(t-s)A}BdW(s).
\]

Take \( \varphi \in D(L) \cap C_b^1(H) \). By taking into account that

\[
X(t, x) = Z_A(t, x) + \int_0^t e^{(t-s)A}F(X(s, x))ds,
\]

by the Taylor formula we have that \( \mathbb{P}\text{-a.s.} \) it holds

\[
\varphi(Z_A(t, x)) = \varphi(Z_A(t, x)) - \varphi(X(t, x)) + \varphi(X(t, x)) = \varphi(X(t, x))
\]

\[
- \int_0^1 \left\langle D\varphi(\xi Z_A(t, x) + (1 - \xi)X(t, x)), \int_0^t e^{(t-s)A}F(X(s, x))ds \right\rangle d\xi.
\]

Then we have

\[
R_t\varphi(x) - \varphi(x) = \mathbb{E}[\varphi(Z_A(t, x))] - \varphi(x) = P_t\varphi(x) - \varphi(x)
\]

\[
-\mathbb{E} \left[ \int_0^1 \left\langle D\varphi(\xi Z_A(t, x) + (1 - \xi)X(t, x)), \int_0^t e^{(t-s)A}F(X(s, x))ds \right\rangle d\xi \right].
\]

Before proceeding, we need the following

**Claim.** For any \( x \in H \) it holds

\[
\lim_{\xi \to 0^+} \frac{1}{t} \mathbb{E} \left[ \int_0^1 \left\langle D\varphi(\xi Z_A(t, x) + (1 - \xi)X(t, x)), \int_0^t e^{(t-s)A}F(X(s, x))ds \right\rangle d\xi \right] = \langle D\varphi(x), F(x) \rangle.
\]  

(20)

For any \( x \in H \) we have

\[
\frac{1}{t} \mathbb{E} \left[ \int_0^1 \left\langle D\varphi(\xi Z_A(t, x) + (1 - \xi)X(t, x)), \int_0^t e^{(t-s)A}F(X(s, x))ds \right\rangle d\xi \right]
\]

\[
= -\langle D\varphi(x), F(x) \rangle.
\]
such that $I$ where $x$ strongly continuous, $I$, Consequently, since $t \to \infty$, exists As easily seen, $x$.

By taking into account that $\phi(t,x)$, we have $0 \to I(t,x)$ and that (20) holds, for any $x \in H$, we have

$$I_{2.1}(t,x) \leq \frac{M}{t} \int_0^t e^{(t-s)x} E[|X(s,x)|] ds$$

$$\leq \frac{\kappa}{t} \int_0^t e^{(t-s)x} E[|X(s,x)|] ds.$$

Consequently, since $E[|X(t,x)|] \to 0$ as $t \to 0^+$, it follows that $I_{2.1}(t,x) \to 0$ as $t \to 0^+$. For $I_{2.2}(t,x)$ we have, by the fact that the semigroup $e^{tA}$, $t \geq 0$ is strongly continuous, $I_{2.2}(t,x) \to 0$ as $t \to 0^+$. Then, $I_2(t,x) = 0$, $\forall x \in H$ as $t \to 0^+$. This prove the claim.

By taking into account that $\varphi \in D(L) \cap C^1_b(H)$ and that (20) holds, for any $x \in H$ we have

$$\lim_{t \to 0^+} \frac{P_t \varphi(x) - \varphi(x)}{t} = L \varphi(x) + \langle D \varphi(x), F(x) \rangle.$$

As easily seen, $x \mapsto L \varphi(x) + \langle D \varphi(x), F(x) \rangle$ is uniformly continuous. Moreover, since $t \to E[|X(t,x)|]$ is continuous and $E[|X(t,x)|] \to 0$ as $t \to 0^+$, there exists $c > 0$ such that

$$\left| \frac{P_t \varphi(x) - \varphi(x)}{t} \right| \leq c(1 + |x|)$$

$$+ c_F \| D \varphi \|_{C_b(H;H)} \frac{M}{t} \int_0^t e^{(t-s)x}(1 + E[|X(s,x)|]) ds$$

$$\leq c \left( 1 + c_F \| D \varphi \|_{C_b(H;H)} \frac{M}{t} \int_0^t e^{(t-s)x} ds \right) (1 + |x|).$$
This implies
\[ \sup_{\varphi \in \mathbb{R}} \frac{\left\| \frac{P_t \varphi - \varphi}{t} \right\|}{\varphi} < \infty. \]

Hence, \( \varphi \in D(K) \cap C_b^1(H) \) and \( K \varphi = L \varphi + (D \varphi, F) \) as claimed. The opposite inclusion follows by interchanging the role of \( R_t \) and \( P_t \) in the Taylor formula.

We need the following approximation result, proved in [9, Proposition 2.7].

**Proposition 4.2.** For any \( \varphi \in C_b(H) \), there exists \( m \in \mathbb{N} \) and an \( m \)-indexed sequence \((\varphi_{n_1,\ldots,n_m})_{n_1,\ldots,n_m \in \mathbb{N}} \subset \mathcal{E}_A(H) \) such that

\[ \lim_{n_1 \to \infty} \cdots \lim_{n_m \to \infty} \varphi_{n_1,\ldots,n_m} \xrightarrow{\pi} \varphi. \quad (21) \]

Moreover, if \( \varphi \in C_b^1(H) \), we can choose the sequence in such a way that (21) holds and

\[ \lim_{n_1 \to \infty} \cdots \lim_{n_m \to \infty} (D \varphi_{n_1,\ldots,n_m}, h) \xrightarrow{\pi} (D \varphi, h), \quad \forall h \in H. \]

Now we are able to prove the following

**Proposition 4.3.** For any \( \varphi \in \mathcal{E}_A(H) \) we have \( \varphi \in D(L) \) and

\[ L \varphi(x) = \frac{1}{2} \text{Tr} \left[ B B^* D^2 \varphi(x) \right] + \langle x, A^* D \varphi(x) \rangle, \quad x \in H. \quad (22) \]

The set \( \mathcal{E}_A(H) \) is a \( \pi \)-core for \((L, D(L))\), and for any \( \varphi \in D(L) \cap C_b^1(H) \) there exists \( m \in \mathbb{N} \) and an \( m \)-indexed sequence \((\varphi_{n_1,\ldots,n_m})_{n_1,\ldots,n_m \in \mathbb{N}} \subset \mathcal{E}_A(H) \) such that

\[ \lim_{n_1 \to \infty} \cdots \lim_{n_m \to \infty} \frac{\varphi_{n_1,\ldots,n_m}}{1 + |\cdot|} \xrightarrow{\pi} \frac{\varphi}{1 + |\cdot|}. \quad (23) \]

\[ \lim_{n_1 \to \infty} \cdots \lim_{n_m \to \infty} \frac{1}{2} \text{Tr} \left[ B B^* D^2 \varphi_{n_1,\ldots,n_m} \right] + \langle \cdot, A^* D \varphi_{n_1,\ldots,n_m} \rangle \xrightarrow{\pi} \frac{L \varphi}{1 + |\cdot|}. \quad (24) \]

Finally, if \( \varphi \in D(L) \cap C_b^1(H) \) we can choose the sequence in such a way that (23), (24) hold and

\[ \lim_{n_1 \to \infty} \cdots \lim_{n_m \to \infty} (D \varphi_{n_1,\ldots,n_m}, h) \xrightarrow{\pi} (D \varphi, h), \quad (25) \]

for any \( h \in H \).

**Proof.** We recall that the proof of (22) may be found in [6], Remark 2.66 (in [6] the result is proved for the semigroup \( R_t, t \geq 0 \) in the space \( C_{b,2}(H) \), but it is clear that the result holds also in \( C_{b,1}(H) \)).

Here we give only a sketch of the proof, which is very similar to the proof given in [9]. Take \( \varphi \in D(L) \). For any \( n_2 \in \mathbb{N} \), set \( \varphi_{n_2}(x) = n_2 \varphi(x)/(n_2 + |x|^2) \).

Clearly, \( \varphi_{n_2} \in C_b(H) \) and \( (1 + |\cdot|)^{-1} \varphi_{n_2} \xrightarrow{\pi} (1 + |\cdot|)^{-1} \varphi \) as \( n_2 \to \infty \). By Proposition 4.2, for any \( n_2 \in \mathbb{N} \) we fix a sequence\(^2\) \((\varphi_{n_2,n_3})_{n_3 \in \mathbb{N}} \subset \mathcal{E}_A(H) \) such that \( \varphi_{n_2,n_3} \xrightarrow{\pi} \varphi_{n_2} \) as \( n_3 \to \infty \). Set now, for any \( n_1, n_2, n_3, n_4 \in \mathbb{N} \)

\[ \varphi_{n_1,n_2,n_3,n_4}(x) = \frac{1}{n_4} \sum_{k=1}^{n_4} R_{n_1-n_k} \varphi_{n_2,n_3}(x). \quad (26) \]

\(^2\) we assume that the sequence has only one index
Since for any $\varphi \in C_b(H)$, $x \in H$ the function $\mathbb{R}^+ \to \mathbb{R}$, $t \mapsto R_t \varphi(x)$ is continuous, a straightforward computation shows that the sequence $(\varphi_{n_1, \ldots, n_4})$ fulfils (23). Similarly, we find that for any $x \in H$ it holds

$$
\lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \lim_{n_3 \to \infty} \lim_{n_4 \to \infty} \frac{1}{2} \text{Tr}[BB^* D^2 \varphi_{n_1, n_2, n_3, n_4}(x)] + \langle x, A^* D \varphi_{n_1, n_2, n_3, n_4}(x) \rangle
= \lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \lim_{n_3 \to \infty} \lim_{n_4 \to \infty} L \varphi_{n_1, n_2, n_3, n_4}(x)
= \lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \lim_{n_3 \to \infty} \lim_{n_4 \to \infty} \int_0^{\frac{1}{n_1}} LR_t \varphi_{n_2, n_3}(x)dt
= \lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \lim_{n_3 \to \infty} \lim_{n_4 \to \infty} \left( R_{\frac{1}{n_1}} \varphi_{n_2, n_3}(x) - \varphi_{n_2, n_3}(x) \right)
= \lim_{n_1 \to \infty} \left( R_{\frac{1}{n_1}} \varphi(x) - \varphi(x) \right) = L \varphi(x).
$$

Here we have used the continuity of $t \mapsto LR_t \varphi_{n_2, n_3}(x)$ and the fact that $LR_t \varphi_{n_2, n_3}(x) = (d/dt)R_t \varphi_{n_2, n_3}(x)$ (cf Proposition 2.2 and Proposition 2.4). The fact that any limit above is equibounded in $C_b(H)$ with respect to the corresponding index follows by the construction of $\varphi_{n_1, n_2, n_3, n_4}(x)$. Hence, (24) follows.

If $\varphi \in D(L) \cap C_b^1(H)$, by Proposition 4.2, there exists a sequence $(\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{E}_A(H)$ such that $\varphi_n \xrightarrow{x} \varphi$ as $n \to \infty$ and $(D \varphi_n, h) \xrightarrow{x} \langle D \varphi, h \rangle$ as $n \to \infty$, for any $h \in H$. Since for any $t > 0$, $n \in \mathbb{N}$ we have

$$
(DR_t \varphi_n, h) = \int_H (D \varphi_n(e^{tA}x + y), h) N_{Q_1}(dy), \quad x \in H
$$

it follows $(DR_t \varphi_n, h) \xrightarrow{x} \langle DR_t \varphi, h \rangle$ as $n \to \infty$, for any $h \in H$. Then, the claim follows by arguing as above.

By Theorem 4.1 and Proposition 4.3 we have

**Corollary 4.4.** $(K, D(K))$ is an extension of $K_0$ and for any $\varphi \in \mathcal{E}_A(H)$ we have $\varphi \in D(K)$ and $K \varphi = K_0 \varphi$.

### 4.2 Approximation of $F$ with smooth functions

It is convenient to introduce an auxiliary Ornstein–Uhlenbeck semigroup

$$
U_t \varphi(x) = \int_H \varphi(e^{tS}x + y) N_{\frac{1}{2} S^{-1}(e^{2iS} - 1)}(dy), \quad \varphi \in C_b(H)
$$

where $S : D(S) \subset H \to H$ is a self-adjoint negative definite operator such that $S^{-1}$ is of trace class. We notice that $U_t$ is strong Feller, and for any $t > 0$, $\varphi \in C_b(H)$, $U_t \varphi$ is infinite times differentiable with bounded differentials (see [6]). We introduce a regularization of $F$ by setting

$$
(F_n(x), h) = \int_H \left( F\left(e^{\frac{i}{2}S}x + y\right), e^{\frac{i}{2}S}h \right) N_{\frac{1}{2} S^{-1}(e^{2iS} - 1)}(dy), \quad n \in \mathbb{N}.
$$

It is easy to check that $F_n$ is infinite times differentiable, with first differential bounded by $\kappa$, for any $n \in \mathbb{N}$. Moreover, $F_n(x) \to F(x)$ as $n \to \infty$ for all $x \in H$ and $|F_n(x)| \leq |F(x)|$, for all $n \in \mathbb{N}$, $x \in H$.

\[\text{Footnote: we assume that the sequence has only one index}\]
Let $P^n_t$ be the transition semigroup

$$P^n_t \varphi(x) = \mathbb{E}[\varphi(X^n(t, x))], \quad \varphi \in C_{b,1}(H)$$

(27)

where $X^n(t, x)$ is the solution of (1) with $F_n$ replacing $F$. It is easy to check

$$\lim_{n \to \infty} \mathbb{E}[|X^n(t, x) - X(t, x)|^2] = 0, \quad t \geq 0, \quad x \in H$$

and

$$\mathbb{E}[|X^n(t, x)|] \leq \mathbb{E}[|X(t, x)|], \quad t \geq 0, \quad x \in H,$$

where $c_0 > 0, \omega_0 \in \mathbb{R}$ are as in Proposition 2.2. This implies

$$\lim_{n \to \infty} \frac{P^n_t \varphi}{1 + |\cdot|} = \frac{P_t \varphi}{1 + |\cdot|},$$

(28)

for any $t \geq 0, \varphi \in C_{b,1}(H)$. We denote by $(K_n, D(K_n))$ the infinitesimal generator of the semigroup $P^n_t$ in $C_{b,1}(H)$, defined as in (6) with $K_n$ replacing $K$ and $P^n_t$ replacing $P_t$. We recall that all the statements of Proposition 2.2, Theorem 3.1 hold also for $P^n_t$ and $(K_n, D(K_n))$. We also recall that the resolvent of $(K, D(K))$ in $C_{b,1}(H)$ is defined for any $\lambda > \omega_0$ by the formula $R(\lambda, K)f(x) = \int_0^\infty e^{-\lambda t} P_t f(x) dt, \quad f \in C_{b,1}(H), \quad x \in H$ (cf Theorem 3.1). Similarly, for a fixed $n \in \mathbb{N}$ the resolvent of $(K_n, D(K_n))$ in $C_{b,1}(H)$ at $\lambda > 0$ is defined by the same formula with $P^n_t$ replacing $P_t$. Since (28) holds, it is straightforward to see that

$$\lim_{n \to \infty} R(\lambda, K_n) \varphi \leq R(\lambda, K) \varphi$$

(29)

for any $\varphi \in C_{b,1}(H), \lambda > \omega_0$.

The following proposition follows by Corollary 4.9 of [9] and by the fact that $\|DF_n\| \leq \kappa$, for any $n \in \mathbb{N}$.

**Proposition 4.5.** For any $n \in \mathbb{N}$, let $(K_n, D(K_n))$ be the infinitesimal generator of the semigroup (27). Then for any $\lambda > \max\{0, \omega + M\kappa\}$, the resolvent $R(\lambda, K_n)$ of $K_n$ at $\lambda$ maps $C^1_0$ into $C^1_0(H)$ and it holds

$$\|DR(\lambda, K_n)f\|_{C_0^1(H)} \leq \frac{M\|DF\|_{C_0^1(H)}}{\lambda - (\omega + M\kappa)}, \quad f \in C^1_0(H).$$

(30)

Corollary 4.4 shows that $K$ is an extension of $K_0$ and that $K\varphi = K_0\varphi, \forall \varphi \in \mathcal{E}_A(H)$. So, in view of the fact that $K P_t \varphi = P_t K_0 \varphi$ for any $\varphi \in \mathcal{E}_A(H)$ (cf (i) of Proposition 2.4), it is not difficult to check that $\{P^n_t \mu\}_{t \geq 0}$ fulfils (7), (10). Now, let $\mu \in \mathcal{M}_1(H)$ and assume that $\{\mu_t, t \geq 0\} \subset \mathcal{M}_1(H)$ fulfils (7), (10). In view of Theorem 3.3, to prove that $\mu_t = P^n_t \mu$, for any $t \geq 0$, it is sufficient to show that $\{\mu_t, t \geq 0\}$ is also a solution of (8). In order to do this, we need an approximation result.

**Lemma 4.6.** The set $\mathcal{E}_A(H)$ is a $\pi$-core for $(K, D(K))$, and for any $\varphi \in D(K)$ there exist $m \in \mathbb{N}$ and an $m$-indexed sequence $(\varphi_{n_1, \ldots, n_m}) \subset \mathcal{E}_A(H)$ such that

$$\lim_{n_1 \to \infty} \cdots \lim_{n_m \to \infty} \frac{\varphi_{n_1, \ldots, n_m}}{1 + |\cdot|} = \frac{\varphi}{1 + |\cdot|},$$

(31)

and

$$\lim_{n_1 \to \infty} \cdots \lim_{n_m \to \infty} \frac{K\varphi_{n_1, \ldots, n_m}}{1 + |\cdot|} = \frac{K\varphi}{1 + |\cdot|}.$$
Proof. **Step 1.** Let $\varphi \in D(K)$, $\lambda > \max\{0, \omega_0, \omega + M\}$ and set $f = \lambda \varphi - K \varphi$. We fix a sequence $(f_{n_1}) \subset C_b^1(H)$ such that

$$\lim_{n_1 \to \infty} \frac{f_{n_1}}{1 + |\cdot|} = \frac{f}{1 + |\cdot|}.$$ 

Set $\varphi_{n_1} = R(\lambda, K)f_{n_1}$. By Proposition 2.4 it follows

$$\lim_{n_1 \to \infty} \frac{\varphi_{n_1}}{1 + |\cdot|} = \varphi, \quad \lim_{n_1 \to \infty} \frac{K\varphi_{n_1}}{1 + |\cdot|} = \frac{K\varphi}{1 + |\cdot|}. \quad (33)$$

**Step 2.** Now set $\varphi_{n_1, n_2} = R(\lambda, K_{n_2})f_{n_1}$, where $K_{n_2}$ is the infinitesimal generator of the semigroup $P_{\gamma_{n_2}}$, introduced in (27). Since $f_{n_1} \in C_b^1(H)$, by Proposition 4.5 we have $\varphi_{n_1, n_2} \in C_b^1(H)$ and

$$\sup_{n_2 \in \mathbb{N}} \|D\varphi_{n_1, n_2}\|_{C_b(H,H)} \leq M \frac{\|Df_{n_1}\|_{C_b(H,H)}}{\lambda - (\omega + MK)}, \quad (34)$$

for any $n_1 \in \mathbb{N}$. Moreover, by (29) it holds

$$\lim_{n_2 \to \infty} \varphi_{n_1, n_2} = \varphi_{n_1}, \quad \lim_{n_2 \to \infty} \frac{K_{n_2}\varphi_{n_1, n_2} - \varphi_{n_1}}{n_2} = K\varphi_{n_1}, \quad (35)$$

for any $n_1 \in \mathbb{N}$. Since $\varphi_{n_1, n_2} \in D(K_{n_2}) \cap C_b^1(H)$, by Theorem 4.1 we have

$$K_{n_2}\varphi_{n_1, n_2} = L\varphi_{n_1, n_2} + \langle D\varphi_{n_1, n_2}, F_{n_2} \rangle.$$ 

**Step 3.** By Proposition 4.3, for any $n_1, n_2$ there exists a sequence $(\varphi_{n_1, n_2, n_3}) \subset \mathcal{E}_A(H)$ (we still assume that it has only one index) such that

$$\lim_{n_3 \to \infty} \varphi_{n_1, n_2, n_3} = \varphi_{n_1, n_2}, \quad \lim_{n_3 \to \infty} \frac{L\varphi_{n_1, n_2, n_3}}{1 + |\cdot|} = \frac{L\varphi_{n_1, n_2}}{1 + |\cdot|}, \quad (36)$$

and

$$\lim_{n_3 \to \infty} \langle D\varphi_{n_1, n_2, n_3}, h \rangle = \langle D\varphi_{n_1, n_2}, h \rangle.$$ 

for any $h \in H$. Notice that the since $F_{n_2}$ is globally Lipschitz, it follows

$$\lim_{n_3 \to \infty} \frac{\langle D\varphi_{n_1, n_2, n_3}, F_{n_2} \rangle}{1 + |\cdot|} = \frac{\langle D\varphi_{n_1, n_2}, F_{n_2} \rangle}{1 + |\cdot|}.$$ 

This, together with (36), implies that the sequence $(\varphi_{n_1, n_2, n_3})$ fulfills

$$\lim_{n_3 \to \infty} \varphi_{n_1, n_2, n_3} = \varphi_{n_1, n_2}, \quad \lim_{n_3 \to \infty} \frac{K_{n_2}\varphi_{n_1, n_2, n_3}}{1 + |\cdot|} = \frac{K_{n_2}\varphi_{n_1, n_2}}{1 + |\cdot|}.$$ 

Since $K$ is an extension of $K_0$ (cf Corollary 4.4), we have

$$K\varphi_{n_1, n_2, n_3} = K_0\varphi_{n_1, n_2, n_3} = K_{n_2}\varphi_{n_1, n_2, n_3} + \langle D\varphi_{n_1, n_2, n_3}, F - F_{n_2} \rangle$$

for any $n_1, n_2, n_3 \in \mathbb{N}$. So we find

$$\lim_{n_3 \to \infty} \frac{K_0\varphi_{n_1, n_2, n_3}}{1 + |\cdot|} = \frac{K_{n_2}\varphi_{n_1, n_2}}{1 + |\cdot|} + \langle D\varphi_{n_1, n_2}, F - F_{n_2} \rangle, \quad (37)$$

\(^4\)the assumption $\lambda > \omega_0$ is necessary to define the resolvent of $K$ (cf Proposition 2.4)
for any $n_1, n_2 \in \mathbb{N}$. Moreover, by (34), we see that
\[
\frac{|\langle D\varphi_{n_1,n_2}(x), F(x) - F_{n_2}(x) \rangle|}{1 + |x|} \leq \frac{M \|Df_n\|_{C_0(H)}}{\lambda - (\omega + M\kappa)} \frac{|F(x) - F_{n_2}(x)|}{1 + |x|}
\]
and consequently
\[
\lim_{n_2 \to \infty} \frac{\langle D\varphi_{n_1,n_2}, F - F_{n_2} \rangle}{1 + |\cdot|} \equiv 0.
\]
This, together with (35) implies
\[
\lim_{n_2 \to \infty} K_{n_2}\varphi_{n_1,n_2} + \langle D\varphi_{n_1,n_2}, F - F_{n_2} \rangle \equiv K\varphi_{n_1,n_2}.
\]
Finally, by virtue of (33), (37); (38), the sequence $(\varphi_{n_1,n_2,n_3}) \subset \mathcal{E}_A(H)$ fulfills
\[
\lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \lim_{n_3 \to \infty} \frac{\varphi_{n_1,n_2,n_3}}{1 + |\cdot|} \equiv \frac{\varphi}{1 + |\cdot|},
\]
\[
\lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \lim_{n_3 \to \infty} \frac{K\varphi_{n_1,n_2,n_3}}{1 + |\cdot|} \equiv \frac{K\varphi}{1 + |\cdot|}.
\]
This concludes the proof.

5 Proof of Theorem 1.4

Let $\varphi \in D(K)$ and assume that $(\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{E}_A(H)$ fulfills (31), (32) (for simplicity the assume that this sequence has only one index: this does not change the generality of the proof). For any $t \geq 0$ we find
\[
\int_H \varphi(x)\mu(dx) - \int_H \varphi(x)\mu(dx) = \lim_{n \to \infty} \left( \int_H \varphi_n(x)\mu(dx) - \int_H \varphi_n(x)\mu(dx) \right)
\]
\[
= \lim_{n \to \infty} \int_0^t \left( \int_H K_0\varphi_n(x)\mu_s(dx) \right) ds,
\]
since $\varphi_n \in D(K)$ and $K\varphi_n = K_0\varphi_n$, for any $n \in \mathbb{N}$ (cf Corollary 4.4). Now observe that since $\sup_{n \in \mathbb{N}} |K_0\varphi_n(x)| \leq c(1 + |x|)$ for some $c > 0$ and since $\mu_s \in \mathcal{M}_1(H)$ for any $s \geq 0$, it holds
\[
\lim_{n \to \infty} \int_H K_0\varphi_n(x)\mu_s(dx) = \int_H K\varphi(x)\mu_s(dx)
\]
and
\[
\sup_{n \in \mathbb{N}} \left| \int_H K_0\varphi_n(x)\mu_s(dx) \right| \leq c \int_H (1 + |x|)|\mu_s|(dx).
\]
Hence, by taking into account (7) we can apply the dominated convergence theorem to obtain
\[
\lim_{n \to \infty} \int_0^t \left( \int_H K_0\varphi_n(x)\mu_s(dx) \right) ds = \int_0^t \left( \int_H K\varphi(x)\mu_s(dx) \right) ds
\]
So, $\{\mu_t, t \geq 0\}$ is also a solution of the Fokker-Planck equation (7), (8) for $(K, D(K))$. Since by Theorem 1.2 such a solution is unique and it is given by $P_t^\ast \mu$, $t \geq 0$, we have $\int_H \varphi(x)P_t^\ast \mu(dx) = \int_H \varphi(x)\mu_t(dx)$, for any $\varphi \in \mathcal{E}_A(H)$. By taking into account that the set $\mathcal{E}_A(H)$ is $\pi$-dense in $C_b(H)$ (cf. Proposition 4.2), we have $\int_H \varphi(x)P_t^\ast \mu(dx) = \int_H \varphi(x)\mu_t(dx)$, for any $\varphi \in C_b(H)$. This, clearly, implies $P_t^\ast \mu = \mu_t$, $\forall t \geq 0$. This concludes the proof.
References


