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Paths Clustering and an Existence Result for Stochastic Vortex Systems

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We prove, via a pathwise analysis, an existence result for stochastic differential equations with singular coefficients that govern stochastic vortex systems. The techniques are self-contained and rely on careful estimates on the displacements of particles, obtained by recursively identifying "vortex clusters" whose mutual interactions can be controlled. This provides a non trivial extension of techniques of Marchioro and Pulvirenti[(7)] for deterministic motion of vortices.

KEY WORDS: vorticity, stochastic differential equations, non-local interaction

AMS subject classification: 60H10, 60K35, 76B47

1. INTRODUCTION

Vorticity has played a fundamental role in the understanding of incompressible fluids, and in numerical approximation methods for the two dimensional Euler or Navier-Stokes equation via the so-called vortex methods. In this work, we are interested in pathwise properties of stochastic differential equations with singular coefficients governing the so-called vortex systems. Consider the so-called *Biot-Savart kernel in* $\mathbb{R}^2$,

$$K(x) = \frac{1}{2\pi}\frac{(-x_2, x_1)}{|x|^2}, \quad x = (x_1, x_2) \in \mathbb{R}^2\backslash\{0\}$$

and the system of stochastic differential equations:

$$X_t^i = X_0^i + \sqrt{2\nu}B_t^i + \int_0^t \sum_{j\neq i} K\left(X_s^i - X_s^j\right) a_j \, ds, \quad i = 1, \ldots, N. \tag{1}$$

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Here, X_0^i and a_i are (possibly random) quantities taking values respectively in $\mathbb{R}^2$ and $\mathbb{R}$, $B^1, \ldots, B^N$ are (independent) 2-dimensional Brownian motions independent of the formers, and $\nu \geq 0$ is the (constant) viscosity coefficient.

Existence of solutions of system (1) is not obvious because of the singular coefficients. In the case where $\nu > 0$ and the "vortex intensities" a_i have all the same sign, it has been proved by Takanobu[14] through a purely probabilistic argument. A proof of existence for general intensities a_i was given by Osada,[12] based on analytic results for generators in generalized divergence form obtained in Osada,[11] and on potential theoretical results.

In the present work, we shall prove an existence result for Eq. (1) through a different, self-contained approach, that exploits from the trajectorial point of view some characteristics of the system and covers a more general situation that the one considered in Ref. 14. We shall prove the following:

Theorem 1.1. *Consider independent standard* 2*-dimensional Brownian motions* $B^1, \ldots, B^N$ *in a filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F})_t, P)$*, and independent* $\mathcal{F}_0$*-measurable random variables* (X_0^i) *having a density with respect to Lebesgue measure in* $\mathbb{R}^2$*. Assume moreover that the (possibly random) intensities* (a_i) *satisfy*

(H) *for all* $I \subseteq \{1, \ldots, N\}$,

$$\sum_{i \in I} a_i \neq 0. \quad (a.s.)$$

Then, there is pathwise existence and uniqueness on $[0, \infty[$ *for the systems of stochastic differential Eq. (1).*

Hypothesis **(H)** is used in Marchioro and Pulvirenti,[7] Chapter 4, to prove existence of solutions of deterministic vortex systems ($\nu = 0$). Their techniques are based on controlling the displacements of particles in each subsystem, say $(X^i)_{i \in I}$ with $I \subset \{1, \ldots N\}$, which is "far away enough" from all other particles, in terms of the displacements of smaller subsystems $J \subset I$. This is possible thanks to the decay of the interaction kernel, but requires a careful control of the influences of the particles not belonging to subsystem I. The analysis is done through a clever recurrence argument that allows to identify different "clusters" of particles occurring at determined distances from each other and for an adequate time lapse, which provides some control of their mutual interaction (for some time intervals). A global control of displacements allows them to prove finiteness of some logarithmic potential along trajectories and deduce *a priori* absence of collisions.

Rather than the statement of Theorem 1.1 itself, our main goal here is to explicitly develop a similar "clustering" argument in the stochastic setting. The identification of the moments when clusters occur requires the knowledge of the past and future displacements of some of the particles. In our case, those displacements are related in a complex way to those of the driving Brownian motions. Many of the arguments used in Ref. 7 in order to define the clusters are rather heuristic or do not provide explicit estimates that could be generalized. We therefore need a considerably more subtle analysis, in order to adapt the general ideas therein and obtain tractable (semi-explicit) estimates in terms of the Brownian motions displacements. These will be fundamental for proving finiteness of the expectation of some functional of the particles trajectories.

We recall that in the mean field case (i.e. when a_j is proportional to $\pm\frac{1}{N}$), if $\nu = 0$ and K is replaced by some regular approximating kernel K_{ε_N}, it is known that system (1) converges (when N goes to infinity and ε_N to 0) to solutions of the 2d-Euler equation (see Marchioro and Pulvirenti[7] or Bertozzi and Majda[1]). Similarly, when $\nu > 0$, convergence of mollified vortex systems has been proved, towards solutions of the incompressible 2d-Navier-Stokes equations (see e.g. Marchioro and Pulvirenti,[8] Méléard[1,9]). A convergence result for the true (non-mollified) particle system (1) has been obtained when $\nu > 0$ is large enough by Osada[10], relying on the results of Refs. 11, 12. However, the probabilistic understanding of that convergence and of the pathwise properties of the system is not satisfactory. It is also worth mentioning that several stochastic particle systems in mean field interaction of singular type, arise in physically relevant models, and also related to spectral measure processes of certain matrix diffusions and generalizations (see for the latter e.g. Rogers and Shi[13], Cépa and Lépingle[2]). Similarly, mean field particle systems with singular interactions can be associated with three dimensional incompressible fluids (see Bertozzi and Majda[1] for the convergence of mollified deterministic particle systems towards the 3d-Euler equation, and Fontbona[4] for stochastic particle approximations of the 3d- Navier-Stokes equations).

We expect that the techniques we present here hopefully provide further insight on the pathwise behavior of the vortex system and related stochastic singular interacting particle systems. Unfortunately, by the moment our techniques do not provide well behaved estimates in terms of N, and our main result excludes the relevant mean field case. Nevertheless, the ideas developed here should allow for refinements in several directions.

This work is presented as follows. In Sec. 2 we develop the "clustering" argument to obtain on mollified vortex systems some uniform (in the mollifying parameter and the initial condition) displacements estimates. In Sec. 3 we use those results to prove Theorem 1.1. In Sec. 4 we discuss the role of assumption **(H)** and limitations of the method.

2. UNIFORM MOMENTS ESTIMATES FOR MOLLIFIED VORTEX SYSTEMS

We first consider vortex systems with regularized interaction kernels, and prove moments estimates which are uniform in the regularizing parameter.

Recall that K is defined as $K(x) = \nabla^{\perp} G(x)$, where $G(x) = -\frac{1}{2\pi} \log |x|$ and $\nabla^{\perp} = (\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1})$.

Let $\log^{\varepsilon}$ be a smooth function such that $\log^{\varepsilon}(r) = \log(r)$ if $r \geq \varepsilon$, and moreover such that $|\frac{d}{dr} \log^{\varepsilon}(r)| \leq \frac{1}{r}$. We define the mollified kernels $K_{\varepsilon}: \mathbb{R}^2 \to \mathbb{R}^2$ by $K_{\varepsilon}(x) = \nabla^{\perp} G_{\varepsilon}(x)$, where $G_{\varepsilon}(x) = -\frac{1}{2\pi} \log^{\varepsilon} |x|$. Then, K_{ε} is an odd function ($K_{\varepsilon}(x) = -K_{\varepsilon}(-x)$) and for all $x \in \mathbb{R}^2$, we have

$$K_{\varepsilon}(x) \leq \frac{1}{2\pi |x|}.$$

Let $0 < s \leq t$ and $x \in \mathbb{R}^2$. We denote by $\xi^{\varepsilon}_{s,t}(x) := (\xi^{i,\varepsilon}_{s,t}(x, \omega))_{i=1}^{N}$ the unique strong solution of the system of stochastic differential equations in $(\mathbb{R}^2)^N$

$$\xi^{\varepsilon,i}_{s,t}(x) = x^i + \sqrt{2\nu}\big(B^i_t - B^i_s\big) + \int_s^t \sum_{j \neq i} K_{\varepsilon}\big(\xi^{\varepsilon,i}_{s,r}(x) - \xi^{\varepsilon,j}_{s,r}(x)\big)\, a_j \, dr \tag{2}$$

for $t \geq s, i = 1, \ldots, N$. We recall that the regularity properties of K_{ε} and standard results on stochastic flows (cf. Kunita[(6)]) imply existence of a continuous version of the three parameter processes $(s, t, x) \to \xi^{\varepsilon}_{s,t}(x)$, which is moreover continuously differentiable in x for all $s \leq t$.

We shall implicity work in a subset $\Omega_0 \subset \Omega$ of full measure where those properties about $\xi^{\varepsilon}_{s,t}(x)$ are everywhere true.

We also fix for the sequel a finite non empty time interval $[0, T]$.

Let us introduce some notation. We write

$$A := \min_{I \subseteq \{1,\ldots,N\}} \Big| \sum_{i \in I} a_i \Big|,$$

which is a strictly positive quantity by assumption **(H)**, and

$$a := \max\{1, |a_1|, \ldots, |a_N|\}.$$

For each $i \in \{1, \ldots, N\}$, we also define the random variables

$$S^i_T := \sup_{t \in [0,T]} \big|B^i_t\big| \quad \text{and} \quad D^i_T := 2\sqrt{2\nu} S^i_T + aT.$$

Finally, for $s \leq t$ we denote by

$$M^I_{s,t}(x) := \sum_{i \in I} a_i \xi^{\varepsilon,i}_{s,t}(x)$$

the bari-center of the subsystem $(\xi^{i,\varepsilon}_{s,t}(x, \omega))_{i \in I}$, and so $M^I_{s,s}(x) = \sum_{i \in I} a_i x^i$.

The following is the first step of a recursive argument:

Lemma 2.1. *Let $i \in \{1, \dots, N\}$ and $\theta_1, \rho_1 \in [0, T]$ with $\theta_1 \leq \rho_1$, and write*

$$\Gamma^{\{i\}}_{\theta_1,\rho_1} := \left\{ (x, \omega) \in (\mathbb{R}^2)^N \times \Omega_0 \colon \sum_{j \neq i} \frac{1}{2\pi} \left| \xi^{\varepsilon,i}_{\theta_1,s}(x, \omega) - \xi^{\varepsilon,j}_{\theta_1,s}(x, \omega) \right|^{-1} \right.$$
$$\left. < \ 1 \forall s \in [\theta_1, \rho_1[\} \right\}.$$

Then, the following inclusion holds:

$$\Gamma^{\{i\}}_{\theta_1,\rho_1} \subseteq \left\{ (x, \omega) \colon \left| \xi^{\varepsilon,i}_{\theta_1,t}(x, \omega) - x^i \right| \leq D^i_T(\omega) \ \forall s \in [\theta_1, \rho_1] \right\}.$$

Proof: On the first set one has for $t \in [\theta_1, \rho_1]$ that

$$\left| \xi^{\varepsilon,i}_{\theta_1,t}(x, \omega) - x^i \right| \leq \sqrt{2\nu} \left| B^i_t(\omega) - B^i_{\theta_1}(\omega) \right| + a(t - \theta_1).$$

□

Before stating and proving the recurrence step, and in order to give its idea in a simpler setting, we next prove explicitly the "$n = 2$ step":

Lemma 2.2. *Let $I = \{i_1, i_2\} \subseteq \{1, \dots N\}$ and $\theta_2, \rho_2 \in [0, T]$ with $\theta_2 \leq \rho_2$, and write*

$$\Gamma^{I}_{\theta_2,\rho_2} := \left\{ (x, \omega) \in (\mathbb{R}^2)^N \times \Omega_0 : \sum_{j \notin I} \frac{1}{2\pi} \left| \xi^{\varepsilon,i}_{\theta_2,s}(x, \omega) - \xi^{\varepsilon,j}_{\theta_2,s}(x, \omega)) \right|^{-1} \right.$$
$$\left. < \ 1 - \frac{1}{N} \ \forall \ i \in I, \forall s \in [\theta_2, \rho_2[\right\}$$

Then, we have the inclusion

$$\Gamma^{I}_{\theta_2,\rho_2} \subseteq \left\{ (x, \omega) \colon \left| \xi^{\varepsilon,i}_{\theta_2,t}(x, \omega) - x^i \right| \right.$$
$$\left. \leq \frac{a}{A} \left((A + 3) \left(D^{i_1}_T + D^{i_2}_T \right) + \frac{N}{\pi} \right) \quad \forall \ i \in I, \forall s \in [\theta_2, \rho_2] \right\}.$$

Proof: We fix $(x, \omega) \in \Gamma^I_{\theta_2,\rho_2}$. For notational simplicity we omit reference to ω in the flow $\xi^\varepsilon(x)$. The function $K_\varepsilon(x)$ being odd, we have for $\theta_2 \leq t$ that

$$M^I_{\theta_2,t}(x) - M^I_{\theta_2,\theta_2}(x) = \sqrt{2\nu} \left(a_{i_1} \left(B^{i_1}_t - B^{i_1}_{\theta_2} \right) + a_{i_2} \left(B^{i_2}_t - B^{i_2}_{\theta_2} \right) \right)$$

$$+ \int_{\theta_2}^{t} \sum_{j \neq i_1, i_2} \big(K_\varepsilon\big(\xi^{\varepsilon,i_1}_{\theta_2,s}(x) - \xi^{\varepsilon,j}_{\theta_2,s}(x)\big)a_{i_i}$$
$$+ K_\varepsilon\big(\xi^{\varepsilon,i_2}_{\theta_2,s}(x) - \xi^{\varepsilon,j}_{\theta_2,s}(x)\big)a_{i_2}\big)a_j \, ds.$$

Therefore, for $\theta_2 \leq t \leq \rho_2$

$$\begin{aligned}\big|M^I_{\theta_2,t}(x) - M^I_{\theta_2,\theta_2}(x)\big| &\leq a\sqrt{2\nu}\big(\big|B^{i_1}_t - B^{i_1}_{\theta_2}\big| + \big|B^{i_2}_t - B^{i_2}_{\theta_2}\big|\big) + 2a^2T\left(1 - \frac{1}{N}\right)\\ &\leq 2a\sqrt{2\nu}\big(S^{i_1}_T + S^{i_2}_T\big) + 2a^2T\\ &= aD^{i_1}_T + aD^{i_2}_T.\end{aligned}$$

Since

$$\begin{aligned}&M^I_{\theta_2,t}(x) - M^I_{\theta_2,\theta_2}(x)\\ &\quad = \big(a_{i_1} + a_{i_2}\big)\big(\xi^{\varepsilon,i_1}_{\theta_2,t}(x) - x^{i_1}\big) + a_{i_2}\big(\big(\xi^{\varepsilon,i_2}_{\theta_2,t}(x) - x^{i_2}\big) - \big(\xi^{\varepsilon,i_1}_{\theta_2,t}(x) - x^{i_1}\big)\big),\end{aligned}$$

we deduce that for $\theta_2 \leq t \leq \rho_2$,

$$A\big|\xi^{\varepsilon,i_1}_{\theta_2,t}(x) - x^{\varepsilon,i_1}\big| \leq a\big(D^{i_1}_T + D^{i_2}_T + \big|\xi^{\varepsilon,i_1}_{\theta_2,t}(x) - \xi^{\varepsilon,i_2}_{\theta_2,t}(x)\big| + \big|x^{i_2} - x^{i_1}\big|\big). \quad (3)$$

Consider the time instant

$$\sigma := \inf\left\{t > \theta_2 \colon A\big|\xi^{\varepsilon,i_1}_{\theta_2,t}(x) - x^{i_1}\big| > a\left(3\big(D^{i_1}_T + D^{i_2}_T\big) + \frac{N}{\pi}\right)\right\}.$$

From continuity of $t \mapsto \xi^{\varepsilon,i_1}_{\theta_2,t}(x)$ and the fact that $\xi^{\varepsilon,i_1}_{\theta_2,t}(x) \to x_{i_1}$ when $t \to \theta_2$, we clearly have $\theta_2 < \sigma \leq \infty$. Furthermore, for all $t \in [\theta_2, \sigma \wedge \rho_2]$, we have

$$\big|\xi^{\varepsilon,i_1}_{\theta_2,t}(x) - x^{i_1}\big| \leq \frac{a}{A}\left(3\big(D^{i_1}_T + D^{i_2}_T\big) + \frac{N}{\pi}\right). \quad (4)$$

We consider now two cases:

Case (a) $\sigma \geq \rho_2$: In this case, the upper bound (4) holds on $[\theta_2, \rho_2]$, and the asserted inclusion is proved.

Case (b) $\sigma < \rho_2$: Since

$$A\big|\xi^{\varepsilon,i_1}_{\theta_2,\sigma}(x) - x^{i_1}\big| = a\left(3\big(D^{i_1}_T + D^{i_2}_T\big) + \frac{N}{\pi}\right), \quad (5)$$

we obtain from (3) that

$$|x^{i_1} - x^{i_2}| + \big|\xi^{\varepsilon,i_1}_{\theta_2,\sigma}(x) - \xi^{\varepsilon,i_2}_{\theta_2,\sigma}(x)\big| \geq 2a\big(D^{i_1}_T + D^{i_2}_T\big) + \frac{N}{\pi}.$$

Consequently, there is $\tau \in \{\theta_2, \sigma\}$ such that

$$\begin{aligned} \left|\xi^{\varepsilon,i_1}_{\theta_2,\tau}(x) - \xi^{\varepsilon,i_2}_{\theta_2,\tau}(x)\right| &\geq a\left(D^{i_1}_T + D^{i_2}_T\right) + \frac{N}{2\pi} \\ &> \frac{N}{2\pi}. \end{aligned} \tag{6}$$

From this and the definition of $\Gamma^I_{\theta_2,\rho_2}$ we get that

$$\sum_{j\neq i_1} \frac{1}{2\pi}\left|\xi^{\varepsilon,i_1}_{\theta_2,\tau}(x) - \xi^{\varepsilon,j}_{\theta_2,\tau}(x)\right|^{-1} < 1,$$

and

$$\sum_{j\neq i_2} \frac{1}{2\pi}\left|\xi^{\varepsilon,i_2}_{\theta_2,\tau}(x) - \xi^{\varepsilon,j}_{\theta_2,\tau}(x)\right|^{-1} < 1.$$

We claim that furthermore, for all $s \in [\tau, \rho_2]$,

$$\sum_{j\neq i_1} \frac{1}{2\pi}\left|\xi^{\varepsilon,i_1}_{\theta_2,s}(x) - \xi^{\varepsilon,j}_{\theta_2,s}(x)\right|^{-1} < 1 \tag{7}$$

and

$$\sum_{j\neq i_2} \frac{1}{2\pi}\left|\xi^{\varepsilon,i_2}_{\theta_2,s}(x) - \xi^{\varepsilon,j}_{\theta_2,s}(x)\right|^{-1} < 1. \tag{8}$$

For suppose there exists $s \in [\tau, \rho_2]$ for which (7) or (8) do not hold. Then, if we set

$$s^* := \inf\left\{s \geq \tau \colon \exists p \in \{1,2\} \text{ s.t. } \sum_{j\neq i_p} \frac{1}{2\pi}\left|\xi^{\varepsilon,i_p}_{\theta_2,s}(x) - \xi^{\varepsilon,j}_{\theta_2,s}(x)\right|^{-1} \geq 1\right\},$$

by continuity we would have

$$\tau \leq s^* \leq \rho_2,$$

and

$$\sum_{j\neq i_p} \frac{1}{2\pi}\left|\xi^{\varepsilon,i_p}_{\theta_2,s^*}(x) - \xi^{\varepsilon,j}_{\theta_2,s^*}(x)\right|^{-1} = 1$$

for $p \in \{1,2\}$ realizing s^*. Note that we must have $\tau \neq s^*$. But from the definition of $\Gamma^I_{\theta_2,\rho_2}$, the previous equality implies that

$$\frac{1}{2\pi}\left|\xi^{\varepsilon,i_1}_{\theta_2,s^*}(x) - \xi^{\varepsilon,i_2}_{\theta_2,s^*}(x)\right|^{-1} > \frac{1}{N}. \tag{9}$$

On the other hand, for all $s \in [\tau, s^*[$ we have from definition of s^* that

$$\sum_{j\neq i_1} \frac{1}{2\pi} \left|\xi^{\varepsilon,i_1}_{\theta_2,s}(x) - \xi^{\varepsilon,j}_{\theta_2,s}(x)\right|^{-1} < 1$$

or, equivalently,

$$\sum_{j\neq i_1} \frac{1}{2\pi} \left|\xi^{\varepsilon,i_1}_{\tau,s}(\xi_{\theta_2,\tau}(x)) - \xi^{\varepsilon,j}_{\tau,s}(\xi_{\theta_2,\tau}(x))\right|^{-1} < 1.$$

Thus, we have $(\xi_{\theta_2,\tau}(x), \omega) \in \Gamma^{\{i_1\}}_{\tau,s^*}$ and Lemma 2.1 then implies that

$$\left|\xi^{\varepsilon,i_1}_{\tau,s}\big(\xi_{\theta_2,\tau}(x)\big) - \xi^{\varepsilon,i_1}_{\theta_2,\tau}(x)\right| = \left|\xi^{\varepsilon,i_1}_{\theta_2,s}(x) - \xi^{\varepsilon,i_1}_{\theta_2,\tau}(x)\right| \leq aD^{i_1}_T \tag{10}$$

for all $s \in [\tau, s^*]$. For $s \in [\tau, s^*[$ we also have

$$\sum_{j\neq i_2} \frac{1}{2\pi} \left|\xi^{\varepsilon,i_2}_{\theta_2,s}(x) - \xi^{\varepsilon,j}_{\theta_2,s}(x)\right|^{-1} < 1,$$

so by an analogous argument we deduce that for all $s \in [\tau, s^*]$,

$$\left|\xi^{\varepsilon,i_2}_{\tau,s}(\xi_{\theta_2,\tau}(x)) - \xi^{\varepsilon,i_2}_{\theta_2,\tau}(x)\right| = \left|\xi^{\varepsilon,i_2}_{\theta_2,s}(x) - \xi^{\varepsilon,i_2}_{\theta_2,\tau}(x)\right| \leq aD^{i_2}_T. \tag{11}$$

From inequalities (10) and (11) with $s = s^*$ and (6) it follows that

$$\begin{aligned}&\left|\xi^{\varepsilon,i_1}_{\theta_2,s^*}(x) - \xi^{\varepsilon,i_2}_{\theta_2,s^*}(x)\right|\\ &\quad\geq \left|\xi^{\varepsilon,i_1}_{\theta_2,\tau}(x) - \xi^{\varepsilon,i_2}_{\theta_2,\tau}(x)\right| - \left|\xi^{\varepsilon,i_1}_{\theta_2,\tau}(x) - \xi^{\varepsilon,i_1}_{\theta_2,s^*}(x)\right| - \left|\xi^{\varepsilon,i_2}_{\theta_2,\tau}(x) - \xi^{\varepsilon,i_2}_{\theta_2,s^*}(x)\right| \geq \frac{N}{2\pi}.\end{aligned}$$

This contradicts (9), and therefore, (7) and (8) must hold for all $s \in [\tau, \rho_2]$ as claimed.

Now, observe that (7) together with (8) mean that $(\xi_{\theta_2,\tau}(x), \omega)$ is an element of $\Gamma^{\{i_1\}}_{\tau,\rho_2}$. A new application of Lemma 2.1 shows that for all $t \in [\tau, \rho_2]$

$$\left|\xi^{\varepsilon,i_1}_{\tau,t}(\xi_{\theta_2,\tau}(x)) - \xi^{\varepsilon,i_1}_{\theta_2,\tau}(x)\right| = \left|\xi^{\varepsilon,i_1}_{\theta_2,t}(x) - \xi^{\varepsilon,i_1}_{\theta_2,\tau}(x)\right| \leq aD^{i_1}_T.$$

If $\tau = \theta_2$, this implies the required upper bound on $[\theta_2, \rho_2]$.

If in turn we have $\tau = \sigma$, we deduce from the previous inequality and from (4) that for all $t \in [\tau, \rho_2]$

$$\begin{aligned}\left|\xi^{\varepsilon,i_1}_{\theta_2,t}(x) - x^{i_1}\right| &\leq \left|\xi^{\varepsilon,i_1}_{\theta_2,t}(x) - \xi^{\varepsilon,i_1}_{\theta_2,\sigma}(x)\right| + \left|\xi^{\varepsilon,i_1}_{\theta_2,\sigma}(x) - x^{i_1}\right|\\ &\leq a\left(D^{i_1}_T + \frac{3}{A}\big(D^{i_1}_T + D^{i_2}_T\big) + \frac{N}{A\pi}\right),\end{aligned} \tag{12}$$

and, also because of (4), inequality (12) holds then for all $t \in [\theta_2, \rho_2]$.

This achieves the proof in *Case (b)*.

Bringing together Case *(a)* and *b)*, we conclude that for all $t \in [\theta_2, \rho_2]$

$$\left|\xi^{\varepsilon,i_1}_{\theta_2,t}(x) - x^{i_1}\right| \leq \frac{a}{A}\left((A+3)\big(D^{i_1}_T + D^{i_2}_T\big) + \frac{N}{\pi}\right).$$

Interchanging the roles of i_1 and i_2 provides the desired upper bound for i_2. □

We introduce next some notation. For each $I \subset \{1, \ldots, N\}$ we define

$$F_T(I) := 0 \quad \text{if } I = \emptyset,$$
$$F_T(I) := aD_T^i \quad \text{if } I = \{i\}$$

and, for all I such that $n := |I| \geq 2$,

$$F_T(I) := \frac{a}{A}\left(\sum_{k\in I} D_T^k + \left(2n(n-1) + \frac{A}{a}\right)\max_{(J_1,J_2)}\{F_T(J_1) + F_T(J_2)\} + \frac{Nn(n-1)^2}{\pi}\right),$$

where the maximum is taken over all non-trivial partitions $\{J_1, J_2\}$ of I into two subsets. The random variable $F_T(I)$ depends only on $N, n = |I|, T$ and on the random variables $\sup_{t\in[0,T]}|B_t^i|$, for $i \in I$. Observe moreover that $F_T(I)$ has finite moments of all orders for all I.

The following elementary fact will be useful:

Remark 2.1. Let $J \subset \mathbb{R}^2$ be a finite set of $n \geq 2$ elements. Suppose there are $y, z \in J$ and $d > 0$ such that $|y - z| \geq d$. Then, there is a non-trivial partition $\{J_1, J_2\}$ of J such that

$$\min_{(y_1,y_2)\in J_i\times J_2} |y_1 - y_2| \geq \frac{d}{n}$$

We can now state and prove main result of this section, which is a generalization of the previous lemma:

Proposition 2.1. *Let $I = \{i_1, \ldots, i_n\} \subseteq \{1, \cdots N\}$, $\theta_n, \rho_n \in [0, T]$ with $\theta_n \leq \rho_n$ and write*

$$\Gamma^I_{\theta_n,\rho_n} := \left\{(x,\omega)\colon \sum_{j\notin I}\frac{1}{2\pi}\left|\xi^{\varepsilon,i}_{\theta_n,s}(x,\omega) - \xi^{\varepsilon,j}_{\theta_n,s}(x,\omega)\right|^{-1} < 1 - \frac{(n-1)}{N} \quad \forall i \in I, \quad \forall s \in [\theta_n, \rho_n[\right\}.$$

Then,

$$\Gamma^I_{\theta_n,\rho_n} \subseteq \left\{(x,\omega)\colon\left|\xi^{\varepsilon,i}_{\theta_n,t}(x,\omega) - x^i\right| \leq F_T(I)\forall i \in I, \forall t \in [\theta_n, \rho_n]\right\}.$$

Proof: The proof is by induction in $n = |I| \in \{1, \dots, N\}$. From previous lemmas, we know that the statement is true for $n = 1$ and $n = 2$. Consider $n \in \{3, \dots N\}$ and suppose the assertion is true for all $m \leq n - 1$. We will prove it also holds for $m = n$.

Let $(x, \omega) \in \Gamma^I_{\theta_n, \rho_n}$, then we have for $t \geq \theta_n$ that

$$
\begin{aligned}
&M^I_{\theta_n,t}(x) - M^I_{\theta_n,\theta_n}(x) \\
&\quad = \sqrt{2\nu} \sum_{k\in I} a_k \big(B^k_t - B^k_{\theta_n}\big) + \int_{\theta_n}^t \sum_{k\in I} \sum_{j\notin I} \big(K_\varepsilon\big(\xi^{\varepsilon,k}_{\theta_n,s}(x) - \xi^{\varepsilon,j}_{\theta_n,s}(x)\big)\big) a_j \, ds.
\end{aligned}
$$

(we omit again ω when writing ξ^ε). Therefore, for $\theta_n \leq t \leq \rho_n$

$$
\begin{aligned}
\big|M^I_{\theta_n,t}(x) - M^I_{\theta_n,\theta_n}(x)\big| &\leq 2a\sqrt{2\nu} \sum_{k\in I} S^k_T + naT\left(1 - \frac{(n-1)}{N}\right) \\
&\leq a \sum_{k\in I} D^k_T.
\end{aligned}
$$

Let us fix $i \in I$. We have

$$
\begin{aligned}
M^I_{\theta_n,t}(x) - M^I_{\theta_n,\theta_n}(x) &= \sum_{k\in I} a_k\big(\xi^{\varepsilon,i}_{\theta_n,t}(x) - x^i\big) \\
&\quad + \sum_{k\in I\setminus\{i\}} a_k\big(\big(\xi^{\varepsilon,k}_{\theta_n,t}(x) - x^k\big) - \big(\xi^{\varepsilon,i}_{\theta_n,t}(x) - x^i\big)\big)
\end{aligned}
$$

from where, for all $\theta_n \leq t \leq \rho_n$,

$$
A\big|\xi^{\varepsilon,i}_{\theta_n,t}(x) - x^i\big| \leq a\left(\sum_{k\in I} D^k_T + \sum_{k\in I\setminus\{i\}} \big(\big|\xi^{\varepsilon,i}_{\theta_n,t}(x) - \xi^{\varepsilon,k}_{\theta_n,t}(x)\big| + |x^i - x^k|\big)\right). \tag{13}
$$

Define

$$
E_T(I) := \max\{F_T(J_1) + F_T(J_2)\}
$$

where the maximum is taken over all non trivial partitions $\{J_1, J_2\}$ of I. Consider the time instant

$$
\begin{aligned}
\sigma := \inf\Bigg\{ &t > \theta_n \colon A\big|\xi^{\varepsilon,i}_{\theta_n,t}(x) - x^i\big| \\
&> a\left(\sum_{k\in I} D^k_T + 2n(n-1)E_T(I) + \frac{Nn(n-1)^2}{\pi}\right)\Bigg\}.
\end{aligned}
$$

Clearly, we have $\theta_n < \sigma \leq \infty$ and for all $t \in [\theta_n, \sigma \wedge \rho_n]$,

$$\left|\xi^{\varepsilon,i}_{\theta_n,t}(x) - x^i\right| \leq \frac{a}{A}\left(\sum_{k\in I} D^k_T + 2n(n-1)E_T(I) + \frac{Nn(n-1)^2}{\pi}\right). \tag{14}$$

Case (a) $\sigma \geq \rho_n$: The upper bound (14) then holds on $[\theta_n, \rho_n]$.

Case (b) $\sigma < \rho_n$: We have

$$A\left|\xi^{\varepsilon,i}_{\theta_n,\sigma}(x) - x^i\right| = a\left(\sum_{k\in I} D^k_T + 2n(n-1)E_T(I) + \frac{Nn(n-1)^2}{\pi}\right),$$

and we obtain from (13)

$$\sum_{k\in I\backslash\{i\}} \left(\left|\xi^{\varepsilon,i}_{\theta_n,\sigma}(x) - \xi^{\varepsilon,k}_{\theta_n,\sigma}(x)\right| + |x^i - x^k|\right) \geq 2n(n-1)E_T(I) + \frac{Nn(n-1)^2}{\pi}$$

Thus, there exists $k_0 \in I\backslash\{i\}$ such that

$$\left|\xi^{\varepsilon,i}_{\theta_n,\sigma}(x) - \xi^{k_0}_{\theta_n,\sigma}(x)\right| + |x^i - x^{k_0}| \geq 2nE_T(I) + \frac{Nn(n-1)}{\pi}$$

and we deduce the existence of some $\tau \in \{\theta_n, \sigma\}$ such that

$$\left|\xi^{\varepsilon,i}_{\theta_n,\tau}(x) - \xi^{k_0}_{\theta_n,\tau}(x)\right| \geq nE_T(I) + \frac{Nn(n-1)}{2\pi}.$$

By Remark 2.1 we have a non trivial partition $\{I_1, I_2\}$ of I, with $i \in I_1$ and $k_0 \in I_2$ such that

$$\begin{aligned}\min_{k_1\in I_1,k_2\in I_2} \left|\xi^{\varepsilon,k_1}_{\theta_n,\tau}(x) - \xi^{\varepsilon,k_2}_{\theta_n,\tau}(x)\right| &\geq \frac{N(n-1)}{2\pi} + E_T(I) \\ &> \frac{N(n-1)}{2\pi}\end{aligned} \tag{15}$$

Consequently, we have for each $k_1 \in I_1$

$$\sum_{k_2\in I_2} \frac{1}{2\pi}\left|\xi^{\varepsilon,k_1}_{\theta_n,\tau}(x) - \xi^{\varepsilon,k_2}_{\theta_n,\tau}(x)\right|^{-1} < \frac{1}{N}$$

and, for each $k_2 \in I_2$,

$$\sum_{k_1\in I_1} \frac{1}{2\pi}\left|\xi^{\varepsilon,k_1}_{\theta_n,\tau}(x) - \xi^{\varepsilon,k_2}_{\theta_n,\tau}(x)\right|^{-1} < \frac{1}{N}.$$

Therefore, since $(x,\omega) \in \Gamma^I_{\theta_n,\rho_n}$, for each $k_1 \in I_1$ we have that

$$\sum_{j\notin I_1} \frac{1}{2\pi}\left|\xi^{\varepsilon,k_1}_{\theta_n,\tau}(x) - \xi^{\varepsilon,j}_{\theta_n,\tau}(x)\right|^{-1} < 1 - \frac{(n-1)-1}{N}$$

and for each $k_2 \in I_2$,

$$\sum_{j \notin I_2} \frac{1}{2\pi} \left| \xi^{\varepsilon,k_2}_{\theta_n,\tau} - \xi^{\varepsilon,j}_{\theta_n,\tau} \right|^{-1} < 1 - \frac{(n-1)-1}{N}$$

Let us check as in the previous lemma that for all $s \in [\tau, \rho_n]$ and $k_1 \in I_1$ and $k_2 \in I_2$,

$$\sum_{j \notin I_1} \frac{1}{2\pi} \left| \xi^{\varepsilon,k_1}_{\theta_n,s} - \xi^{\varepsilon,j}_{\theta_n,s} \right|^{-1} < 1 - \frac{(n-1)-1}{N}, \tag{16}$$

and

$$\sum_{j \notin I_2} \frac{1}{2\pi} \left| \xi^{\varepsilon,k_2}_{\theta_n,s} - \xi^{\varepsilon,j}_{\theta_n,s} \right|^{-1} < 1 - \frac{(n-1)-1}{N}. \tag{17}$$

Suppose there exists $s \in [\tau, \rho_n]$ for which (16) or (17) do not hold. Then, setting

$$s^* := \inf \left\{ s \geq \tau : \exists p \in \{1,2\}, k \in I_p \text{ s.t. } \sum_{j \notin I_p} \frac{1}{2\pi} \left| \xi^{\varepsilon,k}_{\theta_n,s}(x) - \xi^{\varepsilon,j}_{\theta_n,s}(x) \right|^{-1} \geq 1 - \frac{(n-1)-1}{N} \right\}$$

we have

$$\tau \leq s^* \leq \rho_n,$$

and, for some $p \in \{1,2\}$ and $k \in I_p$ realizing s^*,

$$\sum_{j \notin I_p} \frac{1}{2\pi} \left| \xi^{\varepsilon,k}_{\theta_2,s^*}(x) - \xi^{\varepsilon,j}_{\theta_2,s^*}(x) \right|^{-1} = 1 - \frac{(n-1)-1}{N}.$$

Notice that then, we must have $\tau \neq s^*$. The definition of $\Gamma^I_{\theta_n,\rho_n}$ and the previous equality imply that for $q \in \{1,2\} \setminus \{p\}$,

$$\sum_{j \notin I_q} \frac{1}{2\pi} \left| \xi^{\varepsilon,k}_{\theta_2,s^*}(x) - \xi^{\varepsilon,j}_{\theta_2,s^*}(x) \right|^{-1} > \frac{1}{N}. \tag{18}$$

Now, for all $s \in [\tau, s^*[$, we have (from definition of s^*) that

$$\sum_{j \notin I_1} \frac{1}{2\pi} \left| \xi^{\varepsilon,k_1}_{\tau,s}\left(\xi_{\theta_n,\tau}(x)\right) - \xi^{\varepsilon,j}_{\tau,s}(\xi_{\theta_n,\tau}(x)) \right|^{-1} < 1 - \frac{(n-1)-1}{N}$$

and

$$\sum_{j \notin I_2} \frac{1}{2\pi} \left| \xi^{\varepsilon,k_2}_{\tau,s}\left(\xi_{\theta_n,\tau}(x)\right) - \xi^{\varepsilon,j}_{\tau,s}\left(\xi_{\theta_n,\tau}(x)\right) \right|^{-1} < 1 - \frac{(n-1)-1}{N}$$

for all $k_1 \in I_1$ and $k_2 \in I_2$. By induction hypothesis, this implies that for all $s \in [\tau, s^*]$,

$$\left|\xi^{\varepsilon,k_1}_{\tau,s}\left(\xi_{\theta_n,\tau}(x)\right) - \xi^{\varepsilon,k_1}_{\theta_n,\tau}(x)\right| = \left|\xi^{\varepsilon,k_1}_{\theta_n,s}(x) - \xi^{\varepsilon,k_1}_{\theta_n,\tau}(x)\right| \le F_T(I_1)$$

for all $k_1 \in I_1$ and

$$\left|\xi^{\varepsilon,k_2}_{\tau,s}\left(\xi_{\theta_n,\tau}(x)\right) - \xi^{\varepsilon,k_2}_{\theta_n,\tau}(x)\right| = \left|\xi^{\varepsilon,k_2}_{\theta_n,s}(x) - \xi^{\varepsilon,k_2}_{\theta_n,\tau}(x)\right| \le F_T(I_2)$$

for all $k_2 \in I_2$. But then, for $s = s^*$, $k \in I_p$, $I_q \subseteq I$ fixed as before, and any $j \in I_q$, we get

$$\begin{aligned}\left|\xi^{\varepsilon,k}_{\theta_n,s^*}(x)) - \xi^{\varepsilon,j}_{\theta_n,s^*}(x)\right| &\ge \left|\xi^{\varepsilon,k}_{\theta_n,\tau}(x) - \xi^{\varepsilon,j}_{\theta_n,\tau}(x)\right| - \left|\xi^{\varepsilon,k}_{\theta_n,\tau}(x) - \xi^{\varepsilon,k}_{\theta_n,s^*}(x)\right| \\ &\quad - \left|\xi^{\varepsilon,j}_{\theta_n,\tau}(x) - \xi^{\varepsilon,j}_{\theta_n,s^*}(x)\right| \\ &\ge \left|\xi^{\varepsilon,k}_{\theta_n,\tau}(x)) - \xi^{\varepsilon,j}_{\theta_n,\tau}(x)\right| - F_T(I_1) - F_T(I_2) \\ &\ge \left|\xi^{\varepsilon,k}_{\theta_n,\tau}(x)) - \xi^{\varepsilon,j}_{\theta_n,\tau}(x)\right| - E_T(I)\end{aligned}$$

from where we obtain, using also (15),

$$\sum_{j \notin I_p} \frac{1}{2\pi} \left|\xi^{\varepsilon,k}_{\theta_2,s^*}(x) - \xi^{\varepsilon,j}_{\theta_2,s^*}(x)\right|^{-1} \le \frac{1}{N},$$

contradicting (18).

Therefore, (16) and (17) must hold for all $s \in [\tau, \rho_n]$.

Next, from (16) with $k_1 = i$, together with the induction hypothesis applied to $\theta_{n-1} = \tau$ and $\rho_{n-1} = \rho_n$, we get that

$$\left|\xi^{\varepsilon,i}_{\tau,t}(\xi_{\theta_n,\tau}(x)) - \xi^{\varepsilon,i}_{\theta_n,\tau}(x)\right| = \left|\xi^{\varepsilon,i}_{\theta_n,t}(x) - \xi^{\varepsilon,i}_{\theta_n,\tau}(x)\right| \le F_T(I_1)$$

for all $t \in [\tau, \rho_n]$.

In case $\tau = \theta_n$, this implies the required upper bound in $[\theta_n, \rho_n]$.

If $\tau = \sigma$, we use the previous upper bound and (14) to get that

$$\begin{aligned}\left|\xi^{\varepsilon,i_1}_{\theta_n,t}(x) - x^{i_1}\right| &\le \left|\xi^{\varepsilon,i_1}_{\theta_n,t}(x) - \xi^{\varepsilon,i_1}_{\theta_n,\sigma}(x)\right| + \left|\xi^{\varepsilon,i_1}_{\theta_n,\sigma}(x) - x^{i_1}\right| \\ &\le E_T(I) + \frac{a}{A}\left(\sum_{k\in I} D^k_T + 2n(n-1)E_T(I) + \frac{Nn(n-1)^2}{\pi}\right)\end{aligned}$$

for all $t \in [\theta_n, \rho_n]$. This finishes the proof in *Case* (b), and the conclusion follows. □

Corollary 2.1. *For all $i \in \{1, \ldots, N\}$ and $T > 0$, it almost surely holds that*

$$\sup_{0\le s\le t\le T} \sup_{\varepsilon>0} \sup_{x\in(\mathbb{R}^2)^N} \left|\xi^{\varepsilon,i}_{s,t}(x,\omega) - x^i\right| \le F_T(\{1,\ldots,N\})(\omega) < \infty.$$

In particular, we have

$$\sup_{\varepsilon>0}\sup_{x\in(\mathbb{R}^2)^N}\left|\xi^{\varepsilon,i}_{0,t}(x,\omega)-x^i\right|\leq F_t(\{1,\dots,N\})(\omega).$$

Proof: We only have to prove the first statement. This is indeed straightforward, since we can choose in Proposition 2.1 $\theta_N = s$, $\rho_N = t$, and notice that the inequality in the definition of $\Gamma^{\{1,\dots,N\}}_{s,t}$ is simply $0 < \frac{1}{N}$. □

3. PATHWISE EXISTENCE AND UNIQUENESS FOR SOME VORTEX SYSTEMS

The proof of Theorem 1.1 will combine ideas of Ref. 7 with others of Rogers and Shi[(13)] used to show the absence of collision for a system diffusing particles interacting on the real line through a logarithmic potential. The argument relies on obtaining some uniform control on the expectation of the potential for the particles stopped at a sequence of times of "collisions up to distance ε". This allows to prove *a priori* absence of collisions. (The latter indeed is a generalization of an argument for Bessel processes, see e.g. Ref. 5). A different argument, but with a similar underlying idea, is the one given by Ref. 14 for proving existence of the stochastic vortex system under the assumption that all intensities a_i have the same sign.

We remark that in the cases of Refs. 13, 14, the authors rely on the positivity of some quantities arising when evaluating the logarithmic potential at a positive time instant. A supermartingale type argument allows then to control the (expectation of the) potential. In the present case, that control will be consequence of the moments estimates and of the fact that the stochastic flow (2) preserves volume, an argument used in Ref. 7 in the deterministic setting.

More precisely, in the way K_ε was defined, it is clear that $div\ K_\varepsilon = 0$. Then, the drift term of the stochastic flow (2) in $(\mathbb{R}^2)^N$ has null divergence too. The next result can then be proved in a similar way as the classic (deterministic) Liouville theorem (see e.g. Ref. 3).

Lemma 3.1. *Let $J\xi_t$ denote the Jacobian matrix of $\xi_t(x)$. Then $det(J\xi_t) = 1$.*

In the sequel, we write $\mathcal{B}_R$ for the centered ball of radius R in $\mathbb{R}^{2N}$. The following lemma will be used in the proof of Theorem 1.1 and is easily checked:

Lemma 3.2. *There exists a positive constant C_1 such that*

$$\int_{\mathcal{B}_R}\frac{dx}{|x^i-x^j||x^i-x^k|}\leq C_1 R^{(2N-2)}$$

Proof of Theorem 1.1: For each $\varepsilon > 0$, consider the stochastic flow (2) constructed on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathrm{P})$ and write

$$\xi_t^\varepsilon(x,\omega) := \xi_{0,t}^\varepsilon(x,\omega).$$

Let (X_0^i), $i = 1 \dots N$ be i.i.d $\mathbb{R}^2$-valued random variables in the same probability space of law $p_0(y)\,dy$. Denote by $X_t^\varepsilon = (X_t^{\varepsilon,i})_{i=1}^N$ the unique path-wise solution of

$$X_t^{\varepsilon,i} = X_0^i + \sqrt{2\nu} B_t^i + \int_0^t \sum_{j\neq i} K_\varepsilon\big(X_s^{\varepsilon,i} - X_s^{\varepsilon,j}\big) a_j\, ds, \quad i = 1,\dots,N,$$

so $X_t^{\varepsilon,i} = \xi_t^{\varepsilon,i}(X_0)$ P-a.s., where $X_0 = (X_0^1,\dots,X_0^N)$. We consider the $(\mathcal{F}_t)$-stopping time

$$T_\varepsilon := \inf\big\{t > 0 \colon \exists i \neq j \text{ s.t. } \big|X_t^{\varepsilon,i} - X_t^{\varepsilon,j}\big| \leq \varepsilon\big\},$$

and notice that for $\varepsilon' \leq \varepsilon$, it holds that

$$X_t^{\varepsilon,i} = X_t^{i,\varepsilon'} \quad \text{for all } t \leq T_\varepsilon \quad P-\text{ a.s.}$$

since $K_\varepsilon(x) = K_{\varepsilon'}(x)$ for all $|x| \leq \varepsilon'$ and $K_{\varepsilon'}$ is Lipschitz and bounded.

By the latter and by continuity, T_ε increases P-a.s. as $\varepsilon \to 0$. Our goal is to prove that for each $t > 0$,

$$\lim_{\varepsilon\to 0} T_\varepsilon > t, \quad P-\text{ a.s.} \tag{19}$$

We want to take advantage of the volume-preserving property of the flow, but since we are working in the whole space $\mathbb{R}^{2N}$, Lebesgue measure does not have a natural probabilistic sense. We therefore enlarge the probability space as follows: denote by $\mathcal{B}_R^\eta$ the subset of $\mathcal{B}_R$.

$$\mathcal{B}_R^\eta = \{x = (x^1,\dots,x^N) \in \mathbb{R}^{2N} : |x| \leq R \text{ and } \min_{i,j} |x^i - x^j| > \eta\},$$

and let ℓ_R^η be the normalized Lebesgue measure on $\mathcal{B}_R^\eta$.

For each $R > 0$ let the space $\tilde{\Omega}_R := \mathcal{B}_R \times \Omega$ be endowed with the natural completed product sigma-field, with respect to the product measure

$$P_R^\eta = \ell_R^\eta \otimes P.$$

We write E_R^η for the associated expectation, and $\mathcal{G}_t^R$ stands for the smallest sigma-field containing $\mathcal{F}_t \vee \beta_R$ and such that the filtration $(\mathcal{G}_t^R)$ satisfies the usual conditions.

Next, denote by $(Y_0^i)_{i=1}^N$ the random variables defined on $\tilde{\Omega}_R^\eta$ by $Y_0^i(y,\omega) = y$. The vector $(Y_0^i)_{i=1}^N$ has thus the law ℓ_R^η and is independent of the Brownian motions $B^1,\dots,B^N$.

For each ε, consider now on $\tilde{\Omega}_R^\varepsilon$ the process

$$Y_t^{i,\varepsilon}(y,\omega) := \xi_t^{i,\varepsilon}(y,\omega),$$

(which is a $\mathrm{P}_{\mathrm{R}}^{\eta}$- semi-martingale with respect to $(\mathcal{G}_t^R)$), and the $(\mathcal{G}_t^R)$-stopping time

$$\tau_\varepsilon := \inf\left\{t > 0 : \exists i \neq j \text{ s.t. } \left|Y_t^{\varepsilon,i} - Y_t^{\varepsilon,j}\right| \leq \varepsilon\right\}.$$

We will prove that

$$\lim_{\varepsilon\to 0} \tau_\varepsilon > t, \quad P- \text{ a.s.} \tag{20}$$

To that end, we consider the singular "potential" $\Phi(x) = \log(|x|^2)$. We shall establish the existence of a finite constant $C(R, \eta, t, N) > 0$, such that

$$\sup_{0<\varepsilon\leq\eta}\left|E_R^\eta\left(\sum_{i=1}^N\sum_{j\neq i}\Phi\left(Y_{t\wedge\tau_\varepsilon}^{\varepsilon,i} - Y_{t\wedge\tau_\varepsilon}^{\varepsilon,j}\right)\right)\right| < C(R, \eta, t, N). \tag{21}$$

Let $\Phi_\varepsilon(x)$ be a smooth function s.t. $\Phi_\varepsilon(x) = \ln(|x|^2) = \Phi(x)$ for $|x| \geq \varepsilon$. By Itô's formula, we have for all $\eta \geq \varepsilon > 0$ and $t > 0$ that

$$\begin{aligned}\Phi_\varepsilon\left(Y_t^{\varepsilon,i} - Y_t^{\varepsilon,j}\right) = \Phi_\varepsilon\left(Y_0^{\varepsilon,i} - Y_0^{\varepsilon,j}\right) + &\int_0^t \nabla\Phi_\varepsilon\left(Y_s^{\varepsilon,i} - Y_s^{\varepsilon,j}\right). \\ &\times\left\{\sum_{k\neq i} a_k K_\varepsilon\left(Y_s^{\varepsilon,i} - Y_s^{\varepsilon,k}\right) - \sum_{k\neq j} a_k K_\varepsilon\left(Y_s^{\varepsilon,j} - Y_s^{\varepsilon,k}\right)\right\} ds \\ &+ \sqrt{2\nu}\int_0^t \nabla\Phi_\varepsilon\left(Y_s^{\varepsilon,i} - Y_s^{\varepsilon,j}\right)\cdot d\left(B_s^i - B_s^j\right) \\ &+ 2\nu\int_0^t \triangle\Phi_\varepsilon\left(Y_s^{\varepsilon,i} - Y_s^{\varepsilon,j}\right) ds.\end{aligned}$$

Since $\Phi_\varepsilon(x) = \ln(|x|^2) = \Phi(x)$ for $|x| \geq \varepsilon$, the last term vanishes if $t \leq \tau_\varepsilon$. On the other hand, $\nabla\Phi_\varepsilon(x)$ and $K_\varepsilon(x)$ are orthogonal for all x, so we obtain that P_R^η-a.s.,

$$\begin{aligned}\Phi\left(Y_{t\wedge\tau_\varepsilon}^{\varepsilon,i} - Y_{t\wedge\tau_\varepsilon}^{\varepsilon,j}\right) = \Phi\left(Y_0^{\varepsilon,i} - Y_0^{\varepsilon,j}\right) + &\int_0^{t\wedge\tau_\varepsilon} \nabla\Phi\left(Y_s^{\varepsilon,i} - Y_s^{\varepsilon,j}\right). \\ &\times\left\{\sum_{k\neq i,j} a_k K_\varepsilon\left(Y_s^{\varepsilon,i} - Y_s^{\varepsilon,k}\right) - \sum_{k\neq i,j} a_k K_\varepsilon\left(Y_s^{\varepsilon,j} - Y_s^{\varepsilon,k}\right)\right\} ds \\ &+ \sqrt{2\nu}\int_0^{t\wedge\tau_\varepsilon} \nabla\Phi_\varepsilon\left(Y_s^{\varepsilon,i} - Y_s^{\varepsilon,j}\right)\cdot d\left(B_s^i - B_s^j\right).\end{aligned}$$

The last term is a stopped martingale, so summing over $i \neq j$ and taking expectation we get the following bound,

$$\left| E_R^\eta\left(\sum_{i=1}^N \sum_{j\neq i} \Phi\big(Y_{t\wedge\tau_\varepsilon}^{\varepsilon,i} - Y_{t\wedge\tau_\varepsilon}^{\varepsilon,j}\big)\right)\right| \leq \sum_{i=1}^N \sum_{j\neq i} \left|E_R^\eta\big(\Phi(Y_0^{\varepsilon,i} - Y_0^{\varepsilon,j})\right|$$
$$+ C\sum_{i=1}^N \sum_{j\neq i}\sum_{k\neq i,j} E_R^\eta\left[\int_0^{t\wedge\tau_\varepsilon} \frac{ds}{\left|Y_s^{\varepsilon,i} - Y_s^{\varepsilon,j}\right|\left|Y_s^{\varepsilon,i} - Y_s^{k,\varepsilon}\right|}\right]$$

using also the fact that $|K_\varepsilon(x)| \leq C'\frac{1}{|x|}$ for some $C' > 0$.

Now, there is a constant $C(R,\eta) > 0$ such that for $i \neq j$,

$$\left|E_R^\eta\big(\Phi\big(Y_0^{\varepsilon,i} - Y_0^{\varepsilon,j}\big)\right| = \left|\int_{\mathcal{B}_R^\eta} \ln(|x^i - x^j|^2)\ell_R^\eta(dx)\right| \leq C(R,\eta).$$

On the other hand, observe that for different indexes i, j, k, by conditioning on $\mathcal{G}_0$ we have

$$E_R^\eta\left[\int_0^{t\wedge\tau_\varepsilon} \frac{ds}{\left|Y_s^{\varepsilon,i} - Y_s^{\varepsilon,j}\right|\left|Y_s^{\varepsilon,i} - Y_s^{k,\varepsilon}\right|}\right]$$
$$\leq E_R^\eta\left[\int_0^{t} \frac{ds}{\left|Y_s^{\varepsilon,i} - Y_s^{\varepsilon,j}\right|\left|Y_s^{\varepsilon,i} - Y_s^{k,\varepsilon}\right|}\right]$$
$$= \frac{1}{\mathrm{Vol}\big(\mathcal{B}_R^\eta\big)} E\int_0^t \left[\int_{\mathcal{B}_R^\eta} \frac{dx}{\left|\xi_s^{\varepsilon,i}(x) - \xi_s^{\varepsilon,j}(x)\right|\left|\xi_s^{\varepsilon,i}(x) - \xi_s^{k,\varepsilon}(x)\right|}\right] ds$$
$$= \frac{1}{\mathrm{Vol}\big(\mathcal{B}_R^\eta\big)} E\int_0^t \left[\int_{\xi_s^\varepsilon(\mathcal{B}_R^\eta)} \frac{dx}{|x^i - x^j||x^i - x^k|}\right] ds.$$

The last identity is due to the fact that P-a.s., the map $x \mapsto \xi_s^\varepsilon(x)$ is Lebesgue-measure preserving (cf. Lemma 3.1). Since $\mathcal{B}_R^\eta \subseteq \mathcal{B}_R$, we know from Corollary 2.1 that P-a.s., for all $s \in [0,t]$

$$\xi_s^\varepsilon\big(\mathcal{B}_R^\eta\big) \subseteq \mathcal{B}_{R+F_s(\{1,\ldots,N\})} \subseteq \mathcal{B}_{R+F_t(\{1,\ldots,N\})}.$$

This and Lemma 3.2 imply that

$$E_R^\eta\left[\int_0^{t\wedge\tau_\varepsilon} \frac{ds}{\left|Y_s^{\varepsilon,i} - Y_s^{\varepsilon,j}\right|\left|Y_s^{\varepsilon,i} - Y_s^{k,\varepsilon}\right|}\right] \leq CtE\left[\int_{\mathcal{B}_{R+F_t(\{1,\ldots,N\})}} \frac{dx}{|x^i - x^j||x^i - x^k|}\right]$$
$$\leq CtE((R + F_t(\{1,\ldots,N\})^4)$$
$$< \infty,$$

which together with the previous estimates yields (21).

Next, if i_*, j_* denote the random indexes where the inf defining τ_ε is attained, we have

$$E_R^\eta\left(\sum_{i=1}^N\sum_{j\neq i}\Phi\big(Y_{t\wedge\tau_\varepsilon}^{\varepsilon,i}-Y_{t\wedge\tau_\varepsilon}^{\varepsilon,j}\big)\right)$$

$$=2\ln\varepsilon P_R^\eta(\tau_\varepsilon\leq t)+E_R^\eta\left(\sum_{i\neq i_*}\sum_{j\neq i,j_*}\Phi\big(Y_{\tau_\varepsilon}^{\varepsilon,i}-Y_{\tau_\varepsilon}^{\varepsilon,j}\big)\mathbf{1}_{\{\tau_\varepsilon\leq t\}}\right)$$

$$+E_R^\eta\left(\sum_{i=1}^N\sum_{j\neq i}\Phi\big(Y_t^{\varepsilon,i}-Y_t^{\varepsilon,j}\big)\mathbf{1}_{\{\tau_\varepsilon>t\}}\right). \tag{22}$$

On the other hand, from Corollary 2.1, P_R^η-a.s. we have, for all $i\neq j$,

$$\Phi\big(Y_{\tau_\varepsilon}^{\varepsilon,i}-Y_{\tau_\varepsilon}^{\varepsilon,j}\big)\mathbf{1}_{\{\tau_\varepsilon\leq t\}}\leq 2\ln(2R+2F_{\tau_\varepsilon})\mathbf{1}_{\{\tau_\varepsilon\leq t\}}\leq 2\ln_+(2R+2F_t)$$

where $\ln_+(x)=(\ln(x)\vee 0)$. Similarly,

$$\Phi\big(Y_t^{\varepsilon,i}-Y_t^{\varepsilon,j}\big)\mathbf{1}_{\{\tau_\varepsilon>t\}}\leq 2\ln_+(2R+2F_t)$$

From this estimates, (21) and (22), we deduce that for all $\varepsilon>0$,

$$2\ln\varepsilon P_R^\eta(\tau_\varepsilon\leq t)\geq -C(R,\eta,t,N)-4(N^2-N)E(\ln_+(2R+2F_t))>-\infty$$

We conclude that $\lim_{\varepsilon\to 0}P_R^\eta(\tau_\varepsilon\leq t)=0$, from where (20) follows.

Since this is true for all $t>0$, we deduce that

$$\lim_{\varepsilon\to 0}\tau_\varepsilon=+\infty,\quad P_R^\eta-\text{a.s.}$$

or equivalently

$$\int_{\mathcal{B}_R^\eta}P\Big[\lim_{\varepsilon\to 0}\big(\inf\{s>0:\exists i\neq j\text{ s.t. }\big|\xi_s^{\varepsilon,i}(x)-\xi_s^{\varepsilon,j}(x)\big|\leq\varepsilon\}\big)<\infty\Big]\ell_R^\eta(dx)=0.$$

Consequently, η and $R>0$ being arbitrary, we deduce that

$$P\Big[\lim_{\varepsilon\to 0}\big(\inf\{s>0:\exists i\neq j\text{ s.t. }\big|\xi_s^{\varepsilon,i}(x)-\xi_s^{\varepsilon,j}(x)\big|\leq\varepsilon\}\big)<\infty\Big]=0,\quad dx-\text{a.s.}$$

Integrating over $\mathbb{R}^{2N}$ with respect to $(\prod_{i=1}^N p_0(x_i))dx$ we conclude that

$$P\left[\lim_{\varepsilon\to 0}\big(\inf\{s>0:\exists i\neq j\text{ s.t. }\big|X_s^{\varepsilon,i}-X_s^{\varepsilon,j}\big|\leq\varepsilon\}\big)<\infty\right]=0,$$

that is, (19) holds.

We conclude that the following process is P-a.s. well defined for all $t > 0$:

$$X_t = X_t^\varepsilon \quad \text{for all } \varepsilon > 0 \quad \text{such that} \quad t \leq T_\varepsilon.$$

By the Lipschitz property of K_ε for all $\varepsilon > 0$, and since $T_\varepsilon \to +\infty$ when $\varepsilon \to 0$, it is simple to check that X is the unique path-wise solution of (1).

4. SOME FINAL COMMENTS

There are several aspects in which the previous arguments are not satisfactory, or could be improved. First, for the "clustering" argument we only used the decay of the interaction, and not the specific "rotational" form of it. Indeed, the drift induced by a particle, say X^j on a particle X^i is orthogonal to their relative positions (see Ref. 7 for details) and this should be taken into account. It is also likely that some of the "clustering occurrence" events studied in Sec. 2 have small probabilities. Since we need to compute expectations, this could somehow compensate our badly behaved "L^∞"-estimates, but needs a much more careful analysis.

On the other hand, it comes clear from the proof of Lemma 2.2 that roughly, due to assumption **(H)**, whenever some particle in one cluster moves far away from its initial position, then some other particle must have moved away from the first. If **(H)** is violated by some subsystem, then one particle moving away from its initial position only forces some other particle to move away from its own initial position. Thus, nothing prevents such two particles from staying closed to each other. This poses problems when trying to define clusters, and could in principle allow for collisions between the two particles. Nevertheless, in a (very) simple situation, assumption **(H)** can be removed easily:

Lemma 4.1. *If $N = 2$, $a_1 = -a_2 = a$ and $X_0^i, i = 1, 2$ have a densities with respect to Lebesgue measure and are independent of the (independent) Brownian motions B^1, B^2, then there is pathwise existence and uniqueness for (1).*

Proof: We are looking for a solution of

$$\begin{aligned} X_t^1 &= X_0^1 + \sqrt{2\nu} B_t^1 - a \int_0^t K\left(X_s^1 - X_s^2\right) ds, \\ X_t^2 &= X_0^2 + \sqrt{2\nu} B_t^2 + a \int_0^t K\left(X_s^2 - X_s^1\right) ds \end{aligned} \tag{23}$$

with the X_0^i's independent of the B^i's. Consider the Brownian motion

$$Y_t := X_0^1 - X_0^2 + \sqrt{2\nu}\left(B_t^1 - B_t^2\right).$$

Then, almost surely we have $Y_t \neq 0$ for all $t \geq 0$. We can then define

$$X_t^1 = X_0^1 + \sqrt{2\nu} B_t^1 - a \int_0^t K(Y_s)\, ds,$$

$$X_t^2 = X_0^2 + \sqrt{2\nu} B_t^2 + a \int_0^t K(-Y_s)\, ds,$$

and observe that since K is odd, we get $X_t^1 - X_t^2 = Y_t$. Thus, we have a solution of (23). It is similarly seen that this is the unique solution. □

The previous lemma suggests that a cluster I of two particles that do not satisfy **(H)** will not experience collisions if it is far away enough from other particles (this of course needs a proof). At least three particles should then be involved in order to produce a (maybe two-particles) collision. A better understanding of the general case could be gained from a rigorous analysis of a three particle system not satisfying **(H)**. We shall not pursue that problem here.

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