Optimal length estimates for stable CMC surfaces in 3-space-forms

Laurent Mazet

Abstract

In this paper, we study stable constant mean curvature $H$ surfaces in $\mathbb{R}^3$. We prove that, in such a surface, the distance from a point to the boundary is less that $\pi/(2H)$. This upper-bound is optimal and is extended to stable constant mean curvature surfaces in space forms.

1 Introduction

A constant mean curvature (cmc) surface $\Sigma$ in a Riemannian 3-manifold $M^3$ is stable, if its stability operator $L = -\Delta - \text{Ric}(n,n) - |A|^2$ is nonnegative. The nonnegativity of this operator means that $\Sigma$ is a local minimizer of the area functional on surfaces regard to the infinitesimal deformations fixing its boundary.

The stability hypothesis was studied by several authors and has many consequences (see [6] for an overview). For example, D. Fischer-Colbrie and R. Schoen [4] studied the case of complete stable minimal surfaces when $M^3$ has non-negative scalar curvature. They obtain that the universal cover of $\Sigma$ is not conformally equivalent to the disk and, as a consequence, prove that the plane is the only complete stable minimal surface in $\mathbb{R}^3$. From this, R. Schoen [8] has derived a curvature estimate for stable cmc surfaces.

In [2], T. H. Colding and W. P. Minicozzi introduced new technics and obtained area and curvature estimates for stable cmc surfaces. Afterward, these technics were used by P. Castillon [1] to answer a question asked in [4] about the consequences of the positivity of certain elliptic operators. Recently, the same ideas have been used by J. Espinar and H. Rosenberg [3] to obtain similar results.

In [7], A. Ros and H. Rosenberg study constant mean curvature $H$ surfaces in $\mathbb{R}^3$ with $H \neq 0$ : they prove a maximum principle at infinity. One of their tools is a length estimate for stable cmc surface. In fact, they prove
that the intrinsic distance from a point $p$ in a stable cmc surface $\Sigma$ to the boundary of $\Sigma$ is less than $\pi/H$. The aim of this paper is to improve this result. In fact, applying the ideas of [2], we prove that the distance is less than $\pi/(2H)$. This estimate is optimal since, for a hemisphere of radius $1/H$, the distance from the pole to the boundary is $\pi/(2H)$. Actually we prove that the hemisphere of radius $1/H$ is the only stable cmc $H$ surface where the distance $\pi/(2H)$ is reached. We can generalized this result to stable cmc $H$ surfaces in $\mathbb{M}^3(\kappa)$, where $\mathbb{M}^3(\kappa)$ is the 3-space form of sectional curvature $\kappa$. We prove that when $H^2 + \kappa > 0$ such an optimal estimate exists. In fact, it is already known that, when $\kappa \leq 0$ and $H^2 + \kappa \leq 0$, there is no such estimate since there exist complete stable cmc $H$ surfaces. But, in some sense, our results is an extension of the fact that the planes (resp. the horospheres) are the only stable complete constant mean curvature $H$ surfaces in $\mathbb{R}^3$ (resp. $\mathbb{M}^3(\kappa), \kappa < 0$) when $H = 0$ (resp. $H^2 + \kappa = 0$).

2 Definitions

On a constant mean curvature surface $\Sigma$ in a Riemannian 3-manifold $\mathbb{M}^3$, the stability operator is defined by $L = -\Delta - \text{Ric}(n, n) - |A|^2$, where $\Delta$ is the Laplace operator on $\Sigma$, $\text{Ric}$ is the Ricci tensor on $\mathbb{M}^3$, $n$ is the normal to $\Sigma$ and $A$ is the second fundamental form on $\Sigma$. When it is necessary, we will denote the stability operator by $L_f$ to refer to the immersion $f$ of $\Sigma$ in $\mathbb{M}^3$.

The surface $\Sigma$ is called stable if the operator $L$ is nonnegative i.e., for every compactly supported function $u$, we have

$$0 \leq \int_{\Sigma} uL(u)d\sigma = \int_{\Sigma} \|\nabla u\|^2 - (\text{Ric}(n, n) + |A|^2)u^2d\sigma$$

We remark that this property is sometimes called strong stability since it means that the second derivatives of the area functional is nonnegative with respect to any compactly supported infinitesimal deformations $u$ whereas $\Sigma$ is critical for this functional only for compactly supported infinitesimal deformations with vanishing mean value i.e. $\int_{\Sigma} ud\sigma = 0$.

In the following, on a cmc surface, the normal $n$ is always chosen such that $H$ is non-negative.

We will denote by $d_\Sigma$ the intrinsic distance on $\Sigma$ and by $K$ the sectional curvature of the surface.
3 Results

The main result of this paper is the following theorem.

**Theorem 1.** Let $H$ be positive. Let $\Sigma$ be a stable constant mean curvature $H$ surface in $\mathbb{R}^3$. Then, for $p \in \Sigma$, we have:

$$d_\Sigma(p, \partial \Sigma) \leq \frac{\pi}{2H}$$

(1)

Moreover, if the equality is satisfied, $\Sigma$ is a hemisphere.

In $\mathbb{R}^3$, the stability operator can be written $L = -\Delta - 4H^2 + 2K$.

**Proof.** We denote by $R_0$ the distance $d_\Sigma(p, \partial \Sigma)$ and assume that $R_0 \geq \pi/(2H)$. If $R_0 < \pi/H$ we denote by $I$ the segment $[\pi/(2H), R_0]$, otherwise $I = [\pi/(2H), \pi/H]$. In fact, because of the work of Ros and Rosenberg [7], we already know that $R_0 \leq \pi/H$. Let $R$ be in $I$.

The surface $\Sigma$ has constant mean curvature $H$ thus its sectional curvature is less than $H^2$. So the exponential map $\exp_p$ is a local diffeomorphism on the disk $D(0, R) \subset T_p \Sigma$ of center $0$ and radius $R$. On this disk, we consider the induced metric and the operator $L = -\Delta - 4H^2 + 2K$. The surface $\Sigma$ is stable so it exists a positive function $g$ on $\Sigma$ such that $L(g) = 0$ (see Theorem 1 in [4]). On $D(0, R)$, the function $\tilde{g} = g \circ \exp_p$ is then positive and satisfies $L(\tilde{g}) = 0$ since $D(0, R)$ and $\Sigma$ are locally isometric. The operator $L$ is thus nonnegative on $D(0, R)$ [4].

For $r \in [0, R]$, we define $l(r)$ as the length of the circle $\{v, \ |v| = r\} \subset D(0, R)$ and $K(r) = \int_{D(0,r)} Kd\sigma$. Since $D(0, R)$ and $\Sigma$ are locally isometric, the sectional curvature $K$ of $D(0, R)$ is less than $H^2$. Then

$$l(r) \geq \frac{2\pi}{H} \sin Hr$$

(2)

By Gauss-Bonnet, we have:

$$K(r) = 2\pi - l'(r)$$

(3)

Let us consider a function $\eta : [0, R] \to [0, 1]$ with $\eta(0) = 1$ and $\eta(R) = 0$. Let us write the nonnegativity of $L$ for the radial function $u = \eta(r)$.

$$0 \leq \int_0^R (\eta'(r))^2 l(r)dr - 4H^2 \int_0^R \eta^2(r)l(r)dr + 2 \int_0^R K'(r)\eta^2(r)dr$$

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Hence, following the ideas in [2] and using (3) and the boundary values of \( \eta \), we have:

\[
\int_0^R (4H^2\eta^2 - \eta'^2) \, dr \leq 2 \left( [K(r)\eta^2(r)]_0^R - \int_0^R K(r)(\eta^2(r))' \, dr \right)
\]

\[
\leq -2 \int_0^R K(r)(\eta^2(r))' \, dr
\]

\[
\leq -2 \int_0^R (2\pi - l'(r))(\eta^2(r))' \, dr
\]

\[
\leq 4\pi + 2 \int_0^R (\eta^2(r))'l'(r) \, dr
\]

\[
\leq 4\pi + [2(\eta^2(r))'l(r)]_0^R - 2 \int_0^R (\eta^2(r))''l(r) \, dr
\]

\[
\leq 4\pi - 2 \int_0^R (\eta^2(r))''l(r) \, dr
\]

Thus we obtain

\[
\int_0^R \left( 4H^2\eta^2 - \eta'^2 + 2(\eta^2)'' \right) \, dr \leq 4\pi
\]

We shall apply this equation to the function \( \eta(r) = \cos \frac{\pi r}{2R} \). In this case we have

\[
\eta'^2 = \frac{\pi^2}{4R^2} \sin^2 \frac{\pi r}{2R}
\]

\[
(\eta^2)'' = -\frac{\pi^2}{2R^2} \left( \cos^2 \frac{\pi r}{2R} - \sin^2 \frac{\pi r}{2R} \right)
\]

Thus

\[
4H^2\eta^2 - \eta'^2 + 2(\eta^2)'' = (4H^2 - \frac{\pi^2}{R^2}) \cos^2 \frac{\pi r}{2R} + \frac{3\pi^2}{4R^2} \sin^2 \frac{\pi r}{2R}
\]

As \( R \geq \frac{\pi}{2H} \), \( 4H^2\eta^2 - \eta'^2 + 2(\eta^2)'' \) is non-negative and, by (2),

\[
\left( 4H^2\eta^2 - \eta'^2 + 2(\eta^2)'' \right) \geq \left( (4H^2 - \frac{\pi^2}{R^2}) \cos^2 \frac{\pi r}{2R} + \frac{3\pi^2}{4R^2} \sin^2 \frac{\pi r}{2R} \right) \frac{2\pi}{H} \sin Hr
\]

\[
\geq \frac{\pi}{H} \left( (4H^2 - \frac{\pi^2}{4R^2}) \sin Hr + (4H^2 - \frac{7\pi^2}{4R^2}) \frac{1}{2} \left( \sin \left( \frac{\pi}{R} + H \right) r - \sin \left( \frac{\pi}{R} - H \right) r \right) \right)
\]
Thus integrating in (4), we obtain (we recall that $R < \pi/H$)

$$4\pi \geq \frac{\pi}{H} \left( (4H^2 - \frac{\pi^2}{4R^2}) \frac{1}{H} (1 - \cos HR) + (4H^2 - \frac{7\pi^2}{4R^2}) \frac{1}{2} \left( \frac{R}{\pi + HR} (1 - \cos (\pi HR)) - \frac{R}{\pi - HR} (1 - \cos (\pi - HR)) \right) \right)$$

After some simplifications in the above expression, we obtain

$$4\pi \geq \pi \left( -\frac{32H^2 R^4}{H^2} + 24\pi^2 H^2 R^2 - \pi^4 \right) - (10\pi^2 H^2 R^2 - \pi^4) \cos HR$$

Now, passing $4\pi$ on the right-hand side of the above inequality and simplifying by $\pi$, we obtain:

$$0 \geq -\frac{(4H^2 R^2 - \pi^2)^2}{H^2 R^2 (\pi^2 - H^2 R^2)} - (10\pi^2 H^2 R^2 - \pi^4) \cos HR$$

We denote by $F(R)$ the right-hand term of the above inequality. Hence we have proved that, for every $R$ in $I$, $F(R) \leq 0$. If we write $R = \pi/(2H) + x$, we compute the Taylor expansion of $F$ and obtain

$$F\left(\frac{\pi}{2H} + x\right) = 2Hx + o(x)$$

which is positive if $x > 0$. Thus, if $R_0 > \pi/(2H)$, we get a contradiction and the inequality (1) is proved.

Now if $R_0 = \pi/(2H)$, we have in fact equality all along the computation, so $l(r) = 2\pi/H \sin Hr$ and $K(r) = 2\pi - l''(r) = 2\pi(1 - \cos Hr)$. But we also know that the sectional curvature is less than $H^2$ thus $K(r) \leq H^2 \int_0^1 l(u) du = 2\pi(1 - \cos Hr)$. Since this inequality is in fact an equality, the sectional curvature is in fact $H^2$ at every point. Thus the principal curvatures of a point in $\Sigma$ are $H$ and $H$ i.e. there are only umbilical points. Hence $\Sigma$ is a piece of a sphere of radius $1/H$ and, since $d_{\Sigma}(p, \partial \Sigma) = \frac{\pi}{2H}$, it contains the hemisphere of pole $p$. A hemisphere can not be strictly contained in a stable subdomain of the sphere, so $\Sigma$ is a hemisphere. 

With this result we have an important corollary.

**Corollary 2.** Let $H \geq 0$ and $\kappa \in \mathbb{R}$ such that $H^2 + \kappa > 0$. Let $\Sigma$ be a stable constant mean curvature $H$ surface in $\mathbb{M}^3(\kappa)$. Then for $p \in \Sigma$, we have:

$$d_{\Sigma}(p, \partial \Sigma) \leq \frac{\pi}{2\sqrt{H^2 + \kappa}}$$

Moreover, if the equality is satisfied, $\Sigma$ is a geodesical hemisphere of $\mathbb{M}^3(\kappa)$. 

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The proof is based on the Lawson’s correspondence between constant mean curvature surfaces in space forms (see [5]).

Proof. First, the case \( \kappa = 0 \) is Theorem 1.

Let \( \Pi : \tilde{\Sigma} \to \Sigma \) be the universal cover of \( \Sigma \). We then have a constant mean curvature immersion of \( \tilde{\Sigma} \) in \( M^3(\kappa) \), let \( \mathcal{L} = -\Delta - 2\kappa - |A|^2 \) be the stability operator on \( \tilde{\Sigma} \). \( \Sigma \) is stable, so there exists a positive function \( g \) on \( \Sigma \) such that \( \mathcal{L}(g) = -\Delta g - (2\kappa + |A|^2)g = 0 \). Thus the function \( \tilde{g} = g \circ \Pi \) is a positive function on \( \tilde{\Sigma} \) satisfying \( \mathcal{L}(\tilde{g}) = 0 \). Hence \( \tilde{\Sigma} \) is stable. Let \( I \) and \( S \) be respectively the first fundamental form and the shape operator on \( \tilde{\Sigma} \). They satisfy the Gauss and Codazzi equations for \( M^3(\kappa) \).

We define \( S' = S + (-H + \sqrt{H^2 + \kappa})\text{id} \) on \( \tilde{\Sigma} \). Then \( I \) and \( S' \) satisfy the Gauss and Codazzi equations for \( M^3(0) = \mathbb{R}^3 \) (see [5]). Hence there exists an immersion \( f \) of \( \tilde{\Sigma} \) in \( \mathbb{R}^3 \) with first fundamental form \( I \) and shape operator \( S' \) (we notice that the induced metric is the same). Its mean curvature is then \( H + (-H + \sqrt{H^2 + \kappa}) = \sqrt{H^2 + \kappa} \) i.e. the immersion has constant mean curvature. The stability operator is

\[
\mathcal{L}_f = -\Delta - ||S'||^2
= -\Delta - (||S||^2 + 4H(-H + \sqrt{H^2 + \kappa}) + 2(-H + \sqrt{H^2 + \kappa})^2)
= -\Delta - (||S||^2 + 2\kappa)
= \mathcal{L}
\]

Hence the surface \( f(\tilde{\Sigma}) \) is stable. So, from Theorem 1, we have

\[
d_{\Sigma}(p, \partial \Sigma) = d_{\tilde{\Sigma}}(\tilde{p}, \partial \tilde{\Sigma}) \leq \frac{\pi}{2\sqrt{H^2 + \kappa}}
\]

where \( \Pi(\tilde{p}) = p \).

The equality case comes from the equality case in Theorem 1 and since the Lawson’s correspondence sends spheres into spheres.

References


