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Strong approximation of the empirical distribution function for absolutely regular sequences in $\mathbb{R}^d$

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Abstract

We prove a strong approximation result with rates for the empirical process associated to an absolutely regular stationary sequence of random variables with values in $\mathbb{R}^d$. As soon as the absolute regular coefficients of the sequence decrease more rapidly than n^{1-p} for some $p \in]2,3]$, we show that the error of approximation between the empirical process and a two-parameter Gaussian process is of order $n^{1/p}(\log n)^{\lambda(d)}$ for some positive $\lambda(d)$ depending on d, both in $\mathbb{L}^1$ and almost surely. The power of n being independent of the dimension, our results are even new in the independent setting, and improve earlier results. In addition, for absolutely regular sequences, we show that the rate of approximation is optimal up to the logarithmic term.

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1 Introduction

Let $(X_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of random variables in $\mathbb{R}^d$ equipped with the usual product order, with common distribution function F. Define the empirical process of $(X_i)_{i\in\mathbb{Z}}$ by

$$R_X(s,t) = \sum_{1\le k\le t} \big(\mathbf{1}_{X_k\le s} - F(s)\big)\,,\ s\in\mathbb{R}^d\,,\ t\in\mathbb{R}^+\,. \tag{1.1}$$

In this paper we are interested in extensions of the results of Kiefer for the process R_X to absolutely regular processes. Let us start by recalling the known results in the case of independent and identically distributed (iid) random variables X_i. Kiefer (1972) obtained the first result in the case $d=1$. He constructed a continuous centered Gaussian process K_X with covariance function

$$\mathbb{E}\big(K_X(s,t)K_X(s',t')\big) = (t\wedge t')(F(s\wedge s') - F(s)F(s'))$$

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in such a way that

$$\sup_{(s,t)\in\mathbb{R}\times[0,1]} |R_X(s,[nt]) - K_X(s,[nt])| = O(a_n) \quad \text{almost surely,} \tag{1.2}$$

with $a_n = n^{1/3}(\log n)^{2/3}$. The two-parameter Gaussian process K_X is known in the literature as the Kiefer process. Csörgö and Révész (1975a) extended Kiefer's result to the multivariate case. For iid random variables with the uniform distribution over $[0,1]^d$, they obtained (1.2) with $a_n = n^{(d+1)/(2d+4)}(\log n)^2$. Next they extended this result to iid random variables in $\mathbb{R}^d$ with a density satisfying some smoothness conditions (see Csörgö and Révész (1975b)).

In the univariate case, a major advance was made by Komlós, Major and Tusnády (1975): they obtained (1.2) with $a_n = (\log n)^2$ (we refer to Castelle and Laurent-Bonvalot (1998) for a detailed proof) via a new method of construction of the Gaussian process. Concerning the strong approximation by a sequence of Gaussian processes in the case $d = 2$, Tusnády (1977) proved that when the random variables X_i are iid with uniform distribution over $[0,1]^2$, then one can construct a sequence of centered continuous Gaussian processes $(G_n)_{n\geq 1}$ in $\mathbb{R}^2$ with covariance function

$$\mathrm{Cov}(G_n(s), G_n(s')) = n((s_1 \wedge s_1')(s_2 \wedge s_2') - s_1 s_2 s_1' s_2'),$$

with $s = (s_1, s_2)$ and $s' = (s_1', s_2')$, such that

$$\sup_{s\in[0,1]^2} |R_X(s,n) - G_n(s)| = O(\log^2 n) \quad \text{almost surely.} \tag{1.3}$$

Adapting the dyadic method of Komlós, Major and Tusnády (sometimes called Hungarian construction), several authors obtained new results in the multivariate case. For iid random variables in $\mathbb{R}^d$ with distribution with dependent components (without regularity conditions on the distribution), Borisov (1982) obtained the almost sure rate of approximation $O(n^{(d-1)/(2d-1)} \log n)$ in the Tusnády strong approximation. Next, starting from the result of Borisov (1982), Csörgö and Horváth (1988) obtained the almost sure rate $O(n^{(2d-1)/(4d)}(\log n)^{3/2})$ for the strong approximation by a Kiefer process. Up to our knowledge, this result has not yet been improved in the case of general distributions with dependent components. For $d \geq 3$ and Tusnády's type results, Rio (1994) obtained the rate $O\big(n^{(d-1)/(2d)}(\log n)^{1/2}\big)$ for random variables with the uniform distribution or more generally with smooth positive density on the unit cube (see also Massart (1989) in the uniform case). Still in the uniform case, concerning the strong approximation by a Kiefer process, Massart (1989) obtained the almost sure rate $O\big(n^{d/(2d+2)}(\log n)^2\big)$ for any $d \geq 2$, which improves the results of Csörgö and Révész (1975a). In fact the results of Massart (1989) and Rio (1994) also apply to Vapnik-Chervonenkis classes of sets with uniformly bounded perimeters, such as the class of Euclidean balls. In that case, Beck (1985) proved that the error term cannot be better than $n^{(d-1)/(2d)}$. Consequently the result of Rio (1994) for Euclidean balls is optimal, up to the factor $\sqrt{\log n}$. However, there is a gap in the lower bounds between the class of Euclidean balls and the class of orthants, which corresponds to the empirical distribution function. Indeed, concerning the lower bounds in Tusnády's type results, Beck (1985) showed that the rate of approximation cannot be less than $c_d(\log n)^{(d-1)/2}$ where c_d is a positive constant depending on d. To be precise, he proved (see his Theorem 2) that when the random variables X_i are iid with the uniform distribution over $[0,1]^d$, then for any sequence of Brownian bridges $(G_n)_{n\geq 1}$ in $\mathbb{R}^d$,

$$\mathbb{P}\Big(\sup_{s\in[0,1]^d} |R_X(s,n) - G_n(s)| \leq c_d(\log n)^{(d-1)/2}\Big) < e^{-n}.$$

Beck's result implies in particular that, for any $n \geq 2$,

$$(\log n)^{(1-d)/2}\mathbb{E}\Big(\sup_{s\in[0,1]^d}|R_X(s,n)-G_n(s)|\Big) \geq c_d/2\,. \tag{1.4}$$

The results of Beck (1985) motivated new research in the multivariate case. For the empirical distribution function and Tusnády type results, Rio (1996) obtained the rate $O\big(n^{5/12}(\log n)^{c(d)}\big)$ for random variables with the uniform distribution, where $c(d)$ is a positive constant depending on the dimension d, without the help of Hungarian construction. Here the power of n does not depend on the dimension: consequently this result is better than the previous results if $d \geq 7$. It is worth noticing that, although this subject has been treated intensively, up to now, the best known rates for the strong approximation by a Kiefer process in the multivariate case are of the order $n^{1/3}$ for $d=2$, up to some power of $\log n$, even in the uniform case. Furthermore these rates depend on the dimension, contrary to the result of Rio (1996) for Tusnády type approximations.

We now come to the weakly dependent case. Contrary to the iid case, there are only few results concerning the rate of approximation. Up to our knowledge, when $(X_i)_{i\in\mathbb{Z}}$ is a geometrically strongly mixing (in the sense of Rosenblatt (1956)) strictly stationary sequence of random variables in $\mathbb{R}^d$, the best known result concerning rates of convergence, is due to Doukhan and Portal (1987) stating that one can construct a sequence of centered continuous Gaussian processes $(G_n)_{n\geq 1}$ in $\mathbb{R}^d$ with common covariance function

$$\Lambda(s,s') = \sum_{k\in\mathbb{Z}} \mathrm{Cov}(\mathbf{1}_{X_0\leq s}, \mathbf{1}_{X_k\leq s'})\,,$$

such that the Ky-Fan distance between $\{n^{-1/2}R_X(s,n), s\in\mathbb{R}^d\}$ and $\{G_n(s), s\in\mathbb{R}^d\}$ is $o(n^{-a})$ for any $a < 1/(15d+12)$. In their paper, they also give some rates in case of polynomial decay of the mixing coefficients. Concerning the strong approximation by a Kiefer process in the stationary and strongly mixing case, Theorem 3 in Dhompongsa (1984) yields the rate $O(n^{1/2}(\log n)^{-\lambda})$ for some positive λ, under the strong mixing condition $\alpha_n = O(n^{-a})$ for some $a > 2+d$, improving slightly previous results of Phillip and Pinzur (1980) (here λ depends on a and d).

Strong mixing conditions seem to be too poor to get optimal rates of convergence. Now recall that, for irreducible, aperiodic and positively recurrent Markov chains, the coefficients of strong mixing and the coefficients of absolute regularity are of the same order (see for example Rio (2000), chap. 9). Since absolute regularity is a stronger condition, it is more convenient to consider absolute regularity, at least in the case of irreducible Markov chains. Let

$$\beta(\mathcal{A},\mathcal{B}) = \frac{1}{2}\sup\Big\{\sum_{i\in I}\sum_{j\in J}|\mathbb{P}(A_i\cap B_j)-\mathbb{P}(A_i)\mathbb{P}(B_j)|\Big\}\,,$$

the maximum being taken over all finite partitions $(A_i)_{i\in I}$ and $(B_i)_{i\in J}$ of Ω respectively with elements in $\mathcal{A}$ and $\mathcal{B}$. For a strictly stationary sequence $(X_k)_{k\in\mathbb{Z}}$, let $\mathcal{F}_0 = \sigma(X_i, i\leq 0)$ and $\mathcal{G}_k = \sigma(X_i, i\geq k)$. The sequence $(X_k)_{k\in\mathbb{Z}}$ is said to be absolutely regular in the sense of Rozanov and Volkonskii (1959) or β-mixing if

$$\beta_n = \beta(\mathcal{F}_0,\mathcal{G}_n) \to 0\,,\ \text{as } n\to\infty\,.$$

Concerning the strong approximation by a Kiefer process in the stationary and β-mixing case, Theorem 1 in Dhompongsa (1984) yields the rate $O(n^{(1/2)-\lambda})$ for some positive λ, under the assumption $\beta_n = O(n^{-a})$ for some $a > 2+d$. Nevertheless this mixing condition is clearly too restrictive and λ is not explicit.

We now come to our results. For absolutely regular sequences, the finite dimensional convergence of $\{n^{-1/2}R_X(s,n) : s \in \mathbb{R}^d\}$ to a Gaussian process holds under the summability condition $\sum_{k\geq 0}\beta_k < \infty$, and this condition is sharp. Rio (1998) proved that this summability condition also implies the functional central limit theorem for $\{n^{-1/2}R_X(s,n) : s \in \mathbb{R}^d\}$ in the sense of Dudley (1978) for any $d \geq 1$. Assume now that the stronger β-mixing condition

$$\beta_n = O(n^{1-p}) \text{ for some } p > 2 \tag{1.5}$$

holds true. In Section 2, we shall prove that, in the case $d = 1$, one can construct a stationary absolutely regular Markov chain satisfying (1.5), whose marginals are uniformly distributed over $[0,1]$, and such that, for any construction of a sequence $(G_n)_{n>0}$ of continuous Gaussian processes on $[0,1]$,

$$\liminf_{n\to\infty}(n\log n)^{-1/p}\mathbb{E}\Big(\sup_{s\in(0,1]}|R_X(s,n) - G_n(s)|\Big) > 0\,.$$

Concerning the upper bound, Dedecker, Merlevède and Rio (2013) obtain a strong approximation by a Kiefer process under a weak dependence condition which is implied by the above condition, with a power-type rate $O(n^{(1/2)-\delta})$ for some positive δ depending on p. Nevertheless their result holds only for $d = 1$ and the value of δ is far from the optimal value $(1/2) - (1/p)$. This gap motivates the present work. In Section 3, we prove that, if $(X_i)_{i\in\mathbb{Z}}$ is a strictly stationary sequence of random variables in $\mathbb{R}^d$ satisfying (1.5) for $p \in]2,3]$, there exists a two-parameter continuous (with respect to the pseudo metric defined by (3.1)) Kiefer type process K_X such that

$$\mathbb{E}\Big(\sup_{s\in\mathbb{R}^d, t\in[0,1]}|R_X(s,[nt]) - K_X(s,[nt])|\Big) = O(n^{1/p}(\log n)^{\lambda(d)})\,.$$

We also prove that, for another Kiefer process K_X,

$$\sup_{\substack{s\in\mathbb{R}^d\\ k\leq n}}|R_X(s,k) - K_X(s,k)| = O(n^{1/p}(\log n)^{\lambda(d)+\varepsilon+1/p}) \quad \text{almost surely, for any } \varepsilon > 0\,.$$

More precisely, the covariance function Γ_X of K_X is given by

$$\Gamma_X(s,s',t,t') = \min(t,t')\Lambda_X(s,s') \;\text{ where }\; \Lambda_X(s,s') = \sum_{k\in\mathbb{Z}}\mathrm{Cov}(\mathbf{1}_{X_0\leq s}, \mathbf{1}_{X_k\leq s'})\,. \tag{1.6}$$

Our proofs can be adapted to iid random variables with values in $\mathbb{R}^d$ and arbitrary distribution function, for any $d \geq 2$, yielding the error term in the strong approximation $O(n^{1/3}(\log n)^{\varepsilon+(2d+4)/3})$ in the almost sure strong approximation by a Kiefer process. This result improves the results of Csörgö and Horváth (1988) for general distributions as soon as $d \geq 2$ and the result of Massart (1989) concerning the specific case of the uniform law as soon as $d \geq 3$ (recall that Massart's rate is $O(n^{1/3}(\log n)^2)$ for $d = 2$ and $O(n^{3/8}(\log n)^2)$ for $d = 3$).

We now describe our methods of proofs. We shall apply the conditional version of the transport theorem of Kantorovich and Rubinstein (1958) to the trajectories to get a bound on the error. However, in the dependence setting, we do not apply the transport theorem directly. Indeed, we start by approximating the initial process R_X by a Gaussian process with the same covariance structure as R_X, using the conditional Kantorovich-Rubinstein theorem applied in the space of trajectories, together with the Lindeberg method. Next, we use a martingale method to approximate the Gaussian process by the Kiefer process. This step is due to the fact that the Lindeberg method in the space of trajectories applies only to processes with the same covariance structure. In all these steps the error terms can be bounded by $n^{1/p}$ up to some power of $\log n$, which leads to the (nearly) optimal rates of convergence for absolutely regular sequences. Note that the Lindeberg method in the space of trajectories was introduced by Sakhanenko (1988) in the real case to bound up the Prokhorov distance between the partial sum process and the Brownian motion. This result was then extended to random vectors in Banach spaces by Bentkus and Lyubinskas (1987) using smoothing techniques introduced by the first author in his doctoral dissertation. Later, Sakhanenko (2000) improved the results of Bentkus and Lyubinskas (1987) in the specific case of the $\mathbb{L}^\infty$-norm, yielding efficient estimates under some assumptions on the moments of order two and three of the Euclidean norms of the random vectors. Sakhanenko (1988, 2000) also gives some results for martingale differences under very restrictive assumptions on the conditional moments of order two. In our opinion, the smoothing technique used in Sakhanenko (2000) is not suitable in the dependent case. Indeed the assumption on the conditional moments cannot be relaxed. It is noteworthy to indicate that Götze (1986) and Borisov (1988) used the Lindeberg's operators method to derive rates of convergence in the central limit theorem for smooth functionals (at least three times Fréchet differentiable) of the partial sum process associated with independent Banach-space-valued random elements.

Our paper is organized as follows. In Section 2 we give an example of absolutely regular process for which we can derive lower bounds for the rates of approximation by any continuous Gaussian processes. In Section 3 we formulate our main results concerning the upper bounds for the rates of approximation both in the dependent setting and in the independent one. The proofs of these results are given in Sections 4 and 5. Section 6 is devoted to the very technical proofs of key intermediate lemmas leading to our main results. Finally, in Section 7 we collect some auxiliary assertions and general facts.

2 Lower bounds for the rate of approximation

In this section, we give an example of a stationary absolutely regular Markov chain with state space $[0,1]$ and absolute regularity coefficients β_k of the order of k^{1-p} for $p > 2$ which has the following property: with probability one, the error in the strong approximation by Gaussian processes is bounded from below by $(n \log n)^{1/p}$, for any construction of a sequence of continuous Gaussian processes, whereas the $\mathbb{L}^1$-error is bounded from below by $n^{1/p}$.

Theorem 2.1. *For any $p > 2$, there exists a stationary Markov chain $(X_i)_{i\in\mathbb{Z}}$ of random variables with uniform distribution over $[0,1]$ and β-mixing coefficients $(\beta_n)_{n>0}$, such that:*

(i) $0 < \liminf_{n\to+\infty} n^{p-1}\beta_n \le \limsup_{n\to+\infty} n^{p-1}\beta_n < \infty$.

(ii) There exists a positive constant C such that, for any construction of a sequence $(G_n)_{n>0}$ of continuous Gaussian processes on $[0,1]$

$$(a) \qquad \liminf_{n\to\infty} n^{-1/p}\mathbb{E}\Big(\sup_{s\in(0,1]}|R_X(s,n) - G_n(s)|\Big) \ge C\,.$$

Furthermore

$$(b) \qquad \limsup_{n\to\infty}(n\log n)^{-1/p}\sup_{s\in(0,1]}|R_X(s,n) - G_n(s)| > 0 \text{ almost surely.}$$

Before proving this result, we give a second theorem, which proves that the strong approximation of partial sums of functionals of the chain holds with the same error term.

Theorem 2.2. *Let $(X_i)_{i\in\mathbb{Z}}$ be the stationary Markov chain defined in Theorem 2.1 and let f be a map from $[0,1]$ to $\mathbb{R}$, with continuous and strictly positive derivative f' on $[0,1]$. Let*

$$S_n(f) = \sum_{k=1}^{n} f(X_k) - n\int_0^1 f(t)dt\,.$$

Then the series $\operatorname{Var} f(X_0) + 2\sum_{k>0} \operatorname{Cov}(f(X_0), f(X_k))$ is absolutely convergent to some nonnegative $\sigma^2(f)$. Furthermore, for $2 < p < 3$ and any positive ε, one can construct a sequence of iid Gaussian random variables $(g'_k)_{k>0}$ with law $N(0,\sigma^2(f))$ such that

$$(a) \qquad S_n(f) - \sum_{k=1}^{n} g'_k = o(n^{1/p}\sqrt{\log n}(\log\log n)^{(1+\varepsilon)/p}) \text{ almost surely.}$$

In addition, for any $p > 2$ and any stationary and Gaussian centered sequence $(g_k)_{k\in\mathbb{Z}}$ with convergent series of covariances,

$$(b) \qquad \limsup_{n\to\infty}(n\log n)^{-1/p}\Big|S_n(f) - \sum_{k=1}^{n} g_k\Big| > 0 \text{ almost surely.}$$

Note that Part (a) of this theorem was proved in Merlevède and Rio (2012). Part (b) proves that the result in Merlevède and Rio (2012) is optimal up to the factor $(\log n)^{(1/2)-(1/p)}(\log\log n)^{(1+\varepsilon)/p}$. It is worth noticing that the power of the logarithm in the loss tends to 0 as p tends to 2.

Proof of Theorem 2.1. The sequence $(X_i)_{i\in\mathbb{Z}}$ is defined from a strictly stationary Markov chain $(\xi_i)_{i\in\mathbb{Z}}$ on $[0,1]$ as in Rio (2000), Section 9.7. Let λ be the Lebesgue measure, $a = p-1$ and $\nu = (1+a)x^a\mathbf{1}_{[0,1]}\lambda$. The conditional distribution $\Pi(x,.)$ of ξ_{n+1}, given $(\xi_n = x)$, is defined by

$$\Pi(x,.) = \Pi(\delta_x,.) = (1-x)\delta_x + x\nu\,,$$

where δ_x is the Dirac measure at point x. Then the β-mixing coefficients $(\beta_n)_{n>0}$ of the stationary chain $(\xi_i)_{i\in\mathbb{Z}}$ with transition probability $\Pi(x,.)$ satisfy (i) of Theorem 2.1 (see Rio (2000), Section 9.7). Moreover, the stationary distribution π has distribution function $F(x) = x^a$, and consequently setting $X_i = \xi_i^a$ we obtain a stationary Markov chain $(X_i)_{i\in\mathbb{Z}}$ of random variables with uniform distribution over $[0,1]$ and adequate rate of β-mixing. Define then the empirical measure P_n by

$$P_n = n^{-1}\sum_{i=1}^{n}\delta_{X_i}\,.$$

The regeneration times $(T_k)_k$ of the Markov chain $(\xi_i)_{i\in\mathbb{Z}}$ are defined by induction as follows: $T_0 = \inf\{n > 0 : \xi_n \neq \xi_{n-1}\}$ and $T_k = \inf\{n > T_{k-1} : \xi_n \neq \xi_{n-1}\}$. Let $\tau_k = T_{k+1} - T_k$. It follows that the empirical measure at time $T_k - 1$ satisfies the equality

$$(T_k - 1)P_{T_k-1} = (T_0 - 1)\delta_{X_0} + \sum_{j=0}^{k-1}\tau_j\delta_{X_{T_j}}\,. \tag{2.1}$$

Consequently, for $n \geq T_k - 1$ the maximal jump of $R_X(s,n)$ is greater than

$$\Delta_k = \max_{j\in[0,k-1]}\tau_j\,.$$

Next, from the continuity of G_n, for $n \geq T_k - 1$,

$$D_n := \sup_{s\in(0,1]} |R_X(s,n) - G_n(s)| \geq \Delta_k/2\,. \tag{2.2}$$

Now the sequence $(\Delta_k)_k$ is a nondecreasing sequence of positive integers. Notice that the random variables (ξ_{T_k}, τ_k) are independent and identically distributed. Moreover ξ_{T_k} has the distribution ν and the conditional distribution of τ_k given $(\xi_{T_k} = x)$ is the geometric distribution $\mathcal{G}(x)$. Hence,

$$\mathbb{P}(\Delta_k \leq m) = (\mathbb{P}(\tau_0 \leq m))^k\,,$$

and

$$\mathbb{P}(\tau_0 > m) = (1+a)m^{-1-a}\int_0^m (1-y/m)^m y^a dy \sim (1+a)\Gamma(1+a)m^{-1-a} \text{ as } m\uparrow\infty\,. \tag{2.3}$$

From the above facts, it follows that

$$\mathbb{E}(\Delta_k) = \sum_{m\geq 0}\mathbb{P}(\Delta_k > m) \geq c_p k^{1/p}\,. \tag{2.4}$$

In the same way, one can prove that

$$\|\Delta_k\|_{(2+a)/2} \leq C_p k^{1/p}\,. \tag{2.5}$$

Here c_p and C_p are positive constants depending only on p.

Now, by the strong law of large numbers T_k/k converges to $\mathbb{E}(\tau_0)$ almost surely, and therefore in probability. Consequently, for $k = k_n = [n/(2\mathbb{E}(\tau_0))]$,

$$\lim_{n\to\infty}\mathbb{P}(n < T_{k_n} - 1) = 0\,.$$

Now

$$2\mathbb{E}(D_n) \geq \mathbb{E}(\Delta_{k_n}\mathbf{1}_{n\geq T_{k_n}-1}) \geq \mathbb{E}(\Delta_{k_n}) - \|\Delta_{k_n}\|_{\frac{2+a}{2}}(\mathbb{P}(n < T_{k_n} - 1))^{\frac{a}{2+a}}\,.$$

From (2.5), (2.4) and the above inequality, we get that, there exists some positive constant C (depending on p) such that, for n large enough, $\mathbb{E}(D_n) \geq Cn^{1/p}$, which completes the proof of (a) of Theorem 2.1.

To prove (b), we note that, by (2.3) ,

$$\mathbb{P}(\tau_k > (k\ln k)^{1/p}) \sim c_p/(k\log k)\,. \tag{2.6}$$

Since the regeneration times τ_k are independent, by the converse Borel-Cantelli lemma, it follows that

$$\mathbb{P}(\tau_k > (k\log k)^{1/p} \text{ infinitely often }) = 1\,. \tag{2.7}$$

Hence, by (2.2),

$$\limsup_n (n\ln n)^{-1/p} D_{T_n - 1} \geq (1/2) \text{ almost surely.}$$

Both this inequality and the strong law of large numbers for T_n then imply (b) of Theorem 2.1.

Proof of Theorem 2.2. Let b be a real in $]0,1]$ such that $f(b) < \int_0^1 f(t)dt$ (note that such a positive b exists). With the same notations as in the proof of the previous theorem, the random variables (X_{T_k}, τ_k) are independent and identically distributed, and

$$\begin{aligned}\mathbb{P}(\tau_0 > m, X_{T_0} < b) &= (1+a)m^{-1-a}\int_0^{mb^{1/a}} (1-y/m)^m y^a dy \\ &\sim (1+a)\Gamma(1+a)m^{-1-a} \text{ as } m\uparrow\infty.\end{aligned}$$

Consequently, by the converse Borel-Cantelli lemma,

$$\mathbb{P}(\tau_k > (k \log k)^{1/p} \text{ and } X_{T_k} < b \text{ infinitely often }) = 1. \tag{2.8}$$

Since T_n/n converges to $\mathbb{E}(\tau_0)$ almost surely, it follows that, for some positive constant c depending on $\mathbb{E}(\tau_0)$,

$$\limsup_n \sum_{i=n+1}^{n+[c(n\log n)^{1/p}]} (f(b) - f(X_i)) \geq 0 \text{ almost surely.} \tag{2.9}$$

Consider now a stationary and Gaussian centered sequence $(g_k)_{k\in\mathbb{Z}}$ with convergent series of covariances. If follows from both the Borel-Cantelli lemma and the usual tail inequality for Gaussian random variables that, for any positive θ,

$$\liminf_n \sum_{i=n+1}^{n+[c(n\log n)^{1/p}]} (g_i + \theta) \geq 0 \text{ almost surely.}$$

Taking $\theta = (\int_0^1 f(t)dt - f(b))/2$ in the above inequality, we then infer from the two above inequalities that

$$\limsup_{n\to\infty} \frac{1}{[c(n\log n)^{1/p}]} \sum_{i=n+1}^{n+[c(n\log n)^{1/p}]} \Big(g_i + \int_0^1 f(t)dt - f(X_i) \Big) \geq \theta \ \text{ almost surely,}$$

which implies Theorem 2.2.

3 Upper bounds for the rate of approximation

In this section, we state the main result of this paper, which is a Kiefer type approximation theorem for absolutely regular sequences. In all this section, we assume that the underlying probability space $(\Omega, \mathcal{A}, \mathbb{P})$, is rich enough to contain a sequence $(U_i)_{i\in\mathbb{Z}} = (\eta_i, \delta_i, \upsilon_i, \epsilon_i)_{i\in\mathbb{Z}}$ of iid random variables with uniform distribution over $[0,1]^4$, independent of $(X_i)_{i\in\mathbb{Z}}$.

Theorem 3.1. *Let $(X_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of random variables in $\mathbb{R}^d$. Let F_j be the distribution function of the j-th marginal of X_0. Assume that the absolutely regular coefficients of $(X_i)_{i\in\mathbb{Z}}$ are such that $\beta_n = O(n^{1-p})$ for some $p \in]2,3]$. Then*

1. *for all (s,s') in $\mathbb{R}^{2d}$, the series $\Lambda_X(s,s')$ defined by (1.6) converges absolutely.*
2. *For any $(s,s') \in \mathbb{R}^{2d}$ and (t,t') in $\mathbb{R}^+ \times \mathbb{R}^+$, let $\Gamma_X(s,s',t,t') = \min(t,t')\Lambda_X(s,s')$. There exists a centered Gaussian process K_X with covariance function Γ_X, whose sample paths are almost surely uniformly continuous with respect to the pseudo metric*

$$d((s,t),(s',t')) = |t-t'| + \sum_{j=1}^{d} |F_j(s_j) - F_j(s'_j)|\,, \tag{3.1}$$

and such that

(a) $$\mathbb{E}\Big(\sup_{s\in\mathbb{R}^d,\, k\leq n} |R_X(s,k) - K_X(s,k)| \Big) = O(n^{1/p}(\log n)^{\lambda(d)})\ .$$

Furthermore, one can construct another centered Gaussian process K_X with the above covariance function in such a way that

(b) $$\sup_{s\in\mathbb{R}^d,\, k\leq n} |R_X(s,k) - K_X(s,k)| = O(n^{1/p}(\log n)^{\lambda(d)+\varepsilon+1/p}) \quad \text{a.s., for any } \varepsilon > 0$$

In both items $\lambda(d) = \big(\frac{3d}{2} + 2 - \frac{2+d}{2p}\big)\mathbf{1}_{p\in]2,3[} + \big(2 + \frac{4d}{3}\big)\mathbf{1}_{p=3}$.

From the above theorem, in the independent setting, the error in the $\mathbb{L}^1$ approximation is bounded up by $n^{1/3}(\log n)^{2+4d/3}$, whereas the almost sure error is bounded up by $n^{1/3}(\log n)^{\varepsilon+(9+4d)/3}$, for any $\varepsilon > 0$. However, in that case, the powers of $\log n$ can be improved as follows.

Theorem 3.2. *Let $(X_i)_{i\in\mathbb{Z}}$ be a sequence of independent and identically distributed random variables in $\mathbb{R}^d$. Then one can construct a centered Gaussian process K_X with covariance function*

$$\Gamma_X(s,s',t,t') = \min(t,t')(F(s\wedge s') - F(s)F(s')) \text{ where } s\wedge s' = (\min(s_1,s'_1),\dots,\min(s_d,s'_d))\,,$$

whose sample paths are almost surely uniformly continuous with respect to the pseudo metric d defined in (3.1)*, and such that*

$$(a) \qquad \mathbb{E}\Big(\sup_{s\in\mathbb{R}^d,\,k\le n}|R_X(s,k)-K_X(s,k)|\Big) = O(n^{1/3}(\log n)^{(2d+3)/3})\,.$$

Furthermore, one can construct another centered Gaussian process K_X with the above covariance function in such a way that

(b)

$$\sup_{s\in\mathbb{R}^d,\,k\le n}|R_X(s,k)-K_X(s,k)| = O(n^{1/3}(\log n)^{\epsilon+(2d+4)/3}) \text{ almost surely, for any } \varepsilon>0\,.$$

Recently, Merlevède and Rio (2012) obtained efficient strong approximation results for partial sums of real-valued random variables. In the bounded case, under the mixing condition $\beta_n = O(n^{1-p})$, they obtain in their Theorem 2.1 (see Item 1(b)) the rate of almost sure approximation $O(n^{1/p}(\log n)^{(1/2)+\varepsilon})$. According to the results of Section 2, the power of n cannot be improved, contrary to the previous papers on the same subject. Starting from Theorem 3.1, we can derive an extension of this result to partial sums of random vectors in $\mathbb{R}^d$, in the same way as Borisov (1983) derives strong approximation of partial sums from the strong Kiefer approximation of Komlós, Major and Tusnády.

Corollary 3.1. *Let $(X_i)_{i\in\mathbb{Z}}$ be a strictly stationary and absolutely regular sequence of bounded random vectors in $\mathbb{R}^d$. Assume that its absolutely regular coefficients are such that $\beta_n = O(n^{1-p})$ for some $p\in]2,3]$. Then the series of covariance matrices $\sum_{k\in\mathbb{Z}}\mathrm{Cov}(X_0,X_k)$ is convergent to a non-negative definite symmetric matrix Γ. Furthermore, there exists a sequence $(Z_i)_{i\ge1}$ of iid random vectors with law $N(0,\Gamma)$ such that, setting $\Delta_k = \sum_{i=1}^k(X_i - \mathbb{E}(X_i) - Z_i)$,*

$$(a) \qquad \mathbb{E}\big(\sup_{k\le n}\|\Delta_k\|\big) = O(n^{1/p}(\log n)^{\lambda(d)})\,.$$

In addition, there exists another sequence $(Z_i)_{i\ge1}$ of iid random vectors with law $N(0,\Gamma)$ such that, for any positive ε,

$$(b) \qquad \sup_{k\le n}\|\Delta_k\| = o(n^{1/p}(\log n)^{\lambda(d)+\varepsilon+1/p}) \text{ almost surely.}$$

In both items, $\lambda(d)$ is defined in Theorem 3.1.

Proof of Corollary 3.1. Adding some constant vector to the initial random vectors if necessary, we may assume that the components of the random vectors X_i are non-positive. For each integer i, we set $X_i = (X_i^{(1)}, X_i^{(2)},\dots,X_i^{(d)})$. From our assumptions, there exists some positive constant M such that, for any integer i and any X_i belongs to $[-M,0]^d$. Then, for any j in $[1,d]$,

$$X_i^{(j)} = -\int_{-M}^0 \mathbf{1}_{X_i^{(j)}\le t}\,dt\,.$$

Let then $K_X^{(j)}(t,k) = K_X((1,\dots,1,t,1,\dots,1),k)$, where t is the j-th component. Define the random vectors Z_k for any positive integer k by

$$Z_k^{(j)} = -\int_{-M}^{0}\Big(K_X^{(j)}(t,k) - K_X^{(j)}(t,k-1)\Big)dt \ \text{ for any } \ j \in [1,d]\,.$$

Then the so defined sequence $(Z_k)_{k>0}$ is a Gaussian sequence (this means that, for any positive integer n, $(Z_1,\dots,Z_n)$ is a Gaussian vector) and, from the definition of the covariance of K_X, the random vectors Z_k are not correlated. It follows that $(Z_k)_{k>0}$ is a sequence of independent Gaussian random vectors. Now, from the definition of Z_k,

$$\mathrm{Cov}(Z_k^{(j)}, Z_k^{(i)}) = \sum_{m\in\mathbb{Z}}\int_{-M}^{0}\int_{-M}^{0}\mathrm{Cov}(\mathbf{1}_{X_0^{(i)}\le t}, \mathbf{1}_{X_m^{(j)}\le s})dtds\,.$$

Hence, interverting the summation and the integral in the above formula, we get that

$$\mathrm{Cov}(Z_k^{(j)}, Z_k^{(i)}) = \sum_{m\in\mathbb{Z}}\mathrm{Cov}(X_0^{(i)}, X_m^{(j)}) = \Gamma_{ij}\,,$$

which implies that Z_k has the prescribed covariance. Next

$$X_k^{(j)} - \mathbb{E}(X_k^{(j)}) - Z_k^{(j)} = \int_{-M}^{0}\big(K_X^{(j)}(t,k) - K_X^{(j)}(t,k-1) + \mathbb{P}(X_k^{(j)}\le t) - \mathbf{1}_{X_k^{(j)}\le t}\big)dt\,.$$

Let then $\Delta_k^{(j)}$ denote the j-th component of Δ_k and

$$R_X^{(j)}(t,k) = R_X((1,\dots,1,t,1,\dots,1),k)\,,$$

where t is the j-th component. From the above identity,

$$\Delta_k^{(j)} = \int_{-M}^{0}(K_X^{(j)}(t,k) - R_X^{(j)}(t,k))dt\,.$$

It follows that, for any integer j in $[1,d]$,

$$\sup_{k\le n}|\Delta_k^{(j)}| \le M \sup_{\substack{s\in\mathbb{R}^d\\ k\le n}}|R_X(s,k) - K_X(s,k)|\,.$$

Part (a) (resp. Part (b)) of Corollary 3.1 follows then from both these inequalities and Part (a) (resp. Part (b)) of Theorem 3.1.

4 Proof of Theorem 3.1

In this section we shall sometimes use the notation $a_n \ll b_n$ to mean that there exists a numerical constant C not depending on n such that $a_n \le Cb_n$, for all positive integers n. We shall also use the notations $\mathcal{F}_k = \sigma(X_j, j\le k)$ and $\mathcal{F}_\infty = \bigvee_{k\in\mathbb{Z}}\mathcal{F}_k$.

For any $(s,s')\in\mathbb{R}^{2d}$, by using Lemma 7.4 with $U = \mathbf{1}_{X_0\le s}$, $V = \mathbf{1}_{X_k\le s'}$, $r=1$ and $s=\infty$, we get that

$$|\mathrm{Cov}(\mathbf{1}_{X_0\le s}, \mathbf{1}_{X_k\le s'})| \le 2\beta_k\,.$$

Since $\sum_{k\ge 0}\beta_k < \infty$, Item 1 of Theorem 3.1 follows.

To prove Item 2, we first transform the random variables X_i. With this aim, for any k in $\mathbb{Z}$ and any j in $\{1,\dots,d\}$, we denote by $X_k^{(j)}$ the j-th marginal of X_k. By Lemma 7.4

applied with $p = 1$ and $q = \infty$, there exists some non-negative random variable $b(x, k)$ with values in $[0, 1]$ such that, for any functions $f : \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R}^d \to [-1, 1]$,

$$\mathrm{Cov}(f(X_0), g(X_k)) \le 2\mathbb{E}(b(X_0, k)|f(X_0)|) \text{ and } \mathbb{E}(b(X_0, k)) \le \beta_k \,. \tag{4.1}$$

Let then

$$b_j(X_0, k) = \mathbb{E}(b(X_0, k) \mid X_0^{(j)}) \,.$$

We now introduce another probability on Ω. Let $\mathbb{P}^*_{0,j}$ be the probability on Ω whose density with respect to $\mathbb{P}$ is

$$C_j^{-1}\big(1 + 4\sum_{k=1}^{\infty} b_j(X_0, k)\big) \text{ with } C_j = 1 + 4\sum_{k=1}^{\infty} \mathbb{E}(b_j(X_0, k)) \,. \tag{4.2}$$

Let P_j be the law of $X_0^{(j)}$. Notice then that the image measure P_j^* of $\mathbb{P}^*_{0,j}$ by $X_0^{(j)}$ is absolutely continuous with respect to P_j with density

$$C_j^{-1}\big(1 + 4\sum_{k=1}^{\infty} b_j(x, k)\big) \,. \tag{4.3}$$

Let $F_{P_j^*}$ be the distribution function of P_j^*, and let $F_{P_j^*}(x-0) = \sup_{z<x} F_{P_j^*}(z)$. Let $(\eta_i)_{i\in\mathbb{Z}}$ be a sequence of iid random variables with uniform distribution over $[0, 1]$, independent of the initial sequence $(X_i)_{i\in\mathbb{Z}}$. Define then

$$Y_i^{(j)} = F_{P_j^*}(X_i^{(j)} - 0) + \eta_i(F_{P_j^*}(X_i^{(j)}) - F_{P_j^*}(X_i^{(j)} - 0)) \text{ and } Y_i = \big(Y_i^{(1)}, \dots, Y_i^{(d)}\big)^t \,. \tag{4.4}$$

Note that $(Y_i)_{i\in\mathbb{Z}}$ forms a strictly stationary sequence of random variables with values in $[0, 1]^d$ whose β-mixing coefficients are also of order $\beta_n = O(n^{1-p})$. In addition, it follows from Lemma F.1. in Rio (2000) that $X_i^{(j)} = F^{-1}_{P_j^*}(Y_i^{(j)})$ almost surely, where $F^{-1}_{P_j^*}$ is the generalized inverse of the cadlag function $F_{P_j^*}$. Hence

$$R_X(\cdot, \cdot) = R_Y\big((F_{P_1^*}(\cdot), \dots, F_{P_d^*}(\cdot)), \cdot\big) \text{ almost surely } ,$$

where

$$R_Y(s, t) = \sum_{1\le k\le t} \big(\mathbf{1}_{Y_k\le s} - \mathbb{E}(\mathbf{1}_{Y_k\le s})\big) \,,\; s \in [0, 1]^d \,,\; t \in \mathbb{R}^+ \,.$$

Furthermore

$$\mathbb{P}(Y_0^{(j)} \in [a, b]) \le C_j \mathbb{P}^*_{0,j}(Y_0^{(j)} \in [a, b]) = C_j(b - a) \tag{4.5}$$

where the last inequality comes from the fact that the random variables $Y_0^{(j)}$ are uniformly distributed on $[0, 1]$ under $\mathbb{P}^*_{0,j}$ (see Item 1 of Lemma 5.1 in Dedecker, Merlevède and Rio (2013)). Hence $Y_0^{(j)}$ has a density with respect to the Lebesgue measure uniformly bounded by C_j.

For a strictly stationary sequence $(Z_i)_{i\in\mathbb{Z}}$ of random variables with values in $\mathbb{R}^d$, let G_Z be a two parameters Gaussian process with covariance function Γ_Z defined as follows: for any $(s, s') \in \mathbb{R}^{2d}$ and $(t, t') \in (\mathbb{R}^+)^2$

$$\Gamma_Z(s, s', t, t') = \min(t, t')\Lambda_Z(s, s') \text{ where } \Lambda_Z(s, s') = \sum_{k\in\mathbb{Z}} \mathrm{Cov}(\mathbf{1}_{Z_0\le s}, \mathbf{1}_{Z_k\le s'}) \,, \tag{4.6}$$

provided that Λ_Z is well defined.

Let us now give an upper bound on the variance of G_Y. Below, we prove that, for any $u = (u_1, \dots, u_d)$ and $v = (v_1, \dots, v_d)$ in $[0, 1]^d$ and any positive integer n,

$$\mathrm{Var}\big(G_Y(u, n) - G_Y(v, n)\big) \le nC(\beta)\sum_{i=1}^{d} |u_i - v_i| \,, \text{ where } C(\beta) = 1 + 4\sum_{k>0} \beta_k \,. \tag{4.7}$$

If $(u_1, \ldots, u_d) = (F_{P_1^*}(s_1), \ldots, F_{P_d^*}(s_d))$ and $(v_1, \ldots, v_d) = (F_{P_1^*}(s_1'), \ldots, F_{P_d^*}(s_d'))$, then the following equalities hold in distribution: $G_Y(u,n) = G_X(s,n)$ and $G_Y(v,n) = G_X(s',n)$. Hence

$$\mathrm{Var}\big(G_Y(u,n) - G_Y(v,n)\big) = \mathrm{Var}\big(G_X(s,n) - G_X(s',n)\big)\,.$$

Now, by definition of the covariance of G_X,

$$n^{-1}\mathrm{Var}\big(G_X(s,n) - G_X(s',n)\big) = \lim_{N\to\infty} N^{-1}\mathrm{Var}\big(R_X(s,N) - R_X(s',N)\big)\,.$$

Hence, by (4.1) and Corollary 1.4 in Rio (2000),

$$n^{-1}\mathrm{Var}\big(G_X(s,n) - G_X(s',n)\big) \le \mathbb{E}\big(|\mathbf{1}_{X_0\le s'} - \mathbf{1}_{X_0\le s}|(1 + 4\sum_{k>0} b(X_0,k)\big)\,.$$

Now

$$|\mathbf{1}_{X_0\le s'} - \mathbf{1}_{X_0\le s}| \le \sum_{j=1}^d |\mathbf{1}_{X_0^{(j)}\le s_j'} - \mathbf{1}_{X_0^{(j)}\le s_j}|\,.$$

Taking into account the definition of $\mathbb{P}_{0,j}^*$, it follows that

$$\mathrm{Var}\big(G_X(s,n) - G_X(s',n)\big) \le n\sum_{j=1}^d C_j \mathbb{P}_{0,j}^*\big(|\mathbf{1}_{X_0^{(j)}\le s_j'} - \mathbf{1}_{X_0^{(j)}\le s_j}|\big)\,.$$

Now, by definition of $\mathbb{P}_j^*$,

$$\mathbb{P}_{0,j}^*\big(\,|\mathbf{1}_{X_0^{(j)}\le s_j'} - \mathbf{1}_{X_0^{(j)}\le s_j}|\,\big) = \mathbb{P}_j^*(]\min(s_j,s_j'),\max(s_j,s_j')]) = |u_i - v_i|\,,$$

whence

$$\mathrm{Var}\big(G_X(s,n) - G_X(s',n)\big) \le n\sum_{j=1}^d C_j|u_j - v_j| \le nC(\beta)\sum_{i=1}^d |u_i - v_i|\,,$$

which completes the proof of (4.7) .

We shall prove in what follows that the conclusion of Theorem 3.1 holds for the stationary sequence $(Y_i)_{i\in\mathbb{Z}}$ and the associated continuous Gaussian process K_Y with covariance function Γ_Y defined by (4.6). This will imply Theorem 3.1 by taking for $s = (s_1, \ldots, s_d)$,

$$K_X(s,t) = K_Y\big((F_{P_1^*}(s_1), \ldots, F_{P_d^*}(s_d)), t\big)\,,$$

since for any $(s,s') = \big((s_1,\ldots,s_d),(s_1',\ldots,s_d')\big) \in \mathbb{R}^{2d}$,

$$\Gamma_X(s,s',t,t') = \Gamma_Y\big((F_{P_1^*}(s_1), \ldots, F_{P_d^*}(s_d)), (F_{P_1^*}(s_1'), \ldots, F_{P_d^*}(s_d')), t, t'\big)\,.$$

We start by a reduction to a grid and a discretization.

4.1 Reduction to a grid

In this section, we consider a strictly stationary sequence of random variables Z_i in $\mathbb{R}^d$ with marginal distributions with support included in $[0,1]$ and bounded densities. Our aim is to compare the maximal deviation over the unit cube with the maximal deviation over a grid. Let A_n denote the set of x in $[0,1]^d$ such that nx is a multivariate integer. The main result of the section is that, if the marginal densities are each bounded by M, then, for any integer $k \le n$,

$$\sup_{s\in[0,1]^d} |R_Z(s,k) - G_Z(s,k)| \le \sup_{s\in A_n} |R_Z(s,k) - G_Z(s,k)| + dM + \sup_{\|s-s'\|_\infty \le 1/n} |G_Z(s,k) - G_Z(s',k)|\,, \quad (4.8)$$

where we recall that G_Z is a two parameters Gaussian process with covariance function Γ_Z defined by (4.6).

We now prove the above inequality. For each $s = (s_1, \ldots, s_d)$, we set $\pi_-(s) = n^{-1}([ns_1], \ldots, [ns_d])$ and $\pi_+(s) = \pi_-(s) + n^{-1}(1, \ldots, 1)$. From the monotonicity of the multivariate distribution function F and the empirical distribution function F_k,

$$kF_k(\pi_-(s)) - kF(\pi_+(s)) \le R_Z(s,k) \le kF_k(\pi_+(s)) - kF(\pi_-(s))\,.$$

Next let F_i denote the distribution function of the i-th coordinate of Z_0. From our assumption

$$F_i(t) - F_i(s) \le M|t - s|\,.$$

Now, for any $s = (s_1, \ldots s_d)$ and $t = (t_1, \ldots, t_d)$ with $s \le t$,

$$0 \le F(t) - F(s) \le \sum_{i=1}^{d} (F_i(t_i) - F_i(s_i))\,,$$

which, together with the above inequality, ensures that

$$0 \le kF(\pi_+(s)) - kF(\pi_-(s)) \le k(Md/n) \le Md$$

since $k \le n$. Hence

$$R_Z(\pi_-(s), k) - dM \le R_Z(s,k) \le R_Z(\pi_+(s), k) + dM\,.$$

Let then

$$D_Z(k,n) = \sup_{\|s-s'\|_\infty \le 1/n} |G_Z(s,k) - G_Z(s',k)|\,.$$

Clearly

$$-G_Z(\pi_-(s), k) - D_Z(k,n) \le -G_Z(s,k) \le -G_Z(\pi_+(s), k) + D_Z(k,n)\,.$$

Let $\Delta_Z = R_Z - G_Z$. Adding the two above inequalities, we now get that

$$\Delta_Z(\pi_-(s), k) - dM - D_Z(k,n) \le \Delta_Z(s,k) \le \Delta_Z(\pi_+(s), k) + dM + D_Z(k,n)\,,$$

which implies immediately (4.8).

4.2 Discretization

We now apply the inequality (4.8) to our problem. Let $N \in \mathbb{N}^*$ and let $k \in]1, 2^{N+1}]$. We first notice that for any construction of a Kiefer process G_Y with covariance function Γ_Y defined by (4.6),

$$\sup_{1\le k\le 2^{N+1}} \sup_{s\in[0,1]^d} \big|R_Y(s,k) - G_Y(s,k)\big| \le \sup_{s\in[0,1]^d} \big|R_Y(s,1) - G_Y(s,1)\big| + \sum_{L=0}^{N} D_L(G_Y)\,, \quad (4.9)$$

where

$$D_L(G_Y) := \sup_{2^L<\ell\le 2^{L+1}} \sup_{s\in[0,1]^d} \big|(R_Y(s,\ell) - R_Y(s,2^L)) - (G_Y(s,\ell) - G_Y(s,2^L))\big|\,. \quad (4.10)$$

Let then

$$D'_L(G_Y) = \sup_{2^L<\ell\le 2^{L+1}} \sup_{s\in A_{2^L}} |R_Y(s,\ell) - R_Y(s,2^L) - (G_Y(s,\ell) - G_Y(s,2^L))|\,, \quad (4.11)$$

where we recall that A_{2^L} is the set of x in $[0,1]^d$ such that $2^L x$ is a multivariate integer. Applying Inequality (4.8) with $n = 2^L$ to the variables $Z_i = Y_{i+2^L}$ and taking into account (4.5), we get that

$$D_L(G_Y) \le D'_L(G_Y) + dC(\beta) + \sup_{\substack{2^L<\ell\le 2^{L+1}\\ \|s-s'\|_\infty\le 2^{-L}}} |(G_Y(s,\ell) - G_Y(s,2^L)) - (G_Y(s',\ell) - G_Y(s',2^L))|\,. \quad (4.12)$$

4.3 Construction of the Kiefer process

We shall construct in this section a Kiefer process K_Y with covariance function Γ_Y defined by (4.6) in such a way that for $G_Y = K_Y$, the terms involved in (4.12) can be suitably handled.

We start with some notations and definitions.

Definition 4.1. *For two positive integers m and n, let $\mathcal{M}_{m,n}(\mathbb{R})$ be the set of real matrices with m lines and n columns. The Kronecker product (or Tensor product) of $A = [a_{i,j}] \in \mathcal{M}_{m,n}(\mathbb{R})$ and $B = [b_{i,j}] \in \mathcal{M}_{p,q}(\mathbb{R})$ is denoted by $A \otimes B$ and is defined to be the block matrix*

$$A \otimes B = \begin{pmatrix} a_{1,1}B & \cdots & a_{1,n}B \\ \vdots & & \vdots \\ a_{m,1}B & \cdots & a_{m,n}B \end{pmatrix} \in \mathcal{M}_{mp,nq}(\mathbb{R}) .$$

For any positive integer k, the k-th Kronecker power $A^{\otimes k}$ is defined inductively by: $A^{\otimes 1} = A$ and $A^{\otimes k} = A \bigotimes A^{\otimes (k-1)}$, and $\bigotimes_{i=1}^{k} A_i = A_1 \bigotimes \big(\bigotimes_{i=2}^{k} A_i \big)$.
We denote by A^t the transposed matrix of A.

Let $L \in \mathbb{N}$. For any $k \in \mathbb{Z}$ and any $\ell \in \{1, \ldots, d\}$, let $\vec{Z}_{k,\ell,L}$ be the column vector of $\mathbb{R}^{2^L}$ defined by

$$\vec{Z}_{k,\ell,L} = \Big(\big(\mathbf{1}_{Y_{k+2^L} \in [0,1]^{\ell-1} \times [0, j2^{-L}] \times [0,1]^{d-\ell}} \big)_{j=1,\ldots,2^L} \Big)^t . \tag{4.13}$$

Let now $\vec{U}_{k,L}$ and $\vec{U}^{(0)}_{k,L}$ be the column vectors of $\mathbb{R}^{2^{dL}}$ defined by

$$\vec{U}_{k,L} = \bigotimes_{\ell=1}^{d} \vec{Z}_{k,\ell,L} \text{ and } \vec{U}^{(0)}_{k,L} = \vec{U}_{k,L} - \mathbb{E}(\vec{U}_{k,L}) . \tag{4.14}$$

For any $k \in \{1, \ldots, 2^L\}$, let $\vec{e}_{k,L}$ be the column vector of $\mathbb{R}^{2^L}$ defined by

$$\vec{e}_{k,L} = \Big(\big(\mathbf{1}_{k \le m} \big)_{m=1,\ldots,2^L} \Big)^t , \tag{4.15}$$

and let $\vec{S}_{L,d}$ the column vector of $\mathbb{R}^{2^{(d+1)L}}$ defined by

$$\vec{S}_{L,d} = \sum_{k=1}^{2^L} \vec{e}_{k,L} \otimes \vec{U}^{(0)}_{k,L} := \sum_{k=1}^{2^L} \vec{V}_{k,L} . \tag{4.16}$$

Let $C_{L,d}$ be the covariance matrix of $\vec{S}_{L,d}$. It is then the matrix of $\mathcal{M}_{2^{(d+1)L},2^{(d+1)L}}(\mathbb{R})$ defined by

$$C_{L,d} = \mathbb{E}\big(\vec{S}_{L,d} \, \vec{S}^t_{L,d} \big) . \tag{4.17}$$

Let us now continue with some other definitions.

Definition 4.2. *Let m be a positive integer. Let P_1 and P_2 be two probabilities on $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$. Let c be a distance on $\mathbb{R}^m$ associated to a norm. The Kantorovich distance (1942) between P_1 and P_2 with respect to the distance c, also called Wasserstein distance of order 1, is defined by*

$$W_c(P_1, P_2) = \inf\{ \mathbb{E}(c(X,Y)), \, (X,Y) \text{ such that } X \sim P_1, \, Y \sim P_2 \} = \sup_{f \in \mathrm{Lip}(c)} \big(P_1(f) - P_2(f) \big) ,$$

where $\mathrm{Lip}(c)$ is the set of functions from $\mathbb{R}^m$ into $\mathbb{R}$ that are Lipschitz with respect to c; namely for any x and y of $\mathbb{R}^m$, $|f(x) - f(y)| \le c(x,y)$.

Definition 4.3. *Let m be a positive integer. For two vectors $x = \big(x^{(1)}, \dots, x^{(2^m)}\big)^t$ and $y = \big(y^{(1)}, \dots, y^{(2^m)}\big)^t$ of $\mathbb{R}^{2^m}$, we define the following distance*

$$c_m(x, y) = \sup_{j \in \{1, \dots, 2^m\}} |x^{(j)} - y^{(j)}| \,.$$

Let $K \in \{0, \dots, L\}$ and define the following set of integers

$$\mathcal{E}(L, K) = \{1, \dots, 2^{L-K}\} \cap (2\mathbb{N} + 1) \,, \tag{4.18}$$

meaning that if $k \in \mathcal{E}(L, K)$ then k is an odd integer in $[1, 2^{L-K}]$.

For $K \in \{0, \dots, L\}$ and $k \in \mathcal{E}(L, K)$, define

$$B_{K,k} = \big](k-1)2^K, k2^K\big] \,.$$

Notice that for any $m \in \{1, \dots, 2^L\}$,

$$\mathbf{1}_{]0,m]} = \sum_{K=0}^{L} \sum_{k_K \in \mathcal{E}(L,K)} b_{K,k_K}(m) \mathbf{1}_{B_{K,k}} \,, \tag{4.19}$$

with $b_{K,k_K}(m) = 0$ or 1. This representation is unique in the sense that, for m fixed, there exists only one vector $\big(b_{(K,k_K)}(m), k_K \in \mathcal{E}(L, K)\big)_{K \in \{0,\dots,L\}}$ satisfying (4.19). In addition, for any $K \in \{0, \dots, L\}$, $\sum_{k \in \mathcal{E}(L,K)} b_{K,k}(m) \le 1$. More precisely, $b_{K,k}(m) = 1$ if and only if $k = [m2^{-K}]$ and $[m2^{-K}]$ is odd. Let $\vec{b}(m, L)$ be the column vector of $\mathbb{R}^{2^L}$ defined by

$$\vec{b}(m, L) = \Big(\big(b_{K,k_K}(m), k_K \in \mathcal{E}(L, K)\big)_{K \in \{0,\dots,L\}}\Big)^t \text{ and}$$
$$\mathbf{P}_L = \Big(\vec{b}(1, L), b(2, L), \dots, \vec{b}(2^L, L)\Big)^t \,. \tag{4.20}$$

$\mathbf{P}_L$ has the following property: it is a square matrix of $\mathbb{R}^{2^L}$ with determinant equal to 1. Let us denote by $\mathbf{P}_L^{-1}$ its inverse. Notice also that for any positive integer m, $(\mathbf{P}_L^{\otimes m})^{-1} = (\mathbf{P}_L^{-1})^{\otimes m}$ (see Corollary 4.2.11 in Horn and Johnson (1991)).

Let $P_{\vec{S}_{L,d}|\mathcal{F}_{2^L}}$ be the conditional law of $\vec{S}_{L,d}$ given $\mathcal{F}_{2^L}$ and $\mathcal{N}_{C_{L,d}}$ denote the $\mathcal{N}(0, C_{L,d})$-law. Let now $(a_L)_{L \ge 0}$ be a sequence of positive reals and $(\vec{G}^*_{a_L})_{L \ge 0}$ be a sequence of independent random vectors in $\mathbb{R}^{2^{(d+1)L}}$ with respective laws $\mathcal{N}(0, a_L^2 \mathrm{I}_{2^{(d+1)L}})$ (here $\mathrm{I}_{2^{(d+1)L}}$ is the identity matrix on $\mathbb{R}^{2^{(d+1)L}}$), and independent of $\mathcal{F}_\infty \vee \sigma(\eta_i, i \in \mathbb{Z})$. Let $\vec{G}_{a_L} = \mathbf{P}_L^{\otimes(d+1)} \vec{G}^*_{a_L}$. Recall that the probability space is assumed to be large enough to contain a sequence $(\delta_i)_{i \in \mathbb{Z}}$ of iid random variables uniformly distributed on $[0, 1]$, independent of the sequences $(X_i)_{i \in \mathbb{Z}}$ and $(\eta_i)_{i \in \mathbb{Z}}$. According to Rüschendorf (1985) (see also Theorem 2 in Dedecker, Prieur and Raynaud de Fitte (2006)), there exists a random vector $\vec{W}_{L,d} = \big(W_{L,d}^{(1)}, \dots, W_{L,d}^{(2^{L(d+1)})}\big)^t$ in $\mathbb{R}^{2^{(d+1)L}}$ with law $\mathcal{N}_{C_{L,d}} * P_{\vec{G}_{a_L}}$ that is measurable with respect to $\sigma(\delta_L) \vee \sigma(\vec{S}_{L,d} + \vec{G}_{a_L}) \vee \mathcal{F}_{2^L}$, independent of $\mathcal{F}_{2^L}$ and such that

$$\begin{aligned} \mathbb{E}\big(c_{(d+1)L}(\vec{S}_{L,d} + \vec{G}_{a_L}, \vec{W}_{L,d})\big) &= \mathbb{E}\big(W_{c_{(d+1)L}}(P_{\vec{S}_{L,d}|\mathcal{F}_{2^L}} * P_{\vec{G}_{a_L}}, \mathcal{N}_{C_{L,d}} * P_{\vec{G}_{a_L}}\big) \qquad (4.21) \\ &= \mathbb{E} \sup_{f \in \mathrm{Lip}(c_{(d+1)L})} \big(\mathbb{E}\big(f(\vec{S}_{L,d} + \vec{G}_{a_L})|\mathcal{F}_{2^L}\big) - \mathbb{E}(f(\vec{W}_{L,d}))\big) \,. \end{aligned}$$

Here and in what follows $*$ stands for the usual convolution product. Recall that the probability space is assumed to be large enough to contain a sequence $(\upsilon_i)_{i \in \mathbb{Z}}$ of iid random variables uniformly distributed on $[0,1]$, independent of the sequences $(X_i)_{i\in\mathbb{Z}}$ and $(\eta_i,\delta_i)_{i\in\mathbb{Z}}$. By the Skorohod lemma (1976), there exists a measurable function h from $\mathbb{R}^{2^{(d+1)L}}\times[0,1]$ into $\mathbb{R}^{2^{(d+1)L}}\times\mathbb{R}^{2^{(d+1)L}}$ such that $h\big(\vec{W}_{L,d},\upsilon_L\big)=\big(\vec{G}'_{a_L},\vec{T}_{L,d}\big)$ satisfies

$$\vec{G}'_{a_L}+\vec{T}_{L,d}=\vec{W}_{L,d}\text{ a.s. and }\mathcal{L}\big(\vec{G}'_{a_L},\vec{T}_{L,d}\big)=P_{\vec{G}_{a_L}}\otimes\mathcal{N}_{C_{L,d}}\,. \tag{4.22}$$

Hence we have constructed a sequence of centered Gaussian random variables $(\vec{T}_{L,d})_{L\in\mathbb{N}}$ in $\mathbb{R}^{2^{(d+1)L}}$ such that $\mathbb{E}\big(\vec{T}_{L,d}\vec{T}^t_{L,d}\big)=C_{L,d}$, and that are mutually independent. The approximating Kiefer process is then constructed from this Gaussian process as we explain in what follows.

Let us write

$$\vec{T}_{L,d}=\big(T^{(1)}_{1,L},\dots,T^{(2^{dL})}_{1,L},T^{(1)}_{2,L},\dots,T^{(2^{dL})}_{2,L},\dots,T^{(1)}_{2^L,L},\dots,T^{(2^{dL})}_{2^L,L}\big)^t\,,$$

so that for $k\in\{1,\dots,2^L\}$ and $i\in\{1,\dots,2^{dL}\}$, $T^{(i)}_{k,L}$ is the $((k-1)2^{dL}+i)$-th coordinate of the vector $\vec{T}_{L,d}$. Now, for any $k\in\{1,\dots,2^L\}$ and any $i\in\{1,\dots,2^{dL}\}$, we set $T^{(i)}_{0,L}=0$,

$$g^{(i)}_{k,L}=T^{(i)}_{k,L}-T^{(i)}_{k-1,L}\ \text{ and }\vec{g}_{k,L}=(g^{(1)}_{k,L},\dots,g^{(2^{dL})}_{k,L})^t\ . \tag{4.23}$$

Notice that since $\mathbb{E}\big(\vec{T}_{L,d}\vec{T}^t_{L,d}\big)=\mathbb{E}\big(\vec{S}_{L,d}\vec{S}^t_{L,d}\big)$, one can easily verify that for any $(k,\ell)\in\{1,\dots,2^L\}^2$ and any $(i,j)\in\{1,\dots,2^{dL}\}^2$

$$\mathrm{Cov}\big(g^{(i)}_{k,L},g^{(j)}_{\ell,L}\big)=\mathrm{Cov}\big(u^{(i)}_{k,L},u^{(j)}_{\ell,L}\big)\,, \tag{4.24}$$

where $u^{(i)}_{k,L}$ is the i-th coordinate of the vector $\vec{U}_{k,L}$ defined in (4.14). For any $k\in\{1,\dots,2^L\}$, we define now the following Gaussian vectors in $\mathbb{R}^{2^{(d+1)L}}$,

$$\vec{G}_{k,L}=\vec{e}_{k,L}\otimes\vec{g}_{k,L}\,, \tag{4.25}$$

where we recall that $\vec{e}_{k,L}$ is defined in (4.15). We observe that

$$\vec{T}_{L,d}=\sum_{k=1}^{2^L}\vec{G}_{k,L}\,. \tag{4.26}$$

We want to extend now the Gaussian vector $(\vec{g}^{\,t}_{k,L})_{k\in\{1,\dots,2^L\}}=(g^{(1)}_{k,L},\dots,g^{(2^{dL})}_{k,L})_{k\in\{1,\dots,2^L\}}$ of $\mathbb{R}^{2^{(d+1)L}}$ into a Gaussian vector of $(\mathbb{R}^{2^{dL}})^{\mathbb{Z}}$ denoted by $(\vec{g}^{\,t}_{k,L})_{k\in\mathbb{Z}}=(g^{(1)}_{k,L},\dots,g^{(2^{dL})}_{k,L})_{k\in\mathbb{Z}}$ in such a way that $(\vec{g}^{\,t}_{k,L})_{k\in\mathbb{Z}}$ is independent of $\mathcal{F}_{2^L}$ and that for any $(k,\ell)\in\mathbb{Z}^2$ and any $(i,j)\in\{1,\dots,2^{dL}\}^2$, the property (4.24) holds. With this aim, we first notice that by the Kolmogorov extension theorem, there exists a sequence denoted by $(B^{(1)}_k,\dots,B^{(2^{dL})}_k)_{k\in\mathbb{Z}}$ of centered Gaussian random variables such that $\mathrm{Cov}\big(B^{(i)}_k,B^{(j)}_\ell\big)=\mathrm{Cov}\big(u^{(i)}_{k,L},u^{(j)}_{\ell,L}\big)$, and we recall that the probability space is assumed to be large enough to contain a sequence $(\epsilon_i)_{i\in\mathbb{Z}}$ of iid random variables uniformly distributed on $[0,1]$, independent of the sequences $(X_i)_{i\in\mathbb{Z}}$ and $(\eta_i,\delta_i,\upsilon_i)_{i\in\mathbb{Z}}$ introduced before. By the Skorohod lemma (1976) (see also Lemma 2.11 of Dudley and Philipp (1983) and its proof), since $(\mathbb{R}^{2^{dL}})^{\mathbb{Z}}$ is a Polish space, there exists a measurable function h from $\mathbb{R}^{2^{(d+1)L}}\times[0,1]$ into $(\mathbb{R}^{2^{dL}})^{\mathbb{Z}}$ such that

$$h\big((\vec{g}^{\,t}_{k,L})_{k\in\{1,\dots,2^L\}},\epsilon_L\big)=(\vec{g}^{\,t}_{k,L})_{k\in\mathbb{Z}\setminus\{1,\dots,2^L\}} \tag{4.27}$$

satisfies $(B^{(1)}_k,\dots,B^{(2^{dL})}_k)_{k\in\mathbb{Z}}=(\vec{g}^{\,t}_{k,L})_{k\in\mathbb{Z}}$ in law. Therefore the vector $(\vec{g}^{\,t}_{k,L})_{k\in\mathbb{Z}}$ of $(\mathbb{R}^{2^{dL}})^{\mathbb{Z}}$ constructed by the relation (4.27) has the desired property and is such that

the random variables $\big((\vec{g}^{\,t}_{k,L})_{k\in\mathbb{Z}}\big)_{L\in\mathbb{N}}$ are mutually independent.
We use now the following notations: for any $k\in\mathbb{Z}$,

$$\mathcal{G}_k=\sigma\big((\vec{g}^{\,t}_{\ell,L})_{\ell\le k}\big)\ ,\ \mathcal{G}_{-\infty}=\bigvee_{k\in\mathbb{Z}}\mathcal{G}_k\ \text{ and }\ \mathcal{P}_k(\cdot)=\mathbb{E}\big(\,\cdot\,|\mathcal{G}_k\big)-\mathbb{E}\big(\,\cdot\,|\mathcal{G}_{k-1}\big)\,.$$

Let us prove that, for any $k\in\mathbb{Z}$ and any $i\in\{1,\dots,2^{dL}\}$, the random variable

$$d^{(i)}_{k,L}=\sum_{\ell\ge k}\mathcal{P}_k(g^{(i)}_{\ell,L}) \tag{4.28}$$

is well defined in $\mathbb{L}^2$. Note first that by stationarity, $\sum_{\ell\ge k}\|\mathcal{P}_k(g^{(i)}_{\ell,L})\|_2=\sum_{\ell\ge 0}\|\mathcal{P}_0(g^{(i)}_{\ell,L})\|_2$. Next using the computations page 1615 in Peligrad and Utev (2006), we get that, for any integer $m\ge 0$,

$$\sum_{\ell\ge 2m}\|\mathcal{P}_0(g^{(i)}_{\ell,L})\|_2\ll\sum_{\ell\ge m}\frac{\|\mathbb{E}\big(g^{(i)}_{\ell,L}|\mathcal{G}_0\big)\|_2}{(\ell+1)^{1/2}}\,. \tag{4.29}$$

We denote now $\mathcal{H}_k=\overline{\mathrm{span}}(1,(\vec{g}^{\,t}_{\ell,L})_{\ell\le k})$ (where the closure is taken in $\mathbb{L}^2$) and $\mathcal{J}_k=\overline{\mathrm{span}}(1,(\vec{u}^{\,t}_{\ell,L})_{\ell\le k})$ where $\vec{u}_{k,L}=(u^{(1)}_{k,L},\dots,u^{(2^{dL})}_{k,L})^t$, with $u^{(i)}_{k,L}$ the i-th coordinate of the vector $\vec{U}_{k,L}$ defined in (4.14). We denote by $\Pi_{\mathcal{H}_k}(\cdot)$ the orthogonal projection on $\mathcal{H}_k$ and by $\Pi_{\mathcal{J}_k}(\cdot)$ the orthogonal projection on $\mathcal{J}_k$. Since $(\vec{g}^{\,t}_{k,L})_{k\in\mathbb{Z}}$ is a Gaussian process, for any $\ell\ge 0$,

$$\mathbb{E}\big(g^{(i)}_{\ell,L}|\mathcal{G}_0\big)=\Pi_{\mathcal{H}_0}\big(g^{(i)}_{\ell,L}\big)\ \text{ a.s. and in } \mathbb{L}^2\,.$$

Since the property (4.24) holds for any $(k,\ell)\in\mathbb{Z}^2$, we observe that

$$\|\Pi_{\mathcal{H}_0}\big(g^{(i)}_{\ell,L}\big)\|_2=\|\Pi_{\mathcal{J}_0}\big(u^{(i)}_{\ell,L}-\mathbb{E}(u^{(i)}_{\ell,L})\big)\|_2\,.$$

Moreover, for any $\ell\ge 0$, we have

$$\|\Pi_{\mathcal{J}_0}\big(u^{(i)}_{\ell,L}-\mathbb{E}(u^{(i)}_{\ell,L})\big)\|_2\le\|\mathbb{E}\big(u^{(i)}_{\ell,L}-\mathbb{E}(u^{(i)}_{\ell,L})|\mathcal{F}_{2^L}\big)\|_2\,.$$

So, overall, for any $\ell\ge 0$,

$$\|\mathbb{E}\big(g^{(i)}_{\ell,L}|\mathcal{G}_0\big)\|_2\le\|\mathbb{E}\big(u^{(i)}_{\ell,L}-\mathbb{E}(u^{(i)}_{\ell,L})|\mathcal{F}_{2^L}\big)\|_2\,. \tag{4.30}$$

Next, notice that $\|\mathbb{E}\big(u^{(i)}_{\ell,L}-\mathbb{E}(u^{(i)}_{\ell,L})|\mathcal{F}_{2^L}\big)\|_2\le\sup_{Z\in B^2(\mathcal{F}_{2^L})}\mathrm{Cov}(Z,u^{(i)}_{\ell,L})$ where $B^2(\mathcal{F}_{2^L})$ is the set of $\mathcal{F}_{2^L}$-measurable random variables such that $\|Z\|_2\le 1$. Observe that $u^{(i)}_{\ell,L}$ is $\sigma(Y_{\ell+2^L})$-measurable and such that $|u^{(i)}_{\ell,L}|\le 1$. Therefore, by applying Lemma 7.4 with $r=\infty$ and $s=1$, we get that there exists a $\mathcal{F}_{2^L}$-measurable random variable $b_{\mathcal{F}_{2^L}}(\ell+2^L)$ such that

$$\begin{aligned}\|\mathbb{E}\big(u^{(i)}_{\ell,L}-\mathbb{E}(u^{(i)}_{\ell,L})|\mathcal{F}_{2^L}\big)\|_2&\le 2\sup_{Z\in B^2(\mathcal{F}_{2^L})}\mathbb{E}\big(|Z|b_{\mathcal{F}_{2^L}}(\ell+2^L)\big)\\&\le 2\Big(\mathbb{E}\big(b_{\mathcal{F}_{2^L}}(\ell+2^L)\big)\Big)^{1/2}=2\beta_\ell^{1/2}\,.\end{aligned} \tag{4.31}$$

Hence, starting from (4.30) and considering (4.31), we get, for any $\ell\ge 0$,

$$\|\mathbb{E}\big(g^{(i)}_{\ell,L}|\mathcal{G}_0\big)\|_2\le 2\beta_\ell^{1/2}\,. \tag{4.32}$$

Therefore, starting from (4.29) and taking into account (4.32), it follows that

$$\sum_{\ell\ge 0}\|\mathcal{P}_0(g^{(i)}_{\ell,L})\|_2\ll\sum_{\ell\ge 0}\frac{\beta_\ell^{1/2}}{(\ell+1)^{1/2}}\,,$$

implying that the series in (4.28) is well defined in $\mathbb{L}^2$ since by our condition on (β_k), $\sum_{\ell\geq 1}\ell^{-1/2}\beta_\ell^{1/2}<\infty$. We define now

$$\vec{d}_{k,L}=(d_{k,L}^{(1)},\dots,d_{k,L}^{(2^{dL})})^t\ ,\ \vec{D}_{k,L}=\vec{e}_{k,L}\otimes\vec{d}_{k,L}\ \text{ and } \vec{M}_{L,d}=\sum_{k=1}^{2^L}\vec{D}_{k,L}\,. \tag{4.33}$$

Since the random vectors $(\vec{D}_{k,L})_{k\geq 1}$ are orthogonal, $\mathbb{E}(\vec{M}_{L,d}\vec{M}_{L,d}^t)=\sum_{k=1}^{2^L}\mathbb{E}(\vec{D}_{k,L}\vec{D}_{k,L}^t)$, and then, by the property of the tensor product (see for instance Lemma 4.2.10 in Horn and Johnson (1991)),

$$\mathbb{E}(\vec{M}_{L,d}\vec{M}_{L,d}^t)=\sum_{k=1}^{2^L}\vec{e}_{k,L}\vec{e}_{k,L}^t\otimes\mathbb{E}\big(\vec{d}_{k,L}\vec{d}_{k,L}^t\big)\,. \tag{4.34}$$

Let us prove now that for any integer k and any $(i,j)\in\{1,\dots,2^{dL}\}^2$,

$$\mathbb{E}(d_{k,L}^{(i)}d_{k,L}^{(j)})=\sum_{\ell\in\mathbb{Z}}\mathrm{Cov}(u_{0,L}^{(i)},u_{\ell,L}^{(j)})\,, \tag{4.35}$$

which, together with (4.34), will imply that

$$\mathbb{E}(\vec{M}_{L,d}\vec{M}_{L,d}^t)=\sum_{k=1}^{2^L}\Lambda_{k,L,d}\,, \tag{4.36}$$

where $\Lambda_{k,L,d}:=\sum_{\ell\in\mathbb{Z}}\vec{e}_{k,L}\vec{e}_{k,L}^t\otimes\mathbb{E}\big(\vec{U}_{0,L}^{(0)}(\vec{U}_{\ell,L}^{(0)})^t\big)=\sum_{\ell\in\mathbb{Z}}\mathbb{E}\big((\vec{e}_{k,L}\otimes\vec{U}_{0,L}^{(0)})(\vec{e}_{k,L}\otimes(\vec{U}_{\ell,L}^{(0)})^t)\big)$ (by the property of the tensor product).

To prove (4.35), we first notice that the following decomposition is valid: $g_{m,L}^{(i)}=\sum_{k=-\infty}^{m}\mathcal{P}_k(g_{m,L}^{(i)})$ (to see this, it suffices to notice that (4.32) implies that $\mathbb{E}\big(g_{m,L}^{(i)}|\mathcal{G}_{-\infty}\big)=0$ a.s.). Hence, using the fact that the property (4.24) holds for any $(k,\ell)\in\mathbb{Z}^2$, we derive by orthogonality followed by stationarity that

$$\begin{aligned}\mathrm{Cov}(u_{0,L}^{(i)},u_{\ell,L}^{(j)})=\mathbb{E}(g_{0,L}^{(i)}g_{\ell,L}^{(j)})&=\sum_{k=-\infty}^{0\wedge\ell}\mathbb{E}\big(\mathcal{P}_k(g_{0,L}^{(i)})\mathcal{P}_k(g_{\ell,L}^{(j)})\big)\\&=\sum_{k=0\vee(-\ell)}^{\infty}\mathbb{E}\big(\mathcal{P}_0(g_{k,L}^{(i)})\mathcal{P}_0(g_{\ell+k,L}^{(j)})\big)\,.\end{aligned}$$

Whence

$$\sum_{\ell\in\mathbb{Z}}\mathrm{Cov}(u_{0,L}^{(i)},u_{\ell,L}^{(j)})=\sum_{m\geq 0}\sum_{k\geq 0}\mathbb{E}\big(\mathcal{P}_0(g_{m,L}^{(i)})\mathcal{P}_0(g_{k,L}^{(j)})\big)\,. \tag{4.37}$$

On the other hand, by the definition (4.28) of $d_{k,L}^{(i)}$ and stationarity, we have

$$\mathbb{E}(d_{k,L}^{(i)}d_{k,L}^{(j)})=\sum_{\ell\geq k}\sum_{m\geq k}\mathbb{E}\big(\mathcal{P}_k(g_{\ell,L}^{(i)})\mathcal{P}_k(g_{m,L}^{(j)})\big)=\sum_{\ell\geq 0}\sum_{m\geq 0}\mathbb{E}\big(\mathcal{P}_0(g_{\ell,L}^{(i)})\mathcal{P}_0(g_{m,L}^{(j)})\big)\,. \tag{4.38}$$

Considering the equalities (4.37) and (4.38), (4.35) follows.

Hence we have constructed Gaussian random variables $(\vec{M}_{L,d})_{L\in\mathbb{N}}$ in $\mathbb{R}^{2^{(d+1)L}}$ that are mutually independent and such that, according to (4.36), for $\ell,m\in\{1,\dots,2^L\}$ and $s_{L,j}=(j_1 2^{-L},\dots,j_d 2^{-L})$ with $j=(j_1,\dots,j_d)\in\{1,\dots,2^L\}^d$ and $s_{L,k}=(k_1 2^{-L},\dots,k_d 2^{-L})$ with $k=(k_1,\dots,k_d)\in\{1,\dots,2^L\}^d$,

$$\begin{aligned}&\mathrm{Cov}\big((\vec{M}_{L,d})_{(\ell-1)2^{dL}+\sum_{i=1}^d(j_i-1)2^{(d-i)L}+1},(\vec{M}_{L,d})_{(m-1)2^{dL}+\sum_{i=1}^d(k_i-1)2^{(d-i)L}+1}\big)\\&\quad=\inf(\ell,m)\sum_{t\in\mathbb{Z}}\mathrm{Cov}(\mathbf{1}_{Y_0\leq s_{L,j}},\mathbf{1}_{Y_t\leq s_{L,k}})=\Gamma_Y(s_{L,j},s_{L,k},\ell,m)\,.\end{aligned} \tag{4.39}$$

Hence, according to Lemma 2.11 of Dudley and Philipp (1983), there exists a Kiefer process K_Y with covariance function Γ_Y defined by (4.6) such that

$$K_Y(s_{L,j},\ell+2^L)-K_Y(s_{L,j},2^L)=\big(\vec{M}_{L,d}\big)_{(\ell-1)2^{dL}+\sum_{i=1}^{d}(j_i-1)2^{(d-i)L}+1}\,. \qquad (4.40)$$

Thus our construction is now complete. In addition recalling the notation (4.11) and the definition 4.3, we have, for any $L\in\mathbb{N}$,

$$D'_L(K_Y)=c_{(d+1)L}(\vec{S}_{L,d},\vec{M}_{L,d}) \\ \leq c_{(d+1)L}(\vec{S}_{L,d}+\vec{G}_{a_L},\vec{T}_{L,d}+\vec{G}'_{a_L})+c_{(d+1)L}(\vec{T}_{L,d},\vec{M}_{L,d})+c_{(d+1)L}(\vec{0},\vec{G}_{a_L})+c_{(d+1)L}(\vec{0},\vec{G}'_{a_L})\,. \qquad (4.41)$$

4.4 Gaussian approximation

Proposition 4.1. *Let $L\in\mathbb{N}$, K_Y defined by* (4.40) *and $D'_L(K_Y)$ by* (4.11). *Under the assumptions of Theorem 3.1 the following inequality holds: there exists a positive constant C depending on p and d but not depending on L, such that*

$$\mathbb{E}\big(D'_L(K_Y)\big)\leq C(L+1)^{3(d+1)}+C2^{L/p}(L+1)^{\frac{3d}{2}+2-\frac{2+d}{2p}}\mathbf{1}_{p\in]2,3[}+C2^{L/3}(L+1)^{2+4d/3}\mathbf{1}_{p=3}\,.$$

Proposition 4.2. *Let $L\in\mathbb{N}^*$, K_Y defined by* (4.40) *and $D'_L(K_Y)$ by* (4.11). *Assume that the assumptions of Theorem 3.1 holds. Then there exists a positive constant $C(d,p)$ depending on d and p such that for any $L\geq C(d,p)$ and any positive real $x_L\in[2^{L(3-p)/(4-p)}L^{2+3d/2},2^LL^{-d/2}]$, the following inequality holds:*

$$\mathbb{P}\big(D'_L(K_Y)\geq x_L\big)\leq\exp\big(-\kappa_1L\big)+\kappa_2\,x_L^{-1}(L+1)^{d+1} \\ +\kappa_2\,x_L^{-p}2^L\frac{L^{p(3d/2+2)}}{L^{(2+d)/2}}+\kappa_2\,x_L^{-3}2^LL^{4d+6}\mathbf{1}_{p=3}\,,$$

where κ_1 and κ_2 depend on p and d but not on L.

Proof of Proposition 4.1. We shall bound up $\mathbb{E}(D'_L(K_Y))$ with the help of Inequality (4.41). So, for any sequence of positive reals $(a_L)_{L\geq0}$,

$$\mathbb{E}(D'_L(K_Y))\leq2\mathbb{E}\big(c_{(d+1)L}(\vec{G}_{a_L},\vec{0})\big)+\mathbb{E}\big(c_{(d+1)L}(\vec{T}_{L,d},\vec{M}_{L,d})\big) \\ +\mathbb{E}\big(c_{(d+1)L}(\vec{S}_{L,d}+\vec{G}_{a_L},\vec{T}_{L,d}+\vec{G}'_{a_L})\big)\,. \qquad (4.42)$$

We start by giving an upper bound for $\mathbb{E}\big(c_{(d+1)L}(\vec{G}_{a_L},\vec{0})\big)$. With this aim we first recall that $\vec{G}_{a_L}=\big(G_{a_L}^{(1)},\dots,G_{a_L}^{(2^{(d+1)L})}\big)^t$ is a centered Gaussian vector with covariance matrix $a_L^2\big(\mathbf{P}_L\mathbf{P}_L^t\big)^{\otimes(d+1)}$ where $\mathbf{P}_L$ is defined in (4.20) (indeed, notice that by Lemma 4.2.10 in Horn and Johnson (1991), $\mathbf{P}_L^{\otimes(d+1)}(\mathbf{P}_L^{\otimes(d+1)})^t=\big(\mathbf{P}_L\mathbf{P}_L^t\big)^{\otimes(d+1)}$). Therefore, for each $m\in\{1,\dots,2^{(d+1)L}\}$, if we denote by $v_{a_L,m}^2$ the variance of $G_{a_L}^{(m)}$, it follows from the definition of the tensor product that there exists $j=(j_1,\dots,j_{d+1})$ in $\{1,\dots,2^L\}^{(d+1)}$ such that

$$v_{a_L,m}^2=a_L^2\prod_{i=1}^{d+1}\Big(\sum_{K=0}^{L}\sum_{k_K\in\mathcal{E}(L,K)}b_{K,k_K}(j_i)\Big)\,, \qquad (4.43)$$

where we recall that the notations $b_{K,k_K}(j_i)$ and $\mathcal{E}(L,K)$ have been respectively introduced in (4.19) and (4.18). According to the inequality (3.6) in Ledoux and Talagrand (1991),

$$\mathbb{E}\big(c_{(d+1)L}(\vec{G}_{a_L},\vec{0})\big)=\mathbb{E}\Big(\max_{m=1,\dots,2^{(d+1)L}}|G_{a_L}^{(m)}|\Big) \\ \leq\Big(2+3(\log(2^{(d+1)L}))^{1/2}\Big)\max_{m=1,\dots,2^{(d+1)L}}v_{a_L,m}\,.$$

Since $v_{a_L,m}^2 \leq a_L^2(L+1)^{d+1}$, we then get that

$$\mathbb{E}\big(c_{(d+1)L}(\vec{G}_{a_L},\vec{0})\big) \leq 5\, a_L(d+1)^{1/2}\,(L+1)^{1+d/2}\,. \tag{4.44}$$

To bound up now the second and third terms in the right hand side of (4.42), we shall use the two following lemmas. The proof of the second lemma being very technical, it is postponed to Appendix A.

Lemma 4.1. *Let $L \in \mathbb{N}$. Under the assumptions of Theorem 3.1 and the notations of Section 4.3, the following inequality holds: there exists a positive constant C_1 not depending on (L,d), such that*

$$\mathbb{E}\big(c_{(d+1)L}(\vec{T}_{L,d},\vec{M}_{L,d})\big) \leq C_1 d^{1/2}(L+1)^{1/2}(L^{1/2}+2^{(3-p)L/2})\,. \tag{4.45}$$

Lemma 4.2. *Let $L \in \mathbb{N}$. Under the assumptions of Theorem 3.1 and the notations of Section 4.3, the following inequality holds: for any $a_L \in [(L+1)^{d+1}, 2^L(L+1)^{d+1}]$, there exists a positive constant C_2 depending on p but not on (L,d), such that*

$$\begin{aligned}&\mathbb{E}\big(c_{(d+1)L}(\vec{S}_{L,d}+\vec{G}_{a_L},\vec{T}_{L,d}+\vec{G}'_{a_L})\big)\\ &\quad\leq C_2\big((L+1)^{d+1}+a_L^{1-p}(L+1)^{p(d+1)}2^L+a_L^{-3}(L+1)^{4(d+1)}2^L\big(2^{L(3-p)}+(L+1)\mathbf{1}_{p=3}\big)\\ &\qquad+a_L^{2-p}(L+1)^{(5-p)(d+1)}+a_L^{-1}(L+1)^{2d+3}\mathbf{1}_{p=3}+a_L^{-2}(L+1)^{3d+4}2^L\mathbf{1}_{p=3}\big)\,.\end{aligned} \tag{4.46}$$

Starting from (4.42) and considering the upper bound (4.44) and the two above lemmas, the proof of Proposition 4.1 is then achieved by selecting

$$a_L=(L+1)^{d+1}\vee\big(2^{L/p}(L+1)^{d+1-(2+d)/(2p)}\mathbf{1}_{p\in]2,3[}+2^{L/3}(L+1)^{1+5d/6}\mathbf{1}_{p=3}\big)$$

in the bounds (4.44) and (4.46). □

Proof of Lemma 4.1. Notice that by construction, $\vec{T}_{L,d}-\vec{M}_{L,d}$ is a Gaussian vector of $\mathbb{R}^{2^{(d+1)L}}$. Therefore, according to the inequality (3.6) in Ledoux and Talagrand (1991),

$$\mathbb{E}\big(c_{(d+1)L}(\vec{T}_{L,d},\vec{M}_{L,d})\big) \leq \Big(2+3(\log(2^{(d+1)L}))^{1/2}\Big)\sup_{\substack{i=1,\dots,2^{(d+1)L}\\ k\leq 2^L}}\|\sum_{\ell=1}^{k}(g_{\ell,L}^{(i)}-d_{\ell,L}^{(i)})\|_2\,.$$

Using stationarity and Theorem 1(ii) in Wu (2007) followed by Inequality (4.29), we derive that

$$\|\sum_{\ell=1}^{k}(g_{\ell,L}^{(i)}-d_{\ell,L}^{(i)})\|_2^2 \ll \sum_{j=1}^{k}\Big(\sum_{\ell\geq j}\|\mathcal{P}_0(g_{\ell,L}^{(i)})\|_2\Big)^2 \ll \sum_{j=1}^{k}\bigg(\sum_{\ell\geq[j/2]}\frac{\|\mathbb{E}(g_{\ell,L}^{(i)}|\mathcal{G}_0)\|_2}{\ell^{1/2}}\bigg)^2\,.$$

Next using (4.32), followed by the fact that $\beta_k=O(k^{1-p})$ for $p\in]2,3]$, we get that

$$\|\sum_{\ell=1}^{k}(g_{\ell,L}^{(i)}-d_{\ell,L}^{(i)})\|_2^2 = O(\ln k + k^{3-p}\mathbf{1}_{p\neq 3})\,. \tag{4.47}$$

So overall, (4.45) follows. This ends the proof of the lemma. □

Proof of Proposition 4.2. Let $y_L = x_L/7$. Starting from the inequality (4.41), we derive that for any sequence of positive reals $(a_L)_{L\geq 0}$,

$$\begin{aligned}&\mathbb{P}\big(D'_L(K_Y)\geq x_L\big)\leq \mathbb{P}\big(c_{(d+1)L}(\vec{T}_{L,d},\vec{M}_{L,d})\geq 2y_L\big)\\ &\qquad+\mathbb{P}\big(c_{(d+1)L}(\vec{S}_{L,d}+\vec{G}_{a_L},\vec{T}_{L,d}+\vec{G}'_{a_L})\geq y_L\big)+2\mathbb{P}\big(c_{(d+1)L}(\vec{G}_{a_L},\vec{0})\geq 2y_L\big)\,.\end{aligned} \tag{4.48}$$

Recalling Inequality (4.44), we then derive that if we select

$$a_L = \frac{x_L}{35(d+1)^{1/2}(L+1)^{1+d/2}}\,, \tag{4.49}$$

then

$$\mathbb{P}\big(c_{(d+1)L}(\vec{G}_{a_L},\vec{0}) \geq 2y_L\big) \leq \mathbb{P}\big(c_{(d+1)L}(\vec{G}_{a_L},\vec{0}) - \mathbb{E}\big(c_{(d+1)L}(\vec{G}_{a_L},\vec{0})\big) \geq y_L\big)\,.$$

Applying the often-called Cirel'son-Ibragimov-Sudakov inequality (1976), we then infer that for any sequence of positive reals $(a_L)_{L\in\mathbb{N}}$ satisfying (4.49),

$$\mathbb{P}\big(c_{(d+1)L}(\vec{G}_{a_L},\vec{0}) \geq 2y_L\big) \leq \exp\Big(\frac{-y_L^2}{2\sigma^2_{a_L,d}}\Big)\,,$$

where $\sigma^2_{a_L,d} = \sup_{1\leq m\leq 2^{(d+1)L}} v^2_{a_L,m}$ and $v^2_{a_L,m}$ is defined in (4.43). Since we have $v^2_{a_L,m} \leq a_L^2(L+1)^{d+1}$, it follows that for any sequence of positive reals $(a_L)_{L\in\mathbb{N}}$ satisfying (4.49),

$$\mathbb{P}\big(c_{(d+1)L}(\vec{G}_{a_L},\vec{0}) \geq 2y_L\big) \leq \exp\big(-dL\big)\,. \tag{4.50}$$

Let now C_1 be the constant defined in Lemma 4.1. Due to the restriction on x_L, there exists a positive constant $C_1(d,p)$ depending only on p and d, such that for $L \geq C_1(d,p)$, $y_L \geq C_1 d^{1/2}(L+1)^{1/2}(L+2^{(3-p)L/2})$. Whence, for $L \geq C_1(d,p)$,

$$\mathbb{P}\big(c_{(d+1)L}(\vec{T}_{L,d},\vec{M}_{L,d}) \geq 2y_L\big) \leq \mathbb{P}\big(c_{(d+1)L}(\vec{T}_{L,d},\vec{M}_{L,d}) - \mathbb{E}\big(c_{(d+1)L}(\vec{T}_{L,d},\vec{M}_{L,d})\big)\big) \geq y_L\big)\,.$$

By construction, $\vec{T}_{L,d} - \vec{M}_{L,d}$ is a Gaussian vector of $\mathbb{R}^{2(d+1)L}$. Therefore, applying again the Cirel'son-Ibragimov-Sudakov inequality (1976), we then infer that

$$\mathbb{P}\big(c_{(d+1)L}(\vec{T}_{L,d},\vec{M}_{L,d}) \geq 2y_L\big) \leq \exp\Big(\frac{-y_L^2}{2u^2_{L,d}}\Big)\,,$$

where

$$u^2_{L,d} = \sup_{\substack{i=1,\ldots,2^{(d+1)L}\\ k\leq 2^L}} \Big\|\sum_{\ell=1}^{k}(g^{(i)}_{\ell,L} - d^{(i)}_{\ell,L})\Big\|_2^2\,.$$

Using (4.47), it follows that $u^2_{L,d} = O(L + 2^{L(3-p)}\mathbf{1}_{p\neq 3})$. Hence, there exists a positive constant $\kappa(d)$ depending on d such that, for $L \geq C_2(d,p)$ where $C_2(d,p)$ is a positive constant depending only on p and d,

$$\mathbb{P}\big(c_{(d+1)L}(\vec{T}_{L,d},\vec{M}_{L,d}) \geq 2y_L\big) \leq \exp\big(-\kappa(d)L\big)\,. \tag{4.51}$$

Notice now that by the conditions on x_L, the choice of a_L given in (4.49) implies that a_L belongs to $[(L+1)^{d+1}, 2^L(L+1)^{d+1}]$ for L larger than a constant depending on d and p. Therefore applying Lemma 4.2, it follows that there exists a positive constant κ_2 not depending on L, such that, for $L \geq C(d,p)$,

$$\mathbb{P}\big(c_{(d+1)L}(\vec{S}_{L,d} + \vec{G}_{a_L}, \vec{T}_{L,d} + \vec{G}'_{a_L}) \geq y_L\big) \leq \kappa_2\, y_L^{-1}(L+1)^{d+1} + \kappa_2\, y_L^{-p}\frac{L^{p(3d/2+2)}2^L}{L^{(2+d)/2}} + \kappa_2\, y_L^{-3}2^L L^{4d+6}\mathbf{1}_{p=3}\,. \tag{4.52}$$

Starting from (4.48) and considering the upper bounds (4.50), (4.51) and (4.52), the proposition follows. □

4.5 End of the proof of Theorem 3.1

We start by proving Item 1. Let K_Y defined by (4.40). Starting from (4.9) with $G_Y = K_Y$, we get

$$\mathbb{E}\Big(\sup_{1\le k\le 2^{N+1}}\sup_{s\in[0,1]^d}\big|R_Y(s,k)-K_Y(s,k)\big|\Big) \le \mathbb{E}\big(\sup_{s\in[0,1]^d}\big|R_Y(s,1)-K_Y(s,1)\big|\big)+\sum_{L=0}^{N}\mathbb{E}(D_L(K_Y))\,, \quad (4.53)$$

where $D_L(K_Y)$ is defined by (4.10).
Notice first that

$$\sup_{s\in[0,1]^d}|R_Y(s,1)-K_Y(s,1)|\le 1+\sup_{s\in[0,1]^d}|K_Y(s,1)|\,.$$

Now, from (4.7), the Gaussian process $K_Y(.,1)$ has a continuous version. Therefore, according to Theorem 11.17 in Ledoux and Talagrand (1991), there exists a positive constant $c(d)$ depending on d, such that

$$\mathbb{E}(\sup_{s\in[0,1]^d}|K_Y(s,1)|)\le c(d)\,,$$

implying that

$$\mathbb{E}\big(\sup_{s\in[0,1]^d}\big|R_Y(s,1)-K_Y(s,1)\big|\big)\le c(d)+1\,. \quad (4.54)$$

We bound now the terms $\mathbb{E}(D_L(K_Y))$ in (4.53). With this aim, we start with the inequality (4.12) with $G_Y = K_Y$. By definition of Λ_Y, the Gaussian processes B_k defined by

$$B_k(s) = K_Y(s,k+1)-K_Y(s,k)$$

are independent and identically distributed, with common covariance function Λ_Y. Hence, by (4.7), for any integers k and ℓ with $k\le\ell$,

$$\mathrm{Var}(K_Y(s,\ell)-K_Y(s,k))-(K_Y(s',\ell)-K_Y(s',k))\le(\ell-k)C(\beta)\|s-s'\|_1\,.$$

Therefrom, starting from Theorem 11.17 in Ledoux and Talagrand (1991), one can prove that there exists a positive constant $C(d)$ depending on d, such that

$$\mathbb{E}\big(\sup_{\substack{2^L<\ell\le 2^{L+1}\\ \|s-s'\|_\infty\le 2^{-L}}}|(K_Y(s,\ell)-K_Y(s,2^L))-(K_Y(s',\ell)-K_Y(s',2^L))|\big)\le C(d)\sqrt{L}\,. \quad (4.55)$$

Hence starting from Inequality (4.12) with $G_Y = K_Y$, we derive that there exists some positive constant $c'(d)$ such that

$$\mathbb{E}(D_L(K_Y))\le\mathbb{E}(D'_L(K_Y))+c'(d)\sqrt{L}\,, \quad (4.56)$$

where $D'_L(K_Y)$ is defined by (4.11). Starting from (4.53) and considering (4.54) and (4.56) together with the upper bound given in Proposition 4.1, Item 1 of Theorem 3.1 then follows.

We turn now to the proof of Item 2. Starting from (4.9) with $G_Y = K_Y$, and considering the upper bound (4.54), we infer that Item 2 of Theorem 3.1 will hold true provided that we can show that for L large enough,

$$D_L(K_Y)=O(2^{L/p}L^{\lambda(d)+\varepsilon+1/p})\quad\text{almost surely, for any } \varepsilon>0\,, \quad (4.57)$$

where $\lambda(d) = \big(\frac{3d}{2} + 2 - \frac{2+d}{2p}\big)\mathbf{1}_{p\in]2,3[} + \big(2 + \frac{4d}{3}\big)\mathbf{1}_{p=3}$ and $D_L(K_Y)$ is defined by (4.10). Starting from Inequality (4.12) with $G_Y = K_Y$ and considering the upper bound (4.55), we infer that (4.57) will hold true provided that one can prove that for L large enough,

$$D'_L(K_Y) = O(2^{L/p} L^{\lambda(d)+\varepsilon+1/p}) \quad \text{almost surely, for any } \varepsilon > 0\,, \tag{4.58}$$

where $D'_L(K_Y)$ is defined by (4.10). But by using Proposition 4.2, we derive that for L large enough, there exist two positive constants κ_1 and κ_2 depending on p and d but not on L, such that

$$\mathbb{P}\big(D'_L(K_Y) \geq 2^{L/p} L^{\lambda(d)+\varepsilon+1/p}\big) \ll \exp\big(-\kappa_1 L\big) + \kappa_2 \frac{1}{L^{1+\varepsilon p}}\,,$$

which proves (4.58) by using Borel-Cantelli Lemma. This ends the proof of Item 2 and then of the theorem. □

5 Proof of Theorem 3.2

As at the beginning of the proof of Theorem 3.1, we first transform the random variables X_i. The transformation in the iid case is more direct since we do not need to introduce another probability. So, for any k in $\mathbb{Z}$ and any j in $\{1, \ldots, d\}$, we still denote by $X_k^{(j)}$ the j-th marginal of X_k and by P_j the law of $X_0^{(j)}$. Let F_{P_j} be the distribution function of P_j, and let $F_{P_j}(x-0) = \sup_{z<x} F_{P_j}(z)$. Let $(\eta_i)_{i\in\mathbb{Z}}$ be a sequence of iid random variables with uniform distribution over $[0,1]$, independent of the initial sequence $(X_i)_{i\in\mathbb{Z}}$. Define then

$$Y_i^{(j)} = F_{P_j}(X_i^{(j)} - 0) + \eta_i(F_{P_j}(X_i^{(j)}) - F_{P_j}(X_i^{(j)} - 0)) \;\text{ and }\; Y_i = \big(Y_i^{(1)}, \ldots, Y_i^{(d)}\big)^t\,. \tag{5.1}$$

Note that $(Y_i)_{i\in\mathbb{Z}}$ forms a sequence of iid random variables with values in $[0,1]^d$. In addition the marginals of Y_i are uniformly distributed on $[0,1]$ and $X_i^{(j)} = F_{P_j}^{-1}(Y_i^{(j)})$ almost surely, where $F_{P_j}^{-1}$ is the generalized inverse of the cadlag function F_{P_j} (see Lemma F.1. in Rio (2000)). Hence

$$R_X(\cdot,\cdot) = R_Y\big((F_{P_1}(\cdot), \ldots, F_{P_d}(\cdot)), \cdot\big) \text{ almost surely,}$$

where

$$R_Y(s,t) = \sum_{1\leq k\leq t} \big(\mathbf{1}_{Y_k\leq s} - \mathbb{E}(\mathbf{1}_{Y_k\leq s})\big)\,,\; s \in [0,1]^d\,,\; t \in \mathbb{R}^+\,.$$

Therefore to prove Theorem 3.2, it suffices to prove that its conclusions hold for the iid sequence $(Y_i)_{i\in\mathbb{Z}}$ defined above and the associated continuous Gaussian process K_Y with covariance function Γ_Y defined as follows: for any $(s,s') \in [0,1]^{2d}$ and $(t,t') \in (\mathbb{R}^+)^{2d}$

$$\Gamma_Y(s,s',t,t') = \min(t,t')\Lambda_Y(s,s') \;\text{ where }\; \Lambda_Y(s,s') = \operatorname{Cov}(\mathbf{1}_{Y_0\leq s}, \mathbf{1}_{Y_0\leq s'})\,. \tag{5.2}$$

This will clearly imply Theorem 3.2 by taking for $s = (s_1, \ldots, s_d)$,

$$K_X(s,t) = K_Y\big((F_{P_1}(s_1), \ldots, F_{P_d}(s_d)), t\big)\,,$$

since for any $(s,s') = \big((s_1, \ldots, s_d), (s'_1, \ldots, s'_d)\big) \in \mathbb{R}^{2d}$,

$$\Gamma_X(s,s',t,t') = \Gamma_Y\big((F_{P_1}(s_1), \ldots, F_{P_d}(s_d)), (F_{P_1}(s'_1), \ldots, F_{P_d}(s'_d)), t, t'\big)\,.$$

According to the proof of Theorem 3.1, the crucial point is to construct a Kiefer process K_Y with covariance function Γ_Y defined by (5.2) in such a way that one can handle both the expectation and the deviation probability of the quantity $D'_L(K_Y)$ with $(Y_i)_{i\in\mathbb{Z}}$ defined by (5.1).

Construction of the Kiefer process. We shall use the same notations and definitions as in Section 4.3. Therefore $\vec{S}_{L,d}$ denotes the column vector of $\mathbb{R}^{2^{(d+1)L}}$ defined by (4.16) with $(Y_i)_{i\in\mathbb{Z}}$ defined by (5.1), and $C_{L,d}$ the covariance matrix of $\vec{S}_{L,d}$. It is then the matrix of $\mathcal{M}_{2^{(d+1)L},2^{(d+1)L}}(\mathbb{R})$ defined by (4.17). Notice that by independence and the properties of the tensor product (see Lemma 4.2.10 in Horn and Johnson (1991)),

$$C_{L,d} = \sum_{k=1}^{2^L} \vec{e}_{k,L}\vec{e}^{\,t}_{k,L} \otimes \mathbb{E}(\vec{U}^{(0)}_{1,L}(\vec{U}^{(0)}_{1,L})^t)\,. \tag{5.3}$$

As in Section 4.3, to construct the Kiefer process, we consider a sequence $(a_L)_{L\geq 0}$ of positive reals and a sequence $(\vec{G}^*_{a_L})_{L\geq 0}$ of independent random vectors in $\mathbb{R}^{2^{(d+1)L}}$ with respective laws $\mathcal{N}(0, a_L^2 \mathrm{I}_{2^{(d+1)L}})$ ($\mathrm{I}_{2^{(d+1)L}}$ being the identity matrix on $\mathbb{R}^{2^{(d+1)L}}$), and independent of $\mathcal{F}_\infty \vee \sigma(\eta_i, i\in\mathbb{Z})$. Moreover we set $\vec{G}_{a_L} = \mathbf{P}_L^{\otimes(d+1)}\vec{G}^*_{a_L}$ where $\mathbf{P}_L$ has been defined in (4.20). Since the probability space has been assumed to be large enough to contain a sequence $(\delta_i)_{i\in\mathbb{Z}}$ of iid random variables uniformly distributed on $[0,1]$, independent of the sequences $(X_i)_{i\in\mathbb{Z}}$ and $(\eta_i)_{i\in\mathbb{Z}}$, according to Rüschendorf (1985), there exists a random vector $\vec{W}_{L,d} = \big(W^{(1)}_{L,d},\dots,W^{(2^{L(d+1)})}_{L,d}\big)^t$ in $\mathbb{R}^{2^{(d+1)L}}$ with law $\mathcal{N}_{C_{L,d}} * P_{\vec{G}_{a_L}}$ that is measurable with respect to $\sigma(\delta_L)\vee\sigma(\vec{S}_{L,d}+\vec{G}_{a_L})\vee\mathcal{F}_{2^L}$, independent of $\mathcal{F}_{2^L}$, and such that

$$\begin{aligned}\mathbb{E}\big(c_{(d+1)L}(\vec{S}_{L,d}+\vec{G}_{a_L},\vec{W}_{L,d})\big) &= W_{c_{(d+1)L}}(P_{\vec{S}_{L,d}} * P_{\vec{G}_{a_L}}, \mathcal{N}_{C_{L,d}} * P_{\vec{G}_{a_L}}) \\ &= \sup_{f\in \mathrm{Lip}(c_{(d+1)L})} \big(\mathbb{E}\big(f(\vec{S}_{L,d}+\vec{G}_{a_L})\big) - \mathbb{E}(f(\vec{W}_{L,d}))\big)\,,\end{aligned} \tag{5.4}$$

where $P_{\vec{S}_{L,d}}$ and and $\mathcal{N}_{C_{L,d}}$ respectively denote the law of $\vec{S}_{L,d}$ and the $\mathcal{N}(0,C_{L,d})$-law. As in Section 4.3, using the Skorohod Lemma (1976), we infer that there exists a measurable function h from $\mathbb{R}^{2^{(d+1)L}}\times[0,1]$ into $\mathbb{R}^{2^{(d+1)L}}\times\mathbb{R}^{2^{(d+1)L}}$ such that $h\big(\vec{W}_{L,d},\upsilon_L\big) = \big(\vec{G}'_{a_L},\vec{T}_{L,d}\big)$ satisfies

$$\vec{G}'_{a_L} + \vec{T}_{L,d} = \vec{W}_{L,d} \text{ a.s. and } \mathcal{L}\big(\vec{G}'_{a_L},\vec{T}_{L,d}\big) = P_{\vec{G}_{a_L}}\otimes\mathcal{N}_{C_{L,d}}\,. \tag{5.5}$$

Hence we have constructed a sequence of centered Gaussian random variables $(\vec{T}_{L,d})_{L\in\mathbb{N}}$ in $\mathbb{R}^{2^{(d+1)L}}$ that are mutually independent and such that $\mathbb{E}\big(\vec{T}_{L,d}\vec{T}^t_{L,d}\big) = C_{L,d}$. In particular, they satisfy for $\ell, m\in\{1,\dots,2^L\}$ and $s_{L,j} = (j_1 2^{-L},\dots,j_d 2^{-L})$ with $j=(j_1,\dots,j_d)\in\{1,\dots,2^L\}^d$ and $s_{L,k} = (k_1 2^{-L},\dots,k_d 2^{-L})$ with $k=(k_1,\dots,k_d)\in\{1,\dots,2^L\}^d$,

$$\begin{aligned}&\mathrm{Cov}\big((\vec{T}_{L,d})_{(\ell-1)2^{dL}+\sum_{i=1}^d (j_i-1)2^{(d-i)L}+1}, (\vec{T}_{L,d})_{(m-1)2^{dL}+\sum_{i=1}^d (k_i-1)2^{(d-i)L}+1}\big) \\ &\quad = \inf(\ell,m)\,\mathrm{Cov}(\mathbf{1}_{Y_0\leq s_{L,j}},\mathbf{1}_{Y_0\leq s_{L,k}}) = \Gamma_Y(s_{L,j},s_{L,k},\ell,m)\,.\end{aligned} \tag{5.6}$$

Hence, according to Lemma 2.11 of Dudley and Philipp (1983), there exists a Kiefer process K_Y with covariance function Γ_Y defined by (5.2) such that

$$K_Y(s_{L,j},\ell+2^L) - K_Y(s_{L,j},2^L) = \big(\vec{T}_{L,d}\big)_{(\ell-1)2^{dL}+\sum_{i=1}^d (j_i-1)2^{(d-i)L}+1}\,. \tag{5.7}$$

Thus our construction is now complete.

End of the proof. Following the proof of Theorem 3.1 (see Section 4.5), to complete the proof of Theorem 3.2, it suffices to prove the following two propositions.

Proposition 5.1. *Let $L \in \mathbb{N}$, K_Y defined by* (5.7) *and $D'_L(K_Y)$ by* (4.11)*. Under the assumptions of Theorem 3.2, the following inequality holds: there exists a positive constant C not depending on (L,d), such that*

$$\mathbb{E}\big(D'_L(K_Y)\big) \le C(d+1)^{1/3}(L+1)^{1+2d/3}2^{L/3}\,.$$

Proposition 5.2. *Let $L \in \mathbb{N}$, K_Y defined by* (5.7) *and $D'_L(K_Y)$ by* (4.11)*. Assume that the assumptions of Theorem 3.2 holds. Then, for any $x_L \ge (L+1)^{d+3/2}$, the following inequality holds:*

$$\mathbb{P}\big(D'_L(K_Y) \ge x_L\big) \le \exp\big(-\kappa_1 L\big) + \kappa_2\, x_L^{-3}(L+1)^{2d+3}2^L\,,$$

where κ_1 and κ_2 depend on d but not on L.

Proof of Proposition 5.1. Recalling the definition 4.3, we have that for any $L \in \mathbb{N}$,

$$\begin{aligned} D'_L(K_Y) &= c_{(d+1)L}(\vec{S}_{L,d}, \vec{T}_{L,d}) \\ &\le c_{(d+1)L}(\vec{S}_{L,d} + \vec{G}_{a_L}, \vec{T}_{L,d} + \vec{G}'_{a_L}) + c_{(d+1)L}(\vec{0}, \vec{G}_{a_L}) + c_{(d+1)L}(\vec{0}, \vec{G}'_{a_L})\,, \end{aligned} \quad (5.8)$$

where $\vec{T}_{L,d}$ and $\vec{G}'_{a_L}$ have been defined in (5.5).

To bound up the expectation of the first term in the right hand side of (5.8), we shall use the following lemma whose proof is postponed to Appendix A. The expectation of the two last terms is handled by using (4.44).

Lemma 5.1. *Let $L \in \mathbb{N}$. Under the assumptions of Theorem 3.2 the following inequality holds: For any sequence $(a_L)_{L\ge 0}$ of positive reals, there exists a positive constant C not depending on (L,d), such that*

$$\mathbb{E}\big(c_{(d+1)L}(\vec{S}_{L,d} + \vec{G}_{a_L}, \vec{T}_{L,d} + \vec{G}'_{a_L})\big) \le C a_L^{-2}(L+1)^{d+1}2^L + C a_L^{-3}(L+1)^{3(d+1)/2}2^L\,. \quad (5.9)$$

Starting from (5.8), taking the expectation and considering the upper bounds (4.44) and (5.9) by selecting $a_L = (d+1)^{-1/6}(L+1)^{d/6}2^{L/3}$, the proposition follows. □

Proof of Proposition 5.2. The proof of this proposition follows the lines of the one's of Proposition 4.2 with obvious modifications. The term $c_{(d+1)L}(\vec{T}_{L,d}, \vec{M}_{L,d})$ is obviously equal to zero, Lemma 5.1 is used instead of Lemma 4.2 and we select $a_L = \frac{x_L}{25(d+1)^{1/2}(L+1)^{1+d/2}}$. □

6 Appendix A

This section is devoted to the proofs of Lemmas 4.1 and 5.1. We keep the same notations as those given in Section 4, and use sometimes the notation $a_n \ll b_n$ to mean that there exists a numerical constant C not depending on n such that $a_n \le C b_n$, for all positive integers n.

6.1 Proof of Lemma 4.2

According to (4.21) and (4.22), we first recall that

$$\mathbb{E}\big(c_{(d+1)L}(\vec{S}_{L,d} + \vec{G}_{a_L}, \vec{T}_{L,d} + \vec{G}'_{a_L})\big) = \mathbb{E}\big(W_{c_{(d+1)L}}(P_{\vec{S}_{L,d}|\mathcal{F}_{2^L}} * P_{\vec{G}_a}, \mathcal{N}_{C_{L,d}} * P_{\vec{G}_a})\big)\,, \quad (6.1)$$

where $\mathcal{N}_{C_{L,d}}$ is the law of $\vec{T}_{L,d} = \sum_{i=1}^{2^L} \vec{G}_{i,L}$ and $\big(\vec{G}^t_{1,L}, \ldots, \vec{G}^t_{2^L,L}\big)^t$ is a Gaussian vector of $\mathbb{R}^{2^{(d+2)L}}$ such that $\mathbb{E}(\vec{G}_{i,L}\vec{G}^t_{j,L}) = \mathbb{E}(\vec{V}_{i,L}\vec{V}^t_{j,L})$ where the random vectors $\vec{V}_{i,L}$ have been defined in (4.16). We consider now a Gaussian vector $\big(\vec{N}^t_{1,L}, \ldots, \vec{N}^t_{2^L,L}\big)^t$ of $\mathbb{R}^{2^{(d+2)L}}$ such that

$$\big(\vec{N}^t_{1,L}, \ldots, \vec{N}^t_{2^L,L}\big)^t = \big(\vec{G}^t_{1,L}, \ldots, \vec{G}^t_{2^L,L}\big)^t \text{ in law}\,, \quad (6.2)$$

and

$$\big(\vec{N}^t_{1,L},\dots,\vec{N}^t_{2^L,L}\big)^t \text{ is independent of } \mathcal{F}_\infty \vee \sigma(\eta_i, i\in\mathbb{Z})\,. \tag{6.3}$$

Define

$$\vec{N}_{L,d} = \vec{N}_{1,L} + \vec{N}_{2,L} + \cdots + \vec{N}_{2^L,L}\,.$$

Notice that we have in particular

$$\mathbb{E}(\vec{N}_{i,L}\vec{N}^t_{j,L}) = \mathbb{E}(\vec{V}_{i,L}\vec{V}^t_{j,L}) \text{ and } \mathbb{E}\big(\vec{N}_{L,d}\vec{N}^t_{L,d}\big) = \mathbb{E}\big(\vec{S}_{L,d}\vec{S}^t_{L,d}\big)\,. \tag{6.4}$$

Let now $\vec{W}^*_{a_L}$ be a random vector in $\mathbb{R}^{2^{(d+1)L}}$ with law $\mathcal{N}(0, a_L^2 \mathrm{I}_{2^{(d+1)L}})$ independent of $\mathcal{F}_\infty \vee \sigma(\vec{N}_{i,L}, 1\le i\le 2^L)\vee\sigma(\eta_i, i\in\mathbb{Z})$. Let $\vec{W}_{a_L} = \mathbf{P}_L^{\otimes(d+1)}\vec{W}^*_{a_L}$ where $\mathbf{P}_L$ is defined in (4.20). With these notations, we can write

$$\begin{aligned}&\mathbb{E}\big(W_{c_{(d+1)L}}(P_{\vec{S}_{L,d}|\mathcal{F}_{2^L}} * P_{\vec{G}_a}, \mathcal{N}_{C_{L,d}} * P_{\vec{G}_a})\big)\\ &\qquad = \mathbb{E}\sup_{f\in \mathrm{Lip}(c_{(d+1)L})}\Big(\mathbb{E}\big(f(\vec{S}_{L,d}+\vec{W}_{a_L})|\mathcal{F}_{2^L}\big) - \mathbb{E}\big(f(\vec{N}_{L,d}+\vec{W}_{a_L})\big)\Big)\,. \end{aligned}\tag{6.5}$$

We introduce now the following additional notations and definitions:

Notation 6.1. *For any $K=(K_0,\dots,K_d)\in\{0,\dots,L\}^{(d+1)}$, we shall denote*

$$\mathcal{E}^{(d+1)}_{L,K} = \prod_{i=0}^d \mathcal{E}(L,K_i)\,,$$

where the $\mathcal{E}(L,K_i)$'s are defined in (4.18). Therefore the notation $k_K\in\mathcal{E}^{(d+1)}_{L,K}$ means $k_K=(k_{K_0},\dots,k_{K_d})\in\prod_{i=0}^d\mathcal{E}(L,K_i)$. In addition, we also denote

$$\mathcal{I}_L^{d+1} = \{0,\dots,L\}^{(d+1)}\,.$$

So the notation $\sum_{K\in\mathcal{I}_L^{d+1}}\sum_{k_K\in\mathcal{E}^{(d+1)}_{L,K}}$ means $\displaystyle\sum_{K_0=0}^L\cdots\sum_{K_d=0}^L\sum_{k_{K_0}\in\mathcal{E}(L,K_0)}\cdots\sum_{k_{K_d}\in\mathcal{E}(L,K_d)}$ and $\sum_{K\in\mathcal{I}_L^{d+1}}\sup_{k_K\in\mathcal{E}^{(d+1)}_{L,K}}$ means $\sum_{K_0=0}^L\cdots\sum_{K_d=0}^L\sup_{k_{K_0}\in\mathcal{E}(L,K_0)}\cdots\sup_{k_{K_d}\in\mathcal{E}(L,K_d)}$.

Definition 6.1. *Let x and y be two vectors of $\mathbb{R}^{2^{(d+1)L}}$ with coordinates*

$$x = \Big(\big(x^{(K,k_K)}, k_K\in\mathcal{E}^{(d+1)}_{L,K}\big)_{K\in\mathcal{I}_L^{d+1}}\Big)^t$$

and

$$y = \Big(\big(y^{(K,k_K)}, k_K\in\mathcal{E}^{(d+1)}_{L,K}\big)_{K\in\mathcal{I}_L^{d+1}}\Big)^t\,.$$

Let $c^*_{(d+1)L}$ be the following distance on $\mathbb{R}^{2^{(d+1)L}}$,

$$c^*_{(d+1)L}(x,y) = \sum_{K\in\mathcal{I}_L^{d+1}}\sup_{k_K\in\mathcal{E}^{(d+1)}_{L,K}}|x^{(K,k_K)} - y^{(K,k_K)}|\,.$$

Let also $\mathrm{Lip}(c^*_{(d+1)L})$ be the set of functions from $\mathbb{R}^{2^{(d+1)L}}$ into $\mathbb{R}$ that are Lipschitz with respect to $c^*_{(d+1)L}$; namely, $|f(x)-f(y)|\le\sum_{K\in\mathcal{I}_L^{d+1}}\sup_{k_K\in\in\mathcal{E}^{(d+1)}_{L,K}}|x^{(K,k_K)}-y^{(K,k_K)}|$.

Let $x=\big(x^{(1)},\dots,x^{(2^{(d+1)L})}\big)^t$ and $y=\big(y^{(1)},\dots,y^{(2^{(d+1)L})}\big)^t$ be two column vectors of $\mathbb{R}^{2^{(d+1)L}}$. Let now $u=\big(\mathbf{P}_L^{\otimes(d+1)}\big)^{-1}x$ and $v=\big(\mathbf{P}_L^{\otimes(d+1)}\big)^{-1}y$ (recall that $\mathbf{P}_L$ has been defined in (4.20), and since $\mathbf{P}_L$ is non singular so is $\mathbf{P}_L^{\otimes(d+1)}$). The vectors u and v of $\mathbb{R}^{2^{(d+1)L}}$ can be rewritten $u = \Big(\big(u^{(K,k_K)}, k_K \in \mathcal{E}_{L,K}^{(d+1)}\big)_{K\in\mathcal{I}_L^{d+1}}\Big)^t$ and $v = \Big(\big(v^{(K,k_K)}, k_K \in \mathcal{E}_{L,K}^{(d+1)}\big)_{K\in\mathcal{I}_L^{d+1}}\Big)^t$. Notice now that if $f \in \mathrm{Lip}(c_{(d+1)L})$, then

$$|f(x) - f(y)| \le c_{(d+1)L}(x,y) = \sup_{m\in\{1,\ldots,2^{(d+1)L}\}} |x^{(m)} - y^{(m)}|\,.$$

In addition, for any $m \in \{1,\ldots,2^{(d+1)L}\}$, there exists an unique $(j_0,\ldots,j_d) \in \{1,\ldots,2^L\}^{d+1}$ such that

$$m = \sum_{i=0}^{d} (j_i - 1)2^{(d-i)L} + 1\,.$$

Therefore,

$$|x^{(m)} - y^{(m)}| = \Big|\big(\otimes_{i=0}^d b(j_i,L)\big)^t u - \big(\otimes_{i=0}^d b(j_i,L)\big)^t v\Big|\,.$$

So overall,

$$\begin{aligned}
&|f(x) - f(y)| \\
&\le \sup_{(j_0,\ldots,j_d)\in\{1,\ldots,2^L\}^{d+1}} \sum_{K_0=0}^{L} \sum_{k_{K_0}\in\mathcal{E}(L,K_0)} \cdots \sum_{K_d=0}^{L} \sum_{k_{K_d}\in\mathcal{E}(L,K_d)} b_{K_0,k_{K_0}}(j_0)\cdots b_{K_d,k_{K_d}}(j_d) \\
&\qquad\qquad \times \big|u^{((K_0,\ldots,K_d),(k_{K_0},\ldots,k_{K_d}))} - v^{((K_0,\ldots,K_d),(k_{K_0},\ldots,k_{K_d}))}\big| \\
&\le \sup_{(j_0,\ldots,j_d)\in\{1,\ldots,2^L\}^{d+1}} \sum_{K_0=0}^{L} \sum_{k_{K_0}\in\mathcal{E}(L,K_0)} \cdots \sum_{K_d=0}^{L} \sum_{k_{K_d}\in\mathcal{E}(L,K_d)} b_{K_0,k_{K_0}}(j_0)\cdots b_{K_d,k_{K_d}}(j_d) \\
&\qquad\qquad \times \sup_{(i_0,\ldots,i_d)\in\prod_{\ell=0}^d \mathcal{E}(L,K_\ell)} |u^{((K_0,\ldots,K_d),(i_0,\ldots,i_d))} - v^{((K_0,\ldots,K_d),(i_0,\ldots,i_d))}|\,.
\end{aligned}$$

Since for any $K \in \{0,\ldots,L\}$ and any $j \in \{0,\ldots,2^L\}$, $\sum_{k\in\mathcal{E}(L,K)} b_{K,k}(j) \le 1$, it follows that if $f \in \mathrm{Lip}(c_{(d+1)L})$,

$$\begin{aligned}
|f(x) - f(y)| &= |f\circ\mathbf{P}_L^{\otimes(d+1)}(u) - f\circ\mathbf{P}_L^{\otimes(d+1)}(v)| \\
&\le \sum_{K_0=0}^{L} \cdots \sum_{K_d=0}^{L} \sup_{(i_0,\ldots,i_d)\in\prod_{\ell=0}^d \mathcal{E}(L,K_\ell)} |u^{((K_0,\ldots,K_d),(i_0,\ldots,i_d))} - v^{((K_0,\ldots,K_d),(i_0,\ldots,i_d))}|\,.
\end{aligned}$$

Whence, if $f \in \mathrm{Lip}(c_{(d+1)L})$,

$$|f(x) - f(y)| \le c^*_{(d+1)L}(u,v)\,. \tag{6.6}$$

Starting from (6.5), considering (6.6), recalling that $\vec{W}_{a_L} = \mathbf{P}_L^{\otimes(d+1)}\vec{W}^*_{a_L}$, and using the notations

$$\vec{S}^*_{L,d} = \big(\mathbf{P}_L^{\otimes(d+1)}\big)^{-1}\vec{S}_{L,d} \text{ and } \vec{N}^*_{L,d} = \big(\mathbf{P}_L^{\otimes(d+1)}\big)^{-1}\vec{N}_{L,d}\,, \tag{6.7}$$

we get

$$\begin{aligned}
&\mathbb{E}\big(W_{c_{(d+1)L}}(P_{\vec{S}_{L,d}|\mathcal{F}_{2^L}} * P_{\vec{G}_a}, \mathcal{N}_{C_{L,d}} * P_{\vec{G}_a})\big) \\
&\qquad\qquad \le \mathbb{E} \sup_{f\in\mathrm{Lip}(c^*_{(d+1)L})} \Big(\mathbb{E}\big(f(\vec{S}^*_{L,d} + \vec{W}^*_{a_L})|\mathcal{F}_{2^L}\big) - \mathbb{E}\big(f(\vec{N}^*_{L,d} + \vec{W}^*_{a_L})\big)\Big)\,. \qquad (6.8)
\end{aligned}$$

Let now $\mathrm{Lip}(c^*_{(d+1)L}, \mathcal{F}_{2^L})$ be the set of measurable functions $g : \mathbb{R}^{2^{(d+1)L}} \times \Omega \to \mathbb{R}$ wrt the σ-fields $\mathcal{B}(\mathbb{R}^{2^{(d+1)L}}) \otimes \mathcal{F}_{2^L}$ and $\mathcal{B}(\mathbb{R})$, such that $f(\cdot,\omega) \in \mathrm{Lip}(c^*_{(d+1)L})$ and $f(0,\omega) = 0$ for any $\omega \in \Omega$. For the sake of brevity, we shall write $g(x)$ in place of $g(x,\omega)$. From Point 2 of Theorem 1 in Dedecker, Prieur and Raynaud de Fitte (2006), the following inequality holds:

$$\mathbb{E} \sup_{f \in \mathrm{Lip}(c^*_{(d+1)L})} \Big(\mathbb{E}\big(f(\vec{S}^*_{L,d} + \vec{W}^*_{a_L}) | \mathcal{F}_{2^L}\big) - \mathbb{E}\big(f(\vec{N}^*_{L,d} + \vec{W}^*_{a_L})\big)\Big) \\ = \sup_{g \in \mathrm{Lip}(c^*_{(d+1)L}, \mathcal{F}_{2^L})} \mathbb{E}\big(g(\vec{S}^*_{L,d} + \vec{W}^*_{a_L})\big) - \mathbb{E}\big(g(\vec{N}^*_{L,d} + \vec{W}^*_{a_L})\big) . \qquad (6.9)$$

To bound up the right-hand side term of the above equality, we shall use the Lindeberg method. Before developing it, let us make some useful comments.

Recall that since $\mathbf{P}_L$ is nonsingular, then so is $\mathbf{P}_L^{\otimes d}$, and $\big(\mathbf{P}_L^{\otimes d}\big)^{-1} = \big(\mathbf{P}_L^{-1}\big)^{\otimes d}$ (see e.g. Corollary 2.2.11 in Horn and Johnson (1991)). Therefore, for any $i \in \mathbb{Z}$, we can define the column vectors $\vec{e}^{\,*}_{i,L}$, $\vec{U}^*_{i,L}$ and $\vec{U}^{*(0)}_{i,L}$ by

$$\vec{e}^{\,*}_{i,L} = \big(\mathbf{P}_L\big)^{-1} \vec{e}_{i,L} \ , \ \vec{U}^*_{i,L} = \big(\mathbf{P}_L^{\otimes d}\big)^{-1} \vec{U}_{i,L} \ \text{ and } \ \vec{U}^{*(0)}_{i,L} = \big(\mathbf{P}_L^{\otimes d}\big)^{-1} \vec{U}^{(0)}_{i,L} . \qquad (6.10)$$

With these notations, we have

$$\vec{S}^*_{L,d} = \sum_{i=1}^{2^L} \vec{e}^{\,*}_{i,L} \otimes \vec{U}^{*(0)}_{i,L} := \sum_{i=1}^{2^L} \vec{V}^*_{i,L} . \qquad (6.11)$$

Clearly $\vec{V}^*_{i,L} = \big(\mathbf{P}_L^{\otimes (d+1)}\big)^{-1} V_{i,L}$ where $V_{i,L}$ is defined in (4.16). The vector $\vec{V}^*_{i,L}$ can be written as follows:

$$\vec{V}^*_{i,L} = \Big(\big(\widetilde{V}^{(K,k_K)}_{i,L}, k_K \in \mathcal{E}^{(d+1)}_{L,K} \big)_{K \in \mathcal{I}_L^{d+1}} \Big)^t ,$$

where

$$\widetilde{V}^{(K,k_K)}_{i,L} = \mathbf{1}_{i \in B_{K_0, k_{K_0}}} \Big(\mathbf{1}_{Y_{i+2^L} \in B^*_{K_1,k_{K_1}} \times \cdots \times B^*_{K_d,k_{K_d}}} - \mathbb{E}\big(\mathbf{1}_{Y_{i+2^L} \in B^*_{K_1,k_{K_1}} \times \cdots \times B^*_{K_d,k_{K_d}}}\big)\Big) , \qquad (6.12)$$

where

$$B_{K,k} = \big](k-1)2^K, k2^K\big] \ \text{ and } \ B^*_{K,k} = \Big] \frac{(k-1)2^K}{2^L}, \frac{k2^K}{2^L} \Big] .$$

Notice now that

$$\vec{N}^*_{L,d} = \sum_{i=1}^{2^L} \vec{N}^*_{i,L} \text{ where } \vec{N}^*_{i,L} = \big(\mathbf{P}_L^{\otimes (d+1)}\big)^{-1} \vec{N}_{i,L} . \qquad (6.13)$$

In addition, the vector $\vec{N}^*_{i,L}$ can be written as follows:

$$\vec{N}^*_{i,L} = \Big(\big(\widetilde{N}^{(K,k_K)}_{i,L}, k_K \in \mathcal{E}^{(d+1)}_{L,K} \big)_{K \in \mathcal{I}_L^{d+1}} \Big)^t ,$$

and we have

$$\mathbb{E}(\widetilde{N}^{(K,k_K)}_{i,L} \widetilde{N}^{(Q,p_Q)}_{j,L}) = \mathbb{E}(\widetilde{V}^{(K,k_K)}_{i,L} \widetilde{V}^{(Q,p_Q)}_{j,L}) , \qquad (6.14)$$

where $\widetilde{V}^{(K,k_K)}_{i,L}$ is defined in (6.12).

Let us now introduce some notations useful to develop the Lindeberg method.

Notation 6.2. *Let φ_{a_L} be the density of $\vec{W}^*_{a_L}$ and let for x in $\mathbb{R}^{2^{(d+1)L}}$,*

$$g * \varphi_{a_L}(x,\omega) = \int g(x+y,\omega) \varphi_{a_L}(y) dy .$$

For the sake of brevity, we shall write $g * \varphi_{a_L}(x)$ *instead of* $g * \varphi_{a_L}(x, \omega)$ *(the partial derivatives will be taken wrt* x*). Let also*

$$\vec{\mathbf{S}}_0 = \vec{0} \text{ and for } j \geq 1 \,, \; \vec{\mathbf{S}}_j = \sum_{i=1}^{j} \vec{V}^*_{i,L} \,,$$

where the $\vec{V}^*_{i,L}$*'s are defined in* (6.11)*, and*

$$\vec{\mathbf{T}}_{2^L+1} = \vec{0} \text{ and for } j \in \{1, \dots, 2^L\} \,, \; \vec{\mathbf{T}}_j = \sum_{i=j}^{2^L} \vec{N}^*_{i,L} \,,$$

where the $\vec{N}^*_{i,L}$*'s are defined in* (6.13).

Let $a \in \big[(L+1)^{(d+1)}, 2^L(L+1)^{(d+1)}\big]$. Starting from (6.1) and considering (6.5), (6.8) and (6.9), we see that to prove (4.46) it suffices to prove the same bound for $\sup_{g \in \mathrm{Lip}(c^*_{(d+1)L}, \mathcal{F}_{2^L})} \mathbb{E}\big(g(\vec{S}^*_{L,d} + \vec{W}^*_{a_L})\big) - \mathbb{E}\big(g(\vec{N}^*_{L,d} + \vec{W}^*_{a_L})\big)$. With this aim, we shall use the Lindeberg method combined with the so-called Stein's identity, as it is described and done in what follows (see also Neumann (2013) for the case of the partial sums of real-valued random variables).

With the above notations, we write that

$$\sup_{g \in \mathrm{Lip}(c^*_{(d+1)L}, \mathcal{F}_{2^L})} \mathbb{E}\big(g(\vec{S}^*_{L,d} + \vec{W}^*_{a_L})\big) - \mathbb{E}\big(g(\vec{N}^*_{L,d} + \vec{W}^*_{a_L})\big)$$
$$\leq \sup_{g \in \mathrm{Lip}(c^*_{(d+1)L}, \mathcal{F}_{2^L})} \sum_{i=1}^{2^L} \mathbb{E}\Big(g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{V}^*_{i,L} + \vec{\mathbf{T}}_{i+1}\big) - g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{N}^*_{i,L} + \vec{\mathbf{T}}_{i+1}\big)\Big) \,.$$

For any $i \in \{1, \dots, 2^L\}$, let

$$\Delta_{1,i,L}(g) = g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{V}^*_{i,L} + \vec{\mathbf{T}}_{i+1}\big) - g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+1}\big) \,, \tag{6.15}$$

and

$$\Delta_{2,i,L}(g) = g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{N}^*_{i,L} + \vec{\mathbf{T}}_{i+1}\big) - g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+1}\big) \,. \tag{6.16}$$

With these notations, it follows that

$$\sup_{g \in \mathrm{Lip}(c^*_{(d+1)L}, \mathcal{F}_{2^L})} \mathbb{E}\big(g(\vec{S}^*_{L,d} + \vec{W}^*_{a_L})\big) - \mathbb{E}\big(g(\vec{N}^*_{L,d} + \vec{W}^*_{a_L})\big)$$
$$\leq \sup_{g \in \mathrm{Lip}(c^*_{(d+1)L}, \mathcal{F}_{2^L})} \sum_{i=1}^{2^L} \big(\mathbb{E}\big(\Delta_{1,i,L}(g)\big) - \mathbb{E}\big(\Delta_{2,i,L}(g)\big)\big) \,. \tag{6.17}$$

Let us introduce the following definition:

Definition 6.2. *Let* m *be a positive integer. If* ∇ *denotes the differentiation operator given by* $\nabla = \big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\big)^t$ *acting on the differentiable functions* $f : \mathbb{R}^m \to \mathbb{R}$*, we define* $\nabla^{\otimes k}$ *in the same way as in Definition 4.1. If* $f : \mathbb{R}^m \to \mathbb{R}$ *is* k*-times differentiable, for any* $x \in \mathbb{R}^m$*, let* $D^k f(x) = \nabla^{\otimes k} f(x)$*, and for any vector* A *of* $\mathbb{R}^m$*, we define* $D^k f(x) . A^{\otimes k}$ *as the usual scalar product in* $\mathbb{R}^{m^k}$ *between* $D^k f(x)$ *and* $A^{\otimes k}$*. We write* $Df(x)$ *in place of* $D^1 f(x)$*.*

We start by analyzing the term $\mathbb{E}\big(\Delta_{1,i,L}(g)\big)$. By Taylor's integral formula,

$$\Big|\mathbb{E}\big(\Delta_{1,i,L}(g)\big) - \mathbb{E}\big(Dg * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+1}\big) . \vec{V}^*_{i,L}\big) - \frac{1}{2}\mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+1}\big) . \vec{V}^{*\otimes 2}_{i,L}\big)\Big|$$
$$\leq \Big| \mathbb{E} \int_0^1 \frac{(1-t)^2}{2} D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+1} + t\vec{V}^*_{i,L}\big) . \vec{V}^{*\otimes 3}_{i,L} \Big| \,.$$

Applying Lemma 7.2 and using the following bounds: $\sup_{k_K\in\mathcal{E}^{d+1}_{L,K}}|\widetilde{V}^{(K,k_K)}_{i,L}|\leq 2$ and $\sum_{k_K\in\mathcal{E}^{d+1}_{L,K}}(\widetilde{V}^{(K,k_K)}_{i,L})^2\leq 2$, we get

$$\big|\mathbb{E}\big(\Delta_{1,i,L}(g)\big)-\mathbb{E}\big(Dg*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big).\vec{V}^{*}_{i,L}\big)-\frac{1}{2}\mathbb{E}\big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big).\vec{V}^{*\otimes 2}_{i,L}\big)\big| \ll a_L^{-2}(L+1)^{2(d+1)}\,. \quad (6.18)$$

Let

$$\Delta(i,j)(g)=D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j}+\vec{\mathbf{T}}_{i+1}\big)-D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j-1}+\vec{\mathbf{T}}_{i+1}\big)\,, \quad (6.19)$$

and

$$u_L=\big[a_L(L+1)^{-(d+1)}\big]\,. \quad (6.20)$$

Clearly with the notation $X^{(0)}=X-\mathbb{E}(X)$,

$$D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big).(\vec{V}^{*\otimes 2}_{i,L})^{(0)} = \sum_{j=1}^{(u_L\wedge i)-1}\Delta(i,j)(g).(\vec{V}^{*\otimes 2}_{i,L})^{(0)}+D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-(u_L\wedge i)}+\vec{\mathbf{T}}_{i+1}\big).(\vec{V}^{*\otimes 2}_{i,L})^{(0)}\,. \quad (6.21)$$

In the rest of the proof, to weaken the notations and when no confusion is possible, we write

$$\sum_{K\in\mathcal{I}^{d+1}_L}\ \sum_{k_K\in\mathcal{E}^{d+1}_{L,K}}=\sum_{K,k_K}\,.$$

For any $j\leq(u_L\wedge i)-1$, notice that

$$\Delta(i,j)(g).(\vec{V}^{*\otimes 2}_{i,L})^{(0)}=\sum_{K,k_K}\sum_{P,p_P}\Big(\frac{\partial^2 g*\varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}}\big(\vec{\mathbf{S}}_{i-j}+\vec{\mathbf{T}}_{i+1}\big)- \frac{\partial^2 g*\varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}}\big(\vec{\mathbf{S}}_{i-j-1}+\vec{\mathbf{T}}_{i+1}\big)\Big)\big(\widetilde{V}^{(K,k_K)}_{i,L}\widetilde{V}^{(P,p_P)}_{i,L}\big)^{(0)}\,.$$

Using Lemma 7.4 with

$$U=\frac{\partial^2 g*\varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}}\big(\vec{\mathbf{S}}_{i-j}+\vec{\mathbf{T}}_{i+1}\big)-\frac{\partial^2 g*\varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}}\big(\vec{\mathbf{S}}_{i-j-1}+\vec{\mathbf{T}}_{i+1}\big)\,,$$

$V=\widetilde{V}^{(K,k_K)}_{i,L}\widetilde{V}^{(P,p_P)}_{i,L}$, $\mathcal{U}=\sigma\big((Y_\ell,\ell\leq i+2^L-j),\vec{\mathbf{T}}_{i+1}\big)$, $\mathcal{V}=\sigma(Y_{i+2^L})$, $r=\infty$ and $s=1$, we get that there exists a $\mathcal{V}$-measurable random variable $b_{\mathcal{V}}(i+2^L-j)$ such that

$$\big|\mathbb{E}\big(\Delta(i,j)(g).(\vec{V}^{*\otimes 2}_{i,L})^{(0)}\big)\big| \leq 2\Big\{\sup_{(K,k_K)}\sup_{(P,p_P)}\Big\|\frac{\partial^2 g*\varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}}\big(\vec{\mathbf{S}}_{i-j}+\vec{\mathbf{T}}_{i+1}\big)-\frac{\partial^2 g*\varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}}\big(\vec{\mathbf{S}}_{i-j-1}+\vec{\mathbf{T}}_{i+1}\big)\Big\|_\infty\Big\} \times\sum_{K,k_K}\sum_{P,p_P}\mathbb{E}\Big(b_{\mathcal{V}}(i+2^L-j)\big|\widetilde{V}^{(K,k_K)}_{i,L}\widetilde{V}^{(P,p_P)}_{i,L}\big|\Big)\,.$$

Since $\sum_{K\in\mathcal{I}^{d+1}_L}\sum_{k_K\in\mathcal{E}^{d+1}_{L,K}}|\widetilde{V}^{(K,k_K)}_{i,L}|\leq 2(L+1)^{d+1}$ and $\mathbb{E}\big(b_{\mathcal{V}}(i+2^L-j)\big)\leq\beta_j$, it follows that

$$\big|\mathbb{E}\big(\Delta(i,j)(g).(\vec{V}^{*\otimes 2}_{i,L})^{(0)}\big)\big|\ll(L+1)^{2(d+1)}\beta_j \times\sup_{(K,k_K)}\sup_{(P,p_P)}\Big\|\frac{\partial^2 g*\varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}}\big(\vec{\mathbf{S}}_{i-j}+\vec{\mathbf{T}}_{i+1}\big)-\frac{\partial^2 g*\varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}}\big(\vec{\mathbf{S}}_{i-j-1}+\vec{\mathbf{T}}_{i+1}\big)\Big\|_\infty\,. \quad (6.22)$$

Next, using the property of the convolution product and Lemma 7.3, we derive

$$\begin{aligned}\Big\|\frac{\partial^2 g*\varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}}\big(\vec{\mathbf{S}}_{i-j}+\vec{\mathbf{T}}_{i+1}\big)-\frac{\partial^2 g*\varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}}\big(\vec{\mathbf{S}}_{i-j-1}+\vec{\mathbf{T}}_{i+1}\big)\Big\|_\infty&\\ \le\Big\|\frac{\partial^2\varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}}\Big\|_1\Big\|\sup_{y\in\mathbb{R}^{(d+1)L}}\big|g(\vec{V}^*_{i-j,L}+y)-g(y)\big|\Big\|_\infty&\\ \ll a_L^{-2}\Big\|\sup_{y\in\mathbb{R}^{(d+1)L}}\big|g(\vec{V}^*_{i-j,L}+y)-g(y)\big|\Big\|_\infty .&\end{aligned}$$

But, since $g\in\mathrm{Lip}(c^*_{(d+1)L},\mathcal{F}_{2^L})$,

$$\big|g(\vec{V}^*_{i-j,L}+y)-g(y)\big|\le\sum_{K\in\mathcal{I}_L^{d+1}}\sup_{k_K\in\,\in\mathcal{E}^{(d+1)}_{L,K}}|\widetilde{V}^{(K,k_K)}_{i-j,L}|\le 2(L+1)^{d+1}\,. \tag{6.23}$$

So overall, we get that

$$\big|\mathbb{E}\big(\Delta(i,j)(g)\boldsymbol{.}(\vec{V}^{*\otimes 2}_{i,L})^{(0)}\big)\big|\ll a_L^{-2}(L+1)^{3(d+1)}\beta_j\,. \tag{6.24}$$

On the other hand, using the same arguments as to get (6.22), we infer that

$$\begin{aligned}&\big|\mathbb{E}\big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-(u_L\wedge i)}+\vec{\mathbf{T}}_{i+1}\big)\boldsymbol{.}(\vec{V}^{*\otimes 2}_{i,L})^{(0)}\big)\big|\\ &\quad\ll(L+1)^{2(d+1)}\beta_{u_L\wedge i}\sup_{(K,k_K)}\sup_{(P,p_P)}\Big\|\frac{\partial^2 g*\varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}}\big(\vec{\mathbf{S}}_{i-(u_L\wedge i)}+\vec{\mathbf{T}}_{i+1}\big)\Big\|_\infty .\end{aligned}$$

Therefore using Lemma 7.3,

$$\big|\mathbb{E}\big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-(u_L\wedge i)}+\vec{\mathbf{T}}_{i+1}\big)\boldsymbol{.}(\vec{V}^{*\otimes 2}_{i,L})^{(0)}\big)\big|\ll a_L^{-1}(L+1)^{2(d+1)}\beta_{u_L\wedge i}\,. \tag{6.25}$$

Hence, starting from (6.21) and taking into account (6.24), (6.25), the choice of u_L, the fact that $a_L\le 2^L(L+1)^{d+1}$ and using that $\beta_k\ll k^{1-p}$ for some $p\in]2,3]$, we derive

$$\begin{aligned}&\sum_{i=1}^{2^L}\mathbb{E}\big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big)\boldsymbol{.}(\vec{V}^{*\otimes 2}_{i,L})^{(0)}\big)\\ &\qquad\ll 2^La_L^{-1}(L+1)^{2(d+1)}\Big(\frac{(L+1)^{(p-1)(d+1)}}{a_L^{p-1}}+\frac{(L+1)^{d+1}}{a_L}\Big)\,.\end{aligned} \tag{6.26}$$

We give now an estimate of the expectation of $Dg*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big)\boldsymbol{.}\vec{V}^*_{i,L}$. With this aim, we write

$$\begin{aligned}&Dg*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big)\\ &\quad=Dg*\varphi_{a_L}\big(\vec{\mathbf{T}}_{i+1}\big)+\sum_{j=1}^{i-1}\big(Dg*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j}+\vec{\mathbf{T}}_{i+1}\big)-Dg*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j-1}+\vec{\mathbf{T}}_{i+1}\big)\big)\,.\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}\big(Dg*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big)\boldsymbol{.}\vec{V}^*_{i,L}\big)&=\mathbb{E}\big(Dg*\varphi_{a_L}\big(\vec{\mathbf{T}}_{i+1}\big)\boldsymbol{.}\vec{V}^*_{i,L}\big)\\ &+\sum_{j=1}^{i-1}\mathbb{E}\big(\big(Dg*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j}+\vec{\mathbf{T}}_{i+1}\big)-Dg*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j-1}+\vec{\mathbf{T}}_{i+1}\big)\big)\boldsymbol{.}\vec{V}^*_{i,L}\big)\,.\end{aligned} \tag{6.27}$$

Notice that

$$\mathbb{E}\big(Dg*\varphi_{a_L}\big(\vec{\mathbf{T}}_{i+1}\big)\boldsymbol{.}\vec{V}^*_{i,L}\big)=\sum_{K\in\mathcal{I}_L^{d+1}}\sum_{k_K\in\,\mathcal{E}^{d+1}_{L,K}}\mathbb{E}\Big(\frac{\partial g*\varphi_{a_L}}{\partial x^{(K,k_K)}}\big(\vec{\mathbf{T}}_{i+1}\big)\widetilde{V}^{(K,k_K)}_{i,L}\Big)\,.$$

Since $\frac{\partial g*\varphi_{a_L}}{\partial x^{(K,k_K)}}\big(\vec{\mathbf{T}}_{i+1}\big)$ is a $\mathcal{F}_{2^L}\vee\sigma\big(\vec{\mathbf{T}}_{i+1}\big)$-measurable random variable, and $\vec{\mathbf{T}}_{i+1}$ is independent of $\widetilde{V}_{i,L}^{(K,k_K)}$, applying Lemma 7.4 with $U=\frac{\partial g*\varphi_{a_L}}{\partial x^{(K,k_K)}}\big(\vec{\mathbf{T}}_{i+1}\big)$, $V=\widetilde{V}_{i,L}^{(K,k_K)}$, $\mathcal{U}=\mathcal{F}_{2^L}\vee\sigma(\vec{\mathbf{T}}_{i+1})$, $\mathcal{V}=\sigma(Y_{i+2^L})$, $r=1$ and $s=\infty$, we get that there exists a $\mathcal{U}$-measurable random variable $b_{\mathcal{U}}(i+2^L)$ such that

$$\big|\mathbb{E}\big(Dg*\varphi_{a_L}\big(\vec{\mathbf{T}}_{i+1}\big).\vec{V}_{i,L}^{*}\big)\big|\ll\sum_{K\in\mathcal{I}_L^{d+1}}\ \sum_{k_K\in\mathcal{E}_{L,K}^{d+1}}\mathbb{E}\Big(\Big|\frac{\partial g*\varphi_{a_L}}{\partial x^{(K,k_K)}}\big(\vec{\mathbf{T}}_{i+1}\big)\Big|b_{\mathcal{U}}(i+2^L)\Big)\,.$$

Notice now that by the inequality (7.1), for any K in $\mathcal{I}_L^{d+1}$,

$$\sum_{k_K\in\mathcal{E}_{L,K}^{d+1}}\Big|\frac{\partial g*\varphi_{a_L}}{\partial x^{(K,k_K)}}\big(\vec{\mathbf{T}}_{i+1}\big)\Big|\le 1\,.$$

In addition $\mathbb{E}\big(b_{\mathcal{U}}(i+2^L)\big)\le\beta_i$. Therefore,

$$\big|\mathbb{E}\big(Dg*\varphi_{a_L}\big(\vec{\mathbf{T}}_{i+1}\big).\vec{V}_{i,L}^{*}\big)\big|\ll(L+1)^{(d+1)}\beta_i\,. \tag{6.28}$$

We give now an estimate of $\displaystyle\sum_{j=1}^{i-1}\mathbb{E}\big(\big(Dg*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j}+\vec{\mathbf{T}}_{i+1}\big)-Dg*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j-1}+\vec{\mathbf{T}}_{i+1}\big)\big).\vec{V}_{i,L}^{*}\big)$.
For any $i\ge j+1$, we first write that

$$\begin{aligned}&\big(Dg*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j}+\vec{\mathbf{T}}_{i+1}\big)-Dg*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j-1}+\vec{\mathbf{T}}_{i+1}\big)\big).\vec{V}_{i,L}^{*}\\&\qquad=\sum_{K,k_K}\Big(\frac{\partial g*\varphi_{a_L}}{\partial x^{(K,k_K)}}\big(\vec{\mathbf{S}}_{i-j}+\vec{\mathbf{T}}_{i+1}\big)-\frac{\partial g*\varphi_{a_L}}{\partial x^{(K,k_K)}}\big(\vec{\mathbf{S}}_{i-j-1}+\vec{\mathbf{T}}_{i+1}\big)\Big)\widetilde{V}_{i,L}^{(K,k_K)}\,.\end{aligned}$$

Using Lemma 7.4 with

$$U=\frac{\partial g*\varphi_{a_L}}{\partial x^{(K,k_K)}}\big(\vec{\mathbf{S}}_{i-j}+\vec{\mathbf{T}}_{i+1}\big)-\frac{\partial g*\varphi_{a_L}}{\partial x^{(K,k_K)}}\big(\vec{\mathbf{S}}_{i-j-1}+\vec{\mathbf{T}}_{i+1}\big)\,,$$

$V=\widetilde{V}_{i,L}^{(K,k_K)}$, $\mathcal{U}=\sigma(Y_\ell,\ell\le i+2^L-j)\vee\sigma(\vec{\mathbf{T}}_{i+1})$, $\mathcal{V}=\sigma(Y_{i+2^L})$, $r=\infty$ and $s=1$, we get that there exists a $\mathcal{V}$-measurable random variable $b_{\mathcal{V}}(i+2^L-j)$ such that

$$\begin{aligned}&\big|\mathbb{E}\big(\big(Dg*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j}+\vec{\mathbf{T}}_{i+1}\big)-Dg*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j-1}+\vec{\mathbf{T}}_{i+1}\big)\big).\vec{V}_{i,L}^{*}\big)\big|\\&\quad\le 2\sup_{(K,k_K)}\Big\|\frac{\partial g*\varphi_{a_L}}{\partial x^{(K,k_K)}}\big(\vec{\mathbf{S}}_{i-j}+\vec{\mathbf{T}}_{i+1}\big)-\frac{\partial g*\varphi_{a_L}}{\partial x^{(K,k_K)}}\big(\vec{\mathbf{S}}_{i-j-1}+\vec{\mathbf{T}}_{i+1}\big)\Big\|_\infty\\&\qquad\times\sum_{P,p_P}\mathbb{E}\Big(b_{\mathcal{V}}(i+2^L-j)\big|\widetilde{V}_{i,L}^{(P,p_P)}\big|\Big)\,.\end{aligned}$$

Since $\sum_{K\in\mathcal{I}_L^{d+1}}\sum_{k_K\in\mathcal{E}_{L,K}^{d+1}}|\widetilde{V}_{i,L}^{(K,k_K)}|\le 2(L+1)^{d+1}$ and $\mathbb{E}\big(b_{\mathcal{V}}(i+2^L-j)\big)\le\beta_j$, it follows that

$$\begin{aligned}&\big|\mathbb{E}\big(\big(Dg*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j}+\vec{\mathbf{T}}_{i+1}\big)-Dg*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j-1}+\vec{\mathbf{T}}_{i+1}\big)\big).\vec{V}_{i,L}^{*}\big)\big|\\&\quad\ll(L+1)^{d+1}\beta_j\sup_{(K,k_K)}\Big\|\frac{\partial g*\varphi_{a_L}}{\partial x^{(K,k_K)}}\big(\vec{\mathbf{S}}_{i-j}+\vec{\mathbf{T}}_{i+1}\big)-\frac{\partial g*\varphi_{a_L}}{\partial x^{(K,k_K)}}\big(\vec{\mathbf{S}}_{i-j-1}+\vec{\mathbf{T}}_{i+1}\big)\Big\|_\infty\,.\end{aligned}$$

Next, using as the property of the convolution product, Lemma 7.3 and the upper bound (6.23), we derive that

$$\begin{aligned}&\Big\|\frac{\partial g*\varphi_{a_L}}{\partial x^{(K,k_K)}}\big(\vec{\mathbf{S}}_{i-j}+\vec{\mathbf{T}}_{i+1}\big)-\frac{\partial g*\varphi_{a_L}}{\partial x^{(K,k_K)}}\big(\vec{\mathbf{S}}_{i-j-1}+\vec{\mathbf{T}}_{i+1}\big)\Big\|_\infty\\&\qquad\le\Big\|\frac{\partial\varphi_{a_L}}{\partial x^{(K,k_K)}}\Big\|_1\Big\|\sup_{y\in\mathbb{R}^{(d+1)L}}\big|g(\vec{V}_{i-j,L}^{*}+y)-g(y)\big|\Big\|_\infty\ll a_L^{-1}(L+1)^{d+1}\,.\end{aligned}$$

It follows that for any $i \geq j+1$,

$$\left|\mathbb{E}\big(\big(Dg*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j}+\vec{\mathbf{T}}_{i+1}\big)-Dg*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j-1}+\vec{\mathbf{T}}_{i+1}\big)\big).\vec{V}^*_{i,L}\big)\right| \ll a_L^{-1}(L+1)^{2(d+1)}\beta_j\,. \quad (6.29)$$

From now on, we assume that $j < i \wedge u_L$. Notice that

$$\begin{aligned}\big(Dg * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j} + \vec{\mathbf{T}}_{i+1}\big) - Dg * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j-1} + \vec{\mathbf{T}}_{i+1}\big)\big).\vec{V}^*_{i,L} \\ - D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j-1} + \vec{\mathbf{T}}_{i+1}\big).(\vec{V}^*_{i-j,L} \otimes \vec{V}^*_{i,L}) \\ = \int_0^1 (1-t) D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j-1} + \vec{\mathbf{T}}_{i+1} + t\vec{V}^*_{i-j,L}\big).(\vec{V}^{*\otimes 2}_{i-j,L} \otimes \vec{V}^*_{i,L}) dt\,.\end{aligned}$$

We first write

$$\begin{aligned}&\left|\mathbb{E}\big(D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j-1} + \vec{\mathbf{T}}_{i+1} + t\vec{V}^*_{i-j,L}\big).\vec{V}^{*\otimes 2}_{i-j,L} \otimes \vec{V}^*_{i,L}\big)\right| \\ &\quad = \mathbb{E}\Big(\sum_{K,k_K}\sum_{P,p_P}\sum_{Q,q_Q} \frac{\partial^3 g * \varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}\partial x^{(Q,q_Q)}}\big(\vec{\mathbf{S}}_{i-j-1} + \vec{\mathbf{T}}_{i+1} + t\vec{V}^*_{i-j,L}\big) \\ &\qquad\qquad \times \widetilde{V}^{(K,k_K)}_{i-j,L}\widetilde{V}^{(P,p_P)}_{i-j,L}\widetilde{V}^{(Q,q_Q)}_{i,L}\Big)\,.\end{aligned}$$

Let

$$W^{(Q,q_Q)}_{i-j,L} = \sum_{K,k_K}\sum_{P,p_P} \frac{\partial^3 g * \varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}\partial x^{(Q,q_Q)}}\big(\vec{\mathbf{S}}_{i-j-1} + \vec{\mathbf{T}}_{i+1} + t\vec{V}^*_{i-j,L}\big) \times \widetilde{V}^{(K,k_K)}_{i-j,L}\widetilde{V}^{(P,p_P)}_{i-j,L}\,.$$

Using Lemma 7.4 with $U = W^{(Q,q_Q)}_{i-j,L}$, $V = \widetilde{V}^{(Q,q_Q)}_{i,L}$, $\mathcal{U} = \sigma(Y_\ell, \ell \leq i + 2^L - j) \vee \sigma(\vec{\mathbf{T}}_{i+1})$, $\mathcal{V} = \sigma(Y_{i+2^L})$, $r = \infty$ and $s = 1$, we get that we get that there exists a $\mathcal{V}$-measurable random variable $b_{\mathcal{V}}(i + 2^L - j)$ such that

$$\begin{aligned}&\sum_{Q\in\mathcal{I}_L^{d+1}}\sum_{q_Q\in\,\mathcal{E}_{L,Q}^{d+1}}\left|\mathbb{E}\big(W^{(Q,q_Q)}_{i-j,L}\widetilde{V}^{(Q,q_Q)}_{i,L}\big)\right| \\ &\qquad \leq 2\sum_{Q\in\mathcal{I}_L^{d+1}}\sum_{q_Q\in\,\mathcal{E}_{L,Q}^{d+1}}\big\|W^{(Q,q_Q)}_{i-j,L}\big\|_\infty \times \mathbb{E}\Big(b_{\mathcal{V}}(i+2^L-j)\big|\widetilde{V}^{(Q,q_Q)}_{i,L}\big|\Big)\,.\end{aligned}$$

Using Lemma 7.3 and the fact that $\sum_{K,k_K}\sum_{P,p_P}\big|\widetilde{V}^{(K,k_K)}_{i-j,L}\widetilde{V}^{(P,p_P)}_{i-j,L}\big| \leq 4(L+1)^{2(d+1)}$, we get

$$\big|W^{(Q,q_Q)}_{i-j,L}\big| \ll a_L^{-2}(L+1)^{2(d+1)}\,.$$

Hence,

$$\begin{aligned}&\sum_{Q\in\mathcal{I}_L^{d+1}}\sum_{q_Q\in\,\mathcal{E}_{L,Q}^{d+1}}\left|\mathbb{E}\big(W^{(Q,q_Q)}_{i-j,L}\widetilde{V}_{i,L}\big)\right| \\ &\qquad \ll a_L^{-2}(L+1)^{2(d+1)}\sum_{Q\in\mathcal{I}_L^{d+1}}\sum_{q_Q\in\,\mathcal{E}_{L,Q}^{d+1}}\mathbb{E}\Big(b_{\mathcal{V}}(i+2^L-j)\big|\widetilde{V}^{(Q,q_Q)}_{i,L}\big|\Big)\,.\end{aligned}$$

Using the facts that $\sum_{Q\in\mathcal{I}_L^{d+1}}\sum_{q_Q\in\,\mathcal{E}_{L,Q}^{d+1}}\big|\widetilde{V}^{(Q,q_Q)}_{i,L}\big| \leq 2(L+1)^{d+1}$ and $\mathbb{E}\big(b_{\mathcal{V}}(i+2^L-j)\big) \leq \beta_j$, we get overall

$$\left|\mathbb{E}\Big(\int_0^1 (1-t)D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j-1} + \vec{\mathbf{T}}_{i+1} + t\vec{V}^*_{i-j,L}\big).\vec{V}^{*\otimes 2}_{i-j,L} \otimes \vec{V}^*_{i,L} dt\Big)\right| \ll a_L^{-2}(L+1)^{3(d+1)}\beta_j\,. \quad (6.30)$$

In order to estimate the term $\mathbb{E}\big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j-1}+\vec{\mathbf{T}}_{i+1}\big).(\vec{V}^*_{i-j,L}\otimes\vec{V}^*_{i,L})\big)$, we use the following decomposition:

$$D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j-1}+\vec{\mathbf{T}}_{i+1}\big)=D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{(i-2j)\vee 0}+\vec{\mathbf{T}}_{i+1}\big)\\ +\sum_{l=1}^{(j-1)\wedge(i-j-1)}\big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j-l}+\vec{\mathbf{T}}_{i+1}\big)-D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j-l-1}+\vec{\mathbf{T}}_{i+1}\big)\big)\,.$$

For any $l\in\{1,\dots,(j-1)\wedge(i-j-1)\}$, we notice that

$$\big|\mathbb{E}\big(\big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j-l}+\vec{\mathbf{T}}_{i+1}\big)-D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j-l-1}+\vec{\mathbf{T}}_{i+1}\big)\big).\vec{V}^*_{i-j,L}\otimes\vec{V}^*_{i,L}\big)\big|\\ =\Big|\mathbb{E}\Big(\int_0^1 D^3g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j-l}+\vec{\mathbf{T}}_{i+1}+t\vec{V}^*_{i-j-l,L}\big).\vec{V}^*_{i-j-l,L}\otimes\vec{V}^*_{i-j,L}\otimes\vec{V}^*_{i,L}dt\Big)\Big|\,,$$

whence, using the same arguments as to get (6.30), we obtain that

$$\big|\mathbb{E}\big(\big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j-l}+\vec{\mathbf{T}}_{i+1}\big)-D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j-l-1}+\vec{\mathbf{T}}_{i+1}\big)\big).\vec{V}^*_{i-j,L}\otimes\vec{V}^*_{i,L}\big)\big|\\ \ll a_L^{-2}(L+1)^{3(d+1)}\beta_j\,. \quad (6.31)$$

As a second step, we bound up $\big|\mathbb{E}\big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{(i-2j)\vee 0}+\vec{\mathbf{T}}_{i+1}\big).(\vec{V}^*_{i-j,L}\otimes\vec{V}^*_{i,L})^{(0)}\big)\big|$. Assume first that $j\le[i/2]$. Clearly, using the notation (6.19),

$$D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-2j}+\vec{\mathbf{T}}_{i+1}\big)=\sum_{l=j}^{(u_L-1)\wedge(i-j-1)}\Delta(i,l+j)(g)+D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{(i-j-u_L)\vee 0}+\vec{\mathbf{T}}_{i+1}\big)\,.$$

Now for any $l\in\{j,\dots,(u_L-1)\wedge(i-j-1)\}$, by using similar arguments as to get (6.24), we infer that

$$\big|\mathbb{E}\big(\Delta(i,l+j).(\vec{V}^*_{i-j,L}\otimes\vec{V}^*_{i,L})^{(0)}\big)\big|\ll a_L^{-2}(L+1)^{3(d+1)}\beta_l\,. \quad (6.32)$$

If $j\le i-u_L$, with similar arguments,

$$\big|\mathbb{E}\big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-j-u_L}+\vec{\mathbf{T}}_{i+1}\big).(\vec{V}^*_{i-j,L}\otimes\vec{V}^*_{i,L})^{(0)}\big)\big|\ll a_L^{-1}(L+1)^{2(d+1)}\beta_{u_L}\,. \quad (6.33)$$

Now if $j>i-u_L$, we infer that

$$\big|\mathbb{E}\big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{T}}_{i+1}\big).(\vec{V}^*_{i-j,L}\otimes\vec{V}^*_{i,L})^{(0)}\big)\big|\ll a_L^{-1}(L+1)^{2(d+1)}\beta_{i-j}\ll a_L^{-1}(L+1)^{2(d+1)}\beta_{[i/2]}\,, \quad (6.34)$$

where we have used the fact that $j\le[i/2]$, for the last inequality. Assume now that $j\ge[i/2]+1$. We then get that

$$\big|\mathbb{E}\big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{T}}_{i+1}\big).\vec{V}^*_{i-j,L}\otimes\vec{V}^*_{i,L}\big|\ll a_L^{-1}(L+1)^{2(d+1)}\beta_j\ll a_L^{-1}(L+1)^{2(d+1)}\beta_{[i/2]}\,. \quad (6.35)$$

Starting from (6.27), adding the inequalities (6.28)-(6.35) and summing on j and l, we then obtain:

$$\Big|\mathbb{E}\big(Dg*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big).\vec{V}^*_{i,L}\big)-\sum_{j=1}^{u_L-1}\mathbb{E}\big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-2j}+\vec{\mathbf{T}}_{i+1}\big)\big).\mathbb{E}\big(\vec{V}^*_{i-j,L}\otimes\vec{V}^*_{i,L}\big)\mathbf{1}_{j\le[i/2]}\Big|\\ \ll(L+1)^{d+1}\beta_i+a_L^{-1}(L+1)^{2(d+1)}\sum_{j=i\wedge u_L}^{i}\beta_j+a_L^{-1}(L+1)^{2(d+1)}u_L\beta_{u_L}\\ +a_L^{-1}(L+1)^{2(d+1)}u_L\beta_{[i/2]}+a_L^{-2}(L+1)^{3(d+1)}\sum_{j=1}^{u_L}j\beta_j\,.$$

Next summing on i and taking into account the fact that $\beta_k \ll k^{1-p}$ for some $p \in]2,3]$, and the choice of u_L, we get

$$\sum_{i=1}^{2^L} \Big| \mathbb{E}\big(Dg * \varphi_{a_L}(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+1}).\vec{V}^*_{i,L}\big) - \sum_{j=1}^{u_L-1} \mathbb{E}\big(D^2 g * \varphi_{a_L}(\vec{\mathbf{S}}_{i-2j} + \vec{\mathbf{T}}_{i+1})\big).\mathbb{E}\big(\vec{V}^*_{i-j,L} \otimes \vec{V}^*_{i,L}\big)\mathbf{1}_{j \le [\frac{i}{2}]}\Big| \\ \ll (L+1)^{d+1} + a_L^{1-p}(L+1)^{p(d+1)}2^L + a_L^{-2}(L+1)^{3(d+1)}2^L \log(u_L)\mathbf{1}_{p=3}\,. \quad (6.36)$$

Hence, starting from (6.18) and considering the upper bounds (6.26) and (6.36) together with the fact that $a_L \ge (L+1)^{d+1}$, we get

$$\sum_{i=1}^{2^L} \mathbb{E}\big(\Delta_{1,i,L}(g)\big) - \sum_{i=1}^{2^L} \sum_{j=1}^{u_L-1} \mathbb{E}\big(D^2 g * \varphi_{a_L}(\vec{\mathbf{S}}_{i-2j} + \vec{\mathbf{T}}_{i+1})\big).\mathbb{E}\big(\vec{V}^*_{i-j,L} \otimes \vec{V}^*_{i,L}\big)\mathbf{1}_{j \le [i/2]} \\ - \frac{1}{2}\sum_{i=1}^{2^L} \mathbb{E}\big(D^2 g * \varphi_{a_L}(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+1})\big).\mathbb{E}\big(\vec{V}^{*\otimes 2}_{i,L}\big) \\ \ll (L+1)^{d+1} + a_L^{1-p}(L+1)^{p(d+1)}2^L + a_L^{-2}(L+1)^{3d+4}2^L\mathbf{1}_{p=3}\,. \quad (6.37)$$

We analyze now the "Gaussian part" in (6.17), namely, the term $\mathbb{E}\big(\Delta_{2,i,L}(g)\big)$. By Taylor's integral formula,

$$\mathbb{E}\big(\Delta_{2,i,L}(g)\big) - \mathbb{E}\big(Dg * \varphi_{a_L}(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+1}).\vec{N}^*_{i,L}\big) - \frac{1}{2}\mathbb{E}\big(D^2 g * \varphi_{a_L}(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+1}).\vec{N}^{*\otimes 2}_{i,L}\big) \\ - \frac{1}{6}\mathbb{E}\big(D^3 g * \varphi_{a_L}(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+1}).\vec{N}^{*\otimes 3}_{i,L}\big) \\ = \frac{1}{6}\int_0^1 (1-t)^3 \mathbb{E}\big(D^4 g * \varphi_{a_L}(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+1} + t\vec{N}^*_{i,L}).\vec{N}^{*\otimes 4}_{i,L}\big)dt\,. \quad (6.38)$$

Applying Lemma 7.2, we derive that, for any $i \in \{1, \dots, 2^L\}$ and any $t \in [0,1]$,

$$\big|\mathbb{E}\big(D^4 g * \varphi_{a_L}(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+1} + t\vec{N}^*_{i,L}).\vec{N}^{*\otimes 4}_{i,L}\big)\big| \\ \ll a_L^{-3}\mathbb{E}\Big(\Big(\sum_{K \in \mathcal{I}_L^{d+1}} \sup_{k_K \in \mathcal{E}_{L,K}^{(d+1)}} |\widetilde{N}_{i,L}^{(K,k_K)}|\Big)\Big(\sum_{K \in \mathcal{I}_L^{d+1}} \sum_{k_K \in \mathcal{E}_{L,K}^{(d+1)}} (\widetilde{N}_{i,L}^{(K,k_K)})^2\Big)^{3/2}\Big) \\ \ll a_L^{-3}\Big(\mathbb{E}\Big(\sum_{K \in \mathcal{I}_L^{d+1}} \sup_{k_K \in \mathcal{E}_{L,K}^{(d+1)}} |\widetilde{N}_{i,L}^{(K,k_K)}|\Big)^4\Big)^{1/4}\Big(\mathbb{E}\Big(\sum_{K \in \mathcal{I}_L^{d+1}} \sum_{k_K \in \mathcal{E}_{L,K}^{(d+1)}} (\widetilde{N}_{i,L}^{(K,k_K)})^2\Big)^2\Big)^{3/4}. \quad (6.39)$$

Notice that

$$\sum_{K \in \mathcal{I}_L^{d+1}} \sup_{k_K \in \mathcal{E}_{L,K}^{(d+1)}} |\widetilde{N}_{i,L}^{(K,k_K)}| \le (L+1)^{(d+1)/2}\Big(\sum_{K \in \mathcal{I}_L^{d+1}} \sum_{k_K \in \mathcal{E}_{L,K}^{(d+1)}} (\widetilde{N}_{i,L}^{(K,k_K)})^2\Big)^{1/2}, \quad (6.40)$$

and

$$\mathbb{E}\Big(\sum_{K \in \mathcal{I}_L^{d+1}} \sum_{k_K \in \mathcal{E}_{L,K}^{(d+1)}} (\widetilde{N}_{i,L}^{(K,k_K)})^2\Big)^2 \le \Big(\sum_{K \in \mathcal{I}_L^{d+1}} \sum_{k_K \in \mathcal{E}_{L,K}^{(d+1)}} \big(\mathbb{E}(\widetilde{N}_{i,L}^{(K,k_K)})^4\big)^{1/2}\Big)^2 \\ \le 3\Big(\sum_{K \in \mathcal{I}_L^{d+1}} \sum_{k_K \in \mathcal{E}_{L,K}^{(d+1)}} \mathbb{E}((\widetilde{N}_{i,L}^{(K,k_K)})^2)\Big)^2. \quad (6.41)$$

Moreover by using (6.14), we get

$$\sum_{K\in\mathcal{I}_L^{d+1}}\ \sum_{k_K\in\,\mathcal{E}_{L,K}^{(d+1)}}\mathbb{E}((\widetilde{N}_{i,L}^{(K,k_K)})^2)=\sum_{K\in\mathcal{I}_L^{d+1}}\ \sum_{k_K\in\,\mathcal{E}_{L,K}^{(d+1)}}\mathbb{E}((\widetilde{V}_{i,L}^{(K,k_K)})^2)\le 2(L+1)^{d+1}\,. \quad (6.42)$$

Therefore, starting from (6.39), taking into account (6.40), (6.41) and (6.42), we derive that, for any $t\in[0,1]$,

$$\big|\mathbb{E}\big(D^4 g*\varphi_{a_L}(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}+t\vec{N}^*_{i,L}).\vec{N}^{*\otimes 4}_{i,L}\big)\big|\ll a_L^{-3}(L+1)^{3(d+1)/2}\,. \quad (6.43)$$

We deal now with the term $\mathbb{E}\big(D^3 g*\varphi_{a_L}(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}).\vec{N}^{*\otimes 3}_{i,L}\big)$. With this aim, we write

$$\begin{aligned}&\mathbb{E}\big(D^3 g*\varphi_{a_L}(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}).\vec{N}^{*\otimes 3}_{i,L}\big)\\&=\sum_{K,k_K}\sum_{P,p_P}\sum_{Q,q_Q}\mathbb{E}\big(\frac{\partial^3 g*\varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}\partial x^{(Q,q_Q)}}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big)\times\widetilde{N}_{i,L}^{(K,k_K)}\widetilde{N}_{i,L}^{(P,p_P)}\widetilde{N}_{i,L}^{(Q,q_Q)}\big)\,.\end{aligned}$$

We shall now use the so-called Stein's identity for Gaussian vectors (see e.g. Lemma 1 in Liu (1994)): for $G=(G_1,\dots,G_k)^t$ a centered Gaussian vector of $\mathbb{R}^k$ and any function $h:\mathbb{R}^k\to\mathbb{R}$ such that its partial derivatives exist almost everywhere and $\mathbb{E}\big|\frac{\partial}{\partial x_i}h(G)\big|<\infty$ for any $i=1,\dots,k$, the following equality holds true:

$$\mathbb{E}\big(G_i\,h(G)\big)=\sum_{\ell=1}^{k}\mathbb{E}\big(G_iG_\ell\big)\mathbb{E}\Big(\frac{\partial h}{\partial x_\ell}(G)\Big)\ \text{ for any } i\in\{1,\dots,k\}\,. \quad (6.44)$$

Therefore using (6.44) and the fact that $(Y_j, j\in\mathbb{Z})$ is independent of $(\vec{N}_i^*, 1\le i\le 2^L)$, we derive

$$\begin{aligned}&\mathbb{E}\big(D^3 g*\varphi_{a_L}(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}).\vec{N}^{*\otimes 3}_{i,L}\big)\\&=2\sum_{K,k_K}\sum_{P,p_P}\sum_{Q,q_Q}\mathbb{E}\Big(\frac{\partial^3 g*\varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}\partial x^{(Q,q_Q)}}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big)\times\widetilde{N}_{i,L}^{(K,k_K)}\Big)\mathbb{E}\big(\widetilde{N}_{i,L}^{(P,p_P)}\widetilde{N}_{i,L}^{(Q,q_Q)}\big)\\&+\sum_{K,k_K}\sum_{P,p_P}\sum_{Q,q_Q}\sum_{R,r_R}\mathbb{E}\Big(\frac{\partial^4 g*\varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}\partial x^{(Q,q_Q)}\partial x^{(R,r_R)}}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big)\times\widetilde{N}_{i,L}^{(K,k_K)}\widetilde{N}_{i,L}^{(P,p_P)}\Big)\\&\qquad\times\sum_{\ell=i+1}^{2^L}\mathbb{E}\big(\widetilde{N}_{\ell,L}^{(R,r_R)}\widetilde{N}_{i,L}^{(Q,q_Q)}\big)\,. \quad (6.45)\end{aligned}$$

Using again (6.44) and the fact that $(Y_j, j\in\mathbb{Z})$ is independent of $(\vec{N}_i^*, 1\le i\le 2^L)$, we have

$$\begin{aligned}&\mathbb{E}\Big(\frac{\partial^3 g*\varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}\partial x^{(Q,q_Q)}}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big)\times\widetilde{N}_{i,L}^{(K,k_K)}\Big)\\&=\sum_{R,r_R}\mathbb{E}\Big(\frac{\partial^4 g*\varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}\partial x^{(Q,q_Q)}\partial x^{(R,r_R)}}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big)\Big)\times\sum_{\ell=i+1}^{2^L}\mathbb{E}\big(\widetilde{N}_{\ell,L}^{(R,r_R)}\widetilde{N}_{i,L}^{(K,k_K)}\big)\,.\end{aligned} \quad (6.46)$$

On the other hand, applying twice (6.44) and taking into account that $(Y_j, j\in\mathbb{Z})$ is independent of $(\vec{N}^*_i, 1 \le i \le 2^L)$, we derive

$$\mathbb{E}\Big(\frac{\partial^4 g * \varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}\partial x^{(Q,q_Q)}\partial x^{(R,r_R)}}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big) \times \widetilde{N}^{(K,k_K)}_{i,L}\widetilde{N}^{(P,p_P)}_{i,L}\Big)$$
$$= \mathbb{E}\Big(\frac{\partial^4 g * \varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}\partial x^{(Q,q_Q)}\partial x^{(R,r_R)}}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big)\Big)\mathbb{E}\big(\widetilde{N}^{(K,k_K)}_{i,L}\widetilde{N}^{(P,p_P)}_{i,L}\big)$$
$$+ \sum_{M,m_M}\mathbb{E}\Big(\frac{\partial^5 g * \varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}\partial x^{(Q,q_Q)}\partial x^{(R,r_R)}\partial x^{(M,m_M)}}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big) \times \widetilde{N}^{(K,k_K)}_{i,L}\Big)$$
$$\times \sum_{\ell=i+1}^{2^L}\mathbb{E}\big(\widetilde{N}^{(M,m_M)}_{\ell,L}\widetilde{N}^{(P,p_P)}_{i,L}\big)$$
$$= \mathbb{E}\Big(\frac{\partial^4 g * \varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}\partial x^{(Q,q_Q)}\partial x^{(R,r_R)}}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big)\Big)\mathbb{E}\big(\widetilde{N}^{(K,k_K)}_{i,L}\widetilde{N}^{(P,p_P)}_{i,L}\big)$$
$$+ \sum_{M,m_M}\sum_{F,f_F}\mathbb{E}\Big(\frac{\partial^6 g * \varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}\partial x^{(Q,q_Q)}\partial x^{(R,r_R)}\partial x^{(M,m_M)}\partial x^{(F,f_F)}}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big)\Big)$$
$$\times\Big(\sum_{k=i+1}^{2^L}\mathbb{E}\big(\widetilde{N}^{(F,f_F)}_{k,L}\widetilde{N}^{(K,k_K)}_{i,L}\big)\Big)\Big(\sum_{\ell=i+1}^{2^L}\mathbb{E}\big(\widetilde{N}^{(M,m_M)}_{\ell,L}\widetilde{N}^{(P,p_P)}_{i,L}\big)\Big). \tag{6.47}$$

Therefore gathering (6.45)-(6.47), using (6.14) and the definition of the tensor product to shorten the notations, we derive

$$\mathbb{E}\big(D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big).\vec{N}^{*\otimes 3}_{i,L}\big)$$
$$= 3\sum_{\ell=i+1}^{2^L}\mathbb{E}\Big(D^4 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big)\Big).\Big(\mathbb{E}\big(\vec{V}^*_{\ell,L}\otimes\vec{V}^*_{i,L}\big)\otimes\mathbb{E}\big(\vec{V}^{*\otimes 2}_{i,L}\big)\Big)$$
$$+\sum_{\ell=i+1}^{2^L}\sum_{k=i+1}^{2^L}\sum_{j=i+1}^{2^L}\mathbb{E}\Big(D^6 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big)\Big).$$
$$\Big(\mathbb{E}\big(\vec{V}^*_{\ell,L}\otimes\vec{V}^*_{i,L}\big)\otimes\mathbb{E}\big(\vec{V}^*_{k,L}\otimes\vec{V}^*_{i,L}\big)\otimes\mathbb{E}\big(\vec{V}^*_{j,L}\otimes\vec{V}^*_{i,L}\big)\Big). \tag{6.48}$$

Using now Lemma 7.5, we get

$$\mathbb{E}\big(D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big).\vec{N}^{*\otimes 3}_{i,L}\big) \le a_L^{-3}(L+1)^{4(d+1)}\sum_{\ell=1}^{2^L}\beta_\ell + a_L^{-5}(L+1)^{6(d+1)}\Big(\sum_{\ell=1}^{2^L}\beta_\ell\Big)^3.$$

Taking into account the condition on the β-coefficients and the fact that $a_L \ge (L+1)^{d+1}$, it follows that

$$\mathbb{E}\big(D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big).\vec{N}^{*\otimes 3}_{i,L}\big) \le a_L^{-3}(L+1)^{4(d+1)}. \tag{6.49}$$

We analyze now the second and third term in the left-hand side of equality (6.38). This will be done by using similar decompositions as done when analyzing the corresponding terms to deal with $\mathbb{E}\big(\Delta_{1,i,L}(g)\big)$.

Let us first analyze $\mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big).\vec{N}^{*\otimes 2}_{i,L}\big)$. Let

$$R(i,j)(g) = D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+j}\big) - D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+j+1}\big), \tag{6.50}$$

and write

$$D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big).\vec{N}^{*\otimes 2}_{i,L}$$
$$= \sum_{j=1}^{u_L\wedge(2^L-i)} R(i,j)(g).\vec{N}^{*\otimes 2}_{i,L} + D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+(u_L\wedge(2^L-i))+1}\big).\vec{N}^{*\otimes 2}_{i,L}, \tag{6.51}$$

where we recall that u_L as been defined in (6.20). We shall use now several times (6.44) together with (6.14) as we did to get (6.48). Therefore, for any $1 \le j \le u_L \wedge (2^L - i)$,

$$\begin{aligned} &\mathbb{E}\big(R(i,j)(g).\vec{N}^{*\otimes 2}_{i,L}\big) = \mathbb{E}\big(R(i,j)(g)\big).\mathbb{E}\big(\vec{V}^{*\otimes 2}_{i,L}\big) \\ &+ \mathbb{E}\Big(D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j}\big).\vec{N}^*_{i,L} \otimes \mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big)\Big) \\ &+ \sum_{\ell=i+j+1}^{2^L} \mathbb{E}\Big(\big(D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j}\big) - D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+1}\big)\big).\vec{N}^*_{i,L} \otimes \mathbb{E}\big(\vec{V}^*_{\ell,L} \otimes \vec{V}^*_{i,L}\big)\Big). \end{aligned} \tag{6.52}$$

Next,

$$\begin{aligned} &\mathbb{E}\Big(D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j}\big).\vec{N}^*_{i,L} \otimes \mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big)\Big) \\ &\qquad = \mathbb{E}\big(D^4 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j}\big)\big).\,\mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big). \end{aligned} \tag{6.53}$$

Writing

$$\begin{aligned} &\big(D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j}\big) - D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+1}\big)\big).\vec{N}^*_{i,L} \otimes \mathbb{E}\big(\vec{V}^*_{\ell,L} \otimes \vec{V}^*_{i,L}\big) \\ &\qquad = \int_0^1 D^4 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,L}(t)\big).\vec{N}^*_{i+j,L} \otimes \vec{N}^*_{i,L} \otimes \mathbb{E}\big(\vec{V}^*_{\ell,L} \otimes \vec{V}^*_{i,L}\big), \end{aligned}$$

where

$$\vec{\mathbf{R}}_{i,j,L}(t) := \vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+1} + t\vec{N}^*_{i+j,L}, \tag{6.54}$$

we get

$$\begin{aligned} &\mathbb{E}\Big(\big(D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j}\big) - D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+1}\big)\big).\vec{N}^*_{i,L} \otimes \mathbb{E}\big(\vec{V}^*_{\ell,L} \otimes \vec{V}^*_{i,L}\big)\Big) \\ &\quad = \int_0^1 \mathbb{E}\big(D^4 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,L}(t)\big)\big).\,\mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{\ell,L} \otimes \vec{V}^*_{i,L}\big) \\ &\quad + \int_0^1 t\,\mathbb{E}\Big(D^5 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,L}(t)\big).\vec{N}^*_{i+j,L} \otimes \mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{\ell,L} \otimes \vec{V}^*_{i,L}\big)\Big) \\ &\quad + \sum_{k=i+j+1}^{2^L} \int_0^1 \mathbb{E}\Big(D^5 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,L}(t)\big).\vec{N}^*_{i+j,L} \otimes \mathbb{E}\big(\vec{V}^*_{k,L} \otimes \vec{V}^*_{i,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{\ell,L} \otimes \vec{V}^*_{i,L}\big)\Big). \end{aligned}$$

Whence,

$$\begin{aligned} &\mathbb{E}\Big(\big(D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j}\big) - D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+1}\big)\big).\vec{N}^*_{i,L} \otimes \mathbb{E}\big(\vec{V}^*_{\ell,L} \otimes \vec{V}^*_{i,L}\big)\Big) \\ &\quad = \int_0^1 \mathbb{E}\big(D^4 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,L}(t)\big)\big).\,\mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{\ell,L} \otimes \vec{V}^*_{i,L}\big) \\ &\quad + \int_0^1 t^2\,\mathbb{E}\big(D^6 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,L}(t)\big)\big).\,\mathbb{E}\big(\vec{V}^{*\otimes 2}_{i+j,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{\ell,L} \otimes \vec{V}^*_{i,L}\big) \\ &+ \sum_{k=i+j+1}^{2^L} \int_0^1 t\,\mathbb{E}\big(D^6 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,L}(t)\big)\big).\,\mathbb{E}\big(\vec{V}^*_{k,L} \otimes \vec{V}^*_{i+j,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{\ell,L} \otimes \vec{V}^*_{i,L}\big) \\ &\quad + \sum_{k=i+j+1}^{2^L} \int_0^1 t\,\mathbb{E}\big(D^6 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,L}(t)\big)\big).\,\mathbb{E}\big(\vec{V}^{*\otimes 2}_{i+j,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{k,L} \otimes \vec{V}^*_{i,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{\ell,L} \otimes \vec{V}^*_{i,L}\big) \\ &+ \sum_{(k,m)\in[i+j+1,2^L]^2} \int_0^1 \mathbb{E}\big(D^6 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,L}(t)\big)\big).\,\mathbb{E}\big(\vec{V}^*_{m,L} \otimes \vec{V}^*_{i+j,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{k,L} \otimes \vec{V}^*_{i,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{\ell,L} \otimes \vec{V}^*_{i,L}\big). \end{aligned} \tag{6.55}$$

Gathering (6.52)-(6.55), using Lemma 7.5 and taking into account the condition on the β coefficients, we then derive

$$\begin{aligned}&\Big|\sum_{j=1}^{u_L\wedge(2^L-i)}\mathbb{E}\big(R(i,j)(g).\big(\vec{N}_{i,L}^{*\otimes 2}-\mathbb{E}(\vec{V}_{i,L}^{*\otimes 2})\big)\big)\Big|\\ &\ll a_L^{-3}(L+1)^{4(d+1)}+a_L^{-5}(L+1)^{6(d+1)}+a_L^{-5}(L+1)^{6(d+1)}\sum_{j=1}^{u_L\wedge(2^L-i)}\Big(\sum_{\ell=j+1}^{2^L-i}\beta_\ell\Big)^2\\ &\ll a_L^{-3}(L+1)^{4(d+1)}+a_L^{-5}(L+1)^{6(d+1)}\sum_{j=1}^{u_L\wedge(2^L-i)}\Big(\sum_{\ell=j+1}^{2^L-i}\beta_\ell\Big)^2, \qquad (6.56)\end{aligned}$$

where for the last inequality we used the fact that $a_L\geq(L+1)^{d+1}$. On the other hand, using once again several times (6.44) together with (6.14) as we did to get (6.48), we derive, for $i<2^L-u_L$,

$$\begin{aligned}&\mathbb{E}\Big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+u_L+1}\big).\vec{N}_{i,L}^{*\otimes 2}\Big)=\mathbb{E}\big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+(u_L\wedge(2^L-i))}\big)\big).\mathbb{E}\big(\vec{V}_{i,L}^{*\otimes 2}\big)\\ &\quad+\sum_{\ell=i+(u_L\wedge(2^L-i))}^{2^L}\mathbb{E}\Big(D^3g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+u_L+1}\big).\vec{N}_{i,L}^{*}\otimes\mathbb{E}\big(\vec{V}_{\ell,L}^{*}\otimes\vec{V}_{i,L}^{*}\big)\Big)\\ &=\mathbb{E}\big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+u_L+1}\big)\big).\mathbb{E}\big(\vec{V}_{i,L}^{*\otimes 2}\big)\\ &+\sum_{\ell=i+u_L+1}^{2^L}\sum_{k=i+u_L+1}^{2^L}\mathbb{E}\big(D^4g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+u_L+1}\big)\big).\mathbb{E}\big(\vec{V}_{m,L}^{*}\otimes\vec{V}_{i,L}^{*}\big)\otimes\mathbb{E}\big(\vec{V}_{\ell,L}^{*}\otimes\vec{V}_{i,L}^{*}\big)\Big).\end{aligned}$$

Hence, using Lemma 7.5, we obtain, for $i<2^L-u_L$,

$$\Big|\mathbb{E}\Big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+u_L+1}\big).\big(\vec{N}_{i,L}^{*\otimes 2}-\mathbb{E}(\vec{V}_{i,L}^{*\otimes 2})\big)\Big)\Big|\ll a_L^{-3}(L+1)^{4(d+1)}\Big(\sum_{\ell=u_L+1}^{2^L-i}\beta_\ell\Big)^2. \qquad (6.57)$$

Assume now that $i\geq 2^L-u_L$. Using the independence between $\vec{\mathbf{S}}_{i-1}$ and $\vec{N}_{i,L}^{*}$, and the relation (6.14), we then notice that

$$\mathbb{E}\Big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}\big).\vec{N}_{i,L}^{*\otimes 2}\Big)=\mathbb{E}\big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}\big)\big).\mathbb{E}\big(\vec{V}_{i,L}^{*\otimes 2}\big). \qquad (6.58)$$

Therefore, starting from (6.51), considering (6.56), (6.57) and (6.58), and using that $\beta_k=O(k^{1-p})$ with $p\in]2,3]$, we get

$$\begin{aligned}&\Big|\sum_{i=1}^{2^L}\mathbb{E}\Big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big).\big(\vec{N}_{i,L}^{*\otimes 2}-\mathbb{E}(\vec{V}_{i,L}^{*\otimes 2})\big)\Big)\Big|\\ &\qquad\ll a_L^{-3}(L+1)^{4(d+1)}2^L+a_L^{-5}(L+1)^{6(d+1)}2^Lu_L^{3-p}.\end{aligned}$$

Whence, taking into account the choice of u_L, we get overall

$$\begin{aligned}&\sum_{i=1}^{2^L}\Big|\mathbb{E}\Big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big).\big(\vec{N}_{i,L}^{*\otimes 2}-\mathbb{E}(\vec{V}_{i,L}^{*\otimes 2})\big)\Big)\\ &\qquad\ll a_L^{-3}(L+1)^{4(d+1)}2^L+a_L^{-2-p}(L+1)^{(3+p)(d+1)}2^L. \qquad (6.59)\end{aligned}$$

We analyze now the term $\mathbb{E}\big(Dg * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+1}\big).\vec{N}^*_{i,L}\big)$ in the left-hand side of equality (6.38). With this aim, we write

$$Dg * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+1}\big) = Dg * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}\big) + \sum_{j=1}^{2^L-i} \big(Dg * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j}\big) - Dg * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+1}\big)\big)\,.$$

Using the independence between $\vec{\mathbf{S}}_{i-1}$ and $\vec{N}^*_{i,L}$, we first notice that

$$\mathbb{E}\big(Dg * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}\big).\vec{N}^*_{i,L}\big) = \mathbb{E}\big(Dg * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}\big)\big).\mathbb{E}\big(\vec{N}^*_{i,L}\big) = 0\,.$$

Hence

$$\mathbb{E}\big(Dg * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+1}\big).\vec{N}^*_{i,L}\big) = \sum_{j=1}^{2^L-i} \mathbb{E}\big(\big(Dg * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j}\big) - Dg * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+1}\big)\big).\vec{N}^*_{i,L}\big)\,. \quad (6.60)$$

Notice now that

$$\mathbb{E}\big(\big(Dg * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j}\big) - Dg * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+1}\big)\big).\vec{N}^*_{i,L}\big) = \int_0^1 \mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,L}(t)\big).\vec{N}^*_{i+j,L} \otimes \vec{N}^*_{i,L}\big)dt\,, \quad (6.61)$$

where we recall that $\vec{\mathbf{R}}_{i,j,L}(t)$ as been defined in (6.54). We use now several times (6.44) together with (6.14) as we did to get (6.48). Hence,

$$\begin{aligned}\mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,L}(t)\big).\vec{N}^*_{i+j,L} \otimes \vec{N}^*_{i,L}\big) &= \mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+1} + t\vec{N}^*_{i+j,L}\big)\big).\mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big)\\ &+ t\,\mathbb{E}\Big(D^3 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,L}(t)\big).\vec{N}^*_{i+j,L} \otimes \mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big)\Big)\\ &+ \sum_{\ell=i+j+1}^{2^L} \mathbb{E}\Big(D^3 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,L}(t)\big).\vec{N}^*_{i+j,L} \otimes \mathbb{E}\big(\vec{V}^*_{\ell,L} \otimes \vec{V}^*_{i,L}\big)\Big)\,.\end{aligned}$$

Next,

$$\begin{aligned}\mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,L}(t)\big).\vec{N}^*_{i+j,L} \otimes \vec{N}^*_{i,L}\big) &= \mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,L}(t)\big)\big).\mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big)\\ &+ t^2\,\mathbb{E}\big(D^4 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,L}(t)\big)\big).\mathbb{E}\big(\vec{V}^{*\otimes 2}_{i+j,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big)\Big)\\ &+ t \sum_{\ell=i+j+1}^{2^L} \mathbb{E}\big(D^4 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,L}(t)\big)\big).\mathbb{E}\big(\vec{V}^*_{\ell,L} \otimes \vec{V}^*_{i+j,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big)\Big)\\ &+ t \sum_{\ell=i+j+1}^{2^L} \mathbb{E}\big(D^4 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,L}(t)\big)\big).\mathbb{E}\big(\vec{V}^{*\otimes 2}_{i+j,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{\ell,L} \otimes \vec{V}^*_{i,L}\big)\Big)\\ &+ \sum_{k=i+j+1}^{2^L} \sum_{\ell=i+j+1}^{2^L} \mathbb{E}\big(D^4 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,L}(t)\big)\big).\mathbb{E}\big(\vec{V}^*_{k,L} \otimes \vec{V}^*_{i+j,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{\ell,L} \otimes \vec{V}^*_{i,L}\big)\Big)\,.\end{aligned}$$

Hence, starting from (6.61), considering the above equalities and using Lemma 7.5, we obtain, for $1 \le j \le 2^L - i$,

$$\Big|\mathbb{E}\big(\big(Dg * \varphi_{a_L}(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j}) - Dg * \varphi_{a_L}(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+1})\big).\vec{N}^*_{i,L}\big)\Big|$$
$$\ll a_L^{-1}(L+1)^{2(d+1)}\beta_j + a_L^{-3}(L+1)^{4(d+1)}\beta_j \sum_{\ell=1}^{2^L}\beta_\ell + a_L^{-3}(L+1)^{4(d+1)}\sum_{\ell=j+1}^{2^L-1}\beta_\ell \sum_{k=1}^{2^L}\beta_k$$
$$\ll a_L^{-1}(L+1)^{2(d+1)}\beta_j + a_L^{-3}(L+1)^{4(d+1)}\sum_{\ell=j+1}^{2^L-1}\beta_\ell\,, \quad (6.62)$$

where for the last inequality we have used that $a_L \ge (L+1)^{d+1}$ and $\sum_{k\ge1}\beta_k < \infty$.

From now on we assume that $j \le (2^L - i)\wedge(u_L - 1)$. Recalling the notation (6.54), we first write

$$\big(Dg * \varphi_{a_L}(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j})$$
$$- Dg * \varphi_{a_L}(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+1})\big).\vec{N}^*_{i,L} - D^2 g * \varphi_{a_L}(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+1}).(\vec{N}^*_{i+j,L}\otimes \vec{N}^*_{i,L})$$
$$= \int_0^1 (1-t) D^3 g * \varphi_{a_L}(\vec{\mathbf{R}}_{i,j,L}(t)).(\vec{N}^{*\otimes 2}_{i+j,L}\otimes \vec{N}^*_{i,L})dt\,. \quad (6.63)$$

Applying (6.44) together with (6.14), we derive

$$\mathbb{E}\Big(D^3 g * \varphi_{a_L}(\vec{\mathbf{R}}_{i,j,L}(t)).(\vec{N}^{*\otimes 2}_{i+j,L}\otimes \vec{N}^*_{i,L})\Big)$$
$$= 2\,\mathbb{E}\Big(D^3 g * \varphi_{a_L}(\vec{\mathbf{R}}_{i,j,L}(t)).\vec{N}^*_{i+j,L}\otimes \mathbb{E}\big(\vec{V}^*_{i+j,L}\otimes \vec{V}^*_{i,L}\big)\Big)$$
$$+ t\,\mathbb{E}\Big(D^4 g * \varphi_{a_L}(\vec{\mathbf{R}}_{i,j,L}(t))).\vec{N}^{*\otimes 2}_{i+j,L}\otimes \mathbb{E}\big(\vec{V}^*_{i+j,L}\otimes \vec{V}^*_{i,L}\big)\Big)$$
$$+ \sum_{\ell=i+j+1}^{2^L}\mathbb{E}\Big(D^4 g * \varphi_{a_L}(\vec{\mathbf{R}}_{i,j,L}(t))).\vec{N}^{*\otimes 2}_{i+j,L}\otimes \mathbb{E}\big(\vec{V}^*_{\ell,L}\otimes \vec{V}^*_{i,L}\big)\Big)\,.$$

Next, applying again (6.44) together with (6.14), we get

$$\mathbb{E}\Big(D^3 g * \varphi_{a_L}(\vec{\mathbf{R}}_{i,j,L}(t)).\vec{N}^*_{i+j,L}\otimes \mathbb{E}\big(\vec{V}^*_{i+j,L}\otimes \vec{V}^*_{i,L}\big)\Big)$$
$$= t\,\mathbb{E}\big(D^4 g * \varphi_{a_L}(\vec{\mathbf{R}}_{i,j,L}(t))\big).\mathbb{E}\big(\vec{V}^{*\otimes 2}_{i+j,L}\big)\otimes \mathbb{E}\big(\vec{V}^*_{i+j,L}\otimes \vec{V}^*_{i,L}\big)\,,$$

and

$$\mathbb{E}\Big(D^4 g * \varphi_{a_L}(\vec{\mathbf{R}}_{i,j,L}(t))).\vec{N}^{*\otimes 2}_{i+j,L}\otimes \mathbb{E}\big(\vec{V}^*_{\ell,L}\otimes \vec{V}^*_{i,L}\big)\Big)$$
$$= \mathbb{E}\big(D^4 g * \varphi_{a_L}(\vec{\mathbf{R}}_{i,j,L}(t))\big).\mathbb{E}\big(\vec{V}^{*\otimes 2}_{i+j,L}\big)\otimes \mathbb{E}\big(\vec{V}^*_{\ell,L}\otimes \vec{V}^*_{i,L}\big)$$
$$+ t^2\,\mathbb{E}\big(D^6 g * \varphi_{a_L}(\vec{\mathbf{R}}_{i,j,L}(t))\big).\,\mathbb{E}\big(\vec{V}^{*\otimes 2}_{i+j,L}\big)\otimes \mathbb{E}\big(\vec{V}^{*\otimes 2}_{i+j,L}\big)\otimes \mathbb{E}\big(\vec{V}^*_{\ell,L}\otimes \vec{V}^*_{i,L}\big)$$
$$+ 2t\sum_{k=i+j+1}^{2^L}\mathbb{E}\big(D^6 g * \varphi_{a_L}(\vec{\mathbf{R}}_{i,j,L}(t))\big).\,\mathbb{E}\big(\vec{V}^{*\otimes 2}_{i+j,L}\big)\mathbb{E}\big(\vec{V}^*_{k,L}\otimes \vec{V}^*_{i+j,L}\big)\otimes \mathbb{E}\big(\vec{V}^*_{\ell,L}\otimes \vec{V}^*_{i,L}\big)$$
$$+ \sum_{k=i+j+1}^{2^L}\sum_{m=i+j+1}^{2^L}\int_0^1 \mathbb{E}\big(D^6 g * \varphi_{a_L}(\vec{\mathbf{R}}_{i,j,L}(t))\big).\,\mathbb{E}\big(\vec{V}^*_{m,L}\otimes \vec{V}^*_{i+j,L}\big)\otimes$$
$$\mathbb{E}\big(\vec{V}^*_{k,L}\otimes \vec{V}^*_{i+j,L}\big)\otimes \mathbb{E}\big(\vec{V}^*_{\ell,L}\otimes \vec{V}^*_{i,L}\big)\,.$$

Gathering the previous equalities and using Lemma 7.5, we derive

$$\mathbb{E}\Big(\int_0^1 (1-t)D^3 g*\varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,L}(t)\big).(\vec{N}^{*\otimes 2}_{i+j,L}\otimes\vec{N}^*_{i,L})dt\Big)$$
$$\ll a_L^{-3}(L+1)^{4(d+1)}\sum_{\ell=j}^{2^L}\beta_\ell + a_L^{-5}(L+1)^{6(d+1)}\sum_{\ell=j+1}^{2^L}\beta_\ell\Big(\sum_{k\geq 1}\beta_k\Big)^2$$
$$\ll a_L^{-3}(L+1)^{4(d+1)}\sum_{\ell=j}^{2^L}\beta_\ell\,, \tag{6.64}$$

where for the last inequality we have used that $a_L \geq (L+1)^{d+1}$ and $\sum_{k\geq 1}\beta_k<\infty$.

In order to estimate the term $\mathbb{E}\big(D^2 g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+j+1}\big).(\vec{N}^*_{i+j,L}\otimes\vec{N}^*_{i,L})\big)$ in the right-hand side of (6.63), we use the following decomposition:

$$\mathbb{E}\big(D^2 g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+j+1}\big).(\vec{N}^*_{i+j,L}\otimes\vec{N}^*_{i,L})\big)$$
$$=\sum_{l=1}^{j\wedge(2^L-i-j)}\mathbb{E}\Big(\big(D^2 g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+j+l}\big)-D^2 g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+j+l+1}\big)\big).\vec{N}^*_{i+j,L}\otimes\vec{N}^*_{i,L}\Big)$$
$$+\mathbb{E}\big(D^2 g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{(i+2j+1)\wedge(2^L+1)}\big).(\vec{N}^*_{i+j,L}\otimes\vec{N}^*_{i,L})\big)\,.$$

For any $l\in\{1,\dots,j\wedge(2^L-i-j)\}$, we write

$$\Big|\mathbb{E}\Big(\big(D^2 g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+j+l}\big)-D^2 g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+j+l+1}\big)\big).\vec{N}^*_{i+j,L}\otimes\vec{N}^*_{i,L}\Big)\Big|$$
$$=\Big|\mathbb{E}\Big(\int_0^1 D^3 g*\varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,l,L}(t)\big).\vec{N}^*_{i+j+l,L}\otimes\vec{N}^*_{i+j,L}\otimes\vec{N}^*_{i,L}dt\Big)\Big|\,,$$

where

$$\vec{\mathbf{R}}_{i,j,l,L}(t):=\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+j+l+1}+t\vec{N}^*_{i+j+l,L}\,.$$

Applying (6.44) together with (6.14), we derive

$$\mathbb{E}\Big(D^3 g*\varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,l,L}(t)\big).\vec{N}^*_{i+j+l,L}\otimes\vec{N}^*_{i+j,L}\otimes\vec{N}^*_{i,L}$$
$$=\mathbb{E}\Big(D^3 g*\varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,l,L}(t)\big).\vec{N}^*_{i+j+l,L}\otimes\mathbb{E}\big(\vec{V}^*_{i+j,L}\otimes\vec{V}^*_{i,L}\big)\Big)$$
$$+\mathbb{E}\Big(D^3 g*\varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,l,L}(t)\big).\vec{N}^*_{i+j,L}\otimes\mathbb{E}\big(\vec{V}^*_{i+j+l,L}\otimes\vec{V}^*_{i,L}\big)\Big)$$
$$+t\,\mathbb{E}\Big(D^4 g*\varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,l,L}(t)\big)\big).\vec{N}^*_{i+j+l,L}\otimes\vec{N}^*_{i+j,L}\otimes\mathbb{E}\big(\vec{V}^*_{i+j+l,L}\otimes\vec{V}^*_{i,L}\big)\Big)$$
$$+\sum_{k=i+j+l+1}^{2^L}\mathbb{E}\Big(D^4 g*\varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,l,L}(t)\big).\vec{N}^*_{i+j+l,L}\otimes\vec{N}^*_{i+j,L}\otimes\mathbb{E}\big(\vec{V}^*_{k,L}\otimes\vec{V}^*_{i,L}\big)\Big)\,.$$

Next, applying again (6.44) together with (6.14), we get

$$\mathbb{E}\Big(D^3 g*\varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,l,L}(t)\big).\vec{N}^*_{i+j+l,L}\otimes\mathbb{E}\big(\vec{V}^*_{i+j,L}\otimes\vec{V}^*_{i,L}\big)\Big)$$
$$=t\,\mathbb{E}\big(D^4 g*\varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,l,L}(t)\big)\big).\mathbb{E}\big(\vec{V}^{*\otimes 2}_{i+j+l,L}\big)\otimes\mathbb{E}\big(\vec{V}^*_{i+j,L}\otimes\vec{V}^*_{i,L}\big)$$
$$+\sum_{k=i+j+l+1}^{2^L}\mathbb{E}\big(D^4 g*\varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,l,L}(t)\big)\big).\mathbb{E}\big(\vec{V}^*_{k,L}\otimes\vec{V}^*_{i+j+l,L}\big)\otimes\mathbb{E}\big(\vec{V}^*_{i+j,L}\otimes\vec{V}^*_{i,L}\big)\,,$$

$$\mathbb{E}\Big(D^3 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,l,L}(t)\big).\vec{N}^*_{i+j,L} \otimes \mathbb{E}\big(\vec{V}^*_{i+j+l,L} \otimes \vec{V}^*_{i,L}\big)\Big)$$
$$= t\,\mathbb{E}\big(D^4 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,l,L}(t)\big)\big).\mathbb{E}\big(\vec{V}^*_{i+j+l,L} \otimes \vec{V}^*_{i+j,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{i+j+l,L} \otimes \vec{V}^*_{i,L}\big)$$
$$+ \sum_{k=i+j+l+1}^{2^L} \mathbb{E}\big(D^4 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,l,L}(t)\big)\big).\mathbb{E}\big(\vec{V}^*_{k,L} \otimes \vec{V}^*_{i+j,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{i+j+l,L} \otimes \vec{V}^*_{i,L}\big)\,,$$

and

$$\mathbb{E}\Big(D^4 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,l,L}(t)\big).\vec{N}^*_{i+j+l,L} \otimes \vec{N}^*_{i+j,L} \otimes \mathbb{E}\big(\vec{V}^*_{k,L} \otimes \vec{V}^*_{i,L}\big)\Big)$$
$$= \mathbb{E}\big(D^4 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,l,L}(t)\big)\big).\mathbb{E}\big(\vec{V}^*_{i+j+l,L} \otimes \vec{V}^*_{i+j,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{k,L} \otimes \vec{V}^*_{i,L}\big)\Big)$$
$$+ t\,\mathbb{E}\Big(D^5 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,l,L}(t)\big).\vec{N}^*_{i+j+l,L} \otimes \mathbb{E}\big(\vec{V}^*_{i+j+l,L} \otimes \vec{V}^*_{i+j,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{k,L} \otimes \vec{V}^*_{i,L}\big)\Big)$$
$$+ \sum_{m=i+j+l+1}^{2^L} \mathbb{E}\Big(D^5 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,l,L}(t)\big).\vec{N}^*_{i+j+l,L} \otimes \mathbb{E}\big(\vec{V}^*_{m,L} \otimes \vec{V}^*_{i+j,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{k,L} \otimes \vec{V}^*_{i,L}\big)\Big)\,.$$

Next,

$$\mathbb{E}\Big(D^5 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,l,L}(t)\big).\vec{N}^*_{i+j+l,L} \otimes \mathbb{E}\big(\vec{V}^*_{m,L} \otimes \vec{V}^*_{i+j,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{k,L} \otimes \vec{V}^*_{i,L}\big)\Big)$$
$$= t\,\mathbb{E}\big(D^6 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,l,L}(t)\big)\big).\mathbb{E}\big(\vec{V}^{*\otimes 2}_{i+j+l,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{m,L} \otimes \vec{V}^*_{i+j,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{k,L} \otimes \vec{V}^*_{i,L}\big)$$
$$+ \sum_{r=i+j+l+1}^{2^L} \mathbb{E}\big(D^6 g * \varphi_{a_L}\big(\vec{\mathbf{R}}_{i,j,l,L}(t)\big)\big).\mathbb{E}\big(\vec{V}^*_{r,L} \otimes \vec{V}^*_{i+j+l,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{m,L} \otimes \vec{V}^*_{i+j,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{k,L} \otimes \vec{V}^*_{i,L}\big)\,.$$

So, gathering the previous equalities, using Lemma 7.5 and the fact that $\sum_{k\geq 1}\beta_k < \infty$, we get overall that, for any $l \in \{1, \ldots, j \wedge (2^L - i - j)\}$,

$$\Big|\mathbb{E}\Big(\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+l}\big) - D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+l+1}\big)\big).\vec{N}^*_{i+j,L} \otimes \vec{N}^*_{i,L}\Big)\Big|$$
$$\ll a_L^{-3}(L+1)^{4(d+1)}\beta_j + a_L^{-3}(L+1)^{4(d+1)}\beta_l \sum_{k=j+l}^{2^L-i}\beta_k + a_L^{-5}(L+1)^{6(d+1)}\sum_{k=j+l}^{2^L-i}\beta_k \sum_{m=l}^{2^L-i-j}\beta_m\,.$$

Therefore, using again that $\sum_{k\geq 1}\beta_k < \infty$,

$$\sum_{l=1}^{j\wedge(2^L-i-j)} \Big|\mathbb{E}\Big(\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+l}\big) - D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+l+1}\big)\big).\vec{N}^*_{i+j,L} \otimes \vec{N}^*_{i,L}\Big)\Big|$$
$$\ll a_L^{-3}(L+1)^{4(d+1)} j\beta_j + a_L^{-3}(L+1)^{4(d+1)} \sum_{k=j}^{2^L-i}\beta_k$$
$$+ a_L^{-5}(L+1)^{6(d+1)} \sum_{l=1}^{(j-1)\wedge(2^L-i-j)} \sum_{k=j+l}^{2^L-i}\beta_k \sum_{m=l}^{2^L-i-j}\beta_m\,. \quad (6.65)$$

We analyze now $\big|\mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{(i+2j+1)\wedge(2^L+1)}\big).(\vec{N}^*_{i+j,L} \otimes \vec{N}^*_{i,L})\big)\big|$. Assume first that $j \leq [(2^L - i)/2]$. Clearly, using the notation (6.50),

$$D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+2j+1}\big) = \sum_{l=j+1}^{(u_L-1)\wedge(2^L-i-j)} R(i, l+j)(g) + D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{(i+j+u_L)\wedge(2^L+1)}\big)\,.$$

Now for any $l \in \{j+1, \dots, (u_L - 1) \wedge (2^L - i - j)\}$, by (6.44) together with (6.14), we get

$$\mathbb{E}\big(R(i, l+j)(g).(\vec{N}^*_{i+j,L} \otimes \vec{N}^*_{i,L})\big) = \mathbb{E}\big(R(i, l+j)(g)\big).\mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big)$$
$$+ \sum_{k=i+j+l}^{2^L} \mathbb{E}\Big(D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+l}\big).\vec{N}^*_{i,L} \otimes \mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{k,L}\big)\Big)$$
$$- \sum_{k=i+j+l+1}^{2^L} \mathbb{E}\Big(D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+l+1}\big).\vec{N}^*_{i,L} \otimes \mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{k,L}\big)\Big) .$$

Therefore,

$$\sum_{l=j+1}^{(u_L-1)\wedge(2^L-i-j)} \Big(\mathbb{E}\big(R(i, l+j)(g).(\vec{N}^*_{i+j,L} \otimes \vec{N}^*_{i,L})\big) - \mathbb{E}\big(R(i, l+j)(g)\big).\mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big)\Big)$$
$$= \sum_{k=i+2j+1}^{2^L} \mathbb{E}\Big(D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+2j+1}\big).\vec{N}^*_{i,L} \otimes \mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{k,L}\big)\Big)$$
$$- \sum_{k=(i+j+u_L)\wedge(2^L+1)}^{2^L} \mathbb{E}\Big(D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{(i+j+u_L)\wedge(2^L+1)}\big).\vec{N}^*_{i,L} \otimes \mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{k,L}\big)\Big) .$$

Whence, using again (6.44) together with (6.14),

$$\sum_{l=j+1}^{(u_L-1)\wedge(2^L-i-j)} \Big(\mathbb{E}\big(R(i, l+j)(g).(\vec{N}^*_{i+j,L} \otimes \vec{N}^*_{i,L})\big) - \mathbb{E}\big(R(i, l+j)(g)\big).\mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big)\Big)$$
$$= \sum_{k=i+2j+1}^{2^L} \sum_{m=i+2j+1}^{2^L} \mathbb{E}\big(D^4 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+2j+1}\big)\big).\mathbb{E}\big(\vec{V}^*_{m,L} \otimes \vec{V}^*_{i,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{k,L}\big)$$
$$- \sum_{k=(i+j+u_L)\wedge(2^L+1)}^{2^L} \sum_{m=(i+j+u_L)\wedge(2^L+1)}^{2^L} \mathbb{E}\big(D^4 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{(i+j+u_L)\wedge(2^L+1)}\big)\big).\mathbb{E}\big(\vec{V}^*_{m,L} \otimes \vec{V}^*_{i,L}\big)$$
$$\otimes \mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{k,L}\big) .$$

Next, using Lemma 7.5 and the fact that $\sum_{k \geq 1} \beta_k < \infty$, we get

$$\Big|\sum_{l=j+1}^{(u_L-1)\wedge(2^L-i-j)} \Big(\mathbb{E}\big(R(i, l+j)(g).(\vec{N}^*_{i+j,L} \otimes \vec{N}^*_{i,L})\big) - \mathbb{E}\big(R(i, l+j)(g)\big).\mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big)\Big)\Big|$$
$$\ll a_L^{-3}(L+1)^{4(d+1)} \sum_{k \geq j} \beta_k + a_L^{-3}(L+1)^{4(d+1)} \sum_{k \geq u_L} \beta_k . \quad (6.66)$$

Still assuming that $j \leq [(2^L - i)/2]$, let us analyze the following term:

$$\mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{(i+j+u_L)\wedge(2^L+1)}\big).(\vec{N}^*_{i+j,L} \otimes \vec{N}^*_{i,L})\big) .$$

Let us first consider the case where $j \leq 2^L - i - u_L$. By (6.44) together with (6.14), we get

$$\mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+u_L}\big).(\vec{N}^*_{i+j,L} \otimes \vec{N}^*_{i,L})\big)$$
$$= \mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+u_L}\big)\big).\mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big)$$
$$+ \sum_{k=i+j+u_L}^{2^L} \mathbb{E}\Big(D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+u_L}\big).\vec{N}^*_{i+j,L} \otimes \mathbb{E}\big(\vec{V}^*_{k,L} \otimes \vec{V}^*_{i,L}\big)\Big) .$$

Therefore using again (6.44) together with (6.14),

$$\begin{aligned}&\mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+u_L}\big).(\vec{N}^*_{i+j,L} \otimes \vec{N}^*_{i,L})\big) \\ &\qquad\qquad - \mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+u_L}\big)\big).\mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big) \\ &= \sum_{k=i+j+u_L}^{2^L} \sum_{m=i+j+u_L}^{2^L} \mathbb{E}\big(D^4 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+u_L}\big)\big).\mathbb{E}\big(\vec{V}^*_{m,L} \otimes \vec{V}^*_{i+j,L}\big) \otimes \mathbb{E}\big(\vec{V}^*_{k,L} \otimes \vec{V}^*_{i,L}\big)\Big).\end{aligned}$$

Hence using Lemma 7.5 and the fact that $\sum_{k \geq 1} \beta_k < \infty$, it follows that

$$\begin{aligned}&\Big|\mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+u_L}\big).(\vec{N}^*_{i+j,L} \otimes \vec{N}^*_{i,L})\big) \\ &\qquad\qquad - \mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+j+u_L}\big)\big).\mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big)\Big| \\ &\qquad\qquad\qquad\qquad \ll a_L^{-3} (L+1)^{4(d+1)} \sum_{k=u_L}^{2^L} \beta_k\,. \qquad (6.67)\end{aligned}$$

Consider now the case where $j \geq 2^L - i - u_L + 1$. Notice then that by independence between $\vec{\mathbf{S}}_{i-1}$ and the random variables $\vec{N}^*_{i,L}$ and $\vec{N}^*_{i+j,L}$, (6.14) entails that

$$\mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}\big).(\vec{N}^*_{i+j,L} \otimes \vec{N}^*_{i,L})\big) = \mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}\big)\big).\mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big)\,. \qquad (6.68)$$

Assume now that $j \geq [(2^L - i)/2] + 1$. Starting from (6.68) and using Lemma 7.5, we get

$$\big|\mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}\big).(\vec{N}^*_{i+j,L} \otimes \vec{N}^*_{i,L})\big| \ll a_L^{-1} (L+1)^{2(d+1)} \beta_j \ll a_L^{-1} (L+1)^{2(d+1)} \beta_{[(2^L - i)/2]}\,. \qquad (6.69)$$

Starting from (6.60), summing the inequalities (6.62), (6.64), (6.65), (6.66), (6.67) and (6.69) in j, adding them, and taking into account that $\beta_k = O(k^{1-p})$ with $p \in]2, 3]$, we then infer that

$$\begin{aligned}&\big|\mathbb{E}\big(Dg * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+1}\big).\vec{N}^*_{i,L}\big) \\ &\qquad - \sum_{j=1}^{u_L - 1} \mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+2j+1}\big)\big).\,\mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big) \mathbf{1}_{j \leq [(2^L - i)/2]}\big| \\ &\ll a_L^{-1} (L+1)^{2(d+1)} \sum_{j=(2^L - i) \wedge u_L}^{2^L - i} j^{1-p} + a_L^{-3} (L+1)^{4(d+1)} 2^{L(3-p)} + a_L^{-3} (L+1)^{4d+5} \mathbf{1}_{p=3} \\ &\quad + a_L^{-5} (L+1)^{6(d+1)} u_L^{6-2p} + a_L^{-5} (L+1)^{6(d+1)} (\log(u_L))^2 \mathbf{1}_{p=3} + a_L^{-3} (L+1)^{4(d+1)} u_L^{3-p} \\ &\qquad + a_L^{-3} (L+1)^{4(d+1)} \log(u_L) \mathbf{1}_{p=3} + a_L^{-1} (L+1)^{2(d+1)} u_L \beta_{[(2^L - i)/2]}\,.\end{aligned}$$

Next summing on i and taking into account the choice of u_L and that $a_L \geq (L+1)^{d+1}$, we get

$$\begin{aligned}&\sum_{i=1}^{2^L} \Big|\mathbb{E}\big(Dg * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+1}\big).\vec{N}^*_{i,L}\big) \\ &\qquad - \sum_{j=1}^{u_L - 1} \mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+2j+1}\big)\big).\mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big) \mathbf{1}_{j \leq [(2^L - i)/2]}\Big| \\ &\qquad\qquad \ll a_L^{1-p} (L+1)^{p(d+1)} 2^L + a_L^{-3} (L+1)^{4(d+1)} 2^L \big(2^{L(3-p)} + L \mathbf{1}_{p=3}\big)\,. \qquad (6.70)\end{aligned}$$

Hence, starting from (6.38) and considering the upper bounds (6.43) and (6.49) together with the fact that $a_L \geq (L+1)^{d+1}$, we get

$$\sum_{i=1}^{2^L} \mathbb{E}\big(\Delta_{2,i,L}(g)\big) - \sum_{i=1}^{2^L} \sum_{j=1}^{u_L-1} \mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+2j+1}\big)\big).\mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big)\mathbf{1}_{j\leq[(2^L-i)/2]}$$
$$- \frac{1}{2}\sum_{i=1}^{2^L} \mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+1}\big)\big).\mathbb{E}\big(\vec{V}^{*\otimes 2}_{i,L}\big)$$
$$\ll a_L^{1-p}(L+1)^{p(d+1)}2^L + a_L^{-3}(L+1)^{4(d+1)}2^L\big(2^{L(3-p)} + L\mathbf{1}_{p=3}\big)\,. \tag{6.71}$$

Gathering (6.37) and (6.71), it follows that

$$\sum_{i=1}^{2^L}\big(\mathbb{E}\big(\Delta_{1,i,L}(g)\big) - \mathbb{E}\big(\Delta_{2,i,L}(g)\big)\big) - \sum_{i=1}^{2^L} R_{i,L}$$
$$\ll (L+1)^{d+1} + a_L^{1-p}(L+1)^{p(d+1)}2^L + a_L^{-2}(L+1)^{3d+4}2^L\mathbf{1}_{p=3}$$
$$+ a_L^{-3}(L+1)^{4(d+1)}2^L\big(2^{L(3-p)} + L\mathbf{1}_{p=3}\big)\,, \tag{6.72}$$

where

$$R_{i,L} := \sum_{j=1}^{u_L-1} \mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-2j} + \vec{\mathbf{T}}_{i+1}\big)\big).\mathbb{E}\big(\vec{V}^*_{i-j,L} \otimes \vec{V}^*_{i,L}\big)\mathbf{1}_{j\leq[i/2]}$$
$$- \sum_{j=1}^{u_L-1} \mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+2j+1}\big)\big).\mathbb{E}\big(\vec{V}^*_{i+j,L} \otimes \vec{V}^*_{i,L}\big)\mathbf{1}_{j\leq[(2^L-i)/2]}\,. \tag{6.73}$$

We get now an upper bound of $\sum_{i=1}^{2^L} R_{i,L}$. We first write that

$$\mathbb{E}\big(D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{\mathbf{T}}_{i+1}\big) - D^2 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-2j} + \vec{\mathbf{T}}_{i+1}\big)\big).\,\mathbb{E}\big(\vec{V}^*_{i-j,L} \otimes \vec{V}^*_{i,L}\big)$$
$$= \sum_{m=i-2j+1}^{i-1} \int_0^1 \mathbb{E}\Big(D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-2j} + \vec{\mathbf{T}}_{i+1} + t(\vec{\mathbf{S}}_{i-1} - \vec{\mathbf{S}}_{i-2j})\big).\vec{V}^*_{m,L} \otimes \mathbb{E}\big(\vec{V}^*_{i-j,L} \otimes \vec{V}^*_{i,L}\big)\Big)dt\,.$$

Next

$$\mathbb{E}\Big(D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-2j} + \vec{\mathbf{T}}_{i+1} + t(\vec{\mathbf{S}}_{i-1} - \vec{\mathbf{S}}_{i-2j})\big).\vec{V}^*_{m,L} \otimes \mathbb{E}\big(\vec{V}^*_{i-j,L} \otimes \vec{V}^*_{i,L}\big)\Big)$$
$$= \sum_{K,k_K}\sum_{P,p_P}\sum_{Q,q_Q} \mathbb{E}\Big(\frac{\partial^3 g * \varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}\partial x^{(Q,q_Q)}}\big(\vec{\mathbf{S}}_{i-2j} + \vec{\mathbf{T}}_{i+1} + t(\vec{\mathbf{S}}_{i-1} - \vec{\mathbf{S}}_{i-2j})\big) \times \widetilde{V}^{(K,k_K)}_{m,L}\Big)$$
$$\times\, \mathbb{E}\big(\widetilde{V}^{(P,p_P)}_{i-j,L}\widetilde{V}^{(Q,q_Q)}_{i,L}\big)\,.$$

Using Lemma 7.4 with $U = \widetilde{V}^{(P,p_P)}_{i-j,L}$,

$$V = \sum_{K,k_K}\sum_{Q,q_Q} \widetilde{V}^{(Q,q_Q)}_{i,L}\mathbb{E}\Big(\frac{\partial^3 g * \varphi_{a_L}}{\partial x^{(K,k_K)}\partial x^{(P,p_P)}\partial x^{(Q,q_Q)}}\big(\vec{\mathbf{S}}_{i-2j} + \vec{\mathbf{T}}_{i+1} + t(\vec{\mathbf{S}}_{i-1} - \vec{\mathbf{S}}_{i-2j})\big) \times \widetilde{V}^{(K,k_K)}_{m,L}\Big)\,,$$

$\mathcal{U} = \sigma(Y_\ell, \ell \leq i + 2^L - j)$, $\mathcal{V} = \sigma(Y_{i+2^L})$, $r = 1$ and $s = \infty$, we get that we get that there exists a $\mathcal{U}$-measurable random variable $b_{\mathcal{U}}(i + 2^L)$ such that

$$\Big|\mathbb{E}\Big(D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-2j} + \vec{\mathbf{T}}_{i+1} + t(\vec{\mathbf{S}}_{i-1} - \vec{\mathbf{S}}_{i-2j})\big).\vec{V}^*_{m,L} \otimes \mathbb{E}\big(\vec{V}^*_{i-j,L} \otimes \vec{V}^*_{i,L}\big)\Big)\Big|$$
$$\leq 2\sum_{P\in\mathcal{I}_L^{d+1}}\ \sum_{p_P\in\mathcal{E}_{L,Q}^{d+1}} \times\mathbb{E}\Big(\widetilde{V}^{(P,p_P)}_{i-j,L} b_{\mathcal{U}}(i + 2^L)\Big)\|V\|_\infty\,.$$

Using Lemma 7.3 and the fact that $\sum_{K,k_K}\sum_{Q,q_Q}|\widetilde{V}_{i,L}^{(Q,q_Q)}\widetilde{V}_{m,L}^{(K,k_K)}| \le 4(L+1)^{2(d+1)}$, we derive that

$$\|V\|_\infty \ll a_L^{-2}(L+1)^{2(d+1)} .$$

On the other hand, $\sum_{P\in\mathcal{I}_L^{d+1}}\sum_{p_P\in\mathcal{E}_{L,Q}^{d+1}}|\widetilde{V}_{i-j,L}^{(P,p_P)}| \le 2(L+1)^{d+1}$ and $\mathbb{E}\big(b_{\mathcal{U}}(i+2^L)\big) \le \beta_j$. Therefore, for any $t\in[0,1]$,

$$\Big|\mathbb{E}\Big(D^3 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-2j}+\vec{\mathbf{T}}_{i+1}+t(\vec{\mathbf{S}}_{i-1}-\vec{\mathbf{S}}_{i-2j})\big).\vec{V}_{m,L}^*\otimes\mathbb{E}\big(\vec{V}_{i-j,L}^*\otimes\vec{V}_{i,L}^*\big)\Big)\Big| \le a_L^{-2}(L+1)^{3(d+1)}\beta_j .$$

So overall,

$$\sum_{j=1}^{u_L-1}\Big|\mathbb{E}\big(D^2 g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big) - D^2 g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-2j}+\vec{\mathbf{T}}_{i+1}\big)\big).\,\mathbb{E}\big(\vec{V}_{i-j,L}^*\otimes\vec{V}_{i,L}^*\big)\Big| \\ \ll a_L^{-2}(L+1)^{3(d+1)}\sum_{j=1}^{u_L} j\beta_j . \quad (6.74)$$

On the other hand, setting $A_{i,j} = \vec{\mathbf{S}}_{i+j-1}+\vec{\mathbf{T}}_{i+j+1}-\vec{\mathbf{S}}_{i-1}-\vec{\mathbf{T}}_{i+2j+1}$, we write

$$\mathbb{E}\big(D^2 g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i+j-1}+\vec{\mathbf{T}}_{i+j+1}\big) - D^2 g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+2j+1}\big)\big).\,\mathbb{E}\big(\vec{V}_{i+j,L}^*\otimes\vec{V}_{i,L}^*\big) \\ = \sum_{m=i}^{i+j-1}\int_0^1 \mathbb{E}\Big(D^3 g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+2j+1}+tA_{i,j}\big).\big(\vec{V}_{m,L}^*-\vec{N}_{m+j+1,L}^*\big)\otimes\mathbb{E}\big(\vec{V}_{i+j,L}^*\otimes\vec{V}_{i,L}^*\big)\Big)dt .$$

By using the arguments leading to (6.74), we infer that, for any $t\in[0,1]$,

$$\sum_{m=i}^{i+j-1}\Big|\mathbb{E}\Big(D^3 g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+2j+1}+tA_{i,j}\big).\vec{V}_{m,L}^*\otimes\mathbb{E}\big(\vec{V}_{i+j,L}^*\otimes\vec{V}_{i,L}^*\big)\Big)\Big| \\ \ll a_L^{-2}(L+1)^{3(d+1)} j\beta_j .$$

On the other hand, using (6.44) together with (6.14), we derive that

$$\mathbb{E}\Big(D^3 g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+2j+1}+tA_{i,j}\big).\vec{N}_{m+j+1,L}^*\otimes\mathbb{E}\big(\vec{V}_{i+j,L}^*\otimes\vec{V}_{i,L}^*\big)\Big) \\ = (1-t)\sum_{\ell=i+2j+1}^{2^L}\mathbb{E}\big(D^4 g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+2j+1}+tA_{i,j}\big)\big).\mathbb{E}\big(\vec{V}_{m+j+1,L}^*\otimes\vec{V}_{\ell,L}^*\big) \\ \otimes\mathbb{E}\big(\vec{V}_{i+j,L}^*\otimes\vec{V}_{i,L}^*\big) \\ + t\sum_{\ell=i+j+1}^{2^L}\mathbb{E}\big(D^4 g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+2j+1}+tA_{i,j}\big)\big).\mathbb{E}\big(\vec{V}_{m+j+1,L}^*\otimes\vec{V}_{\ell,L}^*\big)\otimes\mathbb{E}\big(\vec{V}_{i+j,L}^*\otimes\vec{V}_{i,L}^*\big) .$$

Hence by Lemma 7.5, it follows that

$$\sum_{m=i}^{i+j-1}\Big|\mathbb{E}\Big(D^3 g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+2j+1}+tA_{i,j}\big).\vec{N}_{m+j+1,L}^*\otimes\mathbb{E}\big(\vec{V}_{i+j,L}^*\otimes\vec{V}_{i,L}^*\big)\Big| \\ \ll a_L^{-3}(L+1)^{4(d+1)} j\beta_j\sum_{\ell\ge 0}\beta_\ell .$$

So overall, since $\sum_{\ell\geq 0}\beta_\ell<\infty$ and $a_L\geq (L+1)^{d+1}$,

$$\sum_{j=1}^{u_L-1}\Big|\mathbb{E}\big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i+j-1}+\vec{\mathbf{T}}_{i+j+1}\big)-D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+2j+1}\big)\big).\,\mathbb{E}\big(\vec{V}^*_{i+j,L}\otimes\vec{V}^*_{i,L}\big)\Big| \ll a_L^{-2}(L+1)^{3(d+1)}\sum_{j=1}^{u_L}j\beta_j\,. \quad (6.75)$$

Starting from (6.73) and considering the upper bounds (6.74) and (6.75) together with the assumption that $\beta_k=O(k^{1-p})$ for $p\in]2,3]$ and the choice of u_L, we derive that

$$\Big|\sum_{i=1}^{2^L}(R_{i,L}-\widetilde{R}_{i,L})\Big|\ll a_L^{1-p}(L+1)^{p(d+1)}2^L+a_L^{-2}(L+1)^{3d+4}2^L\mathbf{1}_{p=3}\,, \quad (6.76)$$

where

$$\widetilde{R}_{i,L}:=\sum_{j=1}^{u_L-1}\mathbb{E}\big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big)\big).\mathbb{E}\big(\vec{V}^*_{i-j,L}\otimes\vec{V}^*_{i,L}\big)\mathbf{1}_{j\leq[i/2]} - \sum_{j=1}^{u_L-1}\mathbb{E}\big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i+j-1}+\vec{\mathbf{T}}_{i+j+1}\big)\big).\mathbb{E}\big(\vec{V}^*_{i+j,L}\otimes\vec{V}^*_{i,L}\big)\mathbf{1}_{j\leq[(2^L-i)/2]}\,. \quad (6.77)$$

Simple algebra together with the Schwarz lemma for cross derivatives entail that

$$\sum_{i=1}^{2^L}\widetilde{R}_{i,L}=\sum_{j=1}^{u_L-1}\sum_{i=2^L-j+2}^{2^L}\mathbb{E}\big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big)\big).\mathbb{E}\big(\vec{V}^*_{i-j,L}\otimes\vec{V}^*_{i,L}\big) - \sum_{j=1}^{u_L-1}\sum_{i=j+1}^{2j-1}\mathbb{E}\big(D^2g*\varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}+\vec{\mathbf{T}}_{i+1}\big)\big).\mathbb{E}\big(\vec{V}^*_{i-j,L}\otimes\vec{V}^*_{i,L}\big)\,.$$

Hence by Lemma 7.5, the assumption that $\beta_k=O(k^{1-p})$ for $p\in]2,3]$ and the choice of u_L, it follows that

$$\Big|\sum_{i=1}^{2^L}\widetilde{R}_{i,L}\Big|\ll a_L^{-1}(L+1)^{2(d+1)}\sum_{j=1}^{u_L}j\beta_j\ll a_L^{2-p}(L+1)^{(5-p)(d+1)}+a_L^{-1}(L+1)^{2d+3}\mathbf{1}_{p=3}\,. \quad (6.78)$$

Starting from (6.17) and considering the upper bounds (6.72), (6.76) and (6.78), the inequality (4.46) follows for $\sup_{g\in\mathrm{Lip}(c^*_{(d+1)L},\mathcal{F}_{2^L})}\mathbb{E}\big(g(\vec{S}^*_{L,d}+\vec{W}^*_{a_L})\big)-\mathbb{E}\big(g(\vec{N}^*_{L,d}+\vec{W}^*_{a_L})\big)$. This ends the proof of the lemma. □

6.2 Proof of Lemma 5.1

According to (5.4) and (5.5), recall that

$$\mathbb{E}\big(c_{(d+1)L}(\vec{S}_{L,d}+\vec{G}_{a_L},\vec{T}_{L,d}+\vec{G}'_{a_L})\big)=W_{c_{(d+1)L}}(P_{\vec{S}_{L,d}}*P_{\vec{G}_a},\mathcal{N}_{C_{L,d}}*P_{\vec{G}_a})\,.$$

As in the proof of Lemma 4.2 we shall use the Lindeberg method. With this aim, we consider a sequence of independent centered Gaussian vectors $\big(\vec{N}_{i,L}\big)_{1\leq i\leq 2^L}$ of $\mathbb{R}^{2^{(d+1)L}}$, independent of $\mathcal{F}_\infty\vee\sigma(\eta_i,i\in\mathbb{Z})$ such that

$$\mathbb{E}(\vec{N}_{k,L}\vec{N}^t_{k,L})=\vec{e}_{k,L}\vec{e}^{\,t}_{k,L}\otimes\mathbb{E}(\vec{U}^{(0)}_{1,L}(\vec{U}^{(0)}_{1,L})^t)=\mathbb{E}(\vec{V}_{k,L}\vec{V}^t_{k,L})\,.$$

Defining

$$\vec{N}_{L,d} = \vec{N}_{1,L} + \vec{N}_{2,L} + \cdots + \vec{N}_{2^L,L} \,,$$

we notice that $\mathcal{L}(\vec{T}_{L,d}) = \mathcal{L}(\vec{N}_{L,d})$. Let now $\vec{W}^*_{a_L}$ be a random vector in $\mathbb{R}^{2^{(d+1)L}}$ with law $\mathcal{N}(0, a_L^2 \mathrm{I}_{2^{(d+1)L}})$ independent of $\mathcal{F}_\infty \vee \sigma(\vec{N}_{i,L}, 1 \le i \le 2^L) \vee \sigma(\eta_i, i \in \mathbb{Z})$. Let $\vec{W}_{a_L} = \mathbf{P}_L^{\otimes(d+1)} \vec{W}^*_{a_L}$ where $\mathbf{P}_L$ is defined in (4.20). With these notations, we can write that

$$W_{c_{(d+1)L}}(P_{\vec{S}_{L,d}} * P_{\vec{G}_a}, \mathcal{N}_{C_{L,d}} * P_{\vec{G}_a}) = \sup_{f \in \mathrm{Lip}(c_{(d+1)L})} \Big(\mathbb{E}\big(f(\vec{S}_{L,d} + \vec{W}_{a_L})\big) - \mathbb{E}(f(\vec{N}_{L,d} + \vec{W}_{a_L}))\Big) \,.$$

Using the notations (6.7) and setting

$$f^* = f \circ \mathbf{P}_L^{\otimes(d+1)} \,, \tag{6.79}$$

we get overall that

$$\mathbb{E}\big(c_{(d+1)L}(\vec{S}_{L,d} + \vec{G}_{a_L}, \vec{T}_{L,d} + \vec{G}'_{a_L})\big) = \sup_{f \in \mathrm{Lip}(c_{(d+1)L})} \Big(\mathbb{E}\big(f^*(\vec{S}^*_{L,d} + \vec{W}^*_{a_L})\big) - \mathbb{E}(f^*(\vec{N}^*_{L,d} + \vec{W}^*_{a_L}))\Big) \,.$$

Using Notation 6.2, we then write that

$$\begin{aligned}
&\mathbb{E}\big(c_{(d+1)L}(\vec{S}_{L,d} + \vec{G}_{a_L}, \vec{T}_{L,d} + \vec{G}'_{a_L})\big) \\
&= \sup_{f \in \mathrm{Lip}(c_{(d+1)L})} \sum_{i=1}^{2^L} \mathbb{E}\Big(f^* * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{V}^*_{i,L} + \vec{\mathbf{T}}_{i+1}\big) - f^* * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{N}^*_{i,L} + \vec{\mathbf{T}}_{i+1}\big)\Big) \\
&\qquad \le \sup_{f \in \mathrm{Lip}(c_{(d+1)L})} \sum_{i=1}^{2^L} \mathbb{E}\Big(f^* * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{V}^*_{i,L}\big) - f^* * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{N}^*_{i,L}\big)\Big) \,.
\end{aligned}$$

Recall Definition 6.2 and for any $i \in \{1, \dots, 2^L\}$, let

$$\Delta_{1,i,L}(f) = f^* * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{V}^*_{i,L}\big) - f^* * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}\big) - \frac{1}{2} D^2 f^* * \varphi_a\big(\vec{\mathbf{S}}_{i-1}\big).\vec{V}^{*\otimes 2}_{i,L} \,,$$

and

$$\Delta_{2,i,L}(f) = f^* * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + \vec{N}^*_{i,L}\big) - f^* * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1}\big) - \frac{1}{2} D^2 f^* * \varphi_a\big(\vec{\mathbf{S}}_{i-1}\big).\vec{N}^{*\otimes 2}_{i,L} \,.$$

With these notations, it follows that

$$\mathbb{E}\big(c_{(d+1)L}(\vec{S}_{L,d} + \vec{G}_{a_L}, \vec{T}_{L,d} + \vec{G}'_{a_L})\big) \le \sup_{f \in \mathrm{Lip}(c_{(d+1)L})} \sum_{i=1}^{2^L} \big(\mathbb{E}(\Delta_{1,i,L}(f)) - \mathbb{E}(\Delta_{2,i,L}(f))\big) \,. \tag{6.80}$$

By using Taylor's integral formula, independence and noticing that $\mathbb{E}(\vec{N}^{*\otimes 3}_{i,L}) = 0$, we get

$$\begin{aligned}
\mathbb{E}(\Delta_{1,i,L}(f)) - \mathbb{E}(\Delta_{2,i,L}(f))) &= \mathbb{E} \int_0^1 \frac{(1-t)^2}{2} D^3 f^* * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + t\vec{V}^*_{i,L}\big).\vec{V}^{*\otimes 3}_{i,L} \\
&\quad + \mathbb{E} \int_0^1 \frac{(1-t)^3}{6} D^4 f^* * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + t\vec{N}^*_{i,L}\big).\vec{N}^{*\otimes 4}_{i,L} \,.
\end{aligned} \tag{6.81}$$

Notice first that by the properties of the convolution product,

$$\begin{aligned}
\big|\mathbb{E} D^3 f^* * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + t\vec{V}^*_{i,L}\big).\vec{V}^{*\otimes 3}_{i,L}\big| &= \big|\mathbb{E}\big((Df^*(\cdot).\vec{V}^*_{i,L} * D^2\varphi_{a_L}(\cdot).\vec{V}^{*\otimes 2}_{i,L})(\vec{\mathbf{S}}_{i-1} + t\vec{V}^*_{i,L})\big)\big| \\
&\le \mathbb{E}\Big(\sup_z \big|Df^*(z).\vec{V}^*_{i,L}\big| \int_{\mathbb{R}^{2^{(d+1)L}}} D^2\varphi_{a_L}(z).\vec{V}^{*\otimes 2}_{i,L} dz\Big) \,.
\end{aligned}$$

But

$$Df^*(z).\vec{V}^*_{i,L} = Df\big(\mathbf{P}_L^{\otimes(d+1)}(z)\big).\mathbf{P}_L^{\otimes(d+1)}\vec{V}^*_{i,L} \leq \sup_{u\in\mathbb{R}^{2^{(d+1)L}}} \big|Df(u).\vec{V}_{i,L}\big| \leq c_{(d+1)L}(\vec{0},\vec{V}_{i,L}) \leq 1\,.$$

In addition, according to Lemma 5.4 in Dedecker, Merlevède and Rio (2013), there exists a constant c not depending on d nor on L such that

$$\int_{\mathbb{R}^{2^{(d+1)L}}} D^2\varphi_{a_L}(z).\vec{V}^{*\otimes 2}_{i,L}dz \leq c\,a_L^{-2}\|\vec{V}^*_{i,L}\|^2_{2,d,L}$$

where

$$\|\vec{V}^*_{i,L}\|^2_{2,d,L} = \sum_{K\in\{0,\dots,L\}^{(d+1)}} \sum_{k_K\in\mathcal{E}^{(d+1)}_{L,K}} (\widetilde{V}^{(K,k_K)}_{i,L})^2 \leq 2(L+1)^{d+1}\,.$$

So overall,

$$\big|\mathbb{E}\big(D^3 f^* * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + t\vec{V}^*_{i,L}\big).\vec{V}^{*\otimes 3}_{i,L}\big)\big| \ll a_L^{-2}(L+1)^{d+1}2^L\,. \qquad (6.82)$$

We deal now with the second term in the right hand side of (6.81). With this aim, we notice that (6.79) together with (6.6) imply that if $f\in\mathrm{Lip}(c_{(d+1)L})$ then $f^*\in\mathrm{Lip}(c^*_{(d+1)L})$ where $c^*_{(d+1)L}$ is defined in Definition 6.1. Therefore

$$\sup_{f\in\mathrm{Lip}(c_{(d+1)L})}\big|\mathbb{E}\big(D^4 f^* * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + t\vec{N}^*_{i,L}\big).\vec{N}^{*\otimes 4}_{i,L}\big)\big| \\ \leq \sup_{g\in\mathrm{Lip}(c^*_{(d+1)L})}\big|\mathbb{E}\big(D^4 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + t\vec{N}^*_{i,L}\big).\vec{N}^{*\otimes 4}_{i,L}\big)\big|\,.$$

Applying Lemma 7.2 as we did to get (6.43), we infer that, for any $i\in\{1,\dots,2^L\}$ and any $g\in\mathrm{Lip}(c^*_{(d+1)L})$,

$$\big|\mathbb{E}\big(D^4 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + t\vec{N}^*_{i,L}\big).\vec{N}^{*\otimes 4}_{i,L}\big)\big| \\ \ll a_L^{-3}(L+1)^{(d+1)/2}\Big(\sum_{K\in\mathcal{I}_L^{d+1}}\sum_{k_K\in\mathcal{E}^{(d+1)}_{L,K}}\mathbb{E}\big((\widetilde{N}^{(K,k_K)}_{i,L})^2\big)\Big)^2\,. \qquad (6.83)$$

Since $\mathbb{E}(\vec{N}_{i,L}\vec{N}^{\,t}_{i,L}) = \mathbb{E}(\vec{V}_{i,L}\vec{V}^{\,t}_{i,L})$, we get that

$$\sum_{K\in\mathcal{I}_L^{d+1}}\sum_{k_K\in\mathcal{E}^{(d+1)}_{L,K}}\mathbb{E}((\widetilde{N}^{(K,k_K)}_{i,L})^2) = \sum_{K\in\mathcal{I}_L^{d+1}}\sum_{k_K\in\mathcal{E}^{(d+1)}_{L,K}}\mathbb{E}((\widetilde{V}^{(K,k_K)}_{i,L})^2) \leq 2(L+1)^{d+1}\,.$$

Therefore, starting from (6.83), we derive that for any $i\in\{1,\dots,2^L\}$, any $g\in\mathrm{Lip}(c^*_{(d+1)L})$ and any $t\in[0,1]$,

$$\big|\mathbb{E}\big(D^4 g * \varphi_{a_L}\big(\vec{\mathbf{S}}_{i-1} + t\vec{N}^*_{i,L}\big).\vec{N}^{*\otimes 4}_{i,L}\big)\big| \ll a_L^{-3}(L+1)^{(d+1)/2} \ll a_L^{-3}(L+1)^{3(d+1)/2}\,. \qquad (6.84)$$

Starting from (6.80) and considering (6.81) together with the upper bounds (6.82) and (6.84), the lemma follows. □

7 Appendix B

This section is devoted to various technical lemmas.

7.1 Upper bounds for the partial derivatives

We gather now some lemmas concerning the upper bounds for partial derivatives. Their proofs are omitted since they are based on the same arguments as those used in Appendix A in Dedecker, Merlevède and Rio (2013).
In what follows d and L are nonnegative integers and $K=(K_0,\dots,K_d)\in\{0,\dots,L\}^{(d+1)}$. We shall denote for any $i=0,\dots,d$,

$$\mathcal{E}(L,K_i)=\{1,\dots,2^{L-K_i}\}\cap(2\mathbb{N}+1)\,,$$

and

$$\mathcal{E}^{(d+1)}_{L,K}=\prod_{i=0}^{d}\mathcal{E}(L,K_i)\,.$$

Therefore the notation $k_K\in\mathcal{E}^{(d+1)}_{L,K}$ means that $k_K=(k_{K_0},\dots,k_{K_d})\in\prod_{i=0}^{d}\mathcal{E}(L,K_i)$.

Let x and y be two column vectors of $\mathbb{R}^{(d+1)L}$ with coordinates

$$x=\Big(\big(x^{(K,k_K)},k_K\in\mathcal{E}^{(d+1)}_{L,K}\big)_{K\in\{0,\dots,L\}^{(d+1)}}\Big)^t \quad\text{and}\quad y=\Big(\big(y^{(K,k_K)},k_K\in\mathcal{E}^{(d+1)}_{L,K}\big)_{K\in\{0,\dots,L\}^{(d+1)}}\Big)^t\,.$$

Let f be a function from $\mathbb{R}^{2^{(d+1)L}}$ into $\mathbb{R}$ that is Lipschitz with respect to the distance $c^*_{(d+1)L}$ defined in Definition 6.1. This means that

$$|f(x)-f(y)|\le\sum_{K\in\{0,\dots,L\}^{(d+1)}}\ \sup_{k_K\in\mathcal{E}^{(d+1)}_{L,K}}|x^{(K,k_K)}-y^{(K,k_K)}|\,.$$

Let $a>0$ and φ_a be the density of a centered Gaussian law of $\mathbb{R}^{(d+1)L}$ with covariance $a^2\mathrm{I}_{2^{(d+1)L}}$ ($\mathrm{I}_{2^{(d+1)L}}$ being the identity matrix on $\mathbb{R}^{2^{(d+1)L}}$). Let also

$$\|x\|_{\infty,d,L}=\sum_{K\in\{0,\dots,L\}^{(d+1)}}\ \sup_{k_K\in\mathcal{E}^{(d+1)}_{L,K}}|x^{(K,k_K)}| \quad\text{and}\quad \|x\|_{2,d,L}=\Big(\sum_{K\in\{0,\dots,L\}^{(d+1)}}\ \sum_{k_K\in\mathcal{E}^{(d+1)}_{L,K}}(x^{(K,k_K)})^2\Big)^{1/2}\,.$$

For the statements of the next lemmas, we refer to Definition 6.2.

Lemma 7.1. *The partial derivatives of f exist almost everywhere and the following inequality holds:*

$$\sup_{y\in\mathbb{R}^{2^{(d+1)L}}}\ \sup_{u\in\mathbb{R}^{2^{(d+1)L}},\,\|u\|_{\infty,d,L}\le1}\big|Df(y).u\big|\le1\,.$$

In addition

$$\sup_{K\in\{0,\dots,L\}^{(d+1)}}\ \sum_{k_K\in\mathcal{E}^{(d+1)}_{L,K}}\Big|\frac{\partial f}{\partial x^{(K,k_K)}}(y)\Big|\le1\,. \tag{7.1}$$

Lemma 7.2. *Let X and Y be two random variables with values in $\mathbb{R}^{2^{(d+1)L}}$. For any positive integer m and any $t\in[0,1]$, there exists a positive constant γ_m depending only on m such that*

$$\Big|\mathbb{E}\big(D^m f*\varphi_a(Y+tX).X^{\otimes m}\big)\Big|\le\gamma_m a^{1-m}\mathbb{E}\Big(\|X\|_{\infty,d,L}\times\|X\|^{m-1}_{2,d,L}\Big)\,.$$

Lemma 7.3. *For any integer $m \geq 1$, there exists a positive constant κ_m depending only on m such that*

$$\sup_{(K(i),k_{K(i)}),i=1,\dots,m} \Big\| \frac{\partial^m \varphi_a}{\prod_{i=1}^m \partial x^{(K_i,k_{K_i})}} \Big\|_1 \leq \kappa_m a^{-m} \,.$$

In addition, for any integer $m \geq 1$ and any $y \in \mathbb{R}^{2^{(d+1)L}}$,

$$\sup_{(K(i),k_{K(i)}),i=1,\dots,m} \Big| \frac{\partial^m f * \varphi_a}{\prod_{i=1}^m \partial x^{(K(i),k_{K(i)})}}(y) \Big| \leq \kappa_{m-1} a^{1-m} \,.$$

The supremum above are taken over all the indexes $K(i) \in \{0,\dots,L\}^{(d+1)}$ and $k_{K(i)} \in \mathcal{E}^{(d+1)}_{L,K(i)}$ for any $i = 1,\dots,m$.

7.2 Covariance inequalities

We first recall the following covariance inequality due to Delyon (1990) (see also Theorem 1.4 in Rio (2000)).

Lemma 7.4. *Let r and s in $[1,\infty]$ such that $r^{-1}+s^{-1}=1$. Let U and V be real random variables respectively in $\mathbb{L}^r$ and $\mathbb{L}^s$, that are respectively $\mathcal{U}$ and $\mathcal{V}$ measurable. Then there exist two random variables $b_{\mathcal{U}}$ and $b_{\mathcal{V}}$ with values in $[0,1]$, measurable respectively with respect to $\mathcal{U}$ and to $\mathcal{V}$, such that $\mathbb{E}(b_{\mathcal{U}}) = \mathbb{E}(b_{\mathcal{V}}) = \beta(\mathcal{U},\mathcal{V})$ and*

$$|\mathrm{Cov}(U,V)| \leq 2\Big(\mathbb{E}\big(|U|^r b_{\mathcal{U}}\big)\Big)^{1/r}\Big(\mathbb{E}\big(|V|^s b_{\mathcal{V}}\big)\Big)^{1/s} \,.$$

Notice that if $U = f(X,\eta)$ and $V = g(Y,\delta)$ where X,η,Y,δ are random variables such that (X,Y) is independent of (η,δ) and η is independent of δ, then the random variables $b_{\mathcal{U}}$ and $b_{\mathcal{V}}$ satisfy $\mathbb{E}(b_{\mathcal{U}}) = \mathbb{E}(b_{\mathcal{V}}) = \beta(\sigma(X),\sigma(Y))$.

For the next lemma, we refer to Definition 6.2.

Lemma 7.5. *Let $\vec{\mathbf{Z}}$ be a random variable with values in $\mathbb{R}^{2^{(d+1)L}}$. Let $(\vec{V}^*_{i,L})_i$ be the random variables in $\mathbb{R}^{2^{(d+1)L}}$ defined in* (6.11) *and $(\beta_\ell)_{\ell\geq 0}$ the sequence of absolutely regular coefficients associated to the strictly stationary sequence $(X_i)_{i\in\mathbb{Z}}$. Let m be a positive integer, $(k_i)_{i\geq 1}$ and $(\ell_i)_{i\geq 1}$ two sequences of integers. Then, there exists a positive constant γ_m depending only on m such that*

$$\Big|\mathbb{E}\Big(D^{2m} g * \varphi_a(\vec{\mathbf{Z}})\Big)\boldsymbol{.}\bigotimes_{i=1}^m \mathbb{E}\big(\vec{V}^*_{k_i,L}\otimes \vec{V}^*_{\ell_i,L}\big)\Big| \leq \gamma_m a^{1-2m}(L+1)^{2m(d+1)}\prod_{i=1}^m \beta_{|k_i-\ell_i|}\,.$$

Proof of Lemma 7.5. We use the notation (6.12) and write

$$\begin{aligned}
&\mathbb{E}\Big(D^{2m} g * \varphi_a(\vec{\mathbf{Z}})\Big)\boldsymbol{.}\bigotimes_{i=1}^m \mathbb{E}\big(\vec{V}^*_{k_i,L}\otimes \vec{V}^*_{\ell_i,L}\big)\\
&= \sum_{P,p_P}\sum_{Q,q_Q}\mathbb{E}\Big(\widetilde{V}^{(P,p_P)}_{k_1,L}\widetilde{V}^{(Q,q_Q)}_{\ell_1,L}\mathbb{E}\Big(\frac{\partial^2}{\partial x^{(P,p_P)}\partial x^{(Q,q_Q)}}D^{2m-2} g * \varphi_a(\vec{\mathbf{Z}})\boldsymbol{.}\bigotimes_{i=2}^m \mathbb{E}\big(\vec{V}^*_{k_i,L}\otimes \vec{V}^*_{\ell_i,L}\big)\Big)\Big)\,.
\end{aligned}$$

We apply now Lemma 7.4 with $U = \widetilde{V}^{(P,p_P)}_{k_1,L}$, $\mathcal{U} = \sigma(Y_{k_1+2^L})$,

$$V = \sum_{Q,q_Q}\widetilde{V}^{(Q,q_Q)}_{\ell_1,L}\mathbb{E}\Big(\frac{\partial^2}{\partial x^{(P,p_P)}\partial x^{(Q,q_Q)}}D^{2m-2} g * \varphi_a(\vec{\mathbf{Z}})\boldsymbol{.}\bigotimes_{i=2}^m \mathbb{E}\big(\vec{V}^*_{k_i,L}\otimes \vec{V}^*_{\ell_i,L}\big)\Big)\,,$$

$\mathcal{V} = \sigma(Y_{\ell_1+2^L})$, $r = 1$ and $s = \infty$. Hence, we derive that there exists a $\mathcal{U}$-measurable random variable $b_{\mathcal{U}}(\ell_1 + 2^L)$ such that

$$\Big|\mathbb{E}\Big(D^{2m} g * \varphi_a(\vec{\mathbf{Z}})\Big). \bigotimes_{i=1}^{m} \mathbb{E}\big(\vec{V}^*_{k_i,L} \otimes \vec{V}^*_{\ell_i,L}\big)\Big| \\ \leq 2 \sum_{P,p_P} \mathbb{E}\Big(\big|\widetilde{V}^{(P,p_P)}_{k_1,L}\big| b_{\mathcal{U}}(\ell_1 + 2^L)\Big) \times \Big\| \sum_{Q,q_Q} \big|\widetilde{V}^{(Q,q_Q)}_{\ell_1,L}\big|\Big\|_\infty \\ \times \sup_{(P,p_P),(Q,q_Q)} \Big|\mathbb{E}\Big(\frac{\partial^2}{\partial x^{(P,p_P)} \partial x^{(Q,q_Q)}} D^{2m-2} g * \varphi_a(\vec{\mathbf{Z}}). \bigotimes_{i=2}^{m} \mathbb{E}\big(\vec{V}^*_{k_i,L} \otimes \vec{V}^*_{\ell_i,L}\big)\Big)\Big| .$$

Since $\mathbb{E}(b_{\mathcal{U}}(\ell_1 + 2^L)) \leq \beta_{|k_1-\ell_1|}$ and $\sum_{Q,q_Q} \big|\widetilde{V}^{(Q,q_Q)}_{\ell_1,L}\big| \leq 2(L+1)^{d+1}$, we get

$$\Big|\mathbb{E}\Big(D^{2m} g * \varphi_a(\vec{\mathbf{Z}})\Big). \bigotimes_{i=1}^{m} \mathbb{E}\big(\vec{V}^*_{k_i,L} \otimes \vec{V}^*_{\ell_i,L}\big)\Big| \leq 8(L+1)^{2(d+1)} \beta_{|k_1-\ell_1|} \\ \times \sup_{(P,p_P),(Q,q_Q)} \Big|\mathbb{E}\Big(\frac{\partial^2}{\partial x^{(P,p_P)} \partial x^{(Q,q_Q)}} \Big(D^{2m-2} g * \varphi_a(\vec{\mathbf{Z}}). \bigotimes_{i=2}^{m} \vec{V}^*_{k_i,L} \otimes \vec{V}^*_{\ell_i,L}\Big)\Big)\Big| .$$

Next, if $m \geq 2$, we write that

$$\mathbb{E}\Big(\frac{\partial^2}{\partial x^{(P,p_P)} \partial x^{(Q,q_Q)}} D^{2m-2} g * \varphi_a(\vec{\mathbf{Z}}). \bigotimes_{i=2}^{m} \mathbb{E}\big(\vec{V}^*_{k_i,L} \otimes \vec{V}^*_{\ell_i,L}\big)\Big) \\ = \sum_{M,m_M} \sum_{R,r_R} \mathbb{E}\Big(\widetilde{V}^{(M,m_M)}_{k_2,L} \widetilde{V}^{(R,r_R)}_{\ell_2,L} \\ \times \mathbb{E}\Big(\frac{\partial^4}{\partial x^{(P,p_P)} \partial x^{(Q,q_Q)} \partial x^{(M,m_M)} \partial x^{(R,r_R)}} D^{2m-4} g * \varphi_a(\vec{\mathbf{Z}}). \bigotimes_{i=3}^{m} \mathbb{E}\big(\vec{V}^*_{k_i,L} \otimes \vec{V}^*_{\ell_i,L}\big)\Big)\Big) .$$

Applying Lemma 7.4 with $U = \widetilde{V}^{(M,m_M)}_{k_2,L}$, $\mathcal{U} = \sigma(Y_{k_2+2^L})$,

$$V = \\ \sum_{R,r_R} \widetilde{V}^{(R,r_R)}_{\ell_2,L} \mathbb{E}\Big(\frac{\partial^4}{\partial x^{(P,p_P)} \partial x^{(Q,q_Q)} \partial x^{(M,m_M)} \partial x^{(R,r_R)}} D^{2m-4} g * \varphi_a(\vec{\mathbf{Z}}). \bigotimes_{i=3}^{m} \mathbb{E}\big(\vec{V}^*_{k_i,L} \otimes \vec{V}^*_{\ell_i,L}\big)\Big) ,$$

$\mathcal{V} = \sigma(Y_{\ell_2+2^L})$, $r = 1$ and $s = \infty$, we get that there exists a $\mathcal{U}$-measurable random variable $b_{\mathcal{U}}(\ell_2 + 2^L)$ such that

$$\sup_{(P,p_P),(Q,q_Q)} \Big|\mathbb{E}\Big(\frac{\partial^2}{\partial x^{(P,p_P)} \partial x^{(Q,q_Q)}} D^{2m-2} g * \varphi_a(\vec{\mathbf{Z}}). \bigotimes_{i=2}^{m} \mathbb{E}\big(\vec{V}^*_{k_i,L} \otimes \vec{V}^*_{\ell_i,L}\big)\Big)\Big| \\ \leq 2 \sum_{M,m_M} \mathbb{E}\Big(\big|\widetilde{V}^{(M,m_M)}_{k_2,L}\big| b_{\mathcal{U}}(\ell_2 + 2^L)\Big) \times \Big\| \sum_{R,r_R} \big|\widetilde{V}^{(R,r_R)}_{\ell_2,L}\big|\Big\|_\infty \\ \times \sup_{\substack{(P,p_P)\\(Q,q_Q)}} \sup_{\substack{(M,m_M)\\(R,r_R)}} \Big|\mathbb{E}\Big(\frac{\partial^4}{\partial x^{(P,p_P)} \partial x^{(Q,q_Q)} \partial x^{(M,m_M)} \partial x^{(R,r_R)}} D^{2m-4} g * \varphi_a(\vec{\mathbf{Z}}). \bigotimes_{i=3}^{m} \mathbb{E}\big(\vec{V}^*_{k_i,L} \otimes \vec{V}^*_{\ell_i,L}\big)\Big)\Big| .$$

Since $\mathbb{E}(b_{\mathcal{U}}(\ell_2+2^L)) \le \beta_{|k_2-\ell_2|}$ and $\sum_{Q,q_Q} \big|\widetilde{V}^{(Q,q_Q)}_{\ell_2,L}\big| \le 2(L+1)^{d+1}$, we get

$$\sup_{(P,p_P),(Q,q_Q)} \Big|\mathbb{E}\Big(\frac{\partial^2}{\partial x^{(P,p_P)}\partial x^{(Q,q_Q)}} D^{2m-2} g * \varphi_a(\vec{\mathbf{Z}}).\bigotimes_{i=2}^m \mathbb{E}\big(\vec{V}^*_{k_i,L}\otimes\vec{V}^*_{\ell_i,L}\big)\Big)\Big|$$
$$\le 8(L+1)^{2(d+1)}\beta_{|k_2-\ell_2|}$$
$$\times \sup_{\substack{(P,p_P)\\(Q,q_Q)}} \sup_{\substack{(M,m_M)\\(R,r_R)}} \Big|\mathbb{E}\Big(\frac{\partial^4}{\partial x^{(P,p_P)}\partial x^{(Q,q_Q)}\partial x^{(M,m_M)}\partial x^{(R,r_R)}} D^{2m-4} g*\varphi_a(\vec{\mathbf{Z}}).\bigotimes_{i=3}^m \big(\vec{V}^*_{k_i,L}\otimes\vec{V}^*_{\ell_i,L}\big)\Big)\Big|.$$

The lemma follows after $m-2$ additional steps by using Lemma 7.3 at the end of the procedure. □

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