Submitted to Bernoulli

Supplement to "Functional CLT for martingale-like nonstationary dependent structures"

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This supplementary file contains a detailed proof of Corollary 4.6 in [1].

1. Proof of Corollary 4.6

In all the proof, we recall that the chain starts from the origin. To soothe the notation, we will often write $\mathbb{P}$ for $\mathbb{P}_{\phi_0=0}$ when no confusion is possible.

We first note that c^2 is well defined. Indeed, the series $\sum_{m\geq 1}\mathbb{E}(\zeta_0\zeta_m)\sum_{j=1}^m (P^j)_{0,m-j}$ is absolutely convergent. To see this, we note that since π is the stationary distribution, we have that for any non-negative integers m and i, $\pi_m=\sum_{j\geq 0}\pi_j(P^i)_{j,m}$. Therefore $(P^j)_{0,m-j}\leq \pi_0^{-1}\pi_{m-j}$ which entails that

$$\sum_{m\geq 1}\left|\mathbb{E}(\zeta_0\zeta_m)\right|\sum_{j=1}^m (P^j)_{0,m-j}\leq \sum_{m\geq 1}\left|\mathbb{E}(\zeta_0\zeta_m)\right|\sum_{j\geq 1} jp_j=\mathbb{E}(\tau)\sum_{m\geq 1}\left|\mathbb{E}(\zeta_0\zeta_m)\right|.$$

So the absolute convergence of the series $\sum_{m\geq 1}\mathbb{E}(\zeta_0\zeta_m)\sum_{j=1}^m (P^j)_{0,m-j}$ follows from the fact that $\sum_{m\geq 1}\left|\mathbb{E}(\zeta_0\zeta_m)\right|<\infty$, which holds under the first part of condition (4.7) of [1]. Indeed, the first part of condition (4.7) of [1] implies that $\sum_{k\geq 0}\|\mathbb{E}(\zeta_k|\mathcal{G}_0)-\mathbb{E}(\zeta_k|\mathcal{G}_{-1})\|_2<\infty$ (see Corollary 2 in [2]), which in turn implies $\sum_{m\geq 1}\left|\mathbb{E}(\zeta_0\zeta_m)\right|<\infty$.

The invariance principle will be proved by an application of Theorem 2.1 together with Proposition 3.2 of [1]. In order to apply it we need to introduce a non-decreasing filtration $(\mathcal{F}_k)_{k\geq 0}$ for which the process $(X_k)_{k\geq 1}$ is adapted. Let

$$\mathcal{A}=\sigma(Y_k,k\geq 0)$$

and for any $k\in\mathbb{Z}$, we define

$$\mathcal{F}_k=\sigma(\mathcal{A};X_j,1\leq j\leq k).$$

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This filtration encodes all the history of the Markov chain, and all the history up to time k of the sampled time scenery $\zeta_{Y_1}, \zeta_{Y_2}, \ldots, \zeta_{Y_k}$. Note that by the above definition we have: $\mathcal{F}_j = \mathcal{A}$, for all $j \leq 0$.

We first notice that by Condition (A_1) of [1] and the independence between the time scenery and the Markov chain, condition (2.2) of [1] is satisfied. It remains to show that conditions (2.6), (3.6) and (3.7)(with $v_n(t) = [nt]$ and $X_{k,n} = X_k/\sqrt{n}$) of [1] hold. With this aim, we start by introducing some notations.

Let us first define a sequence of stopping times $\nu_1, \nu_2, \ldots$ as the times of return to the origin,

$$\nu_1 \stackrel{\text{def}}{=} \inf\{k > 0 : \phi_k = 0\}, \quad \nu_{n+1} \stackrel{\text{def}}{=} \inf\{k > \nu_n : \phi_k = 0\}, \quad n \geq 1\,.$$

Define also the random times $\tau_1, \tau_2, \ldots$ as the interarrival times to the origin, i.e. the times between successive returns to the origin,

$$\tau_1 = \nu_1, \quad \tau_{n+1} = \nu_{n+1} - \nu_n, \quad n \geq 1\,,$$

and the renewal process

$$N_k = \max\{j \geq 0 : \nu_j \leq k\} + 1\,, \tag{1.1}$$

which basically counts the number of visits to the origin up to time k, including the renewal at 0. Let us start with the two following facts that will be useful to prove that conditions (2.6), (3.6) and (3.7) of [1] are satisfied. Their proof will be given later.

***Fact* 1.1.** Let k and m be in $\mathbb{N}$ and such that $m \leq k$. Then

$$\mathbb{P}(N_{k-m} = N_k) \leq \mathbb{P}(\tau > m)\,.$$

***Fact* 1.2.** For any non-negative integer k, let

$$b(k) = \|\mathbb{E}(\zeta_k|\mathcal{G}_0)\|_2 \text{ and } c(k) = \sup_{j \geq i \geq k} \|\mathbb{E}(\zeta_i\zeta_j|\mathcal{G}_0) - \mathbb{E}(\zeta_i\zeta_j)\|_1\,.$$

Then

$$\sup_{k \geq 0} \|\mathbb{E}(X_{k+m}|\mathcal{F}_k)\|_2^2 \leq b^2([m/2]) + b^2(0)\mathbb{P}(\tau > [m/2])$$

and

$$\sup_{k,\ell \geq 0} \|\mathbb{E}(X_{k+m}X_{k+m+\ell}|\mathcal{F}_k) - \mathbb{E}_{\mathcal{A}}(X_{k+m}X_{k+m+\ell})\|_1 \leq c([m/2]) + c(0)\mathbb{P}(\tau > [m/2])\,.$$

The second inequality of Fact 1.2 together with the second part of condition (4.7) of [1] and the fact that $\mathbb{E}(\tau) < \infty$ implies that condition (3.6) of [1] is satisfied. On another hand, by the first inequality of Fact 1.2 and taking into account the first part of condition (4.7) of [1], it follows that condition (2.6) will be satisfied provided that $\sum_{m\geq 1} m^{-1/2}\sqrt{\mathbb{P}(\tau > [m/2])} < \infty$. This last condition obviously holds since $\mathbb{E}(\tau^2) < \infty$ is equivalent to $\sum_{i\geq 1} i\,\mathbb{P}(\tau > i) < \infty$.

It remains to prove that condition (3.7) of [1] holds. Note that it is satisfied if one can prove that, for any $\varepsilon > 0$ and any $t \in [0,1]$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\Big(\frac{1}{m}\sum_{K=1}^{m-1}\Big|\frac{1}{n}\sum_{k=1}^{[nt]}\Big\{\mathbb{E}_{\mathcal{A}}(X_k^2) + 2\sum_{\ell=1}^{K}\mathbb{E}_{\mathcal{A}}(X_k X_{k+\ell})\Big\} - tc^2\Big| > \varepsilon\Big) = 0\,. \quad (1.2)$$

Note that for any integer ℓ in $[0, K]$ where K is a fixed integer, because of the independence between the time scenery and the Markov chain, by stationarity of the random time scenery,

$$\begin{aligned}\mathbb{E}_{\mathcal{A}}(X_k X_{k+\ell}) &= \sum_{m\geq\ell}\mathbf{1}_{\phi_k=m}\mathbb{E}(\zeta_{k+m}^2) + \sum_{i=1}^{\ell}\sum_{m\geq 0}\mathbf{1}_{\phi_k=\ell-i}\mathbf{1}_{\phi_{k+\ell}=m}\mathbb{E}(\zeta_{k+\ell-i}\zeta_{k+\ell+m})\\ &= \mathbb{E}(\zeta_0^2)\mathbf{1}_{\phi_k\geq\ell} + \sum_{i=1}^{\ell}\sum_{m\geq 0}\mathbf{1}_{\phi_k=\ell-i}\mathbf{1}_{\phi_{k+\ell}=m}\mathbb{E}(\zeta_0\zeta_{m+i})\,.\end{aligned}$$

It follows that

$$\begin{aligned}&\Big|\frac{1}{n}\sum_{k=1}^{[nt]}\big(\mathbb{E}_{\mathcal{A}}(X_k^2)+2\sum_{\ell=1}^{K}\mathbb{E}_{\mathcal{A}}(X_kX_{k+\ell})\big)-c^2t\Big| \leq \sigma^2\Big|\frac{[nt]}{n}-t\Big|+2\mathbb{E}(\zeta_0^2)\Big|\frac{1}{n}\sum_{k=1}^{[nt]}\sum_{\ell=1}^{K}\mathbf{1}_{\phi_k\geq\ell}-t\sum_{i\geq 1} i\pi_i\Big|\\ &+2\Big|\frac{1}{n}\sum_{k=1}^{[nt]}\sum_{\ell=1}^{K}\sum_{i=1}^{\ell}\sum_{r\geq 0}U(k,\ell,i,r)-\sum_{r\geq 1}\mathbb{E}(\zeta_0\zeta_r)\sum_{j=1}^{r}(P^j)_{0,r-j}\Big| =: I_n^{(1)}(t)+I_{n,K}^{(2)}(t)+I_{n,K}^{(3)}(t)\,,\end{aligned} \quad (1.3)$$

where

$$U(k,\ell,i,r) := \mathbf{1}_{\phi_k=\ell-i}\mathbf{1}_{\phi_{k+\ell}=r}\mathbb{E}(\zeta_0\zeta_{r+i})\,. \quad (1.4)$$

Let us first prove that, for any $\varepsilon > 0$ and any $t \in [0,1]$,

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathbb{P}\Big(\frac{1}{m}\sum_{K=1}^{m-1}\big|I_{n,K}^{(2)}(t)\big| \geq \varepsilon\Big) = 0\,. \quad (1.5)$$

With this aim, note that since the Markov chain is irreducible and positive recurrent, by the law of large numbers for Markov chains, for any $t \in [0,1]$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{[nt]}\sum_{\ell\geq 1}\mathbf{1}_{\phi_k\geq\ell} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{[nt]}\phi_k = t\sum_{i\geq 1} i\pi_i \quad \text{a.s.}\,, \quad (1.6)$$

whatever the initial law is. In addition, by Fact 1.1,

$$\frac{1}{n}\sum_{k=1}^{[nt]}\sum_{\ell\geq K+1}\mathbb{E}_{\phi_0=0}(\mathbf{1}_{\phi_k\geq \ell})=\frac{1}{n}\sum_{k=1}^{[nt]}\sum_{\ell\geq K+1}\mathbb{P}_{\phi_0=0}(N_{k-1}=N_{k+\ell-1})\leq \sum_{\ell\geq K+1}\mathbb{P}(\tau>\ell)\,.$$

Hence

$$\frac{1}{m}\sum_{K=1}^{m-1}\frac{1}{n}\sum_{k=1}^{[nt]}\sum_{\ell\geq K+1}\mathbb{E}_{\phi_0=0}(\mathbf{1}_{\phi_k\geq \ell})\leq \frac{1}{m}\sum_{K=1}^{m-1}\sum_{\ell\geq K+1}\mathbb{P}(\tau>\ell)\to 0\,,\text{ as } m\to\infty\,. \tag{1.7}$$

The convergence (1.5) follows from (1.6) and (1.7). To end the proof of (1.2) (and then of condition (3.7) in [1]), it remains to show that, for any $\varepsilon>0$ and any $t\in[0,1]$,

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathbb{P}\Big(\frac{1}{m}\sum_{K=1}^{m-1}\big|I^{(3)}_{n,K}(t)\big|\geq\varepsilon\Big)=0\,. \tag{1.8}$$

Let ℓ be a positive fixed integer, by the ergodic Theorem applied to the irreducible and positive recurrent Markov chain $Z_k=(\phi_k,\phi_{k+\ell})$, we have, for $\ell\geq i$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{[nt]}\mathbf{1}_{\phi_k=\ell-i}\mathbf{1}_{\phi_{k+\ell}=r}=t\pi_{\ell-i}(P^\ell)_{\ell-i,r}\quad\text{a.s.}\ ,$$

whatever the initial law is. But, for any $\ell\geq i$, by homogeneity of the Markov chain,

$$\begin{aligned}(P^\ell)_{\ell-i,r}=\mathbb{P}(\phi_\ell=r|\phi_0=\ell-i)&=\mathbb{P}(\phi_\ell=r,\phi_{\ell-i}=0|\phi_0=\ell-i)\\&=\mathbb{P}(\phi_\ell=r|\phi_{\ell-i}=0)=(P^i)_{0,r}\,.\end{aligned}$$

Hence, for any K and L fixed,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{[nt]}\sum_{\ell=1}^{K}\sum_{i=1}^{\ell}\sum_{r=0}^{L}\mathbf{1}_{\phi_k=\ell-i}\mathbf{1}_{\phi_{k+\ell}=r}\mathbb{E}(\zeta_0\zeta_{r+i})=t\sum_{\ell=1}^{K}\sum_{i=1}^{\ell}\pi_{\ell-i}\sum_{r=0}^{L}(P^i)_{0,r}\mathbb{E}(\zeta_0\zeta_{r+i})\quad\text{a.s.} \tag{1.9}$$

whatever the initial law is. On another hand,

$$\begin{aligned}&\mathbb{E}_{\phi_0=0}\Big|\frac{1}{n}\sum_{k=1}^{[nt]}\sum_{\ell=1}^{K}\sum_{i=1}^{\ell}\sum_{r\geq L+1}\mathbf{1}_{\phi_k=\ell-i}\mathbf{1}_{\phi_{k+\ell}=r}\mathbb{E}(\zeta_0\zeta_{r+i})\Big|\\&\qquad\leq\frac{1}{n}\sum_{k=1}^{n}\sum_{\ell=1}^{K}\sum_{i=1}^{\ell}\sum_{r\geq L+1}|\mathbb{E}(\zeta_0\zeta_{r+i})|\mathbb{P}_{\phi_0=0}(\phi_k=\ell-i,\phi_{k+\ell}=r)\\&\qquad\qquad\leq\frac{1}{n}\sum_{k=1}^{n}\sum_{\ell=1}^{K}\sum_{i=1}^{\ell}\sum_{r\geq L+1}|\mathbb{E}(\zeta_0\zeta_{r+i})|(P^\ell)_{\ell-i,r}(P^k)_{0,\ell-i}\,.\end{aligned}$$

But, since π is the stationary distribution, we have that for any non-negative integers m and i, $\pi_m = \sum_{j\geq 0} \pi_j (P^i)_{j,m}$. Therefore $\pi_0 (P^k)_{0,\ell-i} \leq \pi_{\ell-i}$ which implies that

$$\mathbb{E}_{\phi_0=0}\Big|\frac{1}{n}\sum_{k=1}^{[nt]}\sum_{\ell=1}^{K}\sum_{i=1}^{\ell}\sum_{r\geq L+1}\mathbf{1}_{\phi_k=\ell-i}\mathbf{1}_{\phi_{k+\ell}=r}\mathbb{E}(\zeta_0\zeta_{r+i})\Big| \leq \pi_0^{-1}\sum_{\ell=1}^{K}\sum_{i=1}^{\ell}\sum_{r\geq L+1}|\mathbb{E}(\zeta_0\zeta_{r+i})|\pi_{\ell-i}$$
$$\leq K\pi_0^{-1}\sum_{r\geq L+2}|\mathbb{E}(\zeta_0\zeta_r)|\,.$$

But, as already mentioned, $\sum_{m\geq 0}|\mathbb{E}(\zeta_0\zeta_m)| < \infty$ under the first part of condition (4.7) of [1]. Hence, for any positive integer K fixed,

$$\lim_{L\to\infty}\limsup_{n\to\infty}\mathbb{E}_{\phi_0=0}\Big|\frac{1}{n}\sum_{k=1}^{[nt]}\sum_{\ell=1}^{K}\sum_{i=1}^{\ell}\sum_{r\geq L+1}\mathbf{1}_{\phi_k=\ell-i}\mathbf{1}_{\phi_{k+\ell}=r}\mathbb{E}(\zeta_0\zeta_{r+i})\Big| = 0\,. \tag{1.10}$$

Next, we show that, for any positive integer K fixed,

$$\lim_{L\to\infty}\sum_{\ell=1}^{K}\sum_{i=1}^{\ell}\pi_{\ell-i}\sum_{r=L+1}^{\infty}(P^i)_{0,r}\mathbb{E}(\zeta_0\zeta_{r+i}) = 0 \tag{1.11}$$

and

$$\lim_{K\to\infty}\sum_{\ell=K+1}^{\infty}\sum_{i=1}^{\ell}\pi_{\ell-i}\sum_{r\geq 0}(P^i)_{0,r}\mathbb{E}(\zeta_0\zeta_{r+i}) = 0\,. \tag{1.12}$$

Noting that

$$\sum_{\ell=1}^{\infty}\sum_{i=1}^{\ell}\pi_{\ell-i}\sum_{r=0}^{\infty}(P^i)_{0,r}\mathbb{E}(\zeta_0\zeta_{r+i}) = \sum_{i=1}^{\infty}\sum_{r=0}^{\infty}(P^i)_{0,r}\mathbb{E}(\zeta_0\zeta_{r+i}) = \sum_{r\geq 1}\mathbb{E}(\zeta_0\zeta_r)\sum_{j=1}^{r}(P^j)_{0,r-j}\,,$$

and considering (1.9), (1.10), (1.11) and (1.12), we get (1.8) and then the convergence (1.2).

Hence, to end the proof of (1.2), it remains to show (1.11) and (1.12). We start by proving (1.11). We observe that

$$\sum_{\ell=1}^{K}\sum_{i=1}^{\ell}\pi_{\ell-i}\sum_{r=L+1}^{\infty}(P^i)_{0,r}|\mathbb{E}(\zeta_0\zeta_{r+i})| \leq \sum_{i=1}^{K}\sum_{\ell=i}^{K}\pi_{\ell-i}\sum_{r=L+1+i}^{\infty}\mathbb{E}(\zeta_0\zeta_r) \leq K\sum_{r=L+2}^{\infty}|\mathbb{E}(\zeta_0\zeta_r)|\,,$$

which converges to zero as $L\to\infty$ since, as mentioned before, $\sum_{r\geq 0}|\mathbb{E}(\zeta_0\zeta_r)| < \infty$ under the first part of condition (4.7) of [1]. This proves (1.11).

To prove (1.12), we first write the following decomposition

$$\sum_{\ell=K+1}^{\infty}\sum_{i=1}^{\ell}\pi_{\ell-i}\sum_{r=0}^{\infty}(P^i)_{0,r}|\mathbb{E}(\zeta_0\zeta_{r+i})|$$
$$\le \sum_{\ell=K+1}^{\infty}\sum_{i=1}^{[\ell/2]}\pi_{\ell-i}\sum_{r=0}^{\infty}(P^i)_{0,r}|\mathbb{E}(\zeta_0\zeta_{r+i})| + \sum_{\ell=K+1}^{\infty}\sum_{i=[\ell/2]+1}^{\ell}\pi_{\ell-i}\sum_{r=0}^{\infty}(P^i)_{0,r}|\mathbb{E}(\zeta_0\zeta_{r+i})|$$
$$:= I_K + J_K\,.$$

Now, since π is the stationary distribution, $(P^i)_{0,r} \le \pi_0^{-1}\pi_r$ for any positive integer i. Therefore

$$I_K \le \pi_0^{-1}\sum_{\ell=K+1}^{\infty}\sum_{i=1}^{[\ell/2]}\pi_{\ell-i}\sum_{r=0}^{\infty}\pi_r|\mathbb{E}(\zeta_0\zeta_{r+i})|$$
$$\le \pi_0^{-1}\sum_{i=1}^{[(K+1)/2]}\sum_{\ell=K+1}^{\infty}\pi_{\ell-i}\sum_{r=0}^{\infty}\pi_r|\mathbb{E}(\zeta_0\zeta_{r+i})| + \pi_0^{-1}\sum_{i\ge[(K+1)/2]}\sum_{\ell\ge 2i}\pi_{\ell-i}\sum_{r=0}^{\infty}\pi_r|\mathbb{E}(\zeta_0\zeta_{r+i})|$$
$$\le \pi_0^{-1}\sum_{\ell=[(K+1)/2]}^{\infty}\pi_\ell\sum_{r=0}^{\infty}\pi_r\sum_{k\ge1}|\mathbb{E}(\zeta_0\zeta_k)| + \pi_0^{-1}\sum_{\ell\ge[(K+1)/2]}\pi_\ell\sum_{r=0}^{\infty}\pi_r\sum_{j\ge[(K+1)/2]}|\mathbb{E}(\zeta_0\zeta_j)|\,.$$

Hence

$$I_K \le 2\pi_0^{-1}\sum_{\ell\ge[(K+1)/2]}\pi(\ell)\sum_{j\ge1}|\mathbb{E}(\zeta_0\zeta_j)|\,,$$

which converges to zero as $K\to\infty$ using again that $\sum_{r\ge0}|\mathbb{E}(\zeta_0\zeta_r)|<\infty$. On the other hand, since $(P^i)_{0,r}\le\pi_0^{-1}\pi_r$ for any positive integer i,

$$J_K \le \pi_0^{-1}\sum_{i\ge[(K+1)/2]+1}\sum_{\ell=i}^{2i}\pi_{\ell-i}\sum_{r=0}^{\infty}\pi_r|\mathbb{E}(\zeta_0\zeta_{r+i})| \le \pi_0^{-1}\sum_{i\ge[(K+1)/2]+1}\sum_{r=0}^{\infty}\pi_r|\mathbb{E}(\zeta_0\zeta_{r+i})|$$
$$\le \pi_0^{-1}\sum_{j\ge[(K+1)/2]+1}|\mathbb{E}(\zeta_0\zeta_j)|\,,$$

which converges to zero as $K\to\infty$ using once again that $\sum_{r\ge0}|\mathbb{E}(\zeta_0\zeta_r)|<\infty$. This ends the proof of (1.12) and then of (1.2). □

To end the proof of the corollary it remains to prove Facts 1.1 and 1.2.

Proof of Fact 1.1. To prove the fact above, we denote by $\tilde{\nu}$ the last renewal before time

$k-m$ and write

$$\mathbb{P}(N_{k-m}=N_k)=\sum_{r=0}^{k-m}\mathbb{P}(\tilde{\nu}=r,\tau_{\tilde{\nu}+1}>k-r)=\sum_{r=0}^{k-m}\mathbb{P}(\tilde{\nu}=r,\tau_{r+1}>k-r)$$
$$\le\sum_{r=0}^{k-m}\mathbb{P}(\tilde{\nu}=r,\tau_{r+1}>m)\le\mathbb{P}(\tau>m)\,.$$

□

Proof of Fact 1.2. Note first that $Y_j=j+\phi_j=\nu_r$, for $\nu_{r-1}<j\le\nu_r$, and by definition of N_k, we have that $N_k-1=\max\{j\ge1:\nu_j\le k\}$, that is the last time the increasing random walk $\{\nu_j: j\ge0\}$ is below the level k. Thus N_k is the hitting time of the set (k,∞), or the time of first entry in the interval $[k+1,\infty)$ of the random walk $\{\nu_j: j\ge0\}$. Thus $\nu_{N_k-1}<k+1$ and $\nu_{N_k}\ge k+1$. This fact with k replaced by $k-1$ gives

$$\nu_{N_{k-1}-1}<k\le\nu_{N_{k-1}}\,.$$

Since $Y_j=\nu_r$ for $\nu_{r-1}<j\le\nu_r$, we can conclude that $Y_k=\nu_{N_{k-1}}$ for all $k\ge1$. Then once again writing $\mathcal{A}$ for $\sigma(\phi_j,j\in\mathbb{Z})$, $\mathcal{F}_k=\sigma(\mathcal{A},X_j,1\le j\le k)$, and $\mathbb{E}_{\mathcal{A}}(\cdot)$ for $\mathbb{E}(\cdot|\mathcal{A})$, we have

$$\mathbb{E}\big(X_k\big|\mathcal{F}_{k-m}\big)=\mathbb{E}_{\mathcal{A}}\big(\zeta_{Y_k}\big|\zeta_{Y_1},\dots,\zeta_{Y_{k-m}}\big)=\mathbb{E}_{\mathcal{A}}\big(\zeta_{\nu_{N_{k-1}}}\big|\zeta_{\nu_1},\dots,\zeta_{\nu_{N_{k-m-1}}}\big)\,.$$

Now, writing k instead of $k-1$ to simplify notation, we have

$$\Big\|\mathbb{E}_{\mathcal{A}}\big(\zeta_{\nu_{N_k}}\big|\zeta_{\nu_1},\dots,\zeta_{\nu_{N_{k-m}}}\big)\Big\|_2^2\le\Big\|\mathbb{E}_{\mathcal{A}}\big(\zeta_{\nu_{N_k}}\big|\mathcal{G}_{\nu_{N_{k-m}}}\big)\Big\|_2^2\le\mathbb{E}\big[b^2(\nu_{N_k}-\nu_{N_{k-m}})\big]\,.$$

Recall that N_k is the first time the positive random walk (ν_j) exceeds the value k. Thus

$$\nu_{N_k}-\nu_{N_{k-m}}=\nu_{\sigma_k^+}-\nu_{\sigma_{k-m}^+}>k-(\nu_{\sigma_{k-m}^+}-(k-m))-(k-m)=m-(\nu_{\sigma_{k-m}^+}-(k-m))\,,$$

where $\sigma_k^+=\inf\{j\ge0:\nu_j>k\}$, and thus $\nu_{\sigma_{k-m}^+}-(k-m)$ is the overshoot. A quick calculation gives for $l=[m/2]$:

$$\begin{aligned}\mathbb{E}\big[b^2(\nu_{N_k}-\nu_{N_{k-m}})\big]&=\mathbb{E}\big[b^2(\nu_{N_k}-\nu_{N_{k-m}})\mathbf{1}\{N_{k-m}\le N_{k-l}-1\}\big]\\&\quad+\mathbb{E}\big[b^2(\nu_{N_k}-\nu_{N_{k-m}})\mathbf{1}\{N_{k-m}=N_{k-l}\}\big]\\&=:I_1+I_2\,,\end{aligned}$$

where the first event corresponds to there is at least one renewal between times $k-m$ and $k-l$, while the second event corresponds to no renewals. Then, we have

$$\begin{aligned}I_1&=\mathbb{E}\big[b^2\left(\nu_{N_k}-\nu_{N_{k-l}-1}+\nu_{N_{k-l}-1}-\nu_{N_{k-m}}\right)\mathbf{1}\{N_{k-m}\le N_{k-l}-1\}\big]\\&\le b^2([m/2])\mathbb{P}(N_{k-l}>N_{k-m})\le b^2([m/2])\,.\end{aligned}$$

Indeed, by previous considerations, we have that $\nu_{N_k} - \nu_{N_{k-l}-1} > l$, while on the event $\{N_{k-m} \le N_{k-l} - 1\}$, clearly $\nu_{N_{k-l}-1} - \nu_{N_{k-m}} \ge 0$ and thus

$$\nu_{N_k} - \nu_{N_{k-l}-1} + \nu_{N_{k-l}-1} - \nu_{N_{k-m}} \ge l\,.$$

Therefore, since $b(\cdot)$ is non-increasing,

$$b^2\big(\nu_{N_k} - \nu_{N_{k-l}-1} + \nu_{N_{k-l}-1} - \nu_{N_{k-m}}\big) \le b^2(l)\,.$$

For the second term we have $I_2 \le b^2(0)\mathbb{P}(N_{k-m} = N_{k-l})$. So, according to Fact 1.1, we get that for $k \ge m$,

$$I_2 \le b^2(0)\mathbb{P}(\tau \ge m - l) \le b^2(0)\mathbb{P}(\tau \ge [m/2])\,.$$

Taking into account the bounds of I_1 and I_2, the first inequality of Fact 1.2 follows.

To prove the second one, we proceed similarly. Since $Y_k = \zeta_{N_{k-1}}$, for all $k \ge 0$, we have

$$\begin{aligned}&\mathbb{E}(X_{k+m}X_{k+m+\ell}|\mathcal{F}_k) - \mathbb{E}_{\mathcal{A}}(X_{k+m}X_{k+m+\ell})\\ &\qquad = \mathbb{E}_{\mathcal{A}}\big(\zeta_{\nu_{N_{k+m-1}}}\zeta_{\nu_{N_{k+m+\ell-1}}}|\zeta_{\nu_1},\dots,\zeta_{\nu_{N_{k-1}}}\big) - \mathbb{E}_{\mathcal{A}}\big(\zeta_{\nu_{N_{k+m-1}}}\zeta_{\nu_{N_{k+m+\ell-1}}}\big)\,.\end{aligned}$$

Therefore

$$\|\mathbb{E}(X_{k+m}X_{k+m+\ell}|\mathcal{F}_k) - \mathbb{E}_{\mathcal{A}}(X_{k+m}X_{k+m+\ell})\|_1 \le \mathbb{E}\big(c(\nu_{N_{k+m-1}} - \nu_{N_{k-1}})\big)\,,$$

and the second inequality follows from the previous computations with $c(\cdot)$ replacing $b^2(\cdot)$. $\diamond$

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