Almost Sure Invariance Principles via Martingale Approximation

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Abstract

In this paper we estimate the rest of the approximation of a stationary process by a martingale in terms of the projections of partial sums. Then, based on this estimate, we obtain almost sure approximation of partial sums by a martingale with stationary differences. The results are exploited to further investigate the central limit theorem and its invariance principle started at a point, the almost sure central limit theorem, as well as the law of the iterated logarithm via almost sure approximation with a Brownian motion, improving the results available in the literature. The conditions are well suited for a variety of examples; they are easy to verify, for instance, for linear processes and functions of Bernoulli shifts.

Key words: martingale approximation, quenched CLT, almost sure CLT, normal Markov chains, functional CLT, law of the iterated logarithm, almost sure approximation.

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1 Introduction and notations

In recent years there has been an intense effort towards a better understanding of the structure and asymptotic behavior of stochastic processes. For processes with short memory there are two basic techniques: approximation with independent random variables or with martingales. Each of these methods have its own strength. On one hand the classes that can be treated by coupling with an independent sequence exhibit faster rates of convergence in various limit theorems; on the other hand the class of processes that can be treated by a martingale approximation is larger. There are plenty of processes that benefit from approximation with a martingale. Examples are: linear processes with martingale innovations, functions of linear processes, reversible Markov chains, normal Markov chains, various dynamical systems, discrete Fourier transform of general stationary sequences. A martingale approximation provides important information about these structures, because martingales can be embedded into Brownian motion, they satisfy the functional central limit theorem started at a point, the law of the iterated logarithm, and the almost sure central limit theorem. Moreover, martingale approximation provides a simple and unified approach to asymptotic results for many dependence structures. For all these reasons, in recent years martingale approximation, "coupling with a martingale", has gained a prominent role in analyzing dependent data. This is also due to important developments by Liverani (1996), Maxwell and Woodroofe (2000), Derriennic and Lin (2001-a), Wu

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and Woodroofe (2004) and recent developments by Peligrad and Utev (2005), Peligrad, Utev and Wu (2007), Merlevède and Peligrad (2006), Peligrad and Wu (2010) among others. Many of these new results, originally designed for Markov operators, (see Kipnis and Varadhan (1986) and Derriennic and Lin (2007) for a survey) have made their way into limit theorems for stochastic processes.

So far this method has been shown to be well suited to transport from the martingale to the stationary process either the conditional central limit theorem or conditional invariance principle in mean. As a matter of fact, papers by Dedecker-Merlevède-Volný (2006), Volný (2007), Zhao and Woodroofe (2008-a), Gordin and Peligrad (2010), obtain characterizations of stochastic processes that can be approximated by martingales in quadratic mean. These results are useful for treating evolutions in "annealed" media.

In this paper we address the question of almost sure approximation of partial sums by a martingale. These results are useful for obtaining almost sure limit theorems for dependent sequences and also limit theorems started at a point. Limit theorems for stochastic processes that do not start from equilibrium is timely and motivated by evolutions in quenched random environment. Moreover recent discoveries by Ouchti and Volný (2008) and Volný and Woodroofe (2010) show that many of the central limit theorems satisfied by classes of stochastic processes in equilibrium, fail to hold when the processes are started from a point, so, new sharp sufficient conditions should be pointed out for the validity of these types of results. Recent steps in this direction are papers by Wu and Woodroofe (2004), Zhao and Woodroofe (2008-b), Cuny (2009 and 2011), Cuny and Peligrad (2009).

The technical challenge is to estimate the rest of approximation of partial sums by a martingale which leads to almost sure results, ranging from the almost sure central limit theorems, almost sure approximation with a Brownian motion and the law of the iterated logarithm.

We shall develop our results in the framework of stationary processes that can be introduced in several equivalent ways.

We assume that $(\xi_n)_{n\in\mathbb{Z}}$ denotes a stationary Markov chain defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with values in a measurable space $(S,\mathcal{S})$. Let π denote the marginal distribution of ξ_0 and suppose that there is a regular distribution of ξ_1 given ξ_0, say $Q(x,A)=\mathbb{P}(\xi_1\in A|\xi_0=x)$. Next let $\mathbb{L}_0^2(\pi)$ be the set of functions on S such that $\int f^2 d\pi<\infty$ and $\int f d\pi=0$, and for a $f\in\mathbb{L}_0^2(\pi)$ denote $X_i=f(\xi_i)$, $S_n=\sum_{i=0}^{n-1}X_i$ (i.e. $S_1=X_0$, $S_2=X_0+X_1$, ...). In addition Q denotes the operator on $\mathbb{L}_2(\pi)$ acting via $(Qf)(x)=\int_S f(s)Q(x,ds)$. Denote by $\mathcal{F}_k$ the σ–field generated by ξ_i with $i\le k$. For any integrable variable X we denote $\mathbb{E}_k(X)=\mathbb{E}(X|\mathcal{F}_k)$. In our notation $\mathbb{E}_0(X_1)=Qf(\xi_0)=\mathbb{E}(X_1|\xi_0)$.

Notice that any stationary sequence $(X_k)_{k\in\mathbb{Z}}$ can be viewed as a function of a Markov process $\xi_k=(X_i;i\le k)$, for the function $g(\xi_k)=X_k$.

The stationary stochastic processes may also be introduced in the following alternative way. Let $T:\Omega\mapsto\Omega$ be a bijective bi-measurable transformation preserving the probability. Let $\mathcal{F}_0$ be a σ-algebra of $\mathcal{F}$ satisfying $\mathcal{F}_0\subseteq T^{-1}(\mathcal{F}_0)$. We then define the nondecreasing filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ by $\mathcal{F}_i=T^{-i}(\mathcal{F}_0)$. Let X_0 be a random variable which is $\mathcal{F}_0$-measurable, centered, i.e. $\mathbb{E}(X_0)=0$, and square integrable $\mathbb{E}(X_0^2)<\infty$. We then define the stationary sequence $(X_i)_{i\in\mathbb{Z}}$ by $X_i=X_0\circ T^i$. In this paper we shall use both frameworks.

The following notations will be frequently used. We denote by $\|X\|$ the norm in $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{P})$, the space of square integrable functions. We shall also denote by $\|X\|_p$ the norm in $\mathbb{L}^p(\Omega,\mathcal{F},\mathbb{P})$. For any two positive sequences $a_n\ll b_n$ means that for a certain numerical constant C not depending on n, we have $a_n\le Cb_n$ for all n; $[x]$ denotes the largest integer smaller or equal to x. For the law of the iterated logarithm we use the notation $\log_2 n=\log(\log(\max(e,n)))$. The notation $a.s.$ means almost surely, while $\Rightarrow$ denotes convergence in distribution.

The main question addressed is to find sufficient projective conditions such that there is a martingale M_n with stationary differences such that either

$$S_n-M_n=o(n^{1/2})\ \text{ a.s.}\,, \tag{1}$$

or

$$S_n - M_n = o((n \log_2 n)^{1/2}) \ \text{ a.s.} \tag{2}$$

These types of approximations are important to study for instance the limit theorems stated at a point (quenched) and the law of the iterated logarithm.

The "so called" quenched CLT, states that for any function f continuous and bounded

$$\mathbb{E}_0(f(S_n/\sqrt{n})) \to \ \mathbb{E}(f(cN)) \ \text{ a.s.}, \tag{3}$$

where N is a standard normal variable and c is a certain positive constant. By the quenched invariance principle we understand that for any function f continuous and bounded on $D[0,1]$ endowed with uniform topology we have

$$\mathbb{E}_0(f(S_{[nt]}/\sqrt{n})) \to \ \mathbb{E}(f(cW(t))) \ \text{ a.s.} \tag{4}$$

where W is the standard Brownian motion on $[0,1]$. We shall also refer to these types of convergence also as almost sure convergence in distribution under $\mathbb{P}_0$ a.s., where $\mathbb{P}_0(A) = \mathbb{P}(A|\mathcal{F}_0)$.

This conditional form of the CLT is a stable type of convergence that makes possible the change of measure with a majorating measure, as discussed in Billingsley (1999), Rootzén (1976), and Hall and Heyde (1980).

In the Markov chain setting the almost sure convergence in (3) or (4) are presented in a slightly different terminology. Denote by $\mathbb{P}^x$ and $\mathbb{E}^x$ the regular probability and conditional expectation given $X_0 = x$. In this context the quenched CLT is known under the name of CLT started at a point i.e. the CLT or its functional form holds for $\pi-$almost all $x \in S$, under the measure $\mathbb{P}^x$.

Here is a short history of the quenched CLT under projective criteria. A result in Borodin and Ibragimov (1994, Ch 4) states that if $\|\mathbb{E}_0(S_n)\|$ is bounded, then the CLT in its functional form started at a point holds. Later work by Derriennic and Lin (2001-b) improved on this result imposing the condition $\|\mathbb{E}_0(S_n)\| \ll n^{1/2-\epsilon}$ with $\epsilon > 0$ (see also Rassoul-Agha and Seppäläinen, 2008 and 2009). This condition was improved in Zhao and Woodroofe (2008-a) and further improved by Cuny (2011) who imposed the condition $\|\mathbb{E}_0(S_n)\| \ll n^{1/2}(\log n)^{-2}(\log\log n)^{-1-\delta}$ with $\delta > 0$. A result in Cuny and Peligrad (2009) shows that the condition $\sum_{k=1}^{\infty} \|\mathbb{E}_0(X_k)\|/k^{1/2} < \infty$, is sufficient for (3).

We shall prove here that the condition imposed to $\|\mathbb{E}_0(S_n)\|$ can be improved, by requiring less restrictive conditions on the regularity of $\|\mathbb{E}_0(S_n)\|$ than the result in Cuny (2011). Then we shall point out that the condition can be further weakened if we are interested in a result for averages or if finite moments of order larger than 2 are available.

To prove the law of the iterated logarithm we shall develop sufficient conditions for almost sure approximation with a Brownian motion; that is we shall redefine X_n, without changing its distribution, on a richer probability space on which there exists a standard Brownian motion $(W(t), t \geq 0)$ such that for a certain positive constant $c > 0$,

$$S_n - W(cn) = o((n \log_2 n)^{1/2}) \ \text{ a.s.}$$

We shall also develop sufficient conditions in terms of $\|\mathbb{E}_0(S_n)\|$ for the validity of the almost sure central limit theorem, namely: for a certain positive constant $c > 0$ and any real t,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbf{1}_{\{S_k/\sqrt{k} \leq t\}} = \mathbb{P}(cN \leq t) \text{ a.s.}$$

Our method of proof is based on martingale approximation that is valid under the Maxwell-Woodroofe condition:

$$\Delta(X_0) = \sum_{k=1}^{\infty} \frac{\|\mathbb{E}_0(S_k)\|}{k^{3/2}} < \infty \ . \tag{5}$$

The key tool in obtaining our results is the estimate of the rest of the martingale approximation in terms of $\|\mathbb{E}_0(S_k)\|$. We shall establish in Section 2 that there is a unique martingale with stationary and square integrable differences such that

$$\frac{\|S_n - M_n\|}{n^{1/2}} \ll \sum_{k \geq n} \frac{\|\mathbb{E}_0(S_k)\|}{k^{3/2}} . \tag{6}$$

We then further exploit the estimate (6) to derive almost sure martingale approximations of the types (1) and (2).

Our paper is organized as follows: In Section 2 we present the martingale approximation and estimate its rest. In Section 3 we present the almost sure martingale approximation results. Section 4 is dedicated to almost sure limiting results for the stationary processes. Section 5 points out some examples. Several results involving maximal inequalities and several technical lemmas are presented in the Appendix.

2 Martingale approximation with rest

Proposition 1 *For any stationary sequence $(X_k)_{k\in\mathbb{Z}}$ and filtration $(\mathcal{F}_k)_{k\in\mathbb{Z}}$ described above with $\Delta(X_0) < \infty$, there is a martingale $(M_k)_{k\geq 1}$ with stationary and square integrable differences $(D_k)_{k\in\mathbb{Z}}$ adapted to $(\mathcal{F}_{k+1})_{k\in\mathbb{Z}}$, $M_n = \sum_{i=0}^{n-1} D_i$, satisfying (6).*

To prove this proposition we need two preparatory lemmas. It is convenient to use the notation

$$Y_k^m = \frac{1}{m}\mathbb{E}_k(X_{k+1} + ... + X_{k+m}) . \tag{7}$$

As in Zhao and Woodroofe (2008-b), we shall also use the following semi-norm notation. For a stationary process $(X_k)_{k\in\mathbb{Z}}$ define the semi-norm

$$\|X_0\|_+^2 = \limsup_{n\to\infty} \frac{1}{n}\mathbb{E}(S_n^2) . \tag{8}$$

Lemma 2 *Assume $\|Y_0^m\|_+ \to 0$. Then, there is a martingale $(M_k)_{k\geq 1}$ with stationary and square integrable differences adapted to $(\mathcal{F}_{k+1})_{k\in\mathbb{Z}}$ satisfying*

$$\frac{\|S_n - M_n\|}{n^{1/2}} \ll \max_{1\leq k\leq n} \frac{\|\mathbb{E}_0(S_k)\|}{n^{1/2}} + \|Y_0^n\|_+ .$$

Proof of lemma 2. The construction of the martingale decomposition is based on averages. It was introduced by Wu and Woodroofe (2004; see their definition 6 on the page 1677) and further developed in Zhao and Woodroofe (2008-b), extending the construction in Heyde (1974) and Gordin and Lifshitz (1981); see also Theorem 8.1 in Borodin and Ibragimov (1994), and Kipnis and Varadhan (1986). We give the martingale construction with the estimation of the rest.

We introduce a parameter $m \geq 1$ (kept fixed for the moment), and define the stationary sequence of random variables:

$$\theta_0^m = \frac{1}{m}\sum_{i=1}^{m} \mathbb{E}_0(S_i),\ \theta_k^m = \theta_0^m \circ T^k .$$

Set

$$D_k^m = \theta_{k+1}^m - \mathbb{E}_k(\theta_{k+1}^m) \ ; \ M_n^m = \sum_{k=0}^{n-1} D_k^m . \tag{9}$$

Then, $(D_k^m)_{k\in\mathbb{Z}}$ is a sequence of stationary martingale differences such that D_k^m is $\mathcal{F}_{k+1}$-measurable and $(M_n^m)_{n\geq 1}$ is a martingale. So we have

$$X_k = D_k^m + \theta_k^m - \theta_{k+1}^m + \frac{1}{m}\mathbb{E}_k(S_{k+m+1} - S_{k+1})\,,$$

and therefore

$$\begin{aligned} S_k &= M_k^m + \theta_0^m - \theta_k^m + \sum\nolimits_{j=1}^{k} \frac{1}{m}\mathbb{E}_{j-1}(S_{j+m} - S_j) \\ &= M_k^m + \theta_0^m - \theta_k^m + \overline{R}_k^m\,, \end{aligned} \tag{10}$$

where we implemented the notation

$$\overline{R}_k^m = \sum\nolimits_{j=1}^{k} \frac{1}{m}\mathbb{E}_{j-1}(S_{j+m} - S_j)\ .$$

Observe that

$$\overline{R}_k^m = \sum_{j=0}^{k-1} Y_j^m. \tag{11}$$

With the notation

$$R_k^m = \theta_0^m - \theta_k^m + \overline{R}_k^m\,, \tag{12}$$

we have

$$S_k = M_k^m + R_k^m\ . \tag{13}$$

Notice that

$$\|S_n - M_n^n\| \leq 3 \max_{1\leq i\leq n} \|\mathbb{E}_0(S_i)\|\ . \tag{14}$$

Gordin and Peligrad (2009) have shown that if $\|Y_0^m\|_+ \to 0$, then D_0^n converges in $\mathbb{L}_2$ to a martingale difference we shall denote by D_0. Moreover $\max_{1\leq l\leq m}\|\mathbb{E}(S_l|\mathcal{F}_0)\|^2/m \to 0$. Denote D_i the limit of D_i^n and construct the martingale $M_n = \sum_{j=0}^{n-1} D_j$.

Let n and m be two strictly positive integers. By the fact that both D_0^n and D_0^m are martingale differences and using (12) and (13) we deduce

$$\begin{aligned} \|D_0^n - D_0^m\|^2 &= \frac{\|M_m^n - M_m^m\|^2}{m} \\ &\leq \frac{1}{m}\|(\theta_0^n - \theta_m^n + \overline{R}_m^n) - (\theta_0^m - \theta_m^m + \overline{R}_m^m)\|^2\ . \end{aligned}$$

So for n fixed, by the fact that $\sup_{1\leq l\leq m}\|\mathbb{E}(S_l|\mathcal{F}_0)\|^2/m \to 0$ we have that

$$\|D_0^n - D_0\| = \lim_{m\to\infty}\|D_0^n - D_0^m\| \leq \lim_{m\to\infty}\frac{1}{m^{1/2}}\|\overline{R}_m^n\| = \|Y_0^n\|_+\ . \tag{15}$$

We continue the estimate in the following way

$$\begin{aligned} \frac{\|S_n - M_n\|^2}{n} &\leq 2\Big(\frac{\|S_n - M_n^n\|^2}{n} + \frac{\|M_n^n - M_n\|^2}{n}\Big) \\ &\leq 2\Big(\frac{\|S_n - M_n^n\|^2}{n} + \|D_0^n - D_0\|^2\Big)\ . \end{aligned}$$

The lemma follows by combining the estimates in (14) and (15). $\Diamond$

Next we estimate $\|Y_0^n\|_+$.

Lemma 3 *Under the conditions of Proposition 1, for every $n \geq 1$ and any $m \geq 1$, we have*

$$\frac{1}{n^{1/2}} \| \max_{1 \leq j \leq n} | \sum_{k=0}^{j-1} Y_k^m | \, \| \ll \sum_{k=m+1}^{\infty} \frac{\|\mathbb{E}_0(S_k)\|}{k^{3/2}} \,, \tag{16}$$

and

$$\|Y_0^m\|_+ \ll \sum_{k \geq m} \frac{\|\mathbb{E}_0(S_k)\|}{k^{3/2}} \ . \tag{17}$$

Proof of Lemma 3. In order to prove the inequality (16), we apply the maximal inequality in Peligrad and Utev (2005) to the stationary sequence Y_0^m defined by (7), where $m \leq n$. Then

$$\| \max_{1 \leq j \leq n} | \sum_{k=0}^{j-1} Y_k^m | \, \| \ll n^{1/2} (\|Y_0^m\| + \Delta(Y_0^m)) \,,$$

where

$$\Delta(Y_0^m) := \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \|\mathbb{E}_0(Y_0^m + ... + Y_{k-1}^m)\| \,.$$

We first notice that $\|Y_0^m\| \leq m^{-1} \|\mathbb{E}_0(S_m)\|$. We estimate now $\Delta(Y_0^m)$. With this aim, it is convenient to use the decomposition

$$\Delta(Y_0^m) \leq \sum_{k=1}^{m} \frac{1}{k^{3/2}} \|\mathbb{E}_0(Y_0^m + ... + Y_{k-1}^m)\| + \sum_{k=m+1}^{\infty} \frac{1}{k^{3/2}} \|\mathbb{E}_0(Y_0^m + ... + Y_{k-1}^m)\| \,.$$

To estimate the first sum notice that, by the properties of the conditional expectation, we have

$$\|\mathbb{E}_0(Y_0^m + ... + Y_{k-1}^m)\| \leq k \|\mathbb{E}_0(Y_0^m)\| \,,$$

and then, since $\|\mathbb{E}_0(Y_0^m)\| \leq \|\mathbb{E}_0(S_m)\|/m$ we have

$$\sum_{k=1}^{m} \frac{1}{k^{3/2}} \|\mathbb{E}_0(Y_0^m + ... + Y_{k-1}^m)\| \leq \frac{1}{m} \sum_{k=1}^{m} \frac{\|\mathbb{E}_0(S_m)\|}{k^{1/2}} \ll \frac{1}{m^{1/2}} \|\mathbb{E}_0(S_m)\| \ .$$

To estimate the second sum we also apply the properties of the conditional expectation and write this time

$$\|\mathbb{E}_0(Y_0^m + ... + Y_{k-1}^m)\| \leq \|\mathbb{E}_0(S_k)\| \,.$$

Then,

$$\sum_{k=m+1}^{\infty} \frac{1}{k^{3/2}} \|\mathbb{E}_0(Y_0^m + ... + Y_{k-1}^m)\| \leq \sum_{k=m+1}^{\infty} \frac{\|\mathbb{E}_0(S_k)\|}{k^{3/2}} \,,$$

and overall

$$\Delta(Y_0^m) \ll \frac{1}{m^{1/2}} \|\mathbb{E}_0(S_m)\| + \sum_{k=m+1}^{\infty} \frac{\|\mathbb{E}_0(S_k)\|}{k^{3/2}} \,.$$

We conclude that for any strictly positive integers n and m

$$\frac{1}{\sqrt{n}} \| \max_{1 \leq j \leq n} | \sum_{k=0}^{j-1} Y_k^m | \|_2 \ll \frac{\|\mathbb{E}_0(S_m)\|}{m^{1/2}} + \sum_{k=m+1}^{\infty} \frac{\|\mathbb{E}_0(S_k)\|}{k^{3/2}} \,.$$

The estimate (16) of this lemma follows now by using Lemma 19 from the Appendix with $p = 2$ and $\gamma = 1/2$. With the notation (8), by passing to the limit in the inequality (16), we obtain (17). $\lozenge$

Proof of Proposition 1. Notice that (5) implies $\|Y_0^m\|_+ \to 0$. We combine the estimate in Lemma 2 with the estimate of $\|Y_0^m\|_+$ in Lemma 3 to obtain the desired result, via Lemma 19 in Appendix applied with $p = 2$ and $\gamma = 1/2$. $\lozenge$

3 Almost sure martingale approximations

In this section we use the estimate (6) obtained in Proposition 1 to approximate a partial sum by a martingale in the almost sure sense.

Proposition 4 *Assume $(b_n)_{n\geq 1}$ is any nondecreasing positive, slowly varying sequence such that*

$$\sum_{n\geq 1} \frac{b_n}{n}\Big(\sum_{k\geq n} \frac{\|\mathbb{E}_0(S_k)\|}{k^{3/2}}\Big)^2 < \infty \ . \tag{18}$$

Then, there is a martingale $(M_k)_{k\geq 1}$ with stationary and square integrable differences adapted to $(\mathcal{F}_{k+1})_{k\in\mathbb{Z}}$ satisfying

$$\frac{S_n - M_n}{\sqrt{nb_n^*}} \to 0 \quad a.s. \tag{19}$$

where $b_n^* := \sum_{k=1}^{n}(kb_k)^{-1}$.

As an immediate consequence of this proposition we formulate the following corollary:

Corollary 5 *Assume that for a certain sequence of positive numbers $(b_n)_{n\geq 1}$ that is slowly varying, nondecreasing and satisfies $\sum_{n\geq 1}(nb_n)^{-1} < \infty$, the condition (18) is satisfied. Then there is a martingale $(M_k)_{k\geq 1}$ with stationary and square integrable differences adapted to $(\mathcal{F}_{k+1})_{k\in\mathbb{Z}}$ satisfying:*

$$\frac{S_n - M_n}{\sqrt{n}} \to 0 \quad a.s. \tag{20}$$

Example: In Corollary 5 the sequence $(b_n)_{n\geq 3}$ can be taken for instance $b_n = (\log n)(\log_2 n)^{\gamma}$, for some $\gamma > 1$.

Selecting in Proposition 4 the sequence $b_n = \log n$, we obtain:

Corollary 6 *Assume that*

$$\sum_{n\geq 1} \frac{\log n}{n}\Big(\sum_{k\geq n} \frac{\|\mathbb{E}_0(S_k)\|}{k^{3/2}}\Big)^2 < \infty \ . \tag{21}$$

Then there is a martingale $(M_k)_{k\geq 1}$ with stationary and square integrable differences adapted to $(\mathcal{F}_{k+1})_{k\in\mathbb{Z}}$ satisfying:

$$\frac{S_n - M_n}{(n\log_2 n)^{1/2}} \to 0 \ a.s. \tag{22}$$

Proof of Proposition 4. By Corollary 4.2 in Cuny (2011), given in Appendix for the convenience of the reader (see Proposition 20), in order to show that (19) holds, we have to verify that

$$\sum_{n\geq 1} \frac{b_n\|S_n - M_n\|^2}{n^2} < \infty \ .$$

By Proposition 1 we know that

$$\frac{\|S_n - M_n\|}{n^{1/2}} \ll \sum_{k\geq n} \frac{\|\mathbb{E}_0(S_k)\|}{k^{3/2}} \ .$$

Therefore the condition (18) implies the desired martingale approximation. $\Diamond$

Remark. Notice that our condition (18) is implied by the condition in Corollary 5.8 in Cuny (2011). He assumed for the same result $\|\mathbb{E}_0(S_n)\| \ll n^{1/2}(\log n)^{-2}(\log_2)^{-1-\delta}$ with $\delta > 0$, that clearly implies (18). Also (21) is implied by the result in Corollary 5.7 in Cuny (2011) who obtained the same result under the condition $\|\mathbb{E}_0(S_n)\| \ll n^{1/2}(\log n)^{-2}(\log_2 n)^{-\tau}$ with $\tau > 1/2$.

In the next two subsections we propose two ways to improve on the rate of convergence to 0 of $\|\mathbb{E}_0(S_k)\|/\sqrt{k}$ that assure an almost sure martingale approximation in some sense.

3.1 Averaging

In the next proposition we study a Cesàro-type almost sure martingale approximation.

Proposition 7 *Assume that*

$$\sum_{n\geq 1} \frac{1}{n}\Big(\sum_{k\geq n} \frac{\|\mathbb{E}_0(S_k)\|}{k^{3/2}}\Big)^2 < \infty \ . \tag{23}$$

Then there is a martingale $(M_k)_{k\geq 1}$ with stationary and square integrable differences adapted to $(\mathcal{F}_{k+1})_{k\in\mathbb{Z}}$ satisfying:

$$\frac{1}{n}\sum_{k=1}^{n} \frac{|S_k - M_k|}{k^{1/2}} \to 0 \quad a.s. \tag{24}$$

Before proving this proposition we shall formulate the condition (23) in an equivalent form that is due to monotonicity:

$$\sum_{r\geq 0} \Big(\sum_{\ell\geq 2^r} \frac{\|\mathbb{E}_0(S_\ell)\|}{\ell^{3/2}}\Big)^2 < \infty \ . \tag{25}$$

Proof of Proposition 7. We notice that the condition (23) implies by Proposition 1 the existence of a martingale $(M_n)_{n\geq 1}$ with stationary differences such that

$$\sum_{n\geq 1} \frac{\|S_n - M_n\|^2}{n^2} < \infty \ , \tag{26}$$

that further implies

$$\sum_{n\geq 1} \frac{(S_n - M_n)^2}{n^2} < \infty \quad \text{a.s.}$$

Whence, by Kronecker lemma,

$$\frac{1}{n}\sum_{k=1}^{n} \frac{(S_k - M_k)^2}{k} \to 0 \text{ a.s.}$$

and then, by Cauchy-Schwarz inequality

$$\sum_{k=1}^{n} \frac{|S_k - M_k|}{k^{1/2}} \leq \Big(n\sum_{k=1}^{n} \frac{(S_k - M_k)^2}{k}\Big)^{1/2} \ .$$

Therefore

$$\frac{1}{n}\sum_{k=1}^{n} \frac{|S_k - M_k|}{k^{1/2}} \to 0 \quad \text{a.s.}$$

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We can also formulate the following result:

Proposition 8 *Assume that*

$$\sum_{n\geq 1} \frac{1}{n\log n}\Big(\sum_{k\geq n} \frac{\|\mathbb{E}_0(S_k)\|}{k^{3/2}}\Big)^2 < \infty \ . \tag{27}$$

Then there is a martingale $(M_k)_{k\geq 1}$ with stationary and square integrable differences adapted to $(\mathcal{F}_{k+1})_{k\in\mathbb{Z}}$ satisfying:

$$\frac{1}{\log n}\sum_{k=1}^{n} \frac{|S_k - M_k|}{k^{3/2}} \to 0 \quad a.s. \tag{28}$$

Proof of Proposition 8. The condition (27) implies by Proposition 1 the existence of a martingale $(M_n)_{n\geq 1}$ with stationary differences such that

$$\sum_{n\geq 1} \frac{(S_n - M_n)^2}{n^2 \log n} < \infty \quad \text{a.s.}\,, \tag{29}$$

which, by Kronecker lemma, implies

$$\frac{1}{\log n}\sum_{k=1}^{n} \frac{(S_k - M_k)^2}{k^2} \to 0 \text{ a.s.}$$

and then (28), by Cauchy-Schwarz inequality. $\Diamond$

This idea of considering the average approximation can be also applied to Markov chains with normal operators (i.e. $QQ^* = Q^*Q$ on $\mathbb{L}^2(\pi)$). For this case we can replace our Proposition 1 by a result stated in Cuny (2011) for normal Markov chains, namely

$$\frac{\|S_n - M_n\|^2}{n} \ll \frac{1}{n}\sum_{k\leq n} \frac{\|\mathbb{E}_0(S_k)\|^2}{k} + \sum_{k>n} \frac{\|\mathbb{E}_0(S_k)\|^2}{k^2} \ . \tag{30}$$

Then we can replace in the proof of Propositions 7 and 8, the inequality given in our Proposition 1 by the inequality (30). We can then formulate:

Proposition 9 *Let $(\xi_n)_{n\in\mathbb{Z}}$ be a Markov chain with normal operator and stationary distribution π. Let $f \in \mathbb{L}_0^2(\pi)$ and $X_0 = f(\xi_0)$. If the condition*

$$\sum_{n\geq 2} \frac{\log n \|\mathbb{E}_0(S_n)\|^2}{n^2} < \infty\,, \tag{31}$$

is satisfied, then (24) holds. If the condition

$$\sum_{n\geq 2} \frac{\log_2 n \|\mathbb{E}_0(S_n)\|^2}{n^2} < \infty\,, \tag{32}$$

is satisfied, then (28) holds.

We point out that the condition (31) by itself does not imply (20) so the averaging is needed. As a matter of fact, Cuny and Peligrad (2009) commented that there is a stationary and ergodic normal Markov chain and a function f such that

$$\sum_{n\geq 2} \frac{\log n \log_2 n \|\mathbb{E}_0(S_n)\|^2}{n^2} < \infty\,,$$

and such that (20) fails.

3.2 Higher moments

Another way to improve on the rate of convergence to 0 of $\|\mathbb{E}_0(S_k)\|/k^{1/2}$, in order to establish limit theorems started at a point, is to consider the existence of moments larger than 2.

Proposition 10 *Assume that for some $\delta > 0$, $\mathbb{E}(|X_0|^{2+\delta}) < \infty$, and that the condition (23) is satisfied. Then, there is a martingale $(M_k)_{k\geq 1}$ with stationary and square integrable differences adapted to $(\mathcal{F}_{k+1})_{k\in\mathbb{Z}}$ satisfying for every $\varepsilon > 0$*

$$\sum_{n\geq 1} \frac{1}{n}\mathbb{P}(\max_{j\leq n} |S_j - M_j| \geq \varepsilon\sqrt{n}) < \infty \,, \tag{33}$$

and therefore $S_n - M_n = o(n^{1/2})$ a.s.

Proof of Proposition 10. The sequence $(\max_{j\leq n} |S_j - M_j|)_{n\geq 1}$ being nondecreasing, the property (33) is equivalent to: for every $\varepsilon > 0$,

$$\sum_{N\geq 1} \mathbb{P}(\max_{1\leq j\leq 2^N} |S_j - M_j| \geq \varepsilon 2^{N/2}) < \infty \,,$$

which implies that $S_n - M_n = o(n^{1/2})$ almost surely. It remains to prove (33). By assumption (23) it follows that $\sum_{j\geq 1} j^{-3/2}\|\mathbb{E}_0(S_j)\| < \infty$. Therefore, according to Proposition 1, there exists a martingale $(M_k)_{k\geq 1}$ with stationary and square integrable differences $(D_k)_{k\in\mathbb{Z}}$ adapted to $(\mathcal{F}_{k+1})_{k\in\mathbb{Z}}$ such that (6) is satisfied. Applying then Corollary 18 with $\varphi(x) = x^2$, $p = 2$, $Y_i = X_{i-1}$, $Z_i = D_{i-1}$ and $\mathcal{G}_i = \mathcal{F}_i$, and taking into account (6), we get that for every $x > 0$ and any $\alpha \in [0, 1)$,

$$\begin{aligned} \mathbb{P}(\max_{j\leq n} |S_j - M_j| \geq 4x) &\ll \frac{n}{x^2}\Big(\sum_{j\geq n} \frac{1}{j^{3/2}}\|\mathbb{E}_0(S_j)\|\Big)^2 + \frac{n}{x}\mathbb{E}\left(|X_0|\mathbf{1}_{\{|X_0|\geq xn^{-\alpha}\}}\right) \\ &+ \frac{n}{x^2}\Big(\sum_{k\geq [n^\alpha]+1} \frac{\|\mathbb{E}_0(S_k)\|}{k^{3/2}}\Big)^2 . \end{aligned} \tag{34}$$

Choosing now $\alpha = \delta/(2 + 2\delta)$ and $x = \varepsilon\sqrt{n}$, we get by using Fubini theorem that

$$\sum_{n\geq 1} \frac{1}{n^{1/2}}\mathbb{E}\left(|X_0|\mathbf{1}_{\{|X_0|\geq \varepsilon n^{1/2-\alpha}\}}\right) \ll \frac{1}{\varepsilon^{1+\delta}}\mathbb{E}(|X_0|^{2+\delta}) \,. \tag{35}$$

Therefore, starting from (34) and using (35), we infer that (33) holds provided that

$$\sum_{n\geq 1} \frac{1}{n}\Big(\sum_{j\geq [n^{\delta/(2+2\delta)}]} \frac{\|\mathbb{E}_0(S_j)\|}{j^{3/2}}\Big)^2 < \infty \,. \tag{36}$$

Now, by the usual comparison between the series and the integrals, we notice that for any nonincreasing and positive function h on $\mathbb{R}^+$ and any positive γ,

$$\sum_{n\geq 1} n^{-1}h(n^\gamma) < \infty \text{ if and only if } \sum_{n\geq 1} n^{-1}h(n) < \infty. \tag{37}$$

Applying this result with $h(y) = \big(\sum_{j\geq [y]} j^{-3/2}\|\mathbb{E}_0(S_j)\|\big)^2$, it follows that the conditions (23) and (36) are equivalent. This ends the proof of the theorem. $\Diamond$

Next proposition will be useful to transport from the martingale to the stationary sequence the law of iterated logarithm.

Proposition 11 *Assume that* $\mathbb{E}(|X_0|^{2+\delta}) < \infty$ *for some* $\delta > 0$, *and that*

$$\sum_{n\geq 3} \frac{1}{n \log_2 n}\Big(\sum_{j\geq n} \frac{\|\mathbb{E}_0(S_j)\|}{j^{3/2}}\Big)^2 < \infty\,. \tag{38}$$

Then there is a martingale $(M_k)_{k\geq 1}$ *with stationary and square integrable differences adapted to* $(\mathcal{F}_{k+1})_{k\in\mathbb{Z}}$ *satisfying for every* $\varepsilon > 0$

$$\sum_{n\geq 1} \frac{1}{n}\mathbb{P}(\max_{j\leq n} |S_j - M_j| \geq \varepsilon (n \log_2 n)^{1/2}) < \infty\,,$$

and therefore $S_n - M_n = o((n\log_2 n)^{1/2})$ *a.s.*

Proof of Proposition 11. We follow the lines of Proposition 10 with the difference that we select in (34) $x = \varepsilon(n \log_2 n)^{1/2}$ and apply (37) with $h(y) = \big((\log_2 y)^{-1/2} \sum_{j\geq[y]} j^{-3/2}\|\mathbb{E}_0(S_j)\|\big)^2$. $\Diamond$

We shall point now two sets of conditions that satisfy the conditions of these last two propositions. Assume that $\|\mathbb{E}_0(S_n)\| \ll n^{1/2}(\log n)^{-3/2}(\log_2 n)^{\beta}$ for a certain $\beta > 1/2$. Then condition (23) is satisfied. If $\|\mathbb{E}_0(S_n)\| \ll n^{1/2}(\log n)^{-3/2}(\log_2 n)^{-\gamma}$ for a $\gamma > 0$, then the condition (38) is satisfied.

4 Applications to almost sure limit theorems

We shall formulate here a few applications of the almost sure martingale approximations to quenched functional CLT, LIL and almost sure CLT. For simplicity we assume in this section that the stationary sequence is *ergodic* to avoid random normalizers.

Theorem 12 *Assume that the stationary sequence is ergodic and the conditions of Corollary 5 or Proposition 10 hold. Then*

$$S_{[nt]}/\sqrt{n} \Rightarrow \sigma W(t) \text{ under } \mathbb{P}_0 \text{ a.s.}\,,$$

where $\sigma = \|D_0\|$ *and* D_0 *is defined by (9).*

Proof of Theorem 12. The conditions of Corollary 5 or Proposition 10 imply that for every $\varepsilon > 0$

$$\mathbb{P}_0(\max_{1\leq k\leq n} |S_k - M_k| > \varepsilon\sqrt{n}) \to 0 \text{ a.s.}$$

that further implies

$$\mathbb{P}_0(\sup_{0\leq t\leq 1} |S_{[nt]} - M_{[nt]}| > \varepsilon\sqrt{n}) \to 0 \text{ a.s.}$$

According to Theorem 3.1 in Billingsley (1999), the limiting distribution of $S_{[nt]}|/\sqrt{n}$ is the same as of $M_{[nt]}/\sqrt{n}$ under $\mathbb{P}_0$ a.s. It was shown in Derriennic and Lin (2001-a) in details that

$$M_{[nt]}/\sqrt{n} \Rightarrow \sigma W(t) \text{ under } \mathbb{P}_0 \text{ a.s.}\,,$$

and the result follows. $\Diamond$

Theorem 13 *Assume that the stationary sequence is ergodic and the conditions of Proposition 7 are satisfied. Then we have*

$$\frac{1}{n}\sum_{k=1}^{n} \frac{S_k}{k^{1/2}} \Rightarrow \sqrt{\frac{2}{3}}\sigma N \quad \text{under } \mathbb{P}_0 \text{ a.s. }, \tag{39}$$

where $\sigma = \|D_0\|$ *and* D_0 *is defined by (9).*

Proof of Theorem 13. Under the condition (23) we know there is an ergodic martingale (M_n) with stationary and square integrable differences (D_n) satisfying

$$\frac{1}{n}\sum_{k=1}^{n}\frac{|S_k - M_k|}{k^{1/2}} \to 0 \ \text{ a.s.}$$

Then, by Theorem 3.1 in Billingsley (1999), the limiting distribution of

$$\frac{1}{n}\sum_{k=1}^{n}\frac{S_k}{k^{1/2}} \text{ coincides to the limiting distribution of } \frac{1}{n}\sum_{k=1}^{n}\frac{M_k}{k^{1/2}} \text{ under } \mathbb{P}_0 \text{ a.s.}$$

By changing the order of summation we can rewrite $\sum_{k=1}^{n} M_k/k^{1/2}$ as

$$\sum_{i=0}^{n-1}\Big(\sum_{k=i+1}^{n}\frac{1}{k^{1/2}}\Big)D_i\,,$$

and, according to the Raikov method for proving the central limit theorem for martingales, we have to study the limit of the sum of squares. We first write that

$$\frac{4}{n^2}\sum_{i=1}^{n}(\sqrt{n+1}-\sqrt{i})^2 D_i^2 \le \frac{1}{n^2}\sum_{i=0}^{n-1}\Big(\sum_{k=i+1}^{n}\frac{1}{k^{1/2}}\Big)^2 D_i^2 \le \frac{4}{n^2}\sum_{i=0}^{n-1}(\sqrt{n}-\sqrt{i})^2 D_i^2\,. \tag{40}$$

Then, by the Birkhoff ergodic theorem, we have

$$\frac{1}{n}\sum_{i=0}^{n-1} D_i^2 \to \mathbb{E}(D_0^2) = \sigma^2 \quad \text{a.s. and in } \mathbb{L}_1\,.$$

Hence, applying the generalized Toeplitz lemma (see Lemma 21) to the both sides of the inequality (40) with $x_i = D_i^2$ and $c_i = \sqrt{i}$ and then with $c_i = i$, we get that

$$\frac{1}{n^2}\sum_{i=0}^{n-1}(\sum_{k=i+1}^{n}\frac{1}{k^{1/2}})^2 D_i^2 \to \frac{2}{3}\sigma^2 \ \text{ a.s. and in } \mathbb{L}_1.$$

Then, by Theorem 3.6 in Hall and Heyde (1980) we easily obtain the convergence in (39). $\Diamond$

Theorem 14 *Assume that either the conditions of Corollary 6 or of Proposition 11 hold and in addition the sequence is ergodic. Then we can redefine $(X_n)_{n\in\mathbb{Z}}$, without changing its distribution, on a richer probability space on which there exists a standard Brownian motion $(W(t), t \ge 0)$ such that*

$$S_n - W(n\|D_0\|^2) = o((n\log_2 n)^{1/2}) \quad a.s.$$

Therefore, the LIL holds:

$$\limsup_{n\to\infty} \pm\frac{S_n}{(2n\log_2 n)^{1/2}} = \|D_0\| \quad a.s.$$

Proof of Theorem 14. Since by Corollary 6 or by Proposition 11 we have $S_n - M_n = o((n\log_2 n)^{1/2})$ *a.s.* the result follows by the almost sure invariance principle for stationary, ergodic and square integrable martingales (see Strassen, 1967). $\Diamond$

Theorem 15 *Assume that the stationary sequence is ergodic and Condition (27) is satisfied. Then, for any real* t*,*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbf{1}_{\{S_k/\sqrt{k}\le t\}} = \mathbb{P}(\sigma N \le t) \ a.s.\,, \tag{41}$$

where $\sigma = \|D_0\|$ *and* D_0 *is defined by (9).*

Proof of Theorem 15. According to the step (a) of the proof of Theorem 1 in Lacey and Philipp (1990), (41) is equivalent to: for any Lipschitz and bounded function f from $\mathbb{R}$ to $\mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} f(S_k/\sqrt{k}) = \mathbb{E}(f(\sigma N)) \text{ a.s.} \tag{42}$$

Next, by Proposition 8, if the condition (27) is satisfied, there is a martingale $(M_k)_{k\ge 1}$ with stationary and square integrable differences satisfying (28). Therefore, for any Lipschitz function f,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \Big| f(S_k/\sqrt{k}) - f(M_k/\sqrt{k}) \Big| = 0 \text{ a.s.}$$

Notice now that $(M_k)_{k\ge 1}$ is ergodic since $(X_k)_{k\in\mathbb{Z}}$ is. The proof is then completed by the fact that (42) holds with M_k replacing S_k (see Lifshits (2002)). $\Diamond$

5 Examples

We shall mention two examples for which the quantity $\|\mathbb{E}_0(S_n)\|$ is estimated. Then, these estimates introduced in our results will provide new asymptotic results started at a point and LIL, that improve the previous results in the literature.

1. Linear processes.

Let $(\varepsilon_n)_{n\in\mathbb{Z}}$ be a sequence of ergodic martingale differences and consider the linear process

$$X_k = \sum_{i\ge 1} a_i \varepsilon_{k-i}\,,$$

where $(a_i)_{i\ge 1}$ is a sequence of real constants such that $\sum_{i\ge 1} a_i^2 < \infty$. We define

$$S_n = \sum_{i=1}^{n} X_i\ .$$

Denote by

$$b_{nj} = a_{j+1} + \ldots + a_{j+n}\ .$$

Then

$$\|\mathbb{E}_0(S_n)\|^2 = \sum_{j\ge 0} b_{nj}^2\ .$$

For the particular case $a_n \ll 1/(nL(n))$, where $L(\cdot)$ is a positive, nondecreasing, slowly varying function, computations in Zhao and Woodroofe (2008-a) show that $\|\mathbb{E}_0(S_n)\| \ll \sqrt{n}/L(n)$.

2. **Functions of Bernoulli shifts.**

Let $(\varepsilon_k)_{k\in\mathbb{Z}}$ be an i.i.d. sequence of Bernoulli variables, that is $\mathbb{P}(\varepsilon_1 = 0) = 1/2 = \mathbb{P}(\varepsilon_1 = 1)$ and let

$$Y_n = \sum_{k=0}^{\infty} 2^{-k-1}\varepsilon_{n-k}, \quad X_n = g(Y_n) - \int_0^1 g(x)dx, \text{ and } S_n = \sum_{k=1}^{n} X_k \ ,$$

where $g \in \mathbb{L}_2(0,1)$, $(0,1)$ being equipped with the Lebesgue measure. The transform Y_j is usually referred to as the Bernoulli shift of the i.i.d. sequence $(\varepsilon_k)_{k\in\mathbb{Z}}$. Then, following Maxwell and Woodroofe (2000), as in Peligrad, Utev and Wu (2007),

$$\|\mathbb{E}(g(Y_k)|Y_0)\|^2 \le 2^k \int_0^1 \int_0^1 \mathbf{1}_{\{|x-y|\le 2^{-k}\}} |g(x)-g(y)|^2 dydx \,,$$

and then $\|\mathbb{E}(S_n|Y_0)\| \le \sum_{k=1}^n \|\mathbb{E}(g(Y_k)|Y_0)\|$.

6 Appendix

6.1 Maximal inequalities

Following the idea of proof of the maximal inequality given in Proposition 5 of Merlevède and Peligrad (2010), we shall prove the following result:

Proposition 16 *Let $(Y_i)_{1\le i\le 2^r}$ be real random variables where r is a positive integer. Assume that the random variables are adapted to an increasing filtration $(\mathcal{G}_i)_{1\le i\le 2^r}$. Let $(Z_i)_{1\le i\le 2^r}$ be real random variables adapted to $(\mathcal{G}_i)_{1\le i\le 2^r}$ and such that for every i, $\mathbb{E}(Z_i|\mathcal{G}_{i-1}) = 0$ a.s. Let $S_n = Y_1 + \cdots + Y_n$ and $T_n = Z_1 + \cdots + Z_n$. Let φ be a nondecreasing, non negative, convex and even function. Then for any positive real x, any real $p \ge 1$ and any integer $u \in [0, r-1]$, the following inequality holds:*

$$\mathbb{P}(\max_{1\le i\le 2^r} |S_i - T_i| \ge 4x) \le \frac{1}{\varphi(x)}\mathbb{E}(\varphi(S_{2^r} - T_{2^r})) + \frac{1}{x}\sum_{i=1}^{2^r} \mathbb{E}(|Y_i|\mathbf{1}_{\{|Y_i|\ge x/2^u\}})$$
$$+ \frac{1}{x^p}\Big(\sum_{l=u}^{r-1}\Big(\sum_{k=1}^{2^{r-l}-1} \|\mathbb{E}(S_{(k+1)2^l} - S_{k2^l}|\mathcal{G}_{k2^l})\|_p^p\Big)^{1/p}\Big)^p .$$

Remark 17 *When the sequence $(Y_n)_{n\in\mathbb{Z}}$ is stationary as well as the filtration $(\mathcal{G}_n)_{n\in\mathbb{Z}}$, the inequality has the following form:*

$$\mathbb{P}(\max_{1\le i\le 2^r} |S_i - T_i| \ge 4x) \le \frac{1}{\varphi(x)}\mathbb{E}(\varphi(S_{2^r} - T_{2^r})) + \frac{2^r}{x}\mathbb{E}(|Y_1|\mathbf{1}_{\{|Y_1|\ge x/2^u\}})$$
$$+ \frac{2^r}{x^p}\Big(\sum_{l=u}^{r-1} \frac{1}{2^{l/p}}\|\mathbb{E}(S_{2^l}|\mathcal{G}_0)\|_p\Big)^p .$$

Notice now that for any integer $n \in [2^{r-1}, 2^r)$, where r is a positive integer, $\mathbb{E}(\varphi(S_{2^r} - T_{2^r})) \le \max_{n<k<2n} \mathbb{E}(\varphi(S_k - T_k))$. In addition, due to stationarity and the subadditivity of the sequence $\big(\|\mathbb{E}(S_n|\mathcal{G}_0)\|_p\big)_{n\ge1}$, we have that

$$2^k \|\mathbb{E}(S_{2^k}|\mathcal{G}_0)\|_p \le 2\sum_{j=1}^{2^k} \|\mathbb{E}(S_j|\mathcal{G}_0)\|_p \,,$$

implying that for any integer $n \in [2^{r-1}, 2^r)$, where r is a positive integer, and any integer $u \in [0, r-1]$,

$$\sum_{i=u}^{r-1} \frac{1}{2^{i/p}} \|\mathbb{E}(S_{2^i}|\mathcal{G}_0)\|_p \le 2 \sum_{j=1}^{2^{r-1}} \|\mathbb{E}(S_j|\mathcal{G}_0)\|_p \sum_{i:2^i \ge j \vee 2^u} \frac{1}{2^{i(1+1/p)}} ,$$

and then that

$$\sum_{i=u}^{r-1} \frac{1}{2^{i/p}} \|\mathbb{E}(S_{2^i}|\mathcal{G}_0)\|_p \le \frac{2^{2+1/p}}{2^{1+1/p}-1} \Big(\frac{1}{2^{u(1+1/p)}} \sum_{k=1}^{2^u-1} \|\mathbb{E}(S_k|\mathcal{G}_0)\|_p + \sum_{k=2^u}^{n} \frac{\|\mathbb{E}(S_k|\mathcal{G}_0)\|_p}{k^{1+1/p}} \Big) .$$

It remains to apply Lemma 19 below (with $\gamma = 1/p$) to obtain the following corollary:

Corollary 18 *Let $(Y_i)_{i\in\mathbb{Z}}$ be a stationary sequence of real random variables. Assume that the random variables are adapted to an increasing and stationary filtration $(\mathcal{G}_i)_{i\in\mathbb{Z}}$. Let $(Z_i)_{i\in\mathbb{Z}}$ be a sequence of real random variables adapted to $(\mathcal{G}_i)_{i\in\mathbb{Z}}$ and such that for all i, $\mathbb{E}(Z_i|\mathcal{G}_{i-1}) = 0$ a.s. Let $S_n = Y_1 + \cdots + Y_n$ and $T_n = Z_1 + \cdots + Z_n$. Let φ be a nondecreasing, non negative, convex and even function. Then for any positive real x, any positive integer n, any real $p \ge 1$ and any real $\alpha \in [0,1]$, the following inequality holds:*

$$\mathbb{P}(\max_{1\le i\le n} |S_i - T_i| \ge 4x) \le \frac{1}{\varphi(x)} \max_{n<k<2n} \mathbb{E}(\varphi(S_k - T_k)) + \frac{2n}{x} \mathbb{E}(|Y_1| \mathbf{1}_{\{|Y_1| \ge x/n^\alpha\}})$$
$$+ \frac{c_p n}{x^p} \Big(\sum_{k=[n^\alpha]+1}^{\infty} \frac{\|\mathbb{E}(S_k|\mathcal{G}_0)\|_p}{k^{1+1/p}} \Big)^p ,$$

where c_p is a positive constant depending only on p.

Proof of Proposition 16.

Using the fact that $\mathbb{E}(T_n - T_k|\mathcal{G}_k) = 0$ for $0 \le k \le n$, we get, for any $m \in [0, 2^r - 1]$, that

$$S_{2^r-m} - T_{2^r-m} = \mathbb{E}(S_{2^r} - T_{2^r}|\mathcal{G}_{2^r-m}) - \mathbb{E}(S_{2^r} - S_{2^r-m}|\mathcal{G}_{2^r-m}) .$$

So,

$$\max_{1\le i\le 2^r} |S_i - T_i| \le \max_{0\le m\le 2^r-1} |\mathbb{E}(S_{2^r} - T_{2^r}|\mathcal{G}_{2^r-m})| + \max_{0\le m\le 2^r-1} |\mathbb{E}(S_{2^r} - S_{2^r-m}|\mathcal{G}_{2^r-m})| . \tag{43}$$

Since $(\mathbb{E}(S_{2^r} - T_{2^r}|\mathcal{G}_k))_{k\ge1}$ is a martingale, we shall use Doob's maximal inequality (see Theorem 2.1 in Hall and Heyde, 1980) to deal with the first term in the right hand side of (43). Hence, since φ is a nondecreasing, non negative, convex and even function, we get that

$$\mathbb{P}\Big(\max_{0\le m\le 2^r-1} |\mathbb{E}(S_{2^r} - T_{2^r}|\mathcal{G}_{2^r-m})| \ge x \Big) \le \frac{1}{\varphi(x)} \mathbb{E}(\varphi(S_{2^r} - T_{2^r})) . \tag{44}$$

Write now m in basis 2 as follows:

$$m = \sum_{i=0}^{r-1} b_i(m) 2^i, \quad \text{where } b_i(m) = 0 \text{ or } b_i(m) = 1 .$$

Set $m_l = \sum_{i=l}^{r-1} b_i(m) 2^i$. With this notation $m_0 = m$. Let $0 \le u \le r-1$ and write that

$$|\mathbb{E}(S_{2^r} - S_{2^r-m}|\mathcal{G}_{2^r-m})| \le |\mathbb{E}(S_{2^r-m_u} - S_{2^r-m}|\mathcal{G}_{2^r-m})| + |\mathbb{E}(S_{2^r} - S_{2^r-m_u}|\mathcal{G}_{2^r-m})| .$$

Notice first that

$$\mathbb{P}\big(\max_{0\le m\le 2^r-1}|\mathbb{E}(S_{2^r-m_u}-S_{2^r-m}|\mathcal{G}_{2^r-m})|\ge 2x\big)$$
$$\le \mathbb{P}\big(\max_{0\le m\le 2^r-1}\sum_{j=2^r-m+1}^{2^r-m_u}|\mathbb{E}(Y_j|\mathcal{G}_{2^r-m})|\ge 2x\big)\,.$$

Therefore, by using the fact that $m-m_u\le 2^u$ implies

$$\sum_{j=2^r-m+1}^{2^r-m_u}|Y_j|\le x+\sum_{j=2^r-m+1}^{2^r-m_u}|Y_j|\mathbf{1}_{\{|Y_j|\ge x/2^u\}}\,,$$

we derive that

$$\mathbb{P}\big(\max_{0\le m\le 2^r-1}|\mathbb{E}(S_{2^r-m_u}-S_{2^r-m}|\mathcal{G}_{2^r-m})|\ge 2x\big)$$
$$\le \mathbb{P}\big(\max_{0\le m\le 2^r-1}\sum_{j=2^r-m+1}^{2^r-m_u}\mathbb{E}(|Y_j|\mathbf{1}_{\{|Y_j|\ge x/2^u\}}|\mathcal{G}_{2^r-m})\ge x\big)$$
$$\le \mathbb{P}\big(\max_{0\le m\le 2^r-1}\sum_{j=1}^{2^r}\mathbb{E}(|Y_j|\mathbf{1}_{\{|Y_j|\ge x/2^u\}}|\mathcal{G}_{2^r-m})\ge x\big)\,.$$

Noticing then that $\big(\sum_{j=1}^{2^r}\mathbb{E}(|Y_j|\mathbf{1}_{\{|Y_j|\ge x/2^u\}}|\mathcal{G}_k)\big)_{k\ge 1}$ is a martingale, Doob's maximal inequality implies that

$$\mathbb{P}\big(\max_{0\le m\le 2^r-1}|\mathbb{E}(S_{2^r-m_u}-S_{2^r-m}|\mathcal{G}_{2^r-m})|\ge 2x\big)\le x^{-1}\sum_{i=1}^{2^r}\mathbb{E}(|Y_i|\mathbf{1}_{\{|Y_i|\ge x/2^u\}})\,, \tag{45}$$

On the other hand, following the proof of Proposition 5 in Merlevède and Peligrad (2010), for any $m\in\{0,\dots,2^r-1\}$ and any $p\ge 1$, we get that

$$\mathbb{E}(S_{2^r}-S_{2^r-m_u}|\mathcal{G}_{2^r-m})|^p\le\sum_{l=u}^{r-1}\lambda_l^{1-p}|\mathbb{E}(A_{r,l}|\mathcal{G}_{2^r-m})|^p\,, \tag{46}$$

where

$$\lambda_l=\frac{\alpha_l}{\sum_{l=u}^{r-1}\alpha_l}\ \text{ with }\alpha_l=\Big(\sum_{k=1}^{2^{r-l}-1}\|\mathbb{E}(S_{(k+1)2^l}-S_{k2^l}|\mathcal{G}_{k2^l})\|_p^p\Big)^{1/p}\,,$$

and

$$A_{r,l}=\max_{1\le k\le 2^{r-l},k\text{ odd}}|\mathbb{E}(S_{2^r-(k-1)2^l}-S_{2^r-k2^l}|\mathcal{G}_{2^r-k2^l})|\,.$$

Notice now that by Jensen's inequality, $|\mathbb{E}(A_{r,l}|\mathcal{G}_{2^r-m})|^p\le\mathbb{E}(A_{r,l}^p|\mathcal{G}_{2^r-m})$. Hence starting from (46), we get that for any $p\ge 1$,

$$\mathbb{P}\big(\max_{0\le m\le 2^r-1}|\mathbb{E}(S_{2^r}-S_{2^r-m_u}|\mathcal{G}_{2^r-m})|\ge x\big)$$
$$\le \mathbb{P}\big(\max_{0\le m\le 2^r-1}\sum_{l=u}^{r-1}\lambda_l^{1-p}\mathbb{E}(A_{r,l}^p|\mathcal{G}_{2^r-m})\ge x^p\big)\,.$$

Next, since $\big(\sum_{l=u}^{r-1} \lambda_l^{1-p}\mathbb{E}(A_{r,l}^p|\mathcal{G}_k)\big)_{k\geq 1}$ is a martingale, Doob's maximal inequality entails that

$$\mathbb{P}\big(\max_{0\leq m\leq 2^r-1} |\mathbb{E}(S_{2^r} - S_{2^r-m_u}|\mathcal{G}_{2^r-m})| \geq x\big) \leq x^{-p}\sum_{l=u}^{r-1} \lambda_l^{1-p}\mathbb{E}(A_{r,l}^p)\,.$$

Taking into account the fact that $\mathbb{E}(A_{r,l}^p) \leq \alpha_l^p$ together with the definition of α_l and λ_l, we then derive that for any $p \geq 1$,

$$\mathbb{P}\big(\max_{0\leq m\leq 2^r-1} |\mathbb{E}(S_{2^r} - S_{2^r-m_u}|\mathcal{G}_{2^r-m})| \geq x\big)$$
$$\leq x^{-p}\,\Big(\sum_{l=u}^{r-1}\Big(\sum_{k=1}^{2^{r-l}-1} \|\mathbb{E}(S_{(k+1)2^l} - S_{k2^l}|\mathcal{G}_{k2^l})\|_p^p\Big)^{1/p}\Big)^p\,. \tag{47}$$

Starting from (43) and considering the bounds (44), (45) and (47), the proposition follows. $\Diamond$

6.2 Technical results

Lemma 19 *In the context of stationary sequences, for every $\gamma > 0$, $n \geq 1$ and $p \geq 1$,*

$$\frac{1}{n^\gamma}\max_{1\leq k\leq n}\|\mathbb{E}_0(S_k)\|_p \leq c_\gamma \sum_{k=n+1}^{6n}\frac{1}{k^{\gamma+1}}\|\mathbb{E}_0(S_k)\|_p\ ,$$

where $c_\gamma = 2^{3\gamma+3}$.

Proof of lemma 19. Let k be a positive integer and notice first that

$$|\mathbb{E}_0(S_n)| \leq |\mathbb{E}_0(S_{k+n})| + |\mathbb{E}_0(S_{k+n} - S_n)|\ .$$

Then, by the properties of the conditional expectation and stationarity,

$$\|\mathbb{E}_0(S_n)\|_p \leq \|\mathbb{E}_0(S_{k+n})\|_p + \|\mathbb{E}_0(S_k)\|_p\ .$$

So, for any $n \geq 1$,

$$\frac{1}{n^\gamma}\|\mathbb{E}_0(S_n)\|_p = \frac{1}{n^{\gamma+1}}\|\mathbb{E}_0(S_n)\|_p(\sum_{k=n+1}^{2n}1) \leq 2^{\gamma+1}\|\mathbb{E}_0(S_n)\|_p\sum_{k=n+1}^{2n}\frac{1}{k^{\gamma+1}}$$
$$\leq 2^{\gamma+1}\sum_{k=n+1}^{2n}\frac{1}{k^{\gamma+1}}\|\mathbb{E}_0(S_{k+n})\|_p + 2^{\gamma+1}\sum_{k=n+1}^{2n}\frac{1}{k^{\gamma+1}}\|\mathbb{E}_0(S_k)\|_p$$
$$\leq 2^{2\gamma+2}\sum_{k=n+1}^{2n}\frac{1}{(k+n)^{\gamma+1}}\|\mathbb{E}_0(S_{k+n})\|_p + 2^{\gamma+1}\sum_{k=n+1}^{2n}\frac{1}{k^{\gamma+1}}\|\mathbb{E}_0(S_k)\|_p\ .$$

Therefore

$$\frac{1}{n^\gamma}\|\mathbb{E}_0(S_n)\|_p \leq 2^{2\gamma+2}\sum_{l=n+1}^{3n}\frac{1}{l^{\gamma+1}}\|\mathbb{E}_0(S_l)\|_p\ . \tag{48}$$

By writing now, for any positive integer k,

$$|\mathbb{E}_0(S_k)| \leq |\mathbb{E}_0(S_{k+n})| + |\mathbb{E}_0(S_{k+n} - S_k)|\ ,$$

and by using stationarity we obtain

$$\max_{1\leq k\leq n}\|\mathbb{E}_0(S_k)\|_p\leq\max_{n\leq k\leq 2n}\|\mathbb{E}_0(S_k)\|_p+\|\mathbb{E}_0(S_n)\|_p\leq 2\max_{n\leq k\leq 2n}\|\mathbb{E}_0(S_k)\|_p\ ,$$

and the result follows by the inequality (48) applied for each k, $n\leq k\leq 2n$. $\Diamond$

Next result we formulate is Corollary 4.2 in Cuny (2011).

Proposition 20 *Assume $(X_n)_{n\in Z}$ is a stationary sequence of square integrable random variables and $(b_n)_{n\geq 1}$ a positive nondecreasing slowly varying sequence. Assume*

$$\sum_{n\geq 1}\frac{b_n\|S_n\|^2}{n^2}<\infty\ .$$

Then

$$\frac{S_n}{\sqrt{nb_n^*}}\to 0\quad a.s.$$

where $b_n^*:=\sum_{k=1}^n(kb_k)^{-1}$.

We give here a generalized Toeplitz lemma, which is Lemma 5 in M. Peligrad and C. Peligrad (2011).

Lemma 21 *Assume $(x_i)_{i\geq 1}$ and $(c_i)_{i\geq 1}$ are sequences of real numbers such that*

$$\frac{1}{n}\sum_{i=1}^n x_i\to L\ ,\ nc_n\to\infty\ and\ \frac{c_1+...+c_n}{nc_n}\to C<1\ .$$

Then,

$$\frac{\sum_{i=1}^n c_ix_i}{\sum_{i=1}^n c_i}\to L\ .$$

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