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On the Central Limit Theorem and Its Weak Invariance Principle for Strongly Mixing Sequences with Values in a Hilbert Space via Martingale Approximation

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In this paper we not only prove an extension to Hilbert spaces of a sharp central limit theorem for strongly real-valued mixing sequences, but also slightly improve it. The proof is mainly based on the Bernstein blocking technique and approximations by martingale differences. Moreover, we derive also the corresponding functional central limit theorem.

KEY WORDS: Hilbert space; central limit theorem; weak invariance principle; strong mixing sequences; martingale approximation.

1. INTRODUCTION

Central limit theorems (CLT's) and functional central limit theorems (FCLT's) are powerful tools for obtaining asymptotic distribution results for estimators of econometric models. Recently, attention has been increasingly devoted in econometrics to the study of semiparametric and nonparametric estimators. Because such estimators take their values in certain infinite-dimensional spaces, standard finite-dimensional central limit theorems and their weak invariance principles are not applicable. Further, because of the time series nature of much economic data, infinite-dimensional limit theorems for dependent processes are required.

Although the theory of empirical processes mainly deals with the (generally non separable) Banach space $\ell^\infty(\mathscr{F})$ of bounded functionals from

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$\mathscr{F}$ to $\mathbb{R}$, separable Hilbert spaces are sometimes rich enough for statistical applications. We would like to mention the works of Merlevède,[20, 21] Mourid,[25] and Bosq[7] which are in this direction. These authors focus on forecasting and estimation problems for several classes of continuous time processes. To this aim, they deal with linear processes taking their values in separable Hilbert spaces. For these type of processes, limit theorems are derived from the ones available for the sequence generating the linear process; this sequence is often called a white-noise. When the white noise takes its values in a Hilbert space and satisfies certain kinds of dependence, CLT's and FCLT's are already available. For example, a CLT for strictly stationary strong or uniform mixing processes was given by Dehling,[12] later Merlevède *et al.*[22] extended the result of Doukhan *et al.*[14] to the Hilbert spaces. For Hilbert-valued martingale differences, a functional version of the central limit theorem is given by Walk[33] and a triangular version by Jakubowski.[18] Chen and White[10] obtained new Hilbert space CLT's and FCLT's in the mixingale case and gave significant applications for near epoch dependent observations. Recently Dedecker and Merlevède[11] have established a central limit theorem (and also stable convergence) and its weak invariance principle for $\mathbb{H}$-valued strictly stationary sequences under "projective criterion." With their results they recover the special case of $\mathbb{H}$-valued martingale difference sequences and under a strong mixing condition (involving the whole past of the process and just one "future" observation at a time), they give the nonergodic version of the result of Merlevède *et al.*[22]

In this paper we particularly deal with strictly stationary strong mixing processes taking their values in a Hilbert space. For any two σ algebras $\mathscr{A}$ and $\mathscr{B}$, we define the α-mixing coefficient (introduced by Rosenblatt[32]) by

$$\alpha(\mathscr{A}, \mathscr{B}) = \sup_{\substack{A \in \mathscr{A} \\ B \in \mathscr{B}}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\, \mathbb{P}(B)|.$$

Let us start by recalling a sharp result obtained in the real case. Before stating it, we first give the following definition. For any nonnegative real random variable W, define the "upper tail" quantile function via

$$Q_W(u) = \inf\{t \geqslant 0 : \mathbb{P}(W > t) \leqslant u\}.$$

By answering a conjecture of Bradley,[8] Merlevède and Peligrad[23] obtained the following sharp central limit theorem that extends previous results by Ibragimov,[16] Ibragimov and Linnik,[17] and Doukhan *et al.*[14]

Theorem 1.1. Assume that $\{X_k, k \in \mathbb{Z}\}$ is a strictly stationary, centered, real-valued sequence with finite second moment. Set $S_n := \sum_{k=1}^n X_k$,

$\mathscr{M}_0 = \sigma(X_k, k \leqslant 0)$ and $\mathscr{G}^n = \sigma(X_k, k \geqslant n)$. In addition assume that $\alpha_{\infty,\infty}(n) := \alpha(\mathscr{M}_0, \mathscr{G}^n) \longrightarrow 0$, as $n \longrightarrow \infty$,

$$\liminf_{n \to \infty} \frac{\mathbb{E}(S_n)^2}{n} > 0, \tag{1.1}$$

and

$$\int_0^{\alpha_{\infty,\infty}(n)} Q^2_{|X_0|}(u)\, du = o\left(\frac{1}{n}\right) \quad \text{as} \quad n \longrightarrow \infty. \tag{1.2}$$

Then

$$\frac{S_n}{\sqrt{\frac{\pi}{2}}\, \mathbb{E}\, |S_n|} \xrightarrow{\mathscr{D}} N \sim \mathscr{N}(0, 1), \quad \text{as} \quad n \longrightarrow \infty. \tag{1.3}$$

Let us notice that the constant $\sqrt{\frac{\pi}{2}}$ is derived from the fact that $\mathbb{E}\,|N| = \sqrt{\frac{2}{\pi}}$.

The result of the above theorem was predicted by Bradley.[8] In his paper he formulated the following conjecture: Under the assumptions of Theorem 1.1, there exists a sequence $B_n \longrightarrow \infty$ as $n \longrightarrow \infty$, such that $\frac{S_n}{B_n}$ converges in distribution to a standard Gaussian variable.

The proof of Theorem 1.1 in Merlevède and Peligrad[23] is strongly based on the Bernstein-type blocking arguments and on a coupling result for strongly mixing sequences, namely Theorem 4 in Rio.[30] Generally speaking, a coupling result enables us to replace the initial dependent sequence by an independent one, having the same marginals. Moreover, a variable in the newly constructed sequence is independent of the past of the initial sequence and it is close, in some sense, to the variable having the same rank. We refer to Merlevède and Peligrad[24] for a survey on the coupling theorems and their applications to obtain various limit theorems for weakly dependent random variables.

Let us give more details about the proof of Theorem 1.1 given in Merlevède and Peligrad.[23] The index set $\{1, 2, 3, \ldots, n\}$ is partioned into an alternating sequence of "big" blocks of size p_n and "small" blocks of size q_n (in fact we will obtain k_n "big" blocks and k_n "small" blocks, where $k_n = [\frac{n}{p_n + q_n}]$), and the sequence of "big" blocks is approximated by a sequence of independent random variables having the same marginals. Due to Theorem 4 in Rio,[30] this approximation can be carried out in such a way that an upper bound estimate of the distance in $\mathbb{L}_1$ of the sum of the "big" blocks and their approximations can be obtained in term of strong mixing coefficients. We would like to mention that the coefficient of dependence used by this technique a fortiori involves the whole past and the whole future of the process.

In studying the infinite-dimensional case, our question was to what extent Theorem 1.1 remains valid in the new context when we replace X_0 by an infinite-dimensional space valued random variable and the absolute values by the corresponding norms. To see new possible quality effects, we consider a simplest case of infinite dimensional Hilbert space $\mathbb{H}$. To establish the central limit theorem for $\mathbb{H}$-valued random variables, the usual way is first to prove the convergence in distribution for the finite dimensional laws of the normalized partial sums process, and second to prove tightness of this process. We would like to mention that a first idea considered by the author to prove tightness was again to use Bernstein-type blocking arguments and after to approximate the sequence of "big" blocks by an independent one in such a way that we can compute an upper bound estimate of the distance in $\mathbb{L}_1$ (or even in probability) between each summand of these two sequences in terms of strongly mixing coefficients, and then a result of type Theorem 4 in Rio[30] would be needed in the new context of an infinite dimensional space. However, following Dehling,[13] it turns out that such a result has no chance to hold in an infinite dimensional space. The only coupling results in this direction are in terms of absolute regularity coefficients or of φ-coefficients (see Merlevède and Peligrad[24] for a survey of these results). Then using Berbee's lemma[1] and Bryc's construction[9] of approximating variables, it is easy to see that Theorem 1.1 can be extended to Hilbert-space values random variables, the absolute regularity coefficient replacing the strong mixing coefficient and the norm the absolute values. However such a result is not satisfactory.

By wanting to avoid coupling to prove tightness, we may proceed as follows. First we partion the index set $\{1, 2, 3,..., n\}$ into an alternating sequence of "big" blocks of size p_n and "small" blocks of size q_n and then we approximate these blocks by martingale difference arrays. By this method, the central limit theorem will be derived from central limit theorem for martingale difference arrays with values in a Hilbert space.

In this paper we not only prove that Theorem 1.1 can be extended in its full generality but we also slightly sharpen its statement in the real case. To be more precise the coefficients $\alpha_{\infty,\infty}(n)$ can be replaced by the weaker one: $\alpha_{2,\infty}(n) := \sup_{k \geqslant 0} \alpha(\mathscr{M}_0, \sigma(X_n, X_{n+k}))$. This coefficient involves the whole past of the process and only two variables in the future.

The remainder of our paper is organized as follows. Section 2 is devoted to the statements of the results whereas all the proofs are postponed to Section 3.

In all the paper, we use the notation $c_n \ll d_n$ to mean $c_n = O(d_n)$.

2. RESULTS

Let $\mathbb{H}$ be a separable real Hilbert space with the norm $\|\cdot\|_{\mathbb{H}}$ generated by an inner product, $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ and let $\{e_\ell\}_{\ell\geqslant 1}$ be an orthonormal basis in $\mathbb{H}$.

From now on, a strictly stationary sequence $(X_k, k\in\mathbb{Z})$ of random variables defined on $(\Omega, \mathscr{A}, \mathbb{P})$ with values in $\mathbb{H}$ will be called strongly mixing if

$$\alpha_{2,\infty}(n) := \sup_{k\geqslant 0} \alpha(\mathscr{M}_0, \sigma(X_n, X_{n+k})) \longrightarrow 0 \qquad \text{as} \quad n\longrightarrow\infty,$$

where $\mathscr{M}_j := \sigma(X_i, i\leqslant j), j\in\mathbb{Z}$.

This type of mixing coefficients (involving the whole past of the process and just two "future" observations at a time) has been already used by several authors, more often when random fields are considered rather than sequences (see, for instance, Bolthausen[(6)]).

Theorem 2.1. Assume that $(X_k, k\in\mathbb{Z})$ is a strictly stationary, strongly mixing sequence of zero mean random variables defined on $(\Omega, \mathscr{A}, \mathbb{P})$ with values in $\mathbb{H}$ such that $\mathbb{E}\,\|X_0\|^2_{\mathbb{H}}<\infty$. Set $S_n := \sum_{k=1}^n X_k$. In addition assume that

$$\liminf_{n\to\infty} \frac{\mathbb{E}\,\|S_n\|^2_{\mathbb{H}}}{n} > 0, \tag{2.1}$$

$$\int_0^{\alpha_{2,\infty}(n)} Q^2_{\|X_0\|_{\mathbb{H}}}(u)\,du = o\left(\frac{1}{n}\right) \qquad \text{as} \quad n\longrightarrow\infty, \tag{2.2}$$

$$\lim_{n\to\infty} \frac{1}{\mathbb{E}\,\|S_n\|^2_{\mathbb{H}}}\,\mathbb{E}\langle S_n, e_i\rangle_{\mathbb{H}}\,\langle S_n, e_j\rangle_{\mathbb{H}} = \sigma_{ij}, \qquad \text{for all} \quad i,j\geqslant 1, \tag{2.3}$$

and

$$\sum_{i=1}^{\infty} \sigma_{ii} = 1. \tag{2.4}$$

Then for

$$B_n := \frac{\mathbb{E}\,\|S_n\|_{\mathbb{H}}}{\mathbb{E}\,\|\mu\|_{\mathbb{H}}}, \tag{2.5}$$

where μ is a centered Gaussian $\mathbb{H}$-valued random variable with covariance operator $T = (\sigma_{ij})_{i,j\geqslant 1}$, we have

$$\frac{S_n}{B_n} \xrightarrow{\mathscr{D}} \mu. \tag{2.6}$$

Let us make some comments.

1. Observe that if the dimension of the Hilbert space is equal to m, with m finite, then $\limsup_{n\to\infty} \mathbb{E}\,\|S_n\|_{\mathbb{H}}^2 = \sum_{i=1}^{m} \limsup_{n\to\infty} \mathbb{E}\langle S_n, e_i\rangle_{\mathbb{H}}^2$. Hence for a Hilbert space of finite dimension, the condition (2.3) obviously implies (2.4).

2. If ess sup $\|X_0\|_{\mathbb{H}} < \infty$, then (2.2) can be replaced by

$$\lim_{n\to\infty} n\alpha_{2,\infty}(n) = 0$$

and if $\mathbb{E}\,\|X_0\|_{\mathbb{H}}^{2+\delta} < \infty$ for a $\delta > 0$, by

$$\lim_{n\to\infty} n\alpha_{2,\infty}(n)^{\delta/(2+\delta)} = 0.$$

3. A nonnegative linear and self-adjoint operator T on $\mathbb{H}$ is called an $\mathscr{S}(\mathbb{H})$-operator if it has finite trace; i.e., for some (and therefore every) orthonormal basis $\{e_\ell, \ell \geqslant 1\}$ of $\mathbb{H}$, $\sum_{\ell \geqslant 1} \langle Te_\ell, e_\ell\rangle_{\mathbb{H}} < \infty$. In view of this definition and according to (2.4), the operator T involved in Theorem 2.1 is an $\mathscr{S}(\mathbb{H})$-operator.

4. In view of the proof of Theorem 2.1, we believe that the coefficient of dependence $\alpha_{2,\infty}(n)$ cannot be replaced by the weaker one $\alpha_{1,\infty}(n) := \alpha(\mathscr{M}_0, \sigma(X_n))$ without requiring for instance ergodicity or perhaps by replacing the normalizing sequence by a random one!

Let us also notice that requiring that $\alpha_{2,\infty}(n)$ is asymptotically negligible implies that for any, k in $\mathbb{Z}$, $\sigma(X_0, X_k)$ is independent of the σ-algebra of all invariants sets by B, B denoting the bijective bimeasurable transformation preserving the probability $\mathbb{P}$, and also that $\sigma(X_0, X_k)$ is independent of the σ-algebra $\mathscr{M}_{-\infty} = \bigcap_{i\in\mathbb{Z}} \sigma(X_k, k \leqslant i)$.

5. According to the proof of Theorem 2.1, it is not difficult to see that (2.6) also holds for the stable convergence. This concept was first introduced for real valued random variables by Rényi[28] and later generalized by Bingham to locally abelian groups in 1986[4] and to Hilbert spaces in 2000.[5] We do not give more details about this concept here since its is widely studied in Dedecker and Merlevède.[11]

Before stating the functional version of Theorem 2.1, let us start with some considerations.

Let $C_{\mathbb{H}}[0, 1]$ be the set of all continuous $\mathbb{H}$-valued functions on $[0, 1]$. This becomes a separable Banach space under the sup-norm $\|x\|_\infty = \sup\{\|x(t)\|_{\mathbb{H}} : t \in [0, 1]\}$.

Set $S_k := \sum_{j=1}^{k} X_j$ and define the process $\{S_n(t): t \in [0, 1]\}$ by

$$S_n(t) = S_{[nt]} + (nt - [nt])\, X_{[nt]+1},$$

square brackets designating here and throughout the paper the integer part, as usual.

For each ω, $S_n(.)$ is an element of $C_{\mathbb{H}}[0, 1]$. A $C_{\mathbb{H}}[0, 1]$-valued random element W is a Brownian motion in $\mathbb{H}$ if the following conditions are satisfied:

(a) $W(0) = 0$,

(b) the increments on disjoint time intervals are independent,

(c) for all $0 \leqslant t < t+s \leqslant 1$, the increment $W(t+s) - W(t)$ has a Gaussian distribution on $\mathbb{H}$ with mean zero and covariance operator sT, where $T \in \mathscr{S}(\mathbb{H})$ does not depend on t, s.

Theorem 2.2. Assume that the conditions of Theorem 2.1 are satisfied. Then the sequence $\{B_n^{-1} S_n(t): t \in [0, 1]\}$, where B_n is defined by (2.5), converges in distribution in $C_{\mathbb{H}}[0, 1]$ to the Brownian Motion in $\mathbb{H}$, W, having T as covariance operator.

3. PROOFS

3.1. Preparatory Material

In this subsection we collect some preliminary material.

Before stating the first lemma, let us recall the following result which is an extension to Hilbert space valued random variables of Rio's covariance inequality:[29]

For two $\mathbb{H}$-valued random variables, X and Y, with the quantile functions respectively $Q_{\|X\|_{\mathbb{H}}}(x)$ and $Q_{\|Y\|_{\mathbb{H}}}(y)$, Merlevède *et al.*,[22] proved the following covariance inequality:

$$|E\langle X, Y\rangle_{\mathbb{H}} - \langle EX, EY\rangle_{\mathbb{H}}| \leqslant 18 \int_0^{\alpha(\sigma(X), \sigma(Y))} Q_{\|X\|_{\mathbb{H}}}(u)\, Q_{\|Y\|_{\mathbb{H}}}(u)\, du. \tag{3.1}$$

The next lemma refers to the structure of $\mathbb{E}\, \|S_n\|_{\mathbb{H}}^2$, and is an extension to variables with values in a Hilbert space of Lemma 1.2 in Merlevède and Peligrad.[23]

Lemma 3.1. Assume that $\{X_k, k \in \mathbb{Z}\}$ is a strictly stationary sequence of zero mean random variables with values in $\mathbb{H}$, such that $\mathbb{E}\,\|X_0\|_{\mathbb{H}}^2 < \infty$, (2.1) holds and

$$\lim_{n \to \infty} n\mathbb{E}\langle X_0, X_n \rangle_{\mathbb{H}} = 0. \tag{3.2}$$

Then we have the representation

$$\sigma_n^2 = nh(n), \tag{3.3}$$

where $\sigma_n^2 = \mathbb{E}\,\|S_n\|_{\mathbb{H}}^2$ and $h(n)$ is a slowly varying function of n. Moreover, $h(n)$ has an extension to the whole real line which is slowly varying.

Proof of Lemma 3.1. The proof is rigourously the same than the one of Lemma 1.2 in Merlevède and Peligrad[23] by only noticing that by stationarity, we have for $k > 1$

$$\sigma_{nk}^2 = \sigma_{n(k-1)}^2 + \sigma_n^2 + 2\mathbb{E}\left\langle \sum_{i=1}^{n} X_i, \sum_{\ell=n+1}^{nk} X_\ell \right\rangle_{\mathbb{H}}$$

and

$$\left| \mathbb{E}\left\langle \sum_{i=1}^{n} X_i, \sum_{\ell=n+1}^{nk} X_\ell \right\rangle_{\mathbb{H}} \right| \leqslant \sum_{i=1}^{nk-1} i\,|\mathbb{E}\langle X_0, X_i \rangle_{\mathbb{H}}|.$$

Remark 3.1. By using (3.1), it is easy to see that (2.2) implies (3.2). Then it follows that the conclusions of Lemma 3.1 hold under the assumptions of Theorem 2.1.

The next lemma deals with a moment inequality for the sum of *real* random variables. It is straightforward using Theorem 2.5 of Rio.[31]

Lemma 3.2. Assume that $\{\xi_k, k \in \mathbb{Z}\}$ is a strictly stationary sequence of real random variables, such that for each $k \geqslant 1$, $\mathbb{P}(|\xi_k| \leqslant A) = 1$. Set $\alpha_{1,\infty}(n) = \alpha(\sigma(\xi_j, j \leqslant 0), \sigma(\xi_n))$. Then

$$\mathbb{E}\left(\sum_{i=1}^{m} \{\xi_i - \mathbb{E}\xi_i\} \right)^4 \leqslant 8192 A^2 m^2 \sum_{k=0}^{m-1} (k+1) \int_0^{\alpha_{1,\infty}(k)} Q_{|\xi_0|}^2(u)\, du.$$

Proof of Lemma 3.2. By using stationarity together with Theorem 2.5 of Rio,[31] we derive that for all $1 \leqslant i \leqslant j \leqslant m$

$$\mathbb{E}\left(\sum_{k=1}^{m} \{\xi_k - \mathbb{E}\xi_k\}\right)^4 \leqslant (16m)^2 \int_0^1 (\alpha_{1,\infty}^{-1}(u) \wedge m)^2 \, Q^4_{|\xi_0 - \mathbb{E}\xi_0|}(u)\, du,$$

where $\alpha_{1,\infty}^{-1}(u) \wedge m = \sum_{k=0}^{m-1} \mathbb{1}(u < \alpha_{1,\infty}(k))$.

Next using the fact that $|\xi_0| \leqslant A$ combined with relation (C.10) on p. 157 of Rio,[31] we easily derive that

$$\begin{aligned}
&\mathbb{E}\left(\sum_{k=i}^{j} \{\xi_i - \mathbb{E}\xi_i\}\right)^4 \\
&\quad\leqslant 1024Am^2 \sum_{k=0}^{m-1} (k+1) \int_0^{\alpha_{1,\infty}(k)} Q^3_{|\xi_0 - \mathbb{E}\xi_0|}(u)\, du \\
&\quad\leqslant 2048A^2m^2 \sum_{k=0}^{m-1} (k+1) \int_0^{\alpha_{1,\infty}(k)} Q^2_{|\xi_0 - \mathbb{E}\xi_0|}(u)\, du \\
&\quad\leqslant 4096A^2m^2 \sum_{k=0}^{m-1} (k+1) \left\{\int_0^{\alpha_{1,\infty}(k)} Q^2_{|\xi_0|}(u)\, du + \alpha_{1,\infty}(k)\, \mathbb{E}\xi_0^2\right\}.
\end{aligned}$$

Next using the fact that for all $k \geqslant 0$

$$\mathbb{E}\xi_0^2 = \int_0^1 Q^2_{|\xi_0|}(u)\, du \leqslant \alpha_{1,\infty}^{-1}(k) \int_0^{\alpha_{1,\infty}(k)} Q^2_{|\xi_0|}(u)\, du,$$

we get the desired result.

The next lemma deals with the deviation probability of the maximum cumulative sum of *hilbertian* random variables.

Lemma 3.3. There exist positive constants C_1 and C_2 such that the following holds:

Suppose $\{X_k, k \in \mathbb{Z}\}$ is a strictly stationary sequence of random variables with values in $\mathbb{H}$. Suppose T is a positive number, and $\mathbb{P}(\|X_0\|_{\mathbb{H}} \leqslant T) = 1$. Set $S_k = \sum_{i=1}^{k} (X_i - \mathbb{E}(X_i))$. Suppose m and q are positive integers such that $m \geqslant q$. Then for every $\lambda \geqslant 8Tq$, one has that

$$\begin{aligned}
&\mathbb{P}(\sup_{1 \leqslant k \leqslant m} \|S_k\|_{\mathbb{H}} \geqslant \lambda) \\
&\quad\leqslant C_1 \lambda^{-4} m^2 T^2 \mathbb{E} \left\| \sum_{i=1}^{q} (X_i - \mathbb{E}(X_i)) \right\|_{\mathbb{H}}^2 + C_2 \lambda^{-1} m \mathbb{E}\, \|\mathbb{E}(X_q - \mathbb{E}(X_q) \mid \mathscr{M}_0)\|_{\mathbb{H}}.
\end{aligned}$$

Proof of Lemma 3.3. We start the proof by using an inequality similar to the one stated p. 83 line 7 in Rio.[31] To be more precise, define $S_0 = 0_{\mathbb{H}}$ and set $U_i = S_{iq} - S_{iq-q}$ for $i \in \{1, 2, \ldots, [m/q]\}$, and $U_i = 0_{\mathbb{H}}$ for $i \geqslant [m/q]+1$. Since each integer $k \in \{1, \ldots, m\}$ is distant by less than q from some element of $\{q, 2q, \ldots, [m/q]\, q\}$, we get

$$\sup_{1 \leqslant k \leqslant m} \|S_k\|_{\mathbb{H}} \leqslant 2qT + \sup_{1 \leqslant j \leqslant [m/q]} \left\| \sum_{i=1}^{j} U_i \right\|_{\mathbb{H}}.$$

Setting now for all $i \geqslant 1$, $\mathscr{F}_i^U = \sigma(X_j, j \leqslant iq)$, we define a sequence $(\tilde{U}_i)_{i \geqslant 1}$ as follows: for all $i \geqslant 1$, $\tilde{U}_{2i-1} = U_{2i-1} - \mathbb{E}(U_{2i-1} \mid \mathscr{F}^U_{2(i-1)-1})$ and $\tilde{U}_{2i} = U_{2i} - \mathbb{E}(U_{2i} \mid \mathscr{F}^U_{2(i-1)})$. Substituting the variables $\tilde{U}_i$ to the initial variables, we get

$$\sup_{1 \leqslant j \leqslant [m/q]} \left\| \sum_{i=1}^{j} U_i \right\|_{\mathbb{H}} \leqslant \sup_{1 \leqslant j \leqslant [[m/q]/2]} \left\| \sum_{i=1}^{j} \tilde{U}_{2i} \right\|_{\mathbb{H}} + \sup_{1 \leqslant j \leqslant [[m/q]/2]+1} \left\| \sum_{i=1}^{j} \tilde{U}_{2i-1} \right\|_{\mathbb{H}} + \sum_{i=1}^{[m/q]} \|U_i - \tilde{U}_i\|_{\mathbb{H}}.$$

Then since $\lambda \geqslant 8Tq$, we obtain that

$$\mathbb{P}(\sup_{1 \leqslant k \leqslant m} \|S_k\|_{\mathbb{H}} \geqslant \lambda) \leqslant \mathbb{P}\left(\sup_{1 \leqslant j \leqslant [[m/q]/2]} \left\| \sum_{i=1}^{j} \tilde{U}_{2i} \right\|_{\mathbb{H}} \geqslant \lambda/4 \right) + \mathbb{P}\left(\sup_{1 \leqslant j \leqslant [[m/q]/2]+1} \left\| \sum_{i=1}^{j} \tilde{U}_{2i-1} \right\|_{\mathbb{H}} \geqslant \lambda/4 \right) + \mathbb{P}\left(\sum_{i=1}^{[m/q]} \|U_i - \tilde{U}_i\|_{\mathbb{H}} \geqslant \lambda/4 \right).$$

Next applying Markov's inequality we get that

$$\mathbb{P}(\sup_{1 \leqslant k \leqslant m} \|S_k\|_{\mathbb{H}} \geqslant \lambda) \leqslant \frac{4^4}{\lambda^4} \mathbb{E}\left(\sup_{1 \leqslant j \leqslant [[m/q]/2]} \left\| \sum_{i=1}^{j} \tilde{U}_{2i} \right\|_{\mathbb{H}}^4 \right) + \frac{4^4}{\lambda^4} \mathbb{E}\left(\sup_{1 \leqslant j \leqslant [[m/q]/2]+1} \left\| \sum_{i=1}^{j} \tilde{U}_{2i-1} \right\|_{\mathbb{H}}^4 \right) + \frac{4}{\lambda} \sum_{i=1}^{[m/q]} \mathbb{E}\, \|U_i - \tilde{U}_i\|_{\mathbb{H}}.$$

Notice now that $\{\tilde{U}_{2i}\}_{i \geqslant 1}$ (resp. $\{\tilde{U}_{2i-1}\}_{i \geqslant 1}$) is an $\mathbb{H}$-valued martingale difference sequence with respect to the filtration $\{\mathscr{F}^U_{2i}\}_{i \geqslant 1}$ (resp. $\{\mathscr{F}^U_{2i-1}\}_{i \geqslant 1}$).

Then the extension of Burkholder's inequality to Hilbert spaces (see relation (2.10) in Pinelis[27]) entails that there exists a finite constant c such that

$$\mathbb{E}\left(\sup_{1\leqslant j\leqslant [[m/q]/2]}\left\|\sum_{i=1}^{j}\tilde{U}_{2i}\right\|_{\mathbb{H}}^{4}\right)\leqslant c\,\mathbb{E}\left(\sum_{i=1}^{[[m/q]/2]}\|\tilde{U}_{2i}\|_{\mathbb{H}}^{2}\right)^{2}.$$

Now since $\|X_0\|_{\mathbb{H}}\leqslant T$ a.s. it follows that $\|X_0-\mathbb{E}X_0\|_{\mathbb{H}}\leqslant 2T$ a.s. Consequently for all $i\geqslant 1$, $\|U_i\|_{\mathbb{H}}\leqslant 2qT$ a.s. and $\|\tilde{U}_i\|_{\mathbb{H}}\leqslant 4qT$ a.s. Then

$$\begin{aligned}\mathbb{E}\left(\sum_{i=1}^{[[m/q]/2]}\|\tilde{U}_{2i}\|_{\mathbb{H}}^{2}\right)^{2}&\leqslant\frac{m}{2q}\cdot(4qT)^2\cdot\left(\sum_{i=1}^{[[m/q]/2]}\mathbb{E}\,\|\tilde{U}_{2i}\|_{\mathbb{H}}^{2}\right)\\&\leqslant 8mqT^2\cdot\left(\sum_{i=1}^{[[m/q]/2]}\mathbb{E}\,\|U_{2i}\|_{\mathbb{H}}^{2}\right)\\&\leqslant 4m^2T^2\,\mathbb{E}\,\|S_q\|_{\mathbb{H}}^{2}.\end{aligned}$$

Consequently,

$$\mathbb{E}\left(\sup_{1\leqslant j\leqslant [[m/q]/2]}\left\|\sum_{i=1}^{j}\tilde{U}_{2i}\right\|_{\mathbb{H}}^{4}\right)\leqslant 4cm^2T^2\,\mathbb{E}\,\|S_q\|_{\mathbb{H}}^{2}.$$

Obviously a similar inequality holds for $\mathbb{E}(\sup_{1\leqslant j\leqslant [[m/q]/2]+1}\|\sum_{i=1}^{j}\tilde{U}_{2i-1}\|_{\mathbb{H}}^{4})$. On the other hand, using stationarity we derive that

$$\begin{aligned}\sum_{i=1}^{[m/q]}\mathbb{E}\,\|U_i-\tilde{U}_i\|_{\mathbb{H}}&\leqslant (m/q)\,\mathbb{E}\,\|\mathbb{E}(S_{2q}-S_q\mid\mathscr{M}_0)\|_{\mathbb{H}}\\&\leqslant (m/q)\sum_{k=q+1}^{2q}\mathbb{E}\,\|\mathbb{E}(X_k-\mathbb{E}(X_k)\mid\mathscr{M}_0)\|_{\mathbb{H}}\\&\leqslant m\,\mathbb{E}\,\|\mathbb{E}(X_q-\mathbb{E}(X_q)\mid\mathscr{M}_0)\|_{\mathbb{H}}.\end{aligned}$$

Combining all the upper bounds, we obtain the desired result.

3.2. Proof of Theorem 2.1

The proof is first based on the Bernstein-type blocking arguments, namely the index set $\{1, 2, 3,\ldots, n\}$ is partioned into an alternating sequence of "big" blocks of size p_n and "small" blocks of size q_n such that $q_n=o(p_n)$ (in fact we will obtain k_n "big" blocks and k_n "small" blocks, where $k_n=[\frac{n}{p_n+q_n}]$), and then on approximations by martingale differences of these blocks. I would like to mention that this method has been already suggested by other authors; see, for instance, p. 243 in Philipp.[26]

Then first of all we need to define the sequences p_n and q_n used for the blocking arguments. These sequences are selected as in the proof of Merlevède and Peligrad,[23] namely they are somehow implicit solutions of an equation involving the mixing coefficients and a continuous approximation of the quantile function. To be more precise, let us recall this construction since it is needed in what follows.

First the sequence $(\alpha_{2,\infty}(n), n \geqslant 1)$ can be extended to a continuous, nonincreasing function $\alpha(.)\colon [0, \infty) \longrightarrow (0, 1/4]$ with $\alpha(0) = 1/4$, $\alpha(n) = \alpha_{2,\infty}(n)$ for $n = 1, 2, 3,...$ and $\alpha(x)$ is linear on $[n, n+1]$ (if $\alpha_{2,\infty}(n) = 0$ for some $n \geqslant 1$ (and hence for all larger n) then for all such n, replace $\alpha_{2,\infty}(n)$ by a very small positive number, preserving monotonicity and Eq. (2.2)).

On the other hand, according to Lemma 2.3 in Merlevède and Peligrad,[23] there exists a nonincreasing and continuous on (0, 1) function $Q_\star(.)$ such that for all $u \in (0, 1)$, $Q_{\|X_0\|_{\mathbb{H}}}(u) \leqslant Q_\star(u)$ and which under (2.2) satisfies

$$\lim_{x \to \infty} x \int_0^{\alpha(x)} Q_\star^2(u)\, du = 0.$$

Moreover we take a continuous, nondecreasing function $a(.)\colon [0, \infty) \longrightarrow [1, \infty)$ with $a(x) \longrightarrow \infty$ more slowly than $\ln(x)$ and $a(x+1)/a(x) \longrightarrow 1$ as $x \longrightarrow \infty$ and which in addition satisfies the relations (2.7) and (2.8) in Merlevède and Peligrad,[23] namely

$$\lim_{x \to \infty} a(x)\, x \int_0^{\alpha(x)} Q_\star^2(u)\, du = 0 \tag{3.4}$$

and

$$\lim_{x \to \infty} a(x)\, x\alpha(x)\, Q_\star^2(\alpha(x)) = 0. \tag{3.5}$$

Then we set $q_n := [\theta_n] + 1$ and $p_n := [a(\theta_n)\, \theta_n] + 1$, θ_n being the unique positive number such that

$$a^2(\theta_n)\, \theta_n^2 Q_\star^2(\alpha(\theta_n)) = n. \tag{3.6}$$

Note that θ_n is strictly increasing as n increases, and $\theta_n \longrightarrow \infty$ as $n \longrightarrow \infty$.

We now divide the variables $\{X_i\}$ in big blocks of size p_n and small blocks of size q_n in the following way: Let us set $k_n = [\frac{n}{p_n + q_n}]$. For a given positive integer n, the set 1, 2,..., n is being partitioned into blocks of consecutive integers, the blocks being $I_1, J_1,..., I_{k_n}, J_{k_n}, J_{k_n+1}$, such that for each

$1 \leqslant j \leqslant k_n$, I_j contains p_n integers and J_j contains q_n integers, while J_{k_n+1} contains at most $(p_n + q_n - 1)$ integers.

Denote by $Y_{j,n} := \sum_{i \in I_j} X_i$ and $Z_{j,n} := \sum_{i \in J_j} X_i$ for $1 \leqslant j \leqslant k_n$ and $Z_{k_n+1,n} := \sum_{i \in J_{k_n+1}} X_i$ and let us truncate the variables X_i in the following way:

Set $T_n = Q_{\|X_0\|_{\mathbb{H}}}(\alpha(\theta_n))$ and

$$X'_i = \begin{cases} X_i \mathbb{1}(\|X_i\|_{\mathbb{H}} \leqslant T_n) - \mathbb{E} X_i \mathbb{1}(|X_i| \leqslant T_n) & \text{if } X_i \text{ is an unbounded sequence} \\ X_i & \text{if ess.sup } \|X_i\|_{\mathbb{H}} = A \text{ a.s.} \end{cases}$$

and (3.7)

$$X''_i = \begin{cases} X_i \mathbb{1}(\|X_i\|_{\mathbb{H}} > T_n) - \mathbb{E} X_i \mathbb{1}(\|X_i\|_{\mathbb{H}} > T_n) & \text{if } X_i \text{ is an unbounded sequence} \\ 0 & \text{if ess.sup } \|X_i\|_{\mathbb{H}} = A \text{ a.s.} \end{cases}$$

For $j = 1, 2,..., k_n$, set $Y'_{j,n} := \sum_{i \in I_j} X'_i$ and $Z'_{j,n} := \sum_{i \in J_j} X'_i$ and $Z'_{k_n+1,n} := \sum_{i \in J_{k_n+1}} X'_i$. Now we write

$$S_n = \sum_{k=1}^{k_n} Y'_{k,n} + \sum_{k=1}^{k_n+1} Z'_{k,n} + \sum_{i=1}^{n} X''_i.$$

Take $b_n^2 = k_n \sigma_{p_n}^2$ with $\sigma_n^2 := \mathbb{E} \|S_n\|_{\mathbb{H}}^2$ and notice that by using (2.1) together with similar arguments leading to the proof of (3.17) in Merlevède and Peligrad,[23] we derive that

$$\lim_{n \to \infty} \frac{\mathbb{E} \sum_{i=1}^{n} \|X''_i\|_{\mathbb{H}}}{b_n} = 0. \tag{3.8}$$

Now let us focus on the asymptotic behaviour of $\{\frac{\sum_{k=1}^{k_n} Z'_{k,n}}{b_n}\}_{k_n \geqslant 1}$. First set $\mathscr{F}^Z_{k,n} = \sigma(X_j, j \leqslant k(p_n + q_n))$ and notice that we have the following decomposition:

$$\frac{\sum_{k=1}^{k_n} Z'_{k,n}}{b_n} = \frac{\sum_{k=1}^{k_n} [Z'_{k,n} - \mathbb{E}(Z'_{k,n} \mid \mathscr{F}^Z_{k-1,n})]}{b_n} + \frac{\sum_{k=1}^{k_n} \mathbb{E}(Z'_{k,n} \mid \mathscr{F}^Z_{k-1,n})}{b_n}. \tag{3.9}$$

First notice that since $\{V_{k,n} := Z'_{k,n} - \mathbb{E}(Z'_{k,n} \mid \mathscr{F}^Z_{k-1,n})\}_{k \geqslant 1}$ is a martingale difference array with respect to the filtration $\mathscr{F}^Z_{k,n}$, we get

$$\mathbb{E} \left\| \frac{\sum_{k=1}^{k_n} [Z'_{k,n} - \mathbb{E}(Z'_{k,n} \mid \mathscr{F}^Z_{k-1,n})]}{b_n} \right\|_{\mathbb{H}}^2 = \frac{\sum_{k=1}^{k_n} \mathbb{E} \|Z'_{k,n} - \mathbb{E}(Z'_{k,n} \mid \mathscr{F}^Z_{k-1,n})\|_{\mathbb{H}}^2}{b_n^2},$$

whence by using stationarity, the selection of b_n and the fact that $\mathbb{E}\,\|\mathbb{E}(Z'_{2,n} \mid \mathscr{F}^Z_{1,n})\|^2_{\mathbb{H}} \leqslant \mathbb{E}\,\|Z'_{1,n}\|^2_{\mathbb{H}}$, we derive that

$$\mathbb{E}\left\|\frac{\sum_{k=1}^{k_n}[Z'_{k,n}-\mathbb{E}(Z'_{k,n} \mid \mathscr{F}^Z_{k-1,n})]}{b_n}\right\|^2_{\mathbb{H}} \leqslant \frac{4\mathbb{E}\,\|Z'_{1,n}\|^2_{\mathbb{H}}}{\sigma^2_{p_n}} := \frac{4\sigma'^2_{q_n}}{\sigma^2_{p_n}}, \tag{3.10}$$

where $\sigma'^2_n := \mathbb{E}\,\|S'_n\|^2_{\mathbb{H}}$. To compute the upper bound estimate of inequality (3.10), some considerations are needed. First set $S''_n := \sum_{i=1}^n X''_i$ and notice that for $r_n \ll p_n$, stationarity together with inequality (3.1) and Assumption (2.1) yield

$$\begin{aligned}\operatorname{Var}\left(\frac{S''_{r_n}}{\sigma_{p_n}}\right) &= \frac{r_n \operatorname{Var}(X''_1)}{\sigma^2_{p_n}} + 2\,\frac{\sum_{i=1}^{r_n-1}(r_n-i)\operatorname{Cov}(X''_1, X''_{i+1})}{\sigma^2_{p_n}} \\ &\ll \mathbb{E}(\|X_0\|^2_{\mathbb{H}}\,\mathbb{1}(\|X_0\|_{\mathbb{H}} > T_n)) + \sum_{i=1}^{r_n-1}\int_0^{\alpha_{2,\infty}(i)} Q^2_{\|X_0\|_{\mathbb{H}}\,\mathbb{1}(\|X_0\|_{\mathbb{H}} > T_n)}(u)\,du.\end{aligned}$$

(Here and below, for square-integrable $\mathbb{H}$-valued random variables V and W, we use the notations $\operatorname{Var}(V) := \mathbb{E}\,\|V-\mathbb{E}V\|^2_{\mathbb{H}}$ and $\operatorname{Cov}(V, W) := \mathbb{E}\langle V-\mathbb{E}V, W-\mathbb{E}W\rangle_{\mathbb{H}}$.)

The first term in the right-hand side of this above inequality tends obviously to zero as $n \to \infty$, whereas to treat the second one we first observe that

$$Q_{\|X_0\|_{\mathbb{H}}\,\mathbb{1}(\|X_0\|_{\mathbb{H}} > T_n)}(u) \leqslant \begin{cases} Q_{\|X_0\|_{\mathbb{H}}}(u) & \text{if} \quad u < \alpha(\theta_n) \\ 0 & \text{if} \quad u \geqslant \alpha(\theta_n). \end{cases} \tag{3.11}$$

Then by taking into account the selection of p_n, it follows that (for $r_n \ll p_n$)

$$\operatorname{Var}\left(\frac{S''_{r_n}}{\sigma_{p_n}}\right) \ll a(\theta_n)\,\theta_n \int_0^{\alpha(\theta_n)} Q^2_\star(u)\,du,$$

whence according to (3.4) we get for $r_n \ll p_n$,

$$\lim_{n\to\infty} \operatorname{Var}\left(\frac{S''_{r_n}}{\sigma_{p_n}}\right) = 0. \tag{3.12}$$

Then, if we denote by $\sigma''^2_n := \mathbb{E}\,\|S''_n\|^2_{\mathbb{H}}$, we have particularly shown both

$$\frac{\sigma''^2_{q_n}}{\sigma^2_{p_n}} \to 0 \qquad \text{as} \quad n \to \infty, \tag{3.13}$$

and

$$\frac{\sigma''^2_{p_n}}{\sigma^2_{p_n}} \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow \infty. \tag{3.14}$$

In view of (3.13), to prove that the upper bound estimate in inequality (3.10) is tending to zero, it suffices to show that $\lim_{n\to\infty} \frac{\sigma^2_{q_n}}{\sigma^2_{p_n}} = 0$. This result holds according to the representation (3.3) (see the proof in Merlevède and Peligrad[23] for more details). Then we get

$$\frac{\sigma'^2_{q_n}}{\sigma^2_{p_n}} \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow \infty. \tag{3.15}$$

Now we show that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{k_n} \mathbb{E}\, \|\mathbb{E}(Z'_{k,n} \mid \mathscr{F}^Z_{k-1,n})\|_{\mathbb{H}}}{b_n} = 0$$

$$\text{and hence that } \lim_{n\to\infty} \mathbb{E} \left\| \frac{\sum_{k=1}^{k_n} \mathbb{E}(Z'_{k,n} \mid \mathscr{F}^Z_{k-1,n})}{b_n} \right\|_{\mathbb{H}} = 0. \tag{3.16}$$

To this aim we first observe that by stationarity

$$\begin{aligned}
\sum_{k=1}^{k_n} \mathbb{E}\, \|\mathbb{E}(Z'_{k,n} \mid \mathscr{F}^Z_{k-1,n})\|_{\mathbb{H}} &\leqslant \sum_{k=1}^{k_n} \sum_{i=(k-1)(p_n+q_n)+p_n+1}^{k(p_n+q_n)} \mathbb{E}\, \|\mathbb{E}(X'_i \mid \mathscr{M}_{(k-1)(p_n+q_n)})\|_{\mathbb{H}} \\
&= k_n \sum_{i=p_n+1}^{p_n+q_n} \mathbb{E}\, \|\mathbb{E}(X'_i \mid \mathscr{M}_0)\|_{\mathbb{H}}.
\end{aligned} \tag{3.17}$$

Next using the fact that for all $i \geqslant 1$, $\mathbb{E}(X'_i \mid \mathscr{M}_0)$ is an $\mathscr{M}_0$-measurable random variable, we get

$$\begin{aligned}
\mathbb{E}\, \|\mathbb{E}(X'_i \mid \mathscr{M}_0)\|_{\mathbb{H}} &= \mathbb{E} \left\langle \mathbb{E}(X'_i \mid \mathscr{M}_0), \frac{\mathbb{E}(X'_i \mid \mathscr{M}_0)}{\|\mathbb{E}(X'_i \mid \mathscr{M}_0)\|_{\mathbb{H}}} \right\rangle_{\mathbb{H}} \\
&= \mathbb{E} \left\langle X'_i, \frac{\mathbb{E}(X'_i \mid \mathscr{M}_0)}{\|\mathbb{E}(X'_i \mid \mathscr{M}_0)\|_{\mathbb{H}}} \right\rangle_{\mathbb{H}}.
\end{aligned} \tag{3.18}$$

(Interpret $0/0$ to be 0.)

Inequality (3.18) together with the covariance inequality (3.1) entails that for all $i \geqslant 1$,

$$\mathbb{E}\,\|\mathbb{E}(X'_i \mid \mathscr{M}_0)\|_{\mathbb{H}} \leqslant 36 \int_0^{\alpha_{2,\infty}(i)} Q_{\|X_0\|_{\mathbb{H}}}(u)\,du. \tag{3.19}$$

By combining (3.17), (3.19) and using the fact that under (2.1), $\sigma^2_{p_n} \gg p_n$, and that $\alpha_{2,\infty}(n)$ is nonincreasing we derive that

$$\frac{\sum_{k=1}^{k_n} \mathbb{E}\,\|\mathbb{E}(Z'_{k,n} \mid \mathscr{F}^Z_{k-1,n})\|_{\mathbb{H}}}{b_n} \ll \frac{q_n}{p_n}\sqrt{n} \int_0^{\alpha_{2,\infty}(p_n)} Q_{\|X_0\|_{\mathbb{H}}}(u)\,du.$$

Next using the fact that $q_n = o(p_n)$, we get the following more tractable upper bound estimate

$$\frac{\sum_{k=1}^{k_n} \mathbb{E}\,\|\mathbb{E}(Z'_{k,n} \mid \mathscr{F}^Z_{k-1,n})\|_{\mathbb{H}}}{b_n} \ll \sqrt{n} \int_0^{\alpha(\theta_n)} Q_{\|X_0\|_{\mathbb{H}}}(u)\,du. \tag{3.20}$$

Then the Cauchy–Schwarz inequality entails that

$$\frac{\sum_{k=1}^{k_n} \mathbb{E}\,\|\mathbb{E}(Z'_{k,n} \mid \mathscr{F}^Z_{k-1,n})\|_{\mathbb{H}}}{b_n} \ll \sqrt{n\alpha(\theta_n) \int_0^{\alpha(\theta_n)} Q^2_\star(u)\,du}, \tag{3.21}$$

which converges to zero by (3.4) and (3.5) and ends the proof of (3.16).

Gathering (3.10), (3.15), and (3.16), we get that

$$\lim_{n\to\infty} \mathbb{E}\left\|\frac{\sum_{k=1}^{k_n} Z'_{k,n}}{b_n}\right\|_{\mathbb{H}} = 0. \tag{3.22}$$

On the other hand using arguments similar to those in the proof of (3.16) in Merlevède and Peligrad,[23] we easily derive that

$$\lim_{n\to\infty} \mathbb{E}\left\|\frac{Z'_{k_n+1,n}}{b_n}\right\|^2_{\mathbb{H}} = 0. \tag{3.23}$$

Now let us focus on the main part of the proof, namely the asymptotic behaviour of $\{\frac{\sum_{k=1}^{k_n} Y'_{k,n}}{b_n}\}_{k_n \geqslant 1}$. First set $\mathscr{F}^Y_{k,n} = \sigma(X_j, j \leqslant (k-1)(p_n+q_n)+p_n)$ and notice that we have the following decomposition:

$$\frac{\sum_{k=1}^{k_n} Y'_{k,n}}{b_n} = \frac{\sum_{k=1}^{k_n} [Y'_{k,n} - \mathbb{E}(Y'_{k,n} \mid \mathscr{F}^Y_{k-1,n})]}{b_n} + \frac{\sum_{k=1}^{k_n} \mathbb{E}(Y'_{k,n} \mid \mathscr{F}^Y_{k-1,n})}{b_n}. \tag{3.24}$$

First we treat the last term in the right-hand side of the above inequality. Observe that by stationarity

$$\sum_{k=1}^{k_n} \mathbb{E}\, \|\mathbb{E}(Y'_{k,n} \mid \mathscr{F}^{Y}_{k-1,n})\|_{\mathbb{H}} \leqslant \sum_{k=1}^{k_n} \sum_{i=(k-1)(p_n+q_n)+1}^{(k-1)(p_n+q_n)+p_n} \mathbb{E}\, \|\mathbb{E}(X'_i \mid \mathscr{M}_{(k-2)(p_n+q_n)+p_n})\|_{\mathbb{H}}$$

$$= k_n \sum_{i=q_n+1}^{p_n+q_n} \mathbb{E}\, \|\mathbb{E}(X'_i \mid \mathscr{M}_0)\|_{\mathbb{H}}. \tag{3.25}$$

Next by using (3.19), we get an upper bound estimate similar to the one in inequality (3.20), which yields

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{k_n} \mathbb{E}\, \|\mathbb{E}(Y'_{k,n} \mid \mathscr{F}^{Y}_{k-1,n})\|_{\mathbb{H}}}{b_n} = 0$$

$$\text{and hence that } \lim_{n\to\infty} \mathbb{E} \left\| \frac{\sum_{k=1}^{k_n} \mathbb{E}(Y'_{k,n} \mid \mathscr{F}^{Y}_{k-1,n})}{b_n} \right\|_{\mathbb{H}} = 0. \tag{3.26}$$

Let us treat now the first term in decomposition (3.24). Set $W_{k,n} := Y'_{k,n} - \mathbb{E}(Y'_{k,n} \mid \mathscr{F}^{Y}_{k-1,n})$ and notice first that $\{W_{k,n}\}$ is an $\mathbb{H}$-valued martingale difference array with respect to the filtration $\mathscr{F}^{Y}_{k,n}$. Then to study the limiting behaviour of $\{\frac{\sum_{k=1}^{k_n} W_{k,n}}{b_n}\}_{n\geqslant 1}$, we shall apply Theorem C of Jakubowski.[18] To this aim, we are going to prove that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{k_n} \mathbb{E}\, \|W_{k,n}\|^4_{\mathbb{H}}}{b_n^4} = 0, \tag{3.27}$$

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \mathbb{E}\left(\frac{\|W_{k,n}\|^2_{\mathbb{H}}}{b_n^2} \,\middle|\, \mathscr{F}^{Y}_{k-1,n} \right) = 1 \quad \text{in probability}, \tag{3.28}$$

and that for all $i, j \geqslant 1$

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \mathbb{E}\left(\frac{\langle W_{k,n}, e_i\rangle_{\mathbb{H}} \, \langle W_{k,n}, e_j\rangle_{\mathbb{H}}}{b_n^2} \,\middle|\, \mathscr{F}^{Y}_{k-1,n} \right) = \sigma_{ij} \quad \text{in probability}. \tag{3.29}$$

To prove (3.27) we first notice that since $\mathbb{E}\, \|\mathbb{E}(Y'_{k,n} \mid \mathscr{F}_{k-1,n})\|^4_{\mathbb{H}} \leqslant \mathbb{E}\, \|Y'_{k,n}\|^4_{\mathbb{H}}$, it follows that $\mathbb{E}\, \|W_{k,n}\|^4_{\mathbb{H}} \leqslant 16\mathbb{E}\, \|Y'_{k,n}\|^4_{\mathbb{H}}$. Then because of stationarity, in order to verify (3.27), it is enough to show that

$$\lim_{n\to\infty} \frac{k_n \mathbb{E}\, \|Y'_{1,n}\|^4_{\mathbb{H}}}{b_n^4} = 0. \tag{3.30}$$

To this aim we first denote by P^m the projector on the first m components of the orthonormal basis $(e_\ell, \ell \geqslant 1)$ of $\mathbb{H}$ and by $I_{\mathbb{H}}$ the identity projector on $\mathbb{H}$. With these notations, we have the following inequality:

$$\mathbb{E} \, \|Y'_{1,n}\|^4_{\mathbb{H}} \leqslant 2\mathbb{E} \, \|P^m Y'_{1,n}\|^4_{\mathbb{H}} + 2\mathbb{E} \, \|(I_{\mathbb{H}} - P^m) \, Y'_{1,n}\|^4_{\mathbb{H}}. \tag{3.31}$$

Now observe that

$$\begin{aligned} \mathbb{E} \, \|P^m Y'_{1,n}\|^4_{\mathbb{H}} &= \mathbb{E} \left(\sum_{\ell=1}^{m} \langle Y'_{1,n}, e_\ell \rangle^2_{\mathbb{H}} \right)^2 \leqslant m^2 \, \mathbb{E}(\max_{1 \leqslant \ell \leqslant m} \langle Y'_{1,n}, e_\ell \rangle^2_{\mathbb{H}})^2 \\ &\leqslant m^2 \sum_{\ell=1}^{m} \mathbb{E}(\langle Y'_{1,n}, e_\ell \rangle^4_{\mathbb{H}}). \end{aligned} \tag{3.32}$$

Lemma 3.2 together with the fact that $\|X'_0\|_{\mathbb{H}} \leqslant 2T_n$ a.s. entail for all $\ell \geqslant 1$

$$\begin{aligned} \mathbb{E}(\langle Y'_{1,n}, e_\ell \rangle^4_{\mathbb{H}}) &= \mathbb{E} \left(\sum_{i=1}^{p_n} \langle X'_i, e_\ell \rangle_{\mathbb{H}} \right)^4 \\ &\ll p_n^2 T_n^2 \sum_{i=0}^{p_n - 1} (i+1) \int_0^{\alpha_{2,\infty}(i)} Q^2_{\|X_0\|_{\mathbb{H}}}(u) \, du. \end{aligned}$$

Then since $\int_0^{\alpha_{2,\infty}(i)} Q^2_{\|X_0\|_{\mathbb{H}}}(u) \, du = o(\frac{1}{i})$ we obtain for all $\ell \geqslant 1$,

$$\mathbb{E}(\langle Y'_{1,n}, e_\ell \rangle^4_{\mathbb{H}}) = o(p_n^3 T_n^2). \tag{3.33}$$

Gathering (3.33) together with the definition of b_n^2 and (2.1), we derive that for all $\ell \geqslant 1$

$$\frac{k_n \mathbb{E}(\langle Y'_{1,n}, e_\ell \rangle^4_{\mathbb{H}})}{b_n^4} = o\left(\frac{p_n^2 T_n^2}{n} \right). \tag{3.34}$$

Then according to the relation (3.6), the definition of p_n and the inequality (3.32), it follows that

$$\lim_{m \to \infty} \lim_{n \to \infty} \frac{k_n \mathbb{E} \, \|P^m Y'_{1,n}\|^4_{\mathbb{H}}}{b_n^4} = 0. \tag{3.35}$$

Notice now that

$$\mathbb{E} \, \|(I_{\mathbb{H}} - P^m) \, Y'_{1,n}\|^4_{\mathbb{H}} \leqslant 4 p_n^2 T_n^2 \mathbb{E} \, \|(I_{\mathbb{H}} - P^m) \, Y'_{1,n}\|^2_{\mathbb{H}}.$$

Then by using this inequality together with the definition of b_n^2, (2.1) and the relation (3.6), we easily derive that

$$\frac{k_n \mathbb{E}\,\|(I_{\mathbb{H}} - P^m)\,Y'_{1,n}\|_{\mathbb{H}}^4}{b_n^4} \ll \frac{p_n^2 T_n^2}{n}\left(\frac{k_n}{b_n^2}\,\mathbb{E}\,\|(I_{\mathbb{H}} - P^m)(S_{p_n} - S''_{p_n})\|_{\mathbb{H}}^2\right)$$
$$\ll \frac{1}{\sigma_{p_n}^2}\,\mathbb{E}\,\|(I_{\mathbb{H}} - P^m)\,S_{p_n}\|_{\mathbb{H}}^2 + \frac{1}{\sigma_{p_n}^2}\,\mathbb{E}\,\|S''_{p_n}\|_{\mathbb{H}}^2. \tag{3.36}$$

According to (3.14), the last term in the right-hand side of the above inequality converges to zero by letting n tend to infinity; then it remains to compute the first one. By using (2.3), we find that

$$\limsup_{n\to\infty} \frac{1}{\sigma_{p_n}^2}\,\mathbb{E}\,\|(I_{\mathbb{H}} - P^m)\,S_{p_n}\|_{\mathbb{H}}^2 = \limsup_{n\to\infty} \frac{\sigma_{p_n}^2}{\sigma_{p_n}^2} - \sum_{i=1}^{m} \lim_{n\to\infty} \frac{\mathbb{E}\langle S_{p_n}, e_i\rangle_{\mathbb{H}}^2}{\sigma_{p_n}^2}$$
$$= 1 - \sum_{i=1}^{m} \sigma_{ii}.$$

Then by taking the limit in the above expression when m is going to infinity and by using (2.4), we derive that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{1}{\sigma_{p_n}^2}\,\mathbb{E}\,\|(I_{\mathbb{H}} - P^m)\,S_{p_n}\|_{\mathbb{H}}^2 = 0 \tag{3.37}$$

which combined with (3.36) entails that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{k_n \mathbb{E}\,\|(I_{\mathbb{H}} - P^m)\,Y'_{1,n}\|_{\mathbb{H}}^4}{b_n^4} = 0. \tag{3.38}$$

Gathering (3.31), (3.35), and (3.38), we end the proof of (3.30).

We turn now to the proof of (3.28) by proving that the convergence holds in $\mathbb{L}_{\mathbb{H}}^1$. With this aim we first notice that

$$\mathbb{E}(\|W_{k,n}\|_{\mathbb{H}}^2 \mid \mathscr{F}_{k-1,n}^Y) = \mathbb{E}(\|Y'_{k,n}\|_{\mathbb{H}}^2 \mid \mathscr{F}_{k-1,n}^Y) - \|\mathbb{E}(Y'_{k,n} \mid \mathscr{F}_{k-1,n}^Y)\|_{\mathbb{H}}^2. \tag{3.39}$$

By stationarity, we get

$$\sum_{k=1}^{k_n} \mathbb{E}\,\|\mathbb{E}(Y'_{k,n} \mid \mathscr{F}_{k-1,n}^Y)\|_{\mathbb{H}}^2 \leqslant k_n \sum_{i=q_n+1}^{p_n+q_n} \sum_{j=q_n+1}^{p_n+q_n} |\mathbb{E}\langle X'_i, \mathbb{E}(X'_j \mid \mathscr{M}_0)\rangle_{\mathbb{H}}|.$$

The fact that $\|X'_0\|_{\mathbb{H}} \leqslant 2T_n$ a.s. together with inequality (3.19) entail the following upper bound estimate

$$\frac{\sum_{k=1}^{k_n} \mathbb{E}\,\|\mathbb{E}(Y'_{k,n} \mid \mathscr{F}_{k-1,n}^Y)\|_{\mathbb{H}}^2}{b_n^2} \ll \frac{k_n p_n^2 T_n}{b_n^2} \int_0^{\alpha_{2,\infty}(q_n)} Q_{\|X_0\|_{\mathbb{H}}}(u)\,du.$$

Next using the definition of b_n, (2.1) and the relation (3.6), we derive that

$$\frac{\sum_{k=1}^{k_n} \mathbb{E}\,\|\mathbb{E}(Y'_{k,n} \mid \mathscr{F}^Y_{k-1,n})\|^2_{\mathbb{H}}}{b_n^2} \ll \sqrt{n} \int_0^{\alpha_{2,\infty}(q_n)} Q_{\|X_0\|_{\mathbb{H}}}(u)\, du \tag{3.40}$$

which has already been shown to tend to zero. This last consideration together with stationarity, relation (3.39), the fact that (3.14) yields

$$\lim_{n\to\infty} \frac{\mathbb{E}\,\|Y'_{1,n}\|^2_{\mathbb{H}}}{\mathbb{E}\,\|Y_{1,n}\|^2_{\mathbb{H}}} = 1 \tag{3.41}$$

and that $b_n^2 = k_n \mathbb{E}\,\|Y_{1,n}\|^2_{\mathbb{H}}$ imply that to prove (3.28), it is enough to show that

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \mathbb{E}\left|\frac{\mathbb{E}(\|Y'_{k,n}\|^2_{\mathbb{H}} \mid \mathscr{F}^Y_{k-1,n}) - \mathbb{E}\,\|Y'_{k,n}\|^2_{\mathbb{H}}}{b_n^2}\right| = 0. \tag{3.42}$$

By stationarity we derive that

$$\sum_{k=1}^{k_n} \mathbb{E}\left|\frac{\mathbb{E}(\|Y'_{k,n}\|^2_{\mathbb{H}} \mid \mathscr{F}^Y_{k-1,n}) - \mathbb{E}\,\|Y'_{k,n}\|^2_{\mathbb{H}}}{b_n^2}\right|$$
$$= \frac{k_n}{b_n^2}\, \mathbb{E}\,|\mathbb{E}(\|Y'_{2,n}\|^2_{\mathbb{H}} - \mathbb{E}\,\|Y'_{2,n}\|^2_{\mathbb{H}} \mid \mathscr{F}^Y_{1,n})|. \tag{3.43}$$

Now observe that

$$\mathbb{E}\,|\mathbb{E}(\|Y'_{2,n}\|^2_{\mathbb{H}} - \mathbb{E}\,\|Y'_{2,n}\|^2_{\mathbb{H}} \mid \mathscr{F}^Y_{1,n})|$$
$$\leqslant 2 \sum_{i=q_n+1}^{p_n+q_n} \sum_{j=0}^{p_n+q_n-i} \mathbb{E}\,|\mathbb{E}(\langle X'_i, X'_{i+j}\rangle_{\mathbb{H}} - \mathbb{E}\langle X'_i, X'_{i+j}\rangle_{\mathbb{H}} \mid \mathscr{M}_0)|. \tag{3.44}$$

Now set $A_{i,j} := \langle X'_i, X'_{i+j}\rangle_{\mathbb{H}} - \mathbb{E}\langle X'_i, X'_{i+j}\rangle_{\mathbb{H}}$ and observe that

$$\mathbb{E}\,|\mathbb{E}(\langle X'_i, X'_{i+j}\rangle_{\mathbb{H}} - \mathbb{E}\langle X'_i, X'_{i+j}\rangle_{\mathbb{H}} \mid \mathscr{M}_0)|$$
$$= \mathbb{E}\{\mathbb{E}(A_{i,j} \mid \mathscr{M}_0)(\mathbb{1}(\mathbb{E}(A_{i,j} \mid \mathscr{M}_0) \geqslant 0) - \mathbb{1}(\mathbb{E}(A_{i,j} \mid \mathscr{M}_0) < 0))\}$$
$$= \mathbb{E}\{A_{i,j}(\mathbb{1}(\mathbb{E}(A_{i,j} \mid \mathscr{M}_0) \geqslant 0) - \mathbb{1}(\mathbb{E}(A_{i,j} \mid \mathscr{M}_0) < 0))\}$$
$$= \mathbb{E}\{(\langle X'_i, X'_{i+j}\rangle_{\mathbb{H}} - \mathbb{E}\langle X'_i, X'_{i+j}\rangle_{\mathbb{H}})$$
$$\times (\mathbb{1}(\mathbb{E}(A_{i,j} \mid \mathscr{M}_0) \geqslant 0) - \mathbb{1}(\mathbb{E}(A_{i,j} \mid \mathscr{M}_0) < 0))\}. \tag{3.45}$$

Next by using (3.44) combined with (3.45) and Rio's covariance inequality[29] we derive (after a bit of work) that

$$\mathbb{E}\,|\mathbb{E}(\|Y'_{2,n}\|^2_{\mathbb{H}} - \mathbb{E}\,\|Y'_{2,n}\|^2_{\mathbb{H}} \mid \mathscr{F}^Y_{1,n})| \leqslant 8 \sum_{i=q_n+1}^{p_n+q_n} \sum_{j=0}^{p_n+q_n-i} \int_0^{\alpha_{2,\infty}(i)} Q_{|\langle X'_i, X'_{i+j}\rangle_{\mathbb{H}}|}(u)\,du$$

$$\leqslant 64 p_n^2 \int_0^{\alpha_{2,\infty}(q_n)} Q^2_{\|X_0\|_{\mathbb{H}}}(u)\,du, \qquad (3.46)$$

which together with the definition of b_n^2 and (2.1) yield the following upper bound estimate for equality (3.43):

$$(\text{constant}) \cdot a(\theta_n)\,\theta_n \int_0^{\alpha(\theta_n)} Q^2_{\|X_0\|_{\mathbb{H}}}(u)\,du,$$

which converges to zero by (3.4). This last consideration ends the proof of (3.42).

It remains to show now (3.29). This will hold by using similar arguments as for the proof of (3.28), if we can proof that for all $i, j \geqslant 1$,

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{k_n} \mathbb{E}\langle Y'_{k,n}, e_i\rangle_{\mathbb{H}} \langle Y'_{k,n}, e_j\rangle_{\mathbb{H}}}{b_n^2} = \sigma_{ij}. \qquad (3.47)$$

By stationarity we have the following decomposition: for all $i, j \geqslant 1$

$$\frac{\sum_{k=1}^{k_n} \mathbb{E}(\langle Y'_{k,n}, e_i\rangle_{\mathbb{H}} \langle Y'_{k,n}, e_j\rangle_{\mathbb{H}})}{b_n^2}$$

$$= \frac{\mathbb{E}(\langle Y'_{1,n}, e_i\rangle_{\mathbb{H}} \langle Y'_{1,n}, e_j\rangle_{\mathbb{H}})}{\sigma^2_{p_n}}$$

$$= \frac{\mathbb{E}(\langle S_{p_n}, e_i\rangle_{\mathbb{H}} \langle S_{p_n}, e_j\rangle_{\mathbb{H}})}{\sigma^2_{p_n}} + \frac{\mathbb{E}(\langle S''_{p_n}, e_i\rangle_{\mathbb{H}} \langle S''_{p_n}, e_j\rangle_{\mathbb{H}})}{\sigma^2_{p_n}}$$

$$- \frac{\mathbb{E}(\langle S_{p_n}, e_i\rangle_{\mathbb{H}} \langle S''_{p_n}, e_j\rangle_{\mathbb{H}})}{\sigma^2_{p_n}} - \frac{\mathbb{E}(\langle S''_{p_n}, e_i\rangle_{\mathbb{H}} \langle S_{p_n}, e_j\rangle_{\mathbb{H}})}{\sigma^2_{p_n}}$$

$$=: I_{1,n} + I_{2,n} + I_{3,n} + I_{4,n}. \qquad (3.48)$$

By using Cauchy–Schwarz inequality, the two last terms in the above inequality are bounded by

$$\frac{\sqrt{\mathbb{E}\,\|S''_{p_n}\|^2_{\mathbb{H}}}}{\sigma_{p_n}}.$$

Moreover we obviously get

$$I_{2,n} \leqslant \frac{\mathbb{E}\,\|S''_{p_n}\|_{\mathbb{H}}^2}{\sigma_{p_n}^2}.$$

Then by using (3.14), it follows that the three last terms in equality (3.48) converge to zero by letting n tend to infinity. Moreover using (2.3), we clearly have for all $i, j \geqslant 1$,

$$\lim_{n \to \infty} I_{1,n} = \sigma_{ij},$$

which according all the above considerations ends the proof of (3.29).

Then collecting (3.27), (3.28), and (3.29) and by using Theorem C of Jakubowski[18] together with (2.4), we have shown that

$$\frac{\sum_{k=1}^{k_n} W_{k,n}}{b_n} \xrightarrow{\mathscr{D}} \mu,$$

which combined with (3.8), (3.22), (3.23), and (3.26) yields

$$\frac{S_n}{b_n} \xrightarrow{\mathscr{D}} \mu. \tag{3.49}$$

Now we prove that

$$\lim_{n \to \infty} \frac{b_n \mathbb{E}\,\|\mu\|_{\mathbb{H}}}{\mathbb{E}\,\|S_n\|_{\mathbb{H}}} = 1. \tag{3.50}$$

To do this, it is enough to show that $\{\frac{\|S_n\|_{\mathbb{H}}}{b_n}\}_{n \geqslant 1}$ is an uniformly integrable family. Indeed recall that by Theorem 5.4 in Billingsley,[2] if (3.49) holds and if $\{\frac{\|S_n\|_{\mathbb{H}}}{b_n}\}_{n \geqslant 1}$ is an uniformly integrable family, then $\frac{\mathbb{E}\,\|S_n\|_{\mathbb{H}}}{b_n} \longrightarrow \mathbb{E}\,\|\mu\|_{\mathbb{H}}$ as $n \longrightarrow \infty$.

To this aim, first notice that since $\{W_{k,n}\}$ is an $\mathbb{H}$-valued martingale difference array, we have

$$\frac{\mathbb{E}\,\|\sum_{k=1}^{k_n} W_{k,n}\|_{\mathbb{H}}^2}{b_n^2} = \sum_{k=1}^{k_n} \frac{\mathbb{E}\,\|W_{k,n}\|_{\mathbb{H}}^2}{b_n^2}. \tag{3.51}$$

Notice that

$$\sum_{k=1}^{k_n} \frac{\mathbb{E}\,\|W_{k,n}\|_{\mathbb{H}}^2}{b_n^2} = \sum_{k=1}^{k_n} \frac{\mathbb{E}\,\|Y'_{k,n}\|_{\mathbb{H}}^2}{b_n^2} - \sum_{k=1}^{k_n} \frac{\mathbb{E}\,\|\mathbb{E}(Y'_{k,n} \mid \mathscr{F}_{k-1,n}^{Y})\|_{\mathbb{H}}^2}{b_n^2}. \tag{3.52}$$

The last term in the above equality has already been shown to tend to zero (see relation (3.40)), whereas the first one is tending to 1 by using stationarity and (3.41). These last considerations combined with (3.51) yield

$$\lim_{n\to\infty} \frac{\mathbb{E}\,\|\sum_{k=1}^{k_n} W_{k,n}\|_{\mathbb{H}}^2}{b_n^2} = 1, \tag{3.53}$$

which in turn implies that $\{\frac{\|\sum_{k=1}^{k_n} W_{k,n}\|_{\mathbb{H}}}{b_n}\}_{n\geqslant 1}$ is an uniformly integrable family. This last consideration together with (3.8), (3.22), (3.23), and (3.26) yield that $\{\frac{\|S_n\|_{\mathbb{H}}}{b_n}\}_{n\geqslant 1}$ is an uniformly integrable family, which ends the proof of (3.50), and of Theorem 2.1.

3.3. Proof of Theorem 2.2

We shall make use of the same notation from the proof of Theorem 2.1.

Classically, the first part of the proof consists to show that the finite-dimensional distributions the $b_n^{-1} S_n(\cdot)$ converge to those of $W(\cdot)$. This can be done by using the proof of Theorem 2.1 combined with classical arguments and is then left to the reader. To complete the proof, it suffices to derive tightness in $C_{\mathbb{H}}[0,1]$. To this aim, according to relation (3.6) in Kuelbs,[19] it is enough to show that, for each positive ε,

$$\lim_{\delta\to 0}\ \limsup_{n\to\infty}\ \mathbb{P}\left(\sup_{|t-s|\leqslant\delta} \left\|\frac{S_n(t)-S_n(s)}{b_n}\right\|_{\mathbb{H}} \geqslant \varepsilon\right) = 0. \tag{3.54}$$

Notice first that Markov's inequality combined with stationarity, the definition of b_n and (2.1) yields for all $\varepsilon > 0$,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1} \frac{\|(nt-[nt])\,X_{[nt+1]}\|_{\mathbb{H}}}{b_n} \geqslant \varepsilon\right)\\
&\quad\leqslant \mathbb{P}\left(\max_{1\leqslant k\leqslant n} \frac{\|X_k\|_{\mathbb{H}}}{b_n} \geqslant \varepsilon\right)\\
&\quad\leqslant \mathbb{P}\left(\max_{1\leqslant k\leqslant n} \frac{\|X'_k\|_{\mathbb{H}}}{b_n} \geqslant \varepsilon/2\right) + \mathbb{P}\left(\max_{1\leqslant k\leqslant n} \frac{\|X''_k\|_{\mathbb{H}}}{b_n} \geqslant \varepsilon/2\right)\\
&\quad\leqslant \frac{2^4}{\varepsilon^4 b_n^4}\,\mathbb{E}(\max_{1\leqslant k\leqslant n} \|X'_k\|_{\mathbb{H}}^4) + \frac{2}{\varepsilon b_n}\,\mathbb{E}(\max_{1\leqslant k\leqslant n} \|X''_k\|_{\mathbb{H}})\\
&\quad\ll \frac{nT_n^2\,\mathbb{E}\,\|X_0\|_{\mathbb{H}}^2}{n^2} + \sum_{i=1}^{n} \frac{\mathbb{E}\,\|X''_i\|_{\mathbb{H}}}{b_n}.
\end{aligned}$$

Then using (3.6) together with (3.8), we derive that for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{0\leqslant t\leqslant 1} \frac{\|(nt-[nt])\, X_{[nt+1]}\|_{\mathbb{H}}}{b_n} \geqslant \varepsilon\right) = 0.$$

It follows that to prove (3.54) it suffices to show that

$$\lim_{\delta\to 0}\ \limsup_{n\to\infty}\ \mathbb{P}\left(\sup_{|t-s|\leqslant\delta} \left\|\frac{S_{[nt]}-S_{[ns]}}{b_n}\right\|_{\mathbb{H}} \geqslant \varepsilon\right) = 0. \tag{3.55}$$

Denote by $S'_{[nt]} = \sum_{i=1}^{[nt]} X'_i$ and by $S''_{[nt]} = \sum_{i=1}^{[nt]} X''_i$, where X'_i and X''_i designate the truncated variables as described in (3.7).

Let us prove first that $b_n^{-1} S''_{[nt]}$ is negligible for the weak convergence: for all $\varepsilon > 0$, we have

$$\begin{aligned}\mathbb{P}(\sup_{0\leqslant t\leqslant 1} \|b_n^{-1} S''_{[nt]}\|_{\mathbb{H}} \geqslant \varepsilon) &\leqslant \mathbb{P}\left(\frac{\sum_{i=1}^n \|X''_i\|_{\mathbb{H}}}{b_n} \geqslant \varepsilon\right)\\ &\leqslant \frac{1}{\varepsilon}\cdot\frac{\sum_{i=1}^n \mathbb{E}\,\|X''_i\|_{\mathbb{H}}}{b_n},\end{aligned} \tag{3.56}$$

which is convergent to 0 by (3.8).

Now take p_n and q_n as defined in the beginning of the proof of Theorem 2.1. Set $k_{nt} = [\frac{[nt]}{p_n+q_n}]$ and divide the sequence of random variables $\{X'_i\}$ in big and small blocks as in the proof of Theorem 2.1. We obviously have

$$\frac{\sum_{i=1}^{[nt]} X'_i}{b_n} = \frac{\sum_{j=1}^{k_{nt}} Y'_{j,n}}{b_n} + \frac{\sum_{j=1}^{k_{nt}} Z'_{j,n}}{b_n} + \frac{R'_{n,t}}{b_n}, \tag{3.57}$$

where

$$R'_{n,t} := \sum_{i=1}^{[nt]} X'_i - \left(\sum_{j=1}^{k_{nt}} Y'_{j,n} + \sum_{j=1}^{k_{nt}} Z'_{j,n}\right).$$

First we show that for all $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{0\leqslant t\leqslant 1} \frac{\|R'_{n,t}\|_{\mathbb{H}}}{b_n} \geqslant \varepsilon\right) = 0. \tag{3.58}$$

Using stationarity, notice first that for all $\varepsilon > 0$

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1} \frac{\|R'_{n,t}\|_{\mathbb{H}}}{b_n} \geqslant \varepsilon\right) \leqslant (k_n+1)\,\mathbb{P}\left(\max_{1\leqslant j\leqslant p_n+q_n} \frac{\|\sum_{i=1}^j X'_i\|_{\mathbb{H}}}{b_n} \geqslant \varepsilon\right).$$

Now for n large enough, $\varepsilon b_n \geqslant 8q_nT_n$ for all $\varepsilon > 0$. Then we can apply Lemma 3.3 with $\lambda = \varepsilon b_n$, $m = 2p_n$, $T = T_n$, and $q = q_n$. Using the definition of b_n and Equation (3.6), for n large enough, this leads to

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1}\frac{\|R'_{n,t}\|_{\mathbb{H}}}{b_n}\geqslant\varepsilon\right)\ll\frac{\sigma'^2_{q_n}}{\sigma^2_{p_n}}+\sqrt{n}\,\mathbb{E}\,\|\mathbb{E}(X'_{q_n}\mid\mathscr{M}_0)\|_{\mathbb{H}}.$$

Then using (3.19), we get that for all $\varepsilon > 0$

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1}\frac{\|R'_{n,t}\|_{\mathbb{H}}}{b_n}\geqslant\varepsilon\right)\ll\frac{\sigma'^2_{q_n}}{\sigma^2_{p_n}}+\sqrt{n}\int_0^{\alpha_{2,\infty}(q_n)}Q_{\|X_0\|_{\mathbb{H}}}(u)\,du.$$

The first term in the right-hand side is tending to zero according to (3.15). The latter has also previously been shown to converge to zero (see inequalities (3.20) and (3.21)). This ends the proof of (3.58).

Now using the same notations as in the proof of Theorem 2.1, we consider the following decompositions

$$\frac{\sum_{j=1}^{k_{nt}}Y'_{j,n}}{b_n}=\frac{\sum_{j=1}^{k_{nt}}W_{j,n}}{b_n}+\frac{\sum_{j=1}^{k_{nt}}\mathbb{E}(Y'_{j,n}\mid\mathscr{F}^Y_{j-1,n})}{b_n}\tag{3.59}$$

and

$$\frac{\sum_{j=1}^{k_{nt}}Z'_{j,n}}{b_n}=\frac{\sum_{j=1}^{k_{nt}}V_{j,n}}{b_n}+\frac{\sum_{j=1}^{k_{nt}}\mathbb{E}(Z'_{j,n}\mid\mathscr{F}^Z_{j-1,n})}{b_n}.\tag{3.60}$$

Notice that for all $\varepsilon > 0$, Markov's inequality yields

$$\begin{aligned}\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1}\frac{\|\sum_{j=1}^{k_{nt}}\mathbb{E}(Y'_{j,n}\mid\mathscr{F}^Y_{j-1,n})\|_{\mathbb{H}}}{b_n}\geqslant\varepsilon\right)&\leqslant\mathbb{P}\left(\frac{\sum_{j=1}^{k_n}\|\mathbb{E}(Y'_{j,n}\mid\mathscr{F}^Y_{j-1,n})\|_{\mathbb{H}}}{b_n}\geqslant\varepsilon\right)\\&\leqslant\frac{\mathbb{E}(\sum_{j=1}^{k_n}\|\mathbb{E}(Y'_{j,n}\mid\mathscr{F}^Y_{j-1,n})\|_{\mathbb{H}})}{\varepsilon b_n},\end{aligned}\tag{3.61}$$

which converges to 0 by (3.26).

Similarly by using (3.16), for all $\varepsilon > 0$ we derive that

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1}\frac{\|\sum_{j=1}^{k_{nt}}\mathbb{E}(Z'_{j,n}\mid\mathscr{F}^Z_{j-1,n})\|_{\mathbb{H}}}{b_n}\geqslant\varepsilon\right)=0.\tag{3.62}$$

On the other hand since $\{V_{j,n}\}$ is an $\mathbb{H}$-valued martingale difference array with respect to the filtration $\mathscr{F}^Z_{j,n}$, Markov's inequality combined with Doob's inequality for $\mathbb{H}$-valued martingales yields for all $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1}\frac{\|\sum_{j=1}^{k_{nt}}V_{j,n}\|_{\mathbb{H}}}{b_n}\geqslant\varepsilon\right)\leqslant\frac{\mathbb{E}(\sup_{0\leqslant t\leqslant 1}\|\sum_{j=1}^{k_{nt}}V_{j,n}\|_{\mathbb{H}}^2)}{\varepsilon^2 b_n^2}$$
$$\leqslant 4\,\frac{\mathbb{E}(\|\sum_{j=1}^{k_n}V_{j,n}\|_{\mathbb{H}}^2)}{\varepsilon^2 b_n^2}=4\,\frac{\sum_{j=1}^{k_n}\mathbb{E}\,\|V_{j,n}\|_{\mathbb{H}}^2}{\varepsilon^2 b_n^2}$$
$$\leqslant 4\,\frac{\sum_{j=1}^{k_n}\mathbb{E}\,\|Z'_{j,n}\|_{\mathbb{H}}^2}{\varepsilon^2 b_n^2}\tag{3.63}$$

which converges to zero when n tends to infinity by using stationarity together with the definition of b_n and (3.15).

Gathering all these above considerations, we infer that (3.55) will hold provided that, for each positive ε,

$$\lim_{\delta\to 0}\ \limsup_{n\to\infty}\ \mathbb{P}\left(\sup_{|t-s|\leqslant\delta}\left\|\frac{\sum_{j=1}^{k_{nt}}W_{j,n}-\sum_{j=1}^{k_{ns}}W_{j,n}}{b_n}\right\|_{\mathbb{H}}\geqslant\varepsilon\right)=0.\tag{3.64}$$

Now to prove (3.64), it is enough to show that for all $m\geqslant 1$ and each positive ε

$$\lim_{\delta\to 0}\ \limsup_{n\to\infty}\ \mathbb{P}\left(\sup_{|t-s|\leqslant\delta}\left\|\frac{\sum_{j=1}^{k_{nt}}P^m(W_{j,n})-\sum_{j=1}^{k_{ns}}P^m(W_{j,n})}{b_n}\right\|_{\mathbb{H}}\geqslant\varepsilon\right)=0,\tag{3.65}$$

and that for each positive ε,

$$\lim_{m\to\infty}\ \lim\ \sup_{n\to\infty}\ \mathbb{P}\left(\sup_{t\in[0,1]}\left\|\frac{\sum_{j=1}^{k_{nt}}(I_{\mathbb{H}}-P^m)\,W_{j,n}}{b_n}\right\|_{\mathbb{H}}\geqslant\varepsilon\right)=0.\tag{3.66}$$

We first notice that since for all $m\geqslant 1$, $\{(I_{\mathbb{H}}-P^m)\,W_{j,n}\}$ is an $\mathbb{H}$-valued martingale difference array with respect to the filtration $\{\mathscr{F}^Y_{j,n}\}$, we can use arguments similar to those of (3.63) to infer that for all $\varepsilon>0$,

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1}\frac{\|\sum_{j=1}^{k_{nt}}(I_{\mathbb{H}}-P^m)\,W_{j,n}\|_{\mathbb{H}}}{b_n}\geqslant\varepsilon\right)$$
$$\leqslant 4\,\frac{k_n\mathbb{E}\,\|(I_{\mathbb{H}}-P^m)\,Y'_{1,n}\|_{\mathbb{H}}^2}{\varepsilon^2 b_n^2}$$
$$\ll\frac{\mathbb{E}\,\|(I_{\mathbb{H}}-P^m)\,S_{p_n}\|_{\mathbb{H}}^2}{\varepsilon^2\sigma_{p_n}^2}+\frac{\mathbb{E}\,\|(I_{\mathbb{H}}-P^m)\,S''_{p_n}\|_{\mathbb{H}}^2}{\varepsilon^2\sigma_{p_n}^2},$$

and hence (3.66) holds by (3.14) and (3.37).

On the other hand, to prove (3.65) it suffices to prove that for all $\ell \geqslant 1$ and each positive ϵ,

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}\left(\sup_{|t-s| \leqslant \delta} \left\| \frac{\sum_{j=1}^{k_{nt}} \langle W_{j,n}, e_\ell \rangle_{\mathbb{H}} - \sum_{j=1}^{k_{ns}} \langle W_{j,n}, e_\ell \rangle_{\mathbb{H}}}{b_n} \right\|_{\mathbb{H}} \geqslant \varepsilon \right) = 0. \tag{3.67}$$

To this aim, notice first that the proof of (3.35) (see (3.34)) entails that for any positive ϵ,

$$\lim_{n \to \infty} \mathbb{E}\left(\frac{\langle W_{1,n}, e_\ell \rangle_{\mathbb{H}}^2}{\mathbb{E}\, \|Y_{1,n}\|_{\mathbb{H}}^2} \mathbb{1}(|\langle W_{1,n}, e_\ell \rangle_{\mathbb{H}}| > b_n \epsilon) \right) = 0. \tag{3.68}$$

In addition, according to the proof of (3.29), we have shown that for any $t \in [0, 1]$,

$$\lim_{n \to \infty} \sum_{k=1}^{k_{nt}} \mathbb{E}\left(\frac{\langle W_{k,n}, e_\ell \rangle_{\mathbb{H}}^2}{b_n^2} \,\middle|\, \mathscr{F}_{k-1,n}^{Y} \right) = t\sigma_{\ell\ell} \qquad \text{in } \mathbb{L}^1. \tag{3.69}$$

(3.68) together with (3.69) are probably well-known sufficient conditions to get the tightness condition (3.67). A convenient reference for it is for instance Theorem 18.2 in Billingsley.[(3)] Then (3.67) holds and so does (3.65). This in turn combined with (3.66) leads to (3.64) (and hence to (3.55)).

We end the proof by using the fact that $\lim_{n \to \infty} \frac{b_n \mathbb{E}\, \|\mu\|_{\mathbb{H}}}{\mathbb{E}\, \|S_n\|_{\mathbb{H}}} = 1$.

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