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Stochastic Processes and their Applications 108 (2003) 229–262

stochastic processes and their applications

www.elsevier.com/locate/spa

The conditional central limit theorem in Hilbert spaces

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Received 18 March 2002; received in revised form 17 March 2003; accepted 3 July 2003

Abstract

In this paper, we give necessary and sufficient conditions for a stationary sequence of random variables with values in a separable Hilbert space to satisfy the *conditional central limit theorem* introduced in Dedecker and Merlevède (Ann. Probab. 30 (2002) 1044–1081). As a consequence, this theorem implies stable convergence of the normalized partial sums to a mixture of normal distributions. We also establish the functional version of this theorem. Next, we show that these conditions are satisfied for a large class of weakly dependent sequences, including strongly mixing sequences as well as mixingales. Finally, we present an application to linear processes generated by some stationary sequences of $\mathbb{H}$-valued random variables.

MSC: 60F05; 60F17

Keywords: Hilbert space; Central limit theorem; Weak invariance principle; Strictly stationary process; Stable convergence; Strong mixing; Mixingale; Linear processes

1. Introduction

Since Hoffman-Jorgensen and Pisier (1976) and Jain (1977), we know that separable Hilbert spaces are the only infinite-dimensional Banach spaces for which the classical central limit property for i.i.d. sequences is equivalent to the square integrability of the norm of the variables. From a probabilistic point of view, it is therefore natural to extend central limit theorems for dependent random vectors to separable Hilbert spaces.

Although the theory of empirical processes mainly deals with the (generally nonseparable) Banach space $\ell^\infty(\mathscr{F})$ of bounded functionals from $\mathscr{F}$ to $\mathbb{R}$, separable Hilbert

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doi:10.1016/j.spa.2003.07.004

spaces are sometimes rich enough for statistical applications. For instance, if we are interested in Cramér–von Mises statistics, it is natural to consider that the empirical distribution function is a random variable with values in $\mathbb{L}^2(\mu)$ for an appropriate finite measure μ on the real line (see Example 2, Section 2.2). Other examples are given by Bosq (2000) and Merlevède (1995), who study linear processes taking their values in separable Hilbert spaces. These authors focus on forecasting and estimation problems for several classes of continuous time processes.

For Hilbert-valued martingale differences, a functional version of the central limit theorem is given by Walk (1977) and a triangular version by Jakubowski (1980). For strongly mixing sequences we mention the works of Dehling (1983) and Merlevède et al. (1997). The latter extends to Hilbert spaces a well-known result of Doukhan et al. (1994), whose optimality is discussed in Bradley (1997). However, none of these dependence conditions is adapted to describe the behaviour of nonexplosive time series. Starting from this remark, Chen and White (1998) obtained new central limit theorems (and their functional versions) for Hilbert-valued *mixingales*, and gave significant applications. The concept of mixingale introduced by McLeish (1975a) is particularly well adapted to time series, and contains both mixing and martingale difference processes as special cases. To get an idea of the wide range of applications of mixingales (including functions of infinite histories of mixing processes), we refer to McLeish (1975a) and Hall and Heyde (1980, Section 2.3).

In this paper we obtain, as a consequence of a more general result, sufficient conditions for the normalized partial sums of a stationary Hilbert-valued sequence to converge *stably* to a mixture of normal distributions. These conditions are expressed in terms of conditional expectations and are similar to those given by Gordin (1969, 1973) and McLeish (1975a, 1977) for real-valued sequences. To describe our results in more details, we need some preliminary notations.

Notation 1. Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space, and $T : \Omega \mapsto \Omega$ be a bijective bimeasurable transformation preserving the probability $\mathbb{P}$. An element A of $\mathscr{A}$ is said to be invariant if $T(A) = A$. We denote by $\mathscr{I}$ the σ-algebra of all invariant sets. The probability $\mathbb{P}$ is ergodic if each element of $\mathscr{I}$ has measure 0 or 1. Let $\mathscr{M}_0$ be a σ-algebra of $\mathscr{A}$ satisfying $\mathscr{M}_0 \subseteq T^{-1}(\mathscr{M}_0)$, and define the nondecreasing filtration $(\mathscr{M}_i)_{i\in\mathbb{Z}}$ by $\mathscr{M}_i = T^{-i}(\mathscr{M}_0)$.

Notation 2. Let $\mathbb{H}$ be a separable Hilbert space with norm $\|\cdot\|_{\mathbb{H}}$ generated by an inner product, $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ and $(e_\ell)_{\ell\geqslant 1}$ be an orthonormal basis in $\mathbb{H}$. For any real $p \geqslant 1$, denote by $\mathbb{L}^p_{\mathbb{H}}$ the space of $\mathbb{H}$-valued random variables X such that $\|X\|^p_{\mathbb{L}^p_{\mathbb{H}}} = \mathbb{E}(\|X\|^p_{\mathbb{H}})$ is finite.

For any random variable X_0 in $\mathbb{L}^2_{\mathbb{H}}$, set $X_i = X_0 \circ T^i$ and $S_n = X_1 + \cdots + X_n$. When the random variable X_0 is $\mathscr{M}_0$-measurable, we give in Theorem 1 necessary and sufficient conditions for the sequence $n^{-1/2}S_n$ to satisfy the conditional central limit theorem introduced in Dedecker and Merlevède (2002). As a byproduct, we obtain stable convergence in the sense of Rényi (1963) to a mixture of normal distributions in $\mathbb{H}$. Further, assuming that the partial sum process can be well approximated by finite-dimensional projections, we obtain in Theorem 2 the functional version of this result (cf. Theorem 2, Property s1*). From these two general results, we derive sufficient conditions which are easier to satisfy and may be compared to other criteria in the literature. In particular, we show in Corollary 2 that the functional conditional central limit theorem holds as soon as

$$\text{the sequence } \|X_0\|_{\mathbb{H}}\mathbb{E}(S_n|\mathcal{M}_0) \quad \text{converges in } \mathbb{L}^1_{\mathbb{H}}. \tag{1.1}$$

Alternatively, we prove in Corollary 3 that the same property holds under the mixingale-type condition: there exists a sequence $(L_k)_{k>0}$ of positive numbers such that

$$\sum_{i=1}^{\infty}\left(\sum_{k=1}^{i} L_k\right)^{-1} < \infty \quad \text{and} \quad \sum_{k\geqslant 1} L_k\|\mathbb{E}(X_k|\mathcal{M}_0)\|^2_{\mathbb{L}^2_{\mathbb{H}}} < \infty. \tag{1.2}$$

The two preceding conditions extend criteria (1.3) and (1.4) of Dedecker and Merlevède (2002) to separable Hilbert spaces (for real-valued random variables condition (1.1) first appears in Dedecker and Rio (2000)). When X_0 is bounded, criterion (1.1) yields the weak invariance principle for stationary $\mathbb{H}$-valued sequences under the Hilbert analogue of Gordin's criterion (1973). Now, if we control the norm of the conditional expectation in (1.1) with the help of strong mixing coefficients, we obtain the conditional and nonergodic version of the central limit theorem of Merlevède et al. (1997). On the other hand, extending in a natural way the definition of mixingales to Hilbert spaces, we see that criterion (1.2) is satisfied if either condition (2.5) in McLeish (1977) holds or $(X_n, \mathcal{M}_n)$ is a mixingale of size $-1/2$ (cf. McLeish, 1975a, Definitions (2.1) and (2.4)). The optimality of condition (1.2) is discussed in Remark 6, Section 2.2.

If X_0 is no longer $\mathcal{M}_0$-measurable we approximate X_i by $Y_i^k = \mathbb{E}(X_i|\mathcal{M}_{i+k})$ and we assume that the sequence $(Y_i^k)_{i\in\mathbb{Z}}$ satisfies condition (1.1) for the σ-algebra $\mathcal{N}_0 = \mathcal{M}_k$. In order to get back to the initial sequence $(X_i)_{i\in\mathbb{Z}}$, we need to impose additional conditions on some series of residual random variables. More precisely, we obtain in Theorem 3 a conditional central limit theorem under the $\mathbb{L}^q$-criterion

$$X_0 \text{ belongs to } \mathbb{L}^p_{\mathbb{H}}, \quad \sum_{n=0}^{\infty}\mathbb{E}(X_n|\mathcal{M}_0) \text{ and } \sum_{n=0}^{\infty}(X_{-n} - \mathbb{E}(X_{-n}|\mathcal{M}_0)) \text{ converge in } \mathbb{L}^q_{\mathbb{H}}, \tag{1.3}$$

where p and q are two conjugate exponents and p belongs to $[2, \infty]$. For real-valued random variables and the usual central limit theorem, a condition similar to (1.3) is due to Gordin (1969) (see Remark 7, Section 2.3).

To be complete, we present some applications of Corollaries 2 and 3 to linear processes generated by a stationary sequence of $\mathbb{H}$-valued random variables. In Theorem 4 we obtain sufficient conditions for noncausal processes to satisfy the conditional central limit theorem. For causal processes, a functional version of this result is given in Theorem 5.

2. Conditional central limit theorems

2.1. The adapted case

Before stating our main result, we need more notations.

Definition 1. A nonnegative self-adjoint operator Γ on $\mathbb{H}$ will be called an $\mathscr{S}(\mathbb{H})$-operator, if it has finite trace, i.e., for some (and therefore every) orthonormal basis $(e_\ell)_{\ell\geqslant 1}$ of $\mathbb{H}$, $\sum_{\ell\geqslant 1}\langle \Gamma e_\ell, e_\ell\rangle_{\mathbb{H}} < \infty$. A random linear operator Λ from $\mathbb{H}$ to $\mathbb{H}$ is $\mathscr{B}$-measurable if for each i,j in $\mathbb{N}^*$, the random variable $\langle \Lambda e_i, e_j\rangle_{\mathbb{H}}$ is $\mathscr{B}$-measurable. We will say that a random linear operator Λ from $\mathbb{H}$ to $\mathbb{H}$ belongs to $\mathscr{S}(\mathbb{H},\mathscr{B})$ if it is $\mathscr{B}$-measurable, for all $\omega\in\Omega$, $\Lambda(\omega)$ is an $\mathscr{S}(\mathbb{H})$-operator, and $\sum_{\ell\geqslant 1}\mathbb{E}\langle \Lambda e_\ell, e_\ell\rangle_{\mathbb{H}} < +\infty$.

Notation 3. For $\Gamma\in\mathscr{S}(\mathbb{H})$, we denote by P_Γ^ε the law of a centered Gaussian random variable with covariance operator Γ.

Notation 4. Denote by $\mathscr{H}$ be the space of continuous functions φ from $\mathbb{H}$ to $\mathbb{R}$ such that $x\to|(1+\|x\|_{\mathbb{H}}^2)^{-1}\varphi(x)|$ is bounded.

Theorem 1. *Let $\mathscr{M}_0$ be a σ-algebra of $\mathscr{A}$ satisfying $\mathscr{M}_0\subseteq T^{-1}(\mathscr{M}_0)$ and define the nondecreasing filtration $(\mathscr{M}_i)_{i\in\mathbb{Z}}$ by $\mathscr{M}_i=T^{-i}(\mathscr{M}_0)$. Let X_0 be a $\mathscr{M}_0$-measurable, centered random variable with values in $\mathbb{H}$ such that $\mathbb{E}\|X_0\|_{\mathbb{H}}^2<\infty$. Define the sequence $(X_i)_{i\in\mathbb{Z}}$ by $X_i=X_0\circ T^i$. The following statements are equivalent*:

s1. *There exists a random linear operator Λ belonging to $\mathscr{S}(\mathbb{H},\mathscr{M}_0)$ and such that for any φ in $\mathscr{H}$ and any positive integer k,*

$$s1(\varphi):\ \lim_{n\to\infty}\left\|\mathbb{E}\left(\varphi(n^{-1/2}S_n)-\int\varphi(x)P_\Lambda^\varepsilon(\mathrm{d}x)\middle|\mathscr{M}_k\right)\right\|_1=0.$$

s2. (a) *for all i in $\mathbb{N}^*$, the sequence $\langle\mathbb{E}(n^{-1/2}S_n|\mathscr{M}_0),e_i\rangle_{\mathbb{H}}$ tends to 0 in $\mathbb{L}^1$ as n tends to infinity*;
(b) *for all i,j in $\mathbb{N}^*$, there exists a $\mathscr{M}_0$-measurable random variable $\eta_{i,j}$ such that the sequence $\mathbb{E}(\langle n^{-1/2}S_n,e_i\rangle_{\mathbb{H}}\langle n^{-1/2}S_n,e_j\rangle_{\mathbb{H}}|\mathscr{M}_0)$ tends to $\eta_{i,j}$ in $\mathbb{L}^1$ as n tends to infinity*;
(c) *for all i in $\mathbb{N}^*$, the sequence $n^{-1}\langle S_n,e_i\rangle_{\mathbb{H}}^2$ is uniformly integrable*;
(d) $\sum_{i=1}^\infty\mathbb{E}(\eta_{i,i})<\infty$ *and* $\mathbb{E}\|n^{-1/2}S_n\|_{\mathbb{H}}^2$ *converges to* $\sum_{i=1}^\infty\mathbb{E}(\eta_{i,i})$.

Moreover $\langle\Lambda e_i,e_j\rangle_{\mathbb{H}}=\eta_{i,j}$ almost surely and $\eta_{i,j}\circ T=\eta_{i,j}$ almost surely.

Remark 1. If $\mathbb{P}$ is ergodic then Λ is constant and $n^{-1/2}S_n$ converges in distribution to a $\mathbb{H}$-valued Gaussian random variable with covariance operator Λ.

A stationary sequence $(X\circ T^i)_{i\in\mathbb{Z}}$ of $\mathbb{H}$-valued random variables is said to satisfy the conditional central limit theorem (CCLT for short) if it verifies s1. The following result is an important consequence of Theorem 1.

Corollary 1. *Let* $(\mathscr{M}_i)_{i\in\mathbb{Z}}$ *and* $(X_i)_{i\in\mathbb{Z}}$ *be as in Theorem* 1. *If condition* s2 *is satisfied then, for any* φ *in* $\mathscr{H}$, *the sequence* $(\varphi(n^{-1/2}S_n))$ *converges weakly in* $\mathbb{L}^1$ *to* $\int \varphi(x)P^{\varepsilon}_{\Lambda}(\mathrm{d}x)$.

Corollary 1 implies that the sequence $(n^{-1/2}S_n)$ converges *stably* to a mixture of normal distributions in $\mathbb{H}$. We refer to Aldous and Eagleson (1978) for a complete exposition of the concept of stability for real-valued random variables (introduced by Rényi, 1963) and its connection to weak $\mathbb{L}^1$-convergence. This concept has been later used by Bingham (2000) for $\mathbb{H}$-valued random variables. If the covariance operator Λ is constant, the convergence is said to be *mixing*. If $\mathbb{P}$ is ergodic, this result is a consequence of Theorem 4 in Eagleson (1976) (see Application 4.2 therein).

To see the importance of stable convergence, we give the following example.

Example 1. If condition s2 holds then for any y in $\mathbb{H}$, we have

$$\langle y, n^{-1/2}S_n\rangle_{\mathbb{H}} \text{ converges stably to } \langle y, \Lambda y\rangle_{\mathbb{H}}^{1/2}N,$$

where N is a standard real Gaussian random variable independent of Λ. As a consequence of stable convergence, we derive that if Z_n converges in probability to $\langle y, \Lambda y\rangle_{\mathbb{H}}$ and $\mathbb{P}(\langle y, \Lambda y\rangle_{\mathbb{H}} = 0) = 0$, then

$$\frac{\langle y, n^{-1/2}S_n\rangle_{\mathbb{H}}}{\sqrt{Z_n \vee n^{-1}}} \xrightarrow{\mathscr{D}} N, \quad \text{as } n \text{ tends to infinity.}$$

Note that such a Z_n can be built as soon as condition (γ) of Corollary 2 is satisfied.

The next proposition provides sufficient conditions for property s2 to hold.

Proposition 1. *Let* $(\mathscr{M}_i)_{i\in\mathbb{Z}}$ *and* $(X_i)_{i\in\mathbb{Z}}$ *be as in Theorem* 1.

(i) *If for any positive integers* ℓ, m *the sequence* $\langle X_0, e_\ell\rangle_{\mathbb{H}}\mathbb{E}(\langle S_n, e_m\rangle_{\mathbb{H}}|\mathscr{M}_0)$ *converges in* $\mathbb{L}^1$ *then* s2(a)–(c) *hold and the sequence*

$$\begin{aligned}(&\mathbb{E}(\langle X_0, e_\ell\rangle_{\mathbb{H}}\langle X_0, e_m\rangle_{\mathbb{H}}|\mathscr{I}) + \mathbb{E}(\langle X_0, e_\ell\rangle_{\mathbb{H}}\langle S_n, e_m\rangle_{\mathbb{H}}|\mathscr{I})\\ &+\mathbb{E}(\langle X_0, e_m\rangle_{\mathbb{H}}\langle S_n, e_\ell\rangle_{\mathbb{H}}|\mathscr{I}))_{n\geqslant 1}\end{aligned} \tag{2.1}$$

converges in $\mathbb{L}^1$ *to* $\eta_{\ell,m}$.

(ii) *If* $\lim_{N\to\infty}\sup_{M\geqslant N}\sum_{i=1}^{\infty}|\mathbb{E}(\langle X_0, e_i\rangle_{\mathbb{H}}\langle S_M - S_N, e_i\rangle_{\mathbb{H}})| = 0$ *then* s2(d) *holds.*

We turn now to the functional version of Theorem 1. Let $C_{\mathbb{H}}[0,1]$ be the set of all continuous $\mathbb{H}$-valued functions on $[0,1]$. This is a separable Banach space under the sup-norm $\|x\|_\infty = \sup\{\|x(t)\|_{\mathbb{H}} : t\in[0,1]\}$. Define the process $\{W_n(t) : t\in[0,1]\}$ by

$$W_n(t) = S_{[nt]} + (nt - [nt])X_{[nt]+1},$$

$[\,\cdot\,]$ denoting the integer part. Note that for each ω, $W_n(.)$ is an element of $C_{\mathbb{H}}[0,1]$.

Definition 2. Let π_t be the projection from $C_{\mathbb{H}}[0,1]$ to $\mathbb{H}$ such that $\pi_t(x) = x(t)$. For $\Gamma \in \mathscr{S}(\mathbb{H})$, denote by W_Γ the unique measure on $C_{\mathbb{H}}[0,1]$ such that:

(a) $\pi_0 = 0$,
(b) for all $0 \leqslant s < t \leqslant 1$, $\pi_t - \pi_s$ is independent of π_s,
(c) for all $0 \leqslant t < t+s \leqslant 1$, the increment $\pi_{t+s} - \pi_t$ has a Gaussian distribution on $\mathbb{H}$ with mean zero and covariance operator $s\Gamma$, where Γ does not depend on t, s.

Notation 5. Denote by $\mathscr{H}^*$ the space of continuous functions φ from $(C_{\mathbb{H}}([0,1]), \|\cdot\|_\infty)$ to $\mathbb{R}$ such that $x \to |(1+\|x\|_\infty^2)^{-1}\varphi(x)|$ is bounded.

Notation 6. Let $\mathbb{H}_m$ be the subspace generated by the first m components of the orthonormal basis $(e_\ell)_{\ell \geqslant 1}$ of $\mathbb{H}$ and P^m be the projection operator from $\mathbb{H}$ to $\mathbb{H}_m$.

Theorem 2. *Under the notations of Theorem* 1, *the following statements are equivalent*:

s1*. *There exists a random linear operator Λ belonging to $\mathscr{S}(\mathbb{H}, \mathscr{M}_0)$ and such that for any φ in $\mathscr{H}^*$ and any positive integer k,*

$$\text{s1}^*(\varphi)\text{:}\ \lim_{n\to\infty} \left\| \mathbb{E}\left(\varphi(n^{-1/2}W_n) - \int \varphi(x) W_\Lambda(\mathrm{d}x) \middle| \mathscr{M}_k \right) \right\|_1 = 0.$$

s2*. (a) *and* (b) *of* s2 *hold*, *and* (c) *and* (d) *are*, *respectively*, *replaced by*
(c*) *for all $i \geqslant 1$, $n^{-1}(\max_{1\leqslant k\leqslant n} |\langle S_k, e_i\rangle_{\mathbb{H}}|)^2$ is uniformly integrable.*

$$(\text{d}^*)\quad \lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{E}\left(\max_{1\leqslant i\leqslant n} \left(\frac{\|S_i\|_{\mathbb{H}}^2}{n} - \frac{\|P^m S_i\|_{\mathbb{H}}^2}{n} \right) \right) = 0.$$

A stationary sequence $(X \circ T^i)_{i\in\mathbb{Z}}$ of $\mathbb{H}$-valued random variables is said to satisfy the functional conditional central limit theorem if it verifies s1*.

2.2. Application to weakly dependent sequences

In view of applications, the next corollaries give sufficient conditions for property s1* to hold when the sequence satisfies several types of weak dependence. In order to develop our results, we need further definitions.

Definition 3. For two σ-algebras $\mathscr{U}$ and $\mathscr{V}$ of $\mathscr{A}$, the strong mixing coefficient of Rosenblatt (1956) is defined by $\alpha(\mathscr{U},\mathscr{V}) = \sup\{|\mathbb{P}(U\cap V) - \mathbb{P}(U)\mathbb{P}(V)| : U\in\mathscr{U}, V\in\mathscr{V}\}$. For any nonnegative and integrable random variable Y, define the "upper tail" quantile function Q_Y by $Q_Y(u) = \inf\{t \geqslant 0 : \mathbb{P}(Y > t) \leqslant u\}$. Note that, on the set $[0, \mathbb{P}(Y>0)]$, the function $H_Y : x \to \int_0^x Q_Y(u)\,\mathrm{d}u$ is an absolutely continuous and increasing function with values in $[0, \mathbb{E}(Y)]$. Denote by G_Y the inverse of H_Y.

Corollary 2. *Let* $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ *and* $(X_i)_{i\in\mathbb{Z}}$ *be as in Theorem* 1. *Set* $\alpha_k=\alpha(\mathcal{M}_0,\sigma(X_k))$ *and* $\gamma_k=\|\mathbb{E}(X_k|\mathcal{M}_0)\|_{\mathbb{L}^1_{\mathbb{H}}}$. *Consider the conditions*

(α) $\sum_{k\geqslant 1}\int_0^{\alpha_k} Q^2_{\|X_0\|_{\mathbb{H}}}(u)\,\mathrm{d}u<\infty$.
(β) $\sum_{k\geqslant 1}\int_0^{\gamma_k} Q_{\|X_0\|_{\mathbb{H}}}\circ G_{\|X_0\|_{\mathbb{H}}}(u)\,\mathrm{d}u<\infty$.
(δ) $\sum_{k\geqslant 1}\mathbb{E}(\|X_0\|_{\mathbb{H}}\|\mathbb{E}(X_k|\mathcal{M}_0)\|_{\mathbb{H}})<\infty$.
(γ) $\|X_0\|_{\mathbb{H}}\mathbb{E}(S_n|\mathcal{M}_0)$ *converges in* $\mathbb{L}^1_{\mathbb{H}}$.

We have implications (α) ⇒ (β) ⇒ (δ) ⇒ (γ) ⇒ $\mathrm{s}1^*$. *In particular, if* $\|X_0\|_{\mathbb{H}}$ *is bounded,* $\mathrm{s}1^*$ *holds as soon as* $\mathbb{E}(S_n|\mathcal{M}_0)$ *converges in* $\mathbb{L}^1_{\mathbb{H}}$.

Remark 2. Item (α) of Corollary 2 improves on Theorem 4 of Merlevède et al. (1997) in two ways: Firstly, it gives its nonergodic version, since the mixing coefficients we consider here allow to deal with nonergodic sequences. Secondly, it gives its functional and conditional form. Note that, if we consider the slightly more restrictive coefficient $\alpha'_k=\sup_{i>0}\alpha(\mathcal{M}_0,\sigma(X_k,X_{k+i}))$, Merlevède (2003) shows that a central limit theorem still holds under the condition:

$$\text{the sequence } n\int_0^{\alpha'_n} Q^2_{\|X_0\|_{\mathbb{H}}}(u)\,\mathrm{d}u \text{ tends to zero as } n \text{ tends to infinity.}$$

This result extends and slightly improves on the sharp CLT for real-valued random variables given in Merlevède and Peligrad (2000).

Remark 3. Item (γ) extends condition (1.4) of Dedecker and Merlevède (2002) to separable Hilbert spaces. This condition first appears in Dedecker and Rio (2000).

Remark 4. Condition (β) is new to our knowledge. It relies on a result of Dedecker and Doukhan (2003) (see Section 3.2.4). To see the interest of such a condition, let us give the following application: If there exist $r>2$ and $c>0$ such that $\mathbb{P}(\|X_0\|_{\mathbb{H}}>x)\leqslant(c/x)^r$ then (β) (and hence $\mathrm{s}1^*$) holds as soon as $\sum_{k\geqslant 1}(\|\mathbb{E}(X_k|\mathcal{M}_0)\|_{\mathbb{L}^1_{\mathbb{H}}})^{(r-2)/(r-1)}<\infty$.

Example 2 (Asymptotic distribution of Cramér–von Mises statistics). Let $Y=(Y_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of $\mathbb{R}^d$-valued random variables and set $\mathcal{M}_0^Y=\sigma(Y_i,i\leqslant 0)$. Let $\mathbb{F}$ be the distribution function of Y_0: for any $t=(t^{(1)},\dots,t^{(d)})$, $\mathbb{F}(t)=\mathbb{P}(Y_0^{(1)}\leqslant t^{(1)},\dots,Y_0^{(d)}\leqslant t^{(d)})=\mathbb{P}(Y_0\leqslant t)$ and set $X_i(t)=\mathbb{1}_{Y_i\leqslant t}$. Note that for any finite measure μ on $\mathbb{R}^d$, the random variable X_i is $\mathbb{L}^2(\mathbb{R}^d,\mu)$-valued. Moreover for any integer i, we have $\mathbb{E}(X_i)\equiv\mathbb{F}$. Denote by $\mathbb{F}_n$ the empirical distribution function of Y:

$$\text{for any } t \text{ in } \mathbb{R}^d,\quad \mathbb{F}_n(t)=\frac{1}{n}\sum_{i=1}^n X_i(t).$$

If we consider $\sqrt{n}(\mathbb{F}_n-\mathbb{F})$ as a random variable with values in the separable Hilbert space $\mathbb{H}:=\mathbb{L}^2(\mathbb{R}^d,\mu)$, we may apply the results of Corollary 2 to the sequence $(X_i)_{i\in\mathbb{Z}}$.

If the sequence $(Y_i)_{i\in\mathbb{Z}}$ is strongly mixing with mixing coefficients $\alpha_k^Y = \alpha(\mathscr{M}_0^Y, \sigma(Y_k))$, then so is $(X_i)_{i\in\mathbb{Z}}$. Applying item (α) of Corollary 2, we get that if

$$\sum_{k\geqslant 1} \alpha_k^Y < \infty, \tag{2.2}$$

then the $\mathbb{H}$-valued random variable $\sqrt{n}(\mathbb{F}_n - \mathbb{F})$ converges stably to a random variable $\mathbb{G}$ whose conditional distribution with respect to $\mathscr{I}$ is that of a zero mean $\mathbb{H}$-valued Gaussian random variable with covariance function

$$\text{for } (f,g) \text{ in } \mathbb{H}\times\mathbb{H},\ \mathbb{E}(\langle f,\mathbb{G}\rangle_{\mathbb{H}}\langle g,\mathbb{G}\rangle_{\mathbb{H}}) = \int_{\mathbb{R}^{2d}} f(s)g(t)C_{\mathscr{I}}(s,t)\mu(\mathrm{d}t)\mu(\mathrm{d}s), \tag{2.3}$$

where $C_{\mathscr{I}}(s,t) = \mathbb{F}(t\wedge s) - \mathbb{F}(t)\mathbb{F}(s) + 2\sum_{k\geqslant 1}(\mathbb{P}(Y_0 \leqslant t, Y_k \leqslant s|\mathscr{I}) - \mathbb{F}(t)\mathbb{F}(s))$.

Assume now that $Y=(Y_i)_{i\in\mathbb{Z}}$ is a strictly stationary $\mathbb{R}^d$-valued Markov chain. Denote by K its transition kernel and by π its invariant measure. For any integer i, $\mathbb{E}(X_i|\mathscr{M}_0^Y)$ is a $\mathbb{H}$-valued random variable such that $\mathbb{E}(X_i|\mathscr{M}_0^Y)(t) = \mathbb{E}(\mathbb{1}_{Y_i\leqslant t}|Y_0)$. Moreover for t and x in $\mathbb{R}^d$, $\mathbb{E}(\mathbb{1}_{Y_i\leqslant t}|Y_0 = x) = K^i(x, \mathbb{1}_{]-\infty,t]}) =: \mathbb{F}^i(x)(t)$. Applying item ($\gamma$) of Corollary 1, we obtain the same limit as in (2.3) provided that

$$\text{the sequence } \sum_{i=1}^{n}(\mathbb{F}^i(\cdot) - \mathbb{F}) \text{ converges in } \mathbb{L}^1_{\mathbb{H}}(\pi). \tag{2.4}$$

We now give three sufficient conditions for criterion (2.4) to hold:

(a) $\sum_{i=1}^{\infty}\int_{\mathbb{R}} \|\mathbb{F}^i(x) - \mathbb{F}\|_{\mathbb{H}}\,\pi(\mathrm{d}x) < \infty$.
(b) $\sum_{i=1}^{\infty}\int_{\mathbb{R}} \|\mathbb{F}^i(x) - \mathbb{F}\|_{\infty}\,\pi(\mathrm{d}x) < \infty$.
(c) $\sum_{i=1}^{\infty}\int_{\mathbb{R}} \|K^i(x,\cdot) - \pi(\cdot)\|_v\,\pi(\mathrm{d}x) < \infty$, where $\|\cdot\|_v$ is the variation norm.

More precisely, we have implications (c) $\Rightarrow$ (b) $\Rightarrow$ (a) $\Rightarrow$ (2.4). Note that condition (c) means exactly that the β-mixing coefficients of the chain are summable (see Davydov, 1973). Consequently, we also have the implication (c) $\Rightarrow$ (2.2).

Result of type (2.3) yields the asymptotic distribution of $f(\sqrt{n}(\mathbb{F}_n - \mathbb{F}))$ for any continuous functional f from $\mathbb{H}$ to $\mathbb{R}$. In particular for Cramér–von Mises statistics, we have

$$n\int_{\mathbb{R}^d}(\mathbb{F}_n(x) - \mathbb{F}(x))^2\mu(\mathrm{d}x) \text{ converges stably to } \|\mathbb{G}\|^2_{\mathbb{H}}.$$

Cramér–von Mises statistics are useful for the testing of goodness-of-fit. In the i.i.d. case, when $d=1$ the choice $\mu=\mathrm{d}\mathbb{F}$ implies that the distribution of $\|\mathbb{G}\|^2_{\mathbb{H}}$ is the same for every continuous distribution function $\mathbb{F}$. This is no longer true for dependent variables. However we can always write $\|\mathbb{G}\|^2_{\mathbb{H}} = \sum_{i\geqslant 1}\lambda_i(\varepsilon_i)^2$, where (ε_i) is a sequence of i.i.d. standard normal independent of $\mathscr{I}$, and the λ_i's are the eigenvalues of the random operator $C_{\mathscr{I}}$. Since under criteria (2.2) or (2.4), we can always find a positive estimator Z_n of $\mathbb{E}(\|\mathbb{G}\|^2_{\mathbb{H}}|\mathscr{I})$, it follows from the stability of the convergence that

$$\frac{n}{Z_n}\int_{\mathbb{R}^d}(\mathbb{F}_n(x) - \mathbb{F}(x))^2\mu(\mathrm{d}x) \text{ converges in distribution to } U = \frac{\sum_{k\geqslant 1}\lambda_k(\varepsilon_k)^2}{\sum_{k\geqslant 1}\lambda_k}.$$

Using the convexity of the exponential function, it is easy to show that the Laplace transform of U is bounded by the Laplace transform of ε_1^2. Consequently for any $z \geqslant 1$,

$$\mathbb{P}(U \geqslant z) \leqslant \sqrt{z}\exp\left(-\frac{z-1}{2}\right).$$

This upper bound is all the less precise as the variance of U is far from 2. However this bound provides always a critical region at a level α included in the one obtained if all the λ_i's were known. To get more precise critical regions, we need to estimate some of the eigenvalues (see for instance Theorem 4.4 in Bosq (2000) in the particular case of autoregressive processes).

As in Heyde (1974), an alternative approach to Corollary 2 is to consider the projection operator P_i: for any f in $\mathbb{L}^2_{\mathbb{H}}$, $P_i(f)=\mathbb{E}(f|\mathscr{M}_i)-\mathbb{E}(f|\mathscr{M}_{i-1})$. With this notation, we obtain the following extension of Proposition 2 of Dedecker and Merlevède (2002).

Corollary 3. *Let $(\mathscr{M}_i)_{i\in\mathbb{Z}}$ and $(X_i)_{i\in\mathbb{Z}}$ be as in Theorem 1. Define the tail σ-algebra by $\mathscr{M}_{-\infty}=\bigcap_{i\in\mathbb{Z}}\mathscr{M}_i$ and consider the condition*

$$\mathbb{E}(X_0|\mathscr{M}_{-\infty})=0 \text{ a.s.} \quad \text{and} \quad \sum_{i\geqslant 1}\|P_0(X_i)\|_{\mathbb{L}^2_{\mathbb{H}}}<\infty. \tag{2.5}$$

If (2.5) is satisfied then s1* *holds.*

Remark 5. In the two preceding corollaries, the variable $\eta_{\ell,m}=\langle \Lambda e_\ell, e_m\rangle_{\mathbb{H}}$ is the limit in $\mathbb{L}^1$ of the sequence of $\mathscr{I}$-measurable random variables defined in (2.1).

Remark 6. The mixingale-type condition (1.2) implies (2.5). Consequently (2.5) is satisfied if for some positive ε, $\sum_{k\geqslant 1}\ln(k)^{1+\varepsilon}\|\mathbb{E}(X_k|\mathscr{M}_0)\|^2_{\mathbb{L}^2_{\mathbb{H}}}<\infty$. According to Proposition 7 of Dedecker and Merlevède (2002), condition (1.2) is sharp in the sense that the choice $L_k\equiv 1$ is not strong enough to imply weak convergence of $n^{-1/2}S_n$.

2.3. The general case

As a consequence of Corollary 2, we obtain that s1 holds if for two conjugate exponents p and q with p in $[2,+\infty[$

$$X_0 \text{ is } \mathscr{M}_0\text{-measurable, } X_0 \text{ belongs to } \mathbb{L}^p_{\mathbb{H}} \text{ and } \sum_{n=0}^{\infty}\mathbb{E}(X_n|\mathscr{M}_0) \text{ converges in } \mathbb{L}^q_{\mathbb{H}}.$$

The next theorem shows that this result remains valid for nonadapted sequences if in addition we impose the same condition on the series $\sum_{n\geqslant 0}(X_{-n}-\mathbb{E}(X_{-n}|\mathscr{M}_0))$.

Theorem 3. *Let $(\Omega,\mathscr{A},\mathbb{P})$ be a probability space and $(\mathscr{M}_i)_{i\in\mathbb{Z}}$ be as in Theorem 1. Let X_0 be a centered random variable with values in $\mathbb{H}$ such that $\mathbb{E}\|X_0\|^p_{\mathbb{H}}<\infty$ for some p in $[2,+\infty]$, and $X_i=X_0\circ T^i$. If condition (1.3) holds for the conjugate exponent q of p, then there exists a random linear operator Λ belonging to $\mathscr{S}(\mathbb{H},\mathscr{A})$ such that for any φ in $\mathscr{H}$ and any positive integer k, property* s1(φ) *holds. Moreover $\langle \Lambda e_i, e_j\rangle_{\mathbb{H}}=\langle \Lambda e_i, e_j\rangle_{\mathbb{H}}\circ T$ almost surely.*

Remark 7. For real-valued random variables, under the condition

$$X_0 \text{ belongs to } \mathbb{L}^p, \quad \sum_{n=0}^{\infty} \|\mathbb{E}(X_n|\mathscr{M}_0)\|_q < \infty \text{ and } \sum_{n=0}^{\infty} \|X_{-n} - \mathbb{E}(X_{-n}|\mathscr{M}_0)\|_q < \infty,$$

the usual central limit theorem for real-valued random variables is due to Gordin (1969). Even for real-valued random variables, we do not know if s1* holds under condition (1.3).

2.4. Application to $\mathbb{H}$-valued linear processes

Denote by $L(\mathbb{H})$ the class of bounded linear operators from $\mathbb{H}$ to $\mathbb{H}$ and by $\|\cdot\|_{L(\mathbb{H})}$ its usual norm. Let $\{\xi_k\}_{k\in\mathbb{Z}}$ be a strictly stationary sequence of $\mathbb{H}$-valued random variables, and let $\{a_k\}_{k\in\mathbb{Z}}$ be a sequence of operators, $a_k \in L(\mathbb{H})$. We define the causal $\mathbb{H}$-valued linear process by

$$X_k = \sum_{j=0}^{\infty} a_j(\xi_{k-j}) \tag{2.6}$$

and the noncausal $\mathbb{H}$-valued linear process by

$$X_k = \sum_{j=-\infty}^{\infty} a_j(\xi_{k-j}), \tag{2.7}$$

provided the series are convergent in some sense (in the following, we suppress the brackets to soothe the notations). Note that if $\sum_{j\in\mathbb{Z}} \|a_j\|^2_{L(\mathbb{H})} < \infty$ and $\{\xi_k\}_{k\in\mathbb{Z}}$ are i.i.d. centered in $\mathbb{L}^2_{\mathbb{H}}$, then it is well known that the series in (2.7) is convergent in $\mathbb{L}^2_{\mathbb{H}}$ and almost surely (Araujo and Giné, 1980, Chapter 3.2). The sequence $\{X_k\}_{k\geqslant 1}$ is a natural extension of multivariate linear processes (Brockwell and Davis, 1987, Chapter 11). These types of processes with values in functional spaces also facilitate the study of estimation and forecasting problems for several classes of continuous time processes. For more details we mention Merlevède (1995) and Bosq (2000). From now, we use the notations:

$$\mathscr{M}_0^{\xi} = \sigma(\xi_i, i \leqslant 0), \quad \mathscr{M}_k^{\xi} = T^{-k}(\mathscr{M}_0^{\xi}) \quad \text{and} \quad \mathscr{M}_{-\infty}^{\xi} = \bigcap_{i\in\mathbb{Z}} \mathscr{M}_i^{\xi}$$

and for any function f in $\mathbb{L}^2_{\mathbb{H}}(\mathbb{P})$, $P_i(f) = \mathbb{E}(f|\mathscr{M}_i^{\xi}) - \mathbb{E}(f|\mathscr{M}_{i-1}^{\xi})$. Moreover, we assume that the stationary sequence of $\mathbb{H}$-valued random variables $\{\xi_k\}_{k\in\mathbb{Z}}$, satisfies either

$$\mathbb{E}(\xi_0|\mathscr{M}_{-\infty}^{\xi}) = 0 \quad \text{and} \quad \sum_{i\geqslant 1} \|P_0(\xi_i)\|_{\mathbb{L}^2_{\mathbb{H}}} < \infty, \tag{2.8}$$

or

$$\sum_{k\geqslant 1} \mathbb{E}(\|\xi_0\|_{\mathbb{H}} \|\mathbb{E}(\xi_k|\mathscr{M}_0^{\xi})\|_{\mathbb{H}}) < \infty. \tag{2.9}$$

Moreover, we assume that the sequence $a_k \in L(\mathbb{H})$ is summable:

$$\sum_{j=-\infty}^{\infty} \|a_j\|_{L(\mathbb{H})} < \infty. \tag{2.10}$$

If (2.10) is satisfied, set $A := \sum_{j=-\infty}^{\infty} a_j$ and denote by A^* the adjoint operator of A.

Remark 8. According to Corollary 3 (resp. 2), if the strictly stationary sequence of $\mathbb{H}$-valued random variables $\{\xi_k\}_{k\in\mathbb{Z}}$ satisfies (2.8) (resp. (2.9)), there exists a linear random operator Λ^ξ belonging to $\mathscr{S}(\mathbb{H}, \mathscr{M}_0^\xi)$ and such that for all $\ell \geqslant 1$ and $m \geqslant 1$, the sequence $n^{-1}\mathbb{E}(\langle\sum_{i=1}^n \xi_i, e_\ell\rangle_{\mathbb{H}}\langle\sum_{j=1}^n \xi_j, e_m\rangle_{\mathbb{H}}|\mathscr{I})$ converges in $\mathbb{L}^1$ to $\langle\Lambda^\xi e_\ell, e_m\rangle_{\mathbb{H}}$.

Theorem 4. *Let $\{\xi_k\}_{k\in\mathbb{Z}}$ be a strictly stationary sequence of $\mathbb{H}$-valued random variables such that $\mathbb{E}\|\xi_0\|_{\mathbb{H}}^2 < \infty$, and $\{a_k\}_{k\in\mathbb{Z}}$ be a sequence of operators satisfying (2.10). Let $(X_k)_{k\in\mathbb{Z}}$ be the linear process defined by (2.7) and $S_n := \sum_{k=1}^n X_k$. In addition assume that either (2.8) or (2.9) holds. Then for any φ in $\mathscr{H}$ and any positive integer k,*

$$\lim_{n\to\infty} \left\| \mathbb{E}\left(\varphi(n^{-1/2}S_n) - \int \varphi(x) P^{\varepsilon}_{\Lambda_A^\xi}(\mathrm{d}x) \middle| \mathscr{M}_k^\xi \right) \right\|_1 = 0, \tag{2.11}$$

where $\Lambda_A^\xi = A \circ \Lambda^\xi \circ A^*$ and Λ^ξ is defined in Remark 8. According to the definition of Λ^ξ, Λ_A^ξ belongs to $\mathscr{S}(\mathbb{H}, \mathscr{M}_0^\xi)$.

Remark 9. Condition (2.10) is essentially sharp according to the counterexample of Merlevède et al. (1997) (see Theorem 3 therein). When $\{\xi_k\}_{k\in\mathbb{Z}}$ is a sequence of i.i.d. $\mathbb{H}$-valued random variables, they shown that if (2.10) is violated, without any additional assumptions on the behavior of either $\{a_k\}_{k\in\mathbb{Z}}$ or on the covariance operator of ξ_0, the tightness of both $(n^{-1/2}S_n)_{n\geqslant 1}$ and $(S_n/\sqrt{\mathbb{E}\|S_n\|_{\mathbb{H}}^2})_{n\geqslant 1}$ may fail. Hence no analogue of Theorem 18.6.5 of Ibragimov and Linnik (1971) is possible.

The following theorem shows that if the linear process is causal, then we can derive the functional version of Theorem 4 under condition (2.8).

Theorem 5. *Let $(\xi_k)_{k\in\mathbb{Z}}$ be a strictly stationary sequence of $\mathbb{H}$-valued random variables such that $\mathbb{E}\|\xi_0\|_{\mathbb{H}}^2 < \infty$, and $(a_k)_{k\geqslant 0}$ be a sequence of operators satisfying (2.10). Let $(X_k)_{k\in\mathbb{Z}}$ be the linear process defined by (2.6) and set $W_n(t) := \sum_{k=1}^{[nt]} X_k + (nt - [nt])X_{[nt]+1}$. In addition assume that (2.8) holds. Then for any φ in $\mathscr{H}^*$ and any positive integer k,*

$$\lim_{n\to\infty} \left\| \mathbb{E}\left(\varphi(n^{-1/2}W_n) - \int \varphi(x) W_{\Lambda_A^\xi}(\mathrm{d}x) \middle| \mathscr{M}_k^\xi \right) \right\|_1 = 0, \tag{2.12}$$

where $\Lambda_A^\xi = A \circ \Lambda^\xi \circ A^*$ and Λ^ξ is defined in Remark 8.

3. Proofs

3.1. Preparatory material

We first introduce the set $R(\mathscr{M}_k)$ of $\mathscr{M}_k$-measurable Rademacher random variables: $R(\mathscr{M}_k)=\{2\mathbb{1}_A-1 : A\in\mathscr{M}_k\}$. For any random linear operator Λ belonging to $\mathscr{S}(\mathbb{H},\mathscr{M}_0)$ and any bounded random variable Z, let

1. $v_n[Z]$ be the image measure of $Z\,.\,\mathbb{P}$ by the variable $n^{-1/2}S_n$; that is the signed measure defined on $\mathbb{H}$ by: for any continuous bounded function h from $\mathbb{H}$ to $\mathbb{R}$,
$$v_n[Z](h)=\int h(n^{-1/2}S_n(\omega))Z(\omega)\mathbb{P}(\mathrm{d}\omega).$$
2. $v_n^*[Z]$ be the image measure of $Z\,.\,\mathbb{P}$ by the process $n^{-1/2}W_n$; that is the signed measure defined on $C_{\mathbb{H}}([0,1])$ by: for any continuous bounded function h from $C_{\mathbb{H}}([0,1])$ to $\mathbb{R}$,
$$v_n^*[Z](h)=\int h(n^{-1/2}W_n(\omega))Z(\omega)\mathbb{P}(\mathrm{d}\omega).$$
3. $v[Z]$ be the signed measure on $\mathbb{H}$ defined by: for any continuous bounded function h from $\mathbb{H}$ to $\mathbb{R}$,
$$v[Z](h)=\int\left(\int h(x)P^{\varepsilon}_{\Lambda(\omega)}(\mathrm{d}x)\right)Z(\omega)\mathbb{P}(\mathrm{d}\omega).$$
4. $v^*[Z]$ be the signed measure on $C_{\mathbb{H}}([0,1])$ defined by: for any continuous bounded function h from $C_{\mathbb{H}}([0,1])$ to $\mathbb{R}$,
$$v^*[Z](h)=\int\left(\int h(x)W_{\Lambda(\omega)}(\mathrm{d}x)\right)Z(\omega)\mathbb{P}(\mathrm{d}\omega).$$

Firstly, we present the extension to $\mathbb{H}$-valued random variables of Lemma 2 of Dedecker and Merlevède (2002). The proof is unchanged.

Lemma 1. *Let* $\mu_n[Z_n]:=v_n[Z_n]-v[Z_n]$ and $\mu_n^*[Z_n]:=v_n^*[Z_n]-v^*[Z_n]$. *For any* φ *in* $\mathscr{H}$ (*resp.* $\mathscr{H}^*$), *statement* s1(φ) (*resp.* s1*(φ)) *is equivalent to* s3(φ) (*resp.* s3*(φ)): *for any* Z_n *in* $R(\mathscr{M}_k)$, *the sequence* $\mu_n[Z_n](\varphi)$ (*resp.* $\mu_n^*[Z_n](\varphi)$) *tends to zero as n tends to infinity.*

3.2. The adapted case

3.2.1. The operator Λ

In this section, we assume that condition s2 holds. We construct the random linear operator Λ belonging to $\mathscr{S}(\mathbb{H},\mathscr{M}_0)$ as follows.

Let span$\{e_i, i>0\}$ be the space of finite linear combination of $(e_i)_{i>0}$ and L be the unique $\mathscr{M}_0$-measurable random linear operator from span$\{e_i, i>0\}$ to $\mathbb{H}$ satisfying $\langle Le_i, e_j\rangle_{\mathbb{H}}=\eta_{i,j}$. We shall see that L may be almost surely extended to the whole space $\mathbb{H}$. Note first that, for any positive integers p, q,

$$\left| \mathbb{E}\left(\frac{1}{n} \left(\sum_{i=1}^{p} a_i \langle S_n, e_i \rangle_{\mathbb{H}} \right) \left(\sum_{j=1}^{q} b_j \langle S_n, e_j \rangle_{\mathbb{H}} \right) \middle| \mathscr{M}_0 \right) \right|$$

$$\leqslant \left(\sum_{i=1}^{p} a_i^2 \right)^{1/2} \left(\sum_{i=1}^{q} b_i^2 \right)^{1/2} \sum_{i=1}^{p \vee q} \frac{1}{n} \mathbb{E}(\langle S_n, e_i \rangle_{\mathbb{H}}^2 | \mathscr{M}_0) \text{ almost surely.}$$

Taking the limit in $\mathbb{L}^1$ on both sides, we obtain that, almost surely

$$\left| \left\langle \sum_{i=1}^{p} a_i e_i, L\left(\sum_{i=1}^{q} b_i e_i \right) \right\rangle_{\mathbb{H}} \right| = \left| \sum_{i=1}^{p} \sum_{j=1}^{q} a_i b_j \eta_{i,j} \right|$$

$$\leqslant \left(\sum_{i=1}^{p} a_i^2 \right)^{1/2} \left(\sum_{i=1}^{q} b_i^2 \right)^{1/2} \left(\sum_{i=1}^{p \vee q} \eta_{i,i} \right). \tag{3.1}$$

From (3.1), we infer that for any x in $\mathbb{H}$ we have almost surely

$$\left| \left\langle x, L\left(\sum_{i=1}^{q} b_i e_i \right) \right\rangle_{\mathbb{H}} \right| \leqslant \|x\|_{\mathbb{H}} \left(\sum_{i=1}^{q} b_i^2 \right)^{1/2} \left(\sum_{i=1}^{\infty} \eta_{i,i} \right). \tag{3.2}$$

On the other hand, we have

$$\lim_{n \to \infty} \left\| \mathbb{E}\left(\left\langle \sum_{i=1}^{p} a_i e_i, L\left(\sum_{i=1}^{p} a_i e_i \right) \right\rangle_{\mathbb{H}} - \frac{1}{n} \left(\sum_{i=1}^{p} a_i \langle S_n, e_i \rangle_{\mathbb{H}} \right)^2 \middle| \mathscr{M}_0 \right) \right\|_1 = 0. \tag{3.3}$$

From (3.2) and (3.3), we infer that for any x in $\text{span}\{e_i, i > 0\}$ we have almost surely

$$\|Lx\|_{\mathbb{H}} \leqslant \|x\|_{\mathbb{H}} \left(\sum_{i=1}^{\infty} \eta_{i,i} \right) \quad \text{and} \quad \langle x, Lx \rangle_{\mathbb{H}} \geqslant 0. \tag{3.4}$$

Since $\sum_{i>0} \mathbb{E}(\eta_{i,i})$ is finite, then $\sum_{i>0} \eta_{i,i}$ is finite on a set A of probability 1. Let $\mathbb{D}$ be any countable dense subset of $\text{span}\{e_i, i > 0\}$ and let B be the set of probability 1 on which (3.4) holds for any x in $\mathbb{D}$. Clearly if $\omega \in A \cap B$, the operator $L(\omega)$ can be uniquely extended to a continuous linear operator $\bar{L}(\omega)$ defined on $\mathbb{H}$. We now define Λ as follows: $\Lambda = \bar{L}$ on $A \cap B$ and $\Lambda = 0$ on $(A \cap B)^{\mathrm{c}}$. Clearly we have $\langle \Lambda e_i, e_j \rangle_{\mathbb{H}} = \eta_{i,j} \mathbb{1}_{A \cap B}$, and since $A \cap B$ belongs to $\mathscr{M}_0$ the random linear operator Λ is $\mathscr{M}_0$-measurable. By construction Λ belongs to $\mathscr{S}(\mathbb{H}, \mathscr{M}_0)$ and $\langle \Lambda e_i, e_j \rangle_{\mathbb{H}} = \eta_{i,j}$ almost surely.

3.2.2. Proof of Theorem 1

We first show that s1 implies s2. Property s1 applied with $\varphi(.) = \langle ., e_i \rangle_{\mathbb{H}}$ (respectively $\varphi(.) = \langle ., e_i \rangle_{\mathbb{H}} \langle ., e_j \rangle_{\mathbb{H}}$) entails s2(a) (respectively s2(b)). On the other hand, observe that s1 yields the usual central limit theorem which combined with s2(b) leads to s2(c) (see Theorem 5.4 in Billingsley, 1968). Moreover s1 applied with $\varphi(.) = \|.\|_{\mathbb{H}}^2$ implies that

$$\lim_{n\to\infty} \mathbb{E}\left\|\frac{S_n}{\sqrt{n}}\right\|_{\mathbb{H}}^2 = \mathbb{E}\left(\int \|x\|_{\mathbb{H}}^2 P_{\Lambda}^{\varepsilon}(\mathrm{d}x)\right), \tag{3.5}$$

which by definition is equal to $\sum_{i=1}^{\infty} \mathbb{E}\langle \Lambda e_i, e_i\rangle_{\mathbb{H}} = \sum_{i=1}^{\infty} \mathbb{E}(\eta_{i,i})$. This together with (3.5) entails s2(d).

We turn now to the main part of the proof: s2 implies s1. Note first that if the sequence $(\|n^{-1/2}S_n\|_{\mathbb{H}}^2)_{n\geqslant 1}$ is uniformly integrable then it suffices to prove s1(φ) for any continuous bounded functions φ from $\mathbb{H}$ to $\mathbb{R}$. Now s2(d) implies that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \left\|\frac{S_n}{\sqrt{n}} - P^m\left(\frac{S_n}{\sqrt{n}}\right)\right\|_{\mathbb{H}}^2 = 0$$

which together with s2(c) yield the uniform integrability of $(\|n^{-1/2}S_n\|_{\mathbb{H}}^2)_{n\geqslant 1}$.

Consequently, it remains to prove s1(φ) for any continuous bounded function φ. Recall that $\mu_n[Z_n] = v_n[Z_n] - v[Z_n]$, where $Z_n \in R(\mathcal{M}_k)$ and denote by $\mu_n(P^m)^{-1}$ the image measure of μ_n by P^m. With this notation, to prove s3(φ) (and hence s1(φ)) for any continuous bounded function φ, it is enough to show the two following points:

$$\mu_n[Z_n](P^m)^{-1} \text{ converges weakly to } 0 \text{ as } n \to \infty \tag{3.6}$$

$$\mu_n[Z_n] \text{ is relatively compact in } \mathbb{H}. \tag{3.7}$$

We first prove (3.6). Let f be the one to one map from $\mathbb{H}_m$ to $\mathbb{R}^m$ defined by $f(x) = (\langle x, e_1\rangle_{\mathbb{H}}, \ldots, \langle x, e_m\rangle_{\mathbb{H}})$. Clearly, (3.6) is equivalent to: for any positive integer m and any Z_n in $R(\mathcal{M}_k)$, the sequence $\mu_n[Z_n](f \circ P^m)^{-1}$ converges weakly to the null measure as n tends to infinity. Since the measure $\mu_n[Z_n](f \circ P^m)^{-1}$ is a signed measure on $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$, we can apply Lemma 1 in Dedecker and Merlevède (2002). The main point is to prove that for any v in $\mathbb{R}^m$, $\hat{\mu}_n[Z_n](f \circ P^m)^{-1}(v) = \mu_n[Z_n]$ $(f \circ P^m)^{-1}(\exp(\mathrm{i}\langle v, .\rangle_{\mathbb{R}^m}))$ converges to zero as n tends to infinity. Setting $g_v(x) = \langle v, x\rangle_{\mathbb{R}^m}$, it suffices to prove that for any v in $\mathbb{R}^m$, the sequence $\mu_n[Z_n](g_v \circ f \circ P^m)^{-1}$ converges weakly to the null measure. Setting $V_m(x) = v_1\langle x, e_1\rangle_{\mathbb{H}} + \cdots + v_m\langle x, e_m\rangle_{\mathbb{H}}$ and applying Lemma 1, this is equivalent to: for any v in $\mathbb{R}^m$ and any continuous bounded function φ,

$$\lim_{n\to\infty} \left\|\mathbb{E}\left(\varphi(n^{-1/2}V_m(S_n)) - \int \varphi(V_m(x))P_{\Lambda}^{\varepsilon}(\mathrm{d}x)\middle| \mathcal{M}_k\right)\right\|_1 = 0. \tag{3.8}$$

Since $(V_m(X_k))_{k\in\mathbb{Z}}$ is a strictly stationary sequence of square integrable and centered real random variables and $V_m(X_0)$ is $\mathcal{M}_0$-measurable, we may apply Theorem 1 in Dedecker and Merlevède (2002). Firstly s2(a) and (b) entail both

$$\lim_{n\to\infty} \mathbb{E}|\mathbb{E}(n^{-1/2}V_m(S_n)|\mathcal{M}_0)| = 0 \tag{3.9}$$

and

$$\lim_{n\to\infty} \left\|\mathbb{E}\left(n^{-1}(V_m(S_n))^2 - \sum_{p=1}^{m}\sum_{q=1}^{m} v_p v_q \eta_{p,q}\middle| \mathcal{M}_0\right)\right\|_1 = 0. \tag{3.10}$$

Moreover s2(c) implies that

$$\text{the sequence } (n^{-1}(V_m(S_n))^2)_{n\geqslant 1} \text{ is uniformly integrable.} \tag{3.11}$$

From (3.9)–(3.11) and the definition of Λ, Theorem 1 in Dedecker and Merlevède (2002) implies Property (3.8). Consequently $\hat{\mu}_n[Z_n](f\circ P^m)^{-1}(v)$ tends to zero as n tends to infinity. According to Lemma 1 in Dedecker and Merlevède (2002), to prove that $\mu_n[Z_n](f\circ P^m)^{-1}$ converges weakly to the null measure it remains to see that the total variation measure $|\mu_n[Z_n](f\circ P^m)^{-1}|$ of $\mu_n[Z_n](f\circ P^m)^{-1}$ is tight. By definition of $\mu_n[Z_n](f\circ P^m)^{-1}$, we have $|\mu_n[Z_n](f\circ P^m)^{-1}|\leqslant v_n[1](f\circ P^m)^{-1}+v[1](f\circ P^m)^{-1}$. From (3.8) and Lemma 1, we infer that $v_n[1](f\circ P^m)^{-1}$ converges weakly to $v[1](f\circ P^m)^{-1}$. Since $v_n[1](f\circ P^m)^{-1}$ is a sequence of probability measures, it is tight and so is $|\mu_n[Z_n](f\circ P^m)^{-1}|$. This completes the proof of (3.6).

It remains to prove (3.7), namely that the sequence $(\mu_n[Z_n])_{n>0}$ is relatively compact with respect to the topology of weak convergence on $\mathbb{H}$. That is, for any increasing function f from $\mathbb{N}$ to $\mathbb{N}$, there exists an increasing function g with values in $f(\mathbb{N})$ and a signed measure μ on $\mathbb{H}$ such that $(\mu_{g(n)}[Z_{g(n)}])_{n>0}$ converges weakly to μ.

Let Z_n^+ (resp. Z_n^-) be the positive (resp. negative) part of Z_n, and write

$$\mu_n[Z_n]=\mu_n[Z_n^+]-\mu_n[Z_n^-]=v_n[Z_n^+]-v_n[Z_n^-]-v[Z_n^+]+v[Z_n^-].$$

Obviously, it is enough to prove that each sequence of finite positive measures $(v_n[Z_n^+])_{n>0}$, $(v_n[Z_n^-])_{n>0}$, $(v[Z_n^+])_{n>0}$ and $(v[Z_n^-])_{n>0}$ is relatively compact. We prove the result for the sequence $(v_n[Z_n^+])_{n>0}$, the other cases being similar.

Let f be any increasing function from $\mathbb{N}$ to $\mathbb{N}$. Choose an increasing function l with values in $f(\mathbb{N})$ such that

$$\lim_{n\to\infty}\mathbb{E}(Z_{l(n)}^+)=\liminf_{n\to\infty}\mathbb{E}(Z_{f(n)}^+).$$

We must sort out two cases:

1. If $\mathbb{E}(Z_{l(n)}^+)$ converges to zero as n tends to infinity, then, taking $g=l$, the sequence $(v_{g(n)}[Z_{g(n)}^+])_{n>0}$ converges weakly to the null measure.
2. If $\mathbb{E}(Z_{l(n)}^+)$ converges to a positive real number as n tends to infinity, we introduce, for n large enough, the probability measure p_n defined by $p_n=(\mathbb{E}(Z_{l(n)}^+))^{-1}v_{l(n)}[Z_{l(n)}^+]$. Obviously if $(p_n)_{n>0}$ is relatively compact with respect to the topology of weak convergence, then there exists an increasing function g with values in $l(\mathbb{N})$ (and hence in $f(\mathbb{N})$) and a measure v such that $(v_{g(n)}[Z_{g(n)}^+])_{n>0}$ converges weakly to v. According to Prohorov's Theorem, since $(p_n)_{n>0}$ is a family of probability measures, relative compactness is equivalent to tightness. From (3.6), we know that $n^{-1/2}P^m(S_n)$ is tight. According for instance to Lemma 1.8.1 in van der Waart and Wellner (1996), to derive the tightness in $\mathbb{H}$ of the sequence $(p_n)_{n>0}$ it is enough to show that for each positive ε,

$$\lim_{m\to\infty}\limsup_{n\to\infty}p_n(\|x-P^mx\|_{\mathbb{H}}>\varepsilon)=0. \tag{3.12}$$

According to the definition of p_n, we have

$$p_n(\|x - P^m x\|_{\mathbb{H}} > \varepsilon) = \frac{1}{\mathbb{E}(Z^+_{l(n)})} \, v_{l(n)}[Z^+_{l(n)}](\|x - P^m x\|_{\mathbb{H}} > \varepsilon)$$

$$= \frac{1}{\mathbb{E}(Z^+_{l(n)})} \, Z^+_{l(n)}.\mathbb{P}\left(\left\| \frac{S_{l(n)}}{\sqrt{l(n)}} - \frac{P^m S_{l(n)}}{\sqrt{l(n)}} \right\|_{\mathbb{H}} > \varepsilon\right). \quad (3.13)$$

Since both $\mathbb{E}(Z^+_{l(n)})$ converges to a positive number and $Z^+_{l(n)}$ is bounded by one, we infer that (3.12) holds if for each positive ε

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\left\| \frac{S_{l(n)}}{\sqrt{l(n)}} - \frac{P^m S_{l(n)}}{\sqrt{l(n)}} \right\|_{\mathbb{H}} > \varepsilon\right) = 0. \quad (3.14)$$

Markov's inequality together with s2(b) and (d) imply that

$$\limsup_{n\to\infty} \mathbb{P}\left(\left\| \frac{S_{l(n)}}{\sqrt{l(n)}} - \frac{P^m S_{l(n)}}{\sqrt{l(n)}} \right\|_{\mathbb{H}} > \varepsilon\right) \leqslant \frac{1}{\varepsilon^2} \limsup_{n\to\infty} \left(\frac{\mathbb{E}\|S_{l(n)}\|_H^2}{l(n)} - \frac{\mathbb{E}\|P^m S_{l(n)}\|_{\mathbb{H}}^2}{l(n)} \right)$$

$$\leqslant \frac{1}{\varepsilon^2} \sum_{i=m+1}^{\infty} \mathbb{E}(\eta_{i,i}),$$

which according to s2(d) converges to zero as m tends to infinity.

Conclusion. In both cases there exists an increasing function g with values in $f(\mathbb{N})$ and a measure v such that $(v_{g(n)}[Z^+_{g(n)}])_{n>0}$ converges weakly to v. Since this is true for any increasing function f with values in $\mathbb{N}$, we conclude that the sequence $(v_n[Z^+_n])_{n>0}$ is relatively compact with respect to the topology of weak convergence in $\mathbb{H}$. Of course, the same arguments apply to the sequences $(v_n[Z^-_n])_{n>0}$, $(v[Z^+_n])_{n>0}$ and $(v[Z^-_n])_{n>0}$, which implies the relative compactness of the sequence $(\mu_n[Z_n])_{n>0}$.

3.2.3. *Proof of Proposition 1*

Point (i) is a direct consequence of Proposition 3 in Dedecker and Merlevède (2002). It remains to show (ii). By stationarity

$$\frac{\mathbb{E}\|S_n\|_{\mathbb{H}}^2}{n} = \mathbb{E}\|X_0\|_{\mathbb{H}}^2 + \frac{2}{n} \sum_{k=1}^{n-1} (n-k)\mathbb{E}\langle X_0, X_k\rangle_{\mathbb{H}}.$$

From Cesaro's mean convergence theorem, we infer that $n^{-1}\mathbb{E}\|S_n\|_{\mathbb{H}}^2$ converges to

$$\mathbb{E}\|X_0\|_{\mathbb{H}}^2 + 2\sum_{k=1}^{\infty} \mathbb{E}\langle X_0, X_k\rangle_{\mathbb{H}}, \quad (3.15)$$

provided that $(\sum_{k=1}^{n} \mathbb{E}\langle X_0, X_k\rangle_{\mathbb{H}})_{n\geqslant 1}$ converges. Now assumption (ii) implies that $(\sum_{k=1}^{n} \mathbb{E}\langle X_0, X_k\rangle_{\mathbb{H}})_{n\geqslant 1}$ is a Cauchy sequence.

In the same way (ii) implies that for all $i \geqslant 1$, $(\sum_{k=1}^{n} \mathbb{E}\langle X_0, e_i\rangle_{\mathbb{H}}\langle X_k, e_i\rangle_{\mathbb{H}})_{n\geqslant 1}$ is a Cauchy sequence, whence

$$\mathbb{E}(\eta_{i,i}) = \mathbb{E}\langle X_0, e_i\rangle_{\mathbb{H}}^2 + 2\sum_{k=1}^{\infty} \mathbb{E}\langle X_0, e_i\rangle_{\mathbb{H}}\langle X_k, e_i\rangle_{\mathbb{H}}. \tag{3.16}$$

Now we show that $\sum_{i=1}^{\infty} \mathbb{E}(\eta_{i,i}) < \infty$. According to (ii), for each positive ε, there exists $N(\varepsilon)$ such that

$$\sup_{M \geqslant N(\varepsilon)} \sum_{i=1}^{\infty} \left|\mathbb{E}\left(\langle X_0, e_i\rangle_{\mathbb{H}}\langle S_M - S_{N(\varepsilon)}, e_i\rangle_{\mathbb{H}}\right)\right| \leqslant \varepsilon. \tag{3.17}$$

On the other hand, we obtain from (3.16) that

$$\sum_{i=1}^{\infty}\mathbb{E}(\eta_{i,i}) = \mathbb{E}\|X_0\|_{\mathbb{H}}^2 + 2\sum_{k=1}^{N(\varepsilon)}\sum_{i=1}^{\infty} \mathbb{E}\langle X_0, e_i\rangle_{\mathbb{H}}\langle X_k, e_i\rangle_{\mathbb{H}}$$
$$+2\sum_{i=1}^{\infty}\sum_{k=N(\varepsilon)+1}^{\infty} \mathbb{E}\langle X_0, e_i\rangle_{\mathbb{H}}\langle X_k, e_i\rangle_{\mathbb{H}}. \tag{3.18}$$

From (3.17), we easily infer that

$$\left|\sum_{i=1}^{\infty}\sum_{k=N(\varepsilon)+1}^{\infty} \mathbb{E}\langle X_0, e_i\rangle_{\mathbb{H}}\langle X_k, e_i\rangle_{\mathbb{H}}\right| \leqslant \varepsilon, \tag{3.19}$$

which together with (3.18) and Cauchy–Schwarz's inequality yield

$$\sum_{i=1}^{\infty} \mathbb{E}(\eta_{i,i}) \leqslant (1 + 2N(\varepsilon))\mathbb{E}\|X_0\|_{\mathbb{H}}^2 + 2\varepsilon.$$

This implies that $\sum_{i=1}^{\infty} \mathbb{E}(\eta_{i,i}) < \infty$. Combining (3.15) with (3.18) and (3.19), we infer that $\|n^{-1/2}S_n\|_{\mathbb{H}}^2$ tends to $\sum_{i=1}^{\infty} \mathbb{E}(\eta_{i,i})$ as n tends to infinity. This ends the proof of (ii).

3.2.4. Proof of Theorem 2

We first show that s1* yields s2*. The fact that s1* implies both s2*(a) and s2*(b) is obvious. Here we shall prove that s1* entails s2*(d*) (the fact that s1* implies s2*(c*) can be proved in the same way).

Fix $m \geqslant 1$ and let $f(.) = \sum_{\ell=m+1}^{\infty} \langle ., e_\ell\rangle_{\mathbb{H}}^2$ and $g(x) = \sup_{t\in[0,1]}(x(t))$. Property s1* applied with $\varphi = g \circ f$, ensures that

$$\lim_{n\to\infty} \mathbb{E}\left(\sup_{t\in[0,1]} \frac{\|(I_{\mathbb{H}} - P^m)\sum_{i=1}^{[nt]} X_i\|_{\mathbb{H}}^2}{n}\right)$$
$$= \mathbb{E}\left(\int \sup_{t\in[0,1]} \|(I_{\mathbb{H}} - P^m)(x(t))\|_{\mathbb{H}}^2 W_\Lambda(\mathrm{d}x)\right). \tag{3.20}$$

It follows that s2*(d*) holds as soon as

$$\lim_{m\to\infty} \mathbb{E}\left(\int \sup_{t\in[0,1]} \|(I_{\mathbb{H}} - P^m)(x(t))\|_{\mathbb{H}}^2 W_\Lambda(\mathrm{d}x)\right) = 0. \tag{3.21}$$

For the sake of simplicity, denote by $\mathbb{E}_{W_\Lambda}$ the expectation with respect to the probability measure W_Λ, and write

$$\int \sup_{t\in[0,1]} \|(I_{\mathbb{H}} - P^m)(x(t))\|_{\mathbb{H}}^2 W_\Lambda(\mathrm{d}x) = \mathbb{E}_{W_\Lambda}\left(\sup_{t\in[0,1]} \|(I_{\mathbb{H}} - P^m)\pi_t\|_{\mathbb{H}}^2\right).$$

Now since $\{(I_{\mathbb{H}} - P^m)\pi_t\}_t$ is a continuous martingale in $\mathbb{H}$ with respect to the filtration $\sigma(\pi_s, s \leqslant t)$, we infer from Doob's maximal inequality that

$$\begin{aligned}\mathbb{E}\left(\mathbb{E}_{W_\Lambda}\left(\sup_{t\in[0,1]} \|(I_{\mathbb{H}} - P^m)\pi_t\|_{\mathbb{H}}^2\right)\right) &\leqslant 4\mathbb{E}\left(\mathbb{E}_{W_\Lambda}\|(I_{\mathbb{H}} - P^m)\pi_1\|_{\mathbb{H}}^2\right)\\ &\leqslant 4\sum_{i=m+1}^{\infty} \mathbb{E}(\eta_{i,i}),\end{aligned} \tag{3.22}$$

which tends to zero as m tends to infinity. This ends the proof of (3.21) and s2*(d*) is proved.

We turn now to the main part of the proof, namely: s2* implies s1*. According to Lemma 1 we shall prove that s3* holds. For m in $\mathbb{N}$ and $0 \leqslant t_1 < \cdots < t_d \leqslant 1$, define the function $\pi^m_{t_1\cdots t_d}$ from $C_{\mathbb{H}}([0,1])$ to $\mathbb{H}^d_m$ by: $\pi^m_{t_1\cdots t_d}(x) = (P^m(x(t_1)), \ldots, P^m(x(t_d)))$. Recall that if μ and ν are two signed measures on $(C_{\mathbb{H}}([0,1]), \mathscr{B}(C_{\mathbb{H}}([0,1]))$ such that $\mu(\pi^m_{t_1\cdots t_d})^{-1} = \nu(\pi^m_{t_1\cdots t_d})^{-1}$ for any positive integer m, any positive integer d and any d-tuple $0 \leqslant t_1 < \cdots < t_d \leqslant 1$, then $\mu = \nu$. Consequently, s3* is a consequence of the two following items:

(i) Finite-dimensional convergence: for any positive integer m, any positive integer d, any d-tuple $0 \leqslant t_1 < \cdots < t_d \leqslant 1$ and any Z_n in $R(\mathscr{M}_k)$ the sequence $\mu^*_n[Z_n](\pi^m_{t_1\cdots t_d})^{-1}$ converges weakly to the null measure as n tends to infinity.
(ii) Relative compactness: for any Z_n in $R(\mathscr{M}_k)$, the family $(\mu^*_n[Z_n])_{n>0}$ is relatively compact with respect to the topology of weak convergence on $C_{\mathbb{H}}([0,1])$.

The first item follows straightforwardly from the $\mathbb{R}^m$ analogue of Lemma 4 in Dedecker and Merlevède (2002). It remains to prove that the family $(\mu^*_n[Z_n])_{n>0}$ is relatively compact in $C_{\mathbb{H}}([0,1])$. More precisely, we want to show that, for any increasing function f from $\mathbb{N}$ to $\mathbb{N}$, there exists an increasing function g with values in $f(\mathbb{N})$ and a signed measure μ on $(C_{\mathbb{H}}([0,1]), \mathscr{B}(C_{\mathbb{H}}([0,1])))$ such that $(\mu_{g(n)}[Z_{g(n)}])_{n>0}$ converges weakly to μ.

Let Z^+_n (resp. Z^-_n) be the positive (resp. negative) part of Z_n, and write

$$\mu^*_n[Z_n] = \mu^*_n[Z^+_n] - \mu^*_n[Z^-_n] = \nu^*_n[Z^+_n] - \nu^*_n[Z^-_n] - \nu^*[Z^+_n] + \nu^*[Z^-_n].$$

Obviously, it is enough to prove that each sequence of finite positive measures $(\nu^*_n[Z^+_n])_{n>0}$, $(\nu^*_n[Z^-_n])_{n>0}$, $(\nu^*[Z^+_n])_{n>0}$ and $(\nu^*[Z^-_n])_{n>0}$ is relatively compact in

$C_{\mathbb{H}}([0,1])$. We prove the result for the sequences $(v_n^*[Z_n^+])_{n>0}$ and $(v^*[Z_n^+])_{n>0}$, the other cases being similar.

Let f be any increasing function from $\mathbb{N}$ to $\mathbb{N}$. Choose an increasing function l with values in $f(\mathbb{N})$ such that

$$\lim_{n\to\infty} \mathbb{E}(Z_{l(n)}^+) = \liminf_{n\to\infty} \mathbb{E}(Z_{f(n)}^+).$$

We must sort out two cases:

1. If $\mathbb{E}(Z_{l(n)}^+)$ converges to zero as n tends to infinity, then, taking $g=l$, the sequence $(v_{g(n)}^*[Z_{g(n)}^+])_{n>0}$ converges weakly to the null measure.

2. If $\mathbb{E}(Z_{l(n)}^+)$ converges to a positive real number as n tends to infinity, we introduce, for n large enough, the probability measure p_n defined by $p_n=(\mathbb{E}(Z_{l(n)}^+))^{-1}v_{l(n)}^*[Z_{l(n)}^+]$. Obviously if $(p_n)_{n>0}$ is relatively compact with respect to the topology of weak convergence on $C_{\mathbb{H}}([0,1])$, then there exists an increasing function g with values in $l(\mathbb{N})$ (and hence in $f(\mathbb{N})$) and a measure v such that $(v_{g(n)}^*[Z_{g(n)}^+])_{n>0}$ converges weakly to v. According to Prohorov's Theorem, since $(p_n)_{n>0}$ is a family of probability measures, relative compactness is equivalent to tightness. According to relation (3.6) in Kuelbs (1973), to derive tightness in $C_{\mathbb{H}}([0,1])$ of the sequence $(p_n)_{n>0}$ it is enough to show that, for each positive ε,

$$\lim_{\delta\to 0}\limsup_{n\to\infty} p_n(x : w_{\mathbb{H}}(x,\delta)\geqslant\varepsilon)=0, \tag{3.23}$$

where $w_{\mathbb{H}}(x,\delta)$ is the modulus of continuity of an element x of $C_{\mathbb{H}}([0,1])$, that is

$$w_{\mathbb{H}}(x,\delta)=\sup_{|s-t|<\delta}\|x(s)-x(t)\|_{\mathbb{H}}, \quad 0<\delta\leqslant 1.$$

According to the definition of p_n and since both $\mathbb{E}(Z_{l(n)}^+)$ converges to a positive number and $Z_{l(n)}^+$ is bounded by one, we infer that (3.23) holds if for any positive ε

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}\left(w_{\mathbb{H}}\left(\frac{P^m W_n}{\sqrt{n}},\delta\right)\geqslant\varepsilon\right)=0 \tag{3.24}$$

and

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\sup_{t\in[0,1]}\left\|\frac{S_{[nt]}}{\sqrt{n}}-\frac{P^m S_{[nt]}}{\sqrt{n}}\right\|_{\mathbb{H}}\geqslant\varepsilon\right)=0. \tag{3.25}$$

Using Markov's inequality, (3.25) follows directly from s2*(d*).

It remains to show (3.24). Observe that

$$\mathbb{P}\left(w_{\mathbb{H}}\left(\frac{P^m W_n}{\sqrt{n}},\delta\right)\geqslant\varepsilon\right)\leqslant\sum_{\ell=1}^{m}\mathbb{P}\left(\sup_{|t-s|<\delta}\frac{|\langle W_n(s),e_\ell\rangle_{\mathbb{H}}-\langle W_n(t),e_\ell\rangle_{\mathbb{H}}|}{\sqrt{n}}\geqslant\frac{\varepsilon}{m}\right).$$

From this inequality together with Theorem 8.3 and inequality (8.16) in Billingsley (1968), it suffices to prove that, for any $1\leqslant\ell\leqslant m$ and any positive ε,

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{\delta}\mathbb{P}\left(\max_{1\leqslant i\leqslant n\delta}\frac{|\langle S_i,e_\ell\rangle_{\mathbb{H}}|}{\sqrt{n\delta}}\geqslant\frac{\varepsilon}{m\sqrt{\delta}}\right)=0,$$

which follows straightforwardly from s2*(c*) and Markov's inequality. This together with item 1 complete the proof of the fact that the sequence $(\nu_n^*[Z_n^+])_{n>0}$ is relatively compact in $C_{\mathbb{H}}([0,1])$.

To show that the sequence $(\nu^*[Z_n^+])_{n>0}$ is relatively compact in $C_{\mathbb{H}}([0,1])$, we may proceed in the same way. The only differences are the following: for n large enough, the probability measure p_n defined in item 2 becomes: $p_n^* = (\mathbb{E}(Z_{l(n)}^+))^{-1}\nu^*[Z_{l(n)}^+]$. By definition of the measure $\nu^*[Z_{l(n)}^+]$, we have

$$\nu^*[Z_{l(n)}^+](x : w_{\mathbb{H}}(x,\delta) \geqslant \varepsilon) = \int \left(\int \mathbb{1}\{x : w_{\mathbb{H}}(x,\delta) \geqslant \varepsilon\} W_{\Lambda(\omega)}(\mathrm{d}x) \right) Z_{l(n)}^+(\omega)\mathbb{P}(\mathrm{d}\omega)$$

$$\leqslant \int \mathbb{P}_{W_{\Lambda(\omega)}} \left(\sup_{|s-t|<\delta} \|\pi_t - \pi_s\|_{\mathbb{H}} \geqslant \varepsilon \right) \mathbb{P}(\mathrm{d}\omega). \qquad (3.26)$$

Since for any ω, $W_{\Lambda(\omega)}$ is a probability measure on $C_{\mathbb{H}}([0,1])$, we have

$$\text{for all } \omega \text{ in } \Omega\text{: } \lim_{\delta\to 0} \mathbb{P}_{W_{\Lambda(\omega)}} \left(\sup_{|s-t|<\delta} \|\pi_t - \pi_s\|_{\mathbb{H}} \geqslant \varepsilon \right) = 0.$$

This together with the dominated convergence theorem imply that

$$\lim_{\delta\to 0} \nu^*[Z_{l(n)}^+](x : w_{\mathbb{H}}(x,\delta) \geqslant \varepsilon) = 0. \qquad (3.27)$$

According to the definition of p_n^* and since $\mathbb{E}(Z_{l(n)}^+)$ converges to a positive number, (3.27) implies that the sequence $(\nu^*[Z_n^+])_{n>0}$ is relatively compact in $C_{\mathbb{H}}([0,1])$. This ends the proof of item (ii).

3.2.5. *Proof of Corollary 2*

The fact that (δ) ⇒ (γ) is obvious. Besides, using Proposition 3 in Dedecker and Merlevède (2002), we easily derive that (γ) entails at once s2*(a), (b) and s2*(c*). It remains to show that (γ) yields s2*(d*). To this aim, note that for all m in $\mathbb{N}^*$,

$$\mathbb{E}\left(\max_{1\leqslant i\leqslant n} \frac{\|S_i - P^m S_i\|_{\mathbb{H}}^2}{n} \right) = \mathbb{E}\left(\max_{1\leqslant i\leqslant n} \left(\sum_{\ell=m+1}^{\infty} \frac{\langle S_i, e_\ell\rangle_{\mathbb{H}}^2}{n} \right) \right)$$

$$\leqslant \sum_{\ell=m+1}^{\infty} \mathbb{E}\left(\max_{1\leqslant i\leqslant n} \frac{\langle S_i, e_\ell\rangle_{\mathbb{H}}^2}{n} \right). \qquad (3.28)$$

Now observe that

$$\max_{1\leqslant i\leqslant n} \langle S_i, e_\ell\rangle_{\mathbb{H}}^2 \leqslant (\max\{0, \langle S_1, e_\ell\rangle_{\mathbb{H}}, \ldots, \langle S_n, e_\ell\rangle_{\mathbb{H}}\})^2$$

$$+ (\max\{0, \langle -S_1, e_\ell\rangle_{\mathbb{H}}, \cdots, \langle -S_n, e_\ell\rangle_{\mathbb{H}}\})^2.$$

Starting from this inequality, we apply Proposition 1 in Dedecker and Rio (2000): for each $\ell \geqslant m+1$,

$$\mathbb{E}\left(\max_{1\leqslant i\leqslant n}\frac{\langle S_i, e_\ell\rangle_{\mathbb{H}}^2}{n}\right) \leqslant \frac{8}{n}\sum_{k=1}^{n}\mathbb{E}\langle X_k, e_\ell\rangle_{\mathbb{H}}^2 + \frac{16}{n}\sum_{k=1}^{n-1}\mathbb{E}|\langle X_k, e_\ell\rangle_{\mathbb{H}}\langle \mathbb{E}(S_n - S_k|\mathcal{M}_k), e_\ell\rangle_{\mathbb{H}}|. \tag{3.29}$$

Combining (3.28) with (3.29) and applying Hölder's inequality in ℓ^2, we infer that the quantity $n^{-1}\mathbb{E}(\max_{1\leqslant i\leqslant n}\|S_i - P^m S_i\|_{\mathbb{H}}^2)$ is bounded by

$$8\mathbb{E}\|(I_{\mathbb{H}} - P^m)X_0\|_{\mathbb{H}}^2 + \frac{16}{n}\sum_{k=1}^{n-1}\mathbb{E}(\|(I_{\mathbb{H}} - P^m)X_k\|_{\mathbb{H}}\|\mathbb{E}((I_{\mathbb{H}} - P^m)(S_n - S_k)|\mathcal{M}_k)\|_{\mathbb{H}}),$$

which by stationarity is equal to

$$8\mathbb{E}\|(I_{\mathbb{H}} - P^m)X_0\|_{\mathbb{H}}^2 + \frac{16}{n}\sum_{k=1}^{n-1}\mathbb{E}\left(\|(I_{\mathbb{H}} - P^m)X_0\|_{\mathbb{H}}\left\|\mathbb{E}\left(\sum_{j=1}^{n-k}(I_{\mathbb{H}} - P^m)X_j|\mathcal{M}_0\right)\right\|_{\mathbb{H}}\right). \tag{3.30}$$

The first term in the right-hand side of the above quantity tends to zero as m tends to infinity. To control the second term we proceed as follows: fix $N \geqslant 1$ and write

$$\frac{1}{n}\sum_{k=1}^{n-1}\mathbb{E}\left(\|(I_{\mathbb{H}} - P^m)X_0\|_{\mathbb{H}}\left\|\mathbb{E}\left(\sum_{j=1}^{n-k}(I_{\mathbb{H}} - P^m)X_j|\mathcal{M}_0\right)\right\|_{\mathbb{H}}\right)$$
$$\leqslant \frac{1}{n}\sum_{k=1}^{n-1}\mathbb{E}\left(\|(I_{\mathbb{H}} - P^m)X_0\|_{\mathbb{H}}\left\|\mathbb{E}\left(\sum_{j=1}^{N\wedge(n-k)}(I_{\mathbb{H}} - P^m)X_j|\mathcal{M}_0\right)\right\|_{\mathbb{H}}\right)$$
$$+\frac{1}{n}\sum_{k=1}^{n-1}\mathbb{E}\left(\|(I_{\mathbb{H}} - P^m)X_0\|_{\mathbb{H}}\left\|\mathbb{E}\left(\sum_{j=N\wedge(n-k)+1}^{n-k}(I_{\mathbb{H}} - P^m)X_j|\mathcal{M}_0\right)\right\|_{\mathbb{H}}\right). \tag{3.31}$$

Cauchy–Schwarz's inequality entails that the first term on right-hand is bounded by $N\mathbb{E}\|(I_{\mathbb{H}} - P^m)X_0\|_{\mathbb{H}}^2$, which converges to zero as m tends to infinity. On the other hand, the second term on right-hand side is bounded by

$$\sup_{M>N}\mathbb{E}(\|X_0\|_{\mathbb{H}}\|\mathbb{E}(S_M - S_N|\mathcal{M}_0)\|_{\mathbb{H}}).$$

From condition (γ), we can choose N large enough so that the right-hand term of (3.31) is less than ε. Gathering all these considerations, we infer that (γ) entails s2*(d*).

To prove that (β) implies (δ), we proceed as in Dedecker and Doukhan (2003). Note first that

$$\mathbb{E}(\|X_0\|_{\mathbb{H}}\|\mathbb{E}(X_k|\mathcal{M}_0)\|_{\mathbb{H}}) = \int_0^\infty \mathbb{E}(\|\mathbb{E}(X_k|\mathcal{M}_0)\|_{\mathbb{H}}\mathbb{1}_{\|X_0\|_{\mathbb{H}}>t})\,\mathrm{d}t.$$

Clearly, we have $\mathbb{E}(\|\mathbb{E}(X_k|\mathcal{M}_0)\|_{\mathbb{H}}\mathbb{1}_{\|X_0\|_{\mathbb{H}}>t}) \leqslant \gamma_k \wedge \mathbb{E}(\|X_k\|_{\mathbb{H}}\mathbb{1}_{\|X_0\|_{\mathbb{H}}>t})$. Consequently, setting $R_k(t) = \mathbb{E}(\|X_k\|_{\mathbb{H}}\mathbb{1}_{\|X_0\|_{\mathbb{H}}>t})$, we have the inequality

$$\mathbb{E}(\|X_0\|_{\mathbb{H}}\|\mathbb{E}(X_k|\mathcal{M}_0)\|_{\mathbb{H}}) \leqslant \int_0^\infty \left(\int_0^{\gamma_k} \mathbb{1}_{u<R_k(t)}\,\mathrm{d}u\right)\mathrm{d}t. \tag{3.32}$$

Now, applying Fréchet's inequality (1957) we obtain, with the notations of Definition 3:

$$R_k(t) \leqslant \int_0^{\mathbb{P}(\|X_0\|_{\mathbb{H}}>t)} Q_{\|X_k\|_{\mathbb{H}}}(u)\,\mathrm{d}u,$$

Since the random variable X_0 has the same distribution as X_k, this means exactly that $R_k(t) \leqslant H_{\|X_0\|_{\mathbb{H}}}(\mathbb{P}(\|X_0\|_{\mathbb{H}} > t))$. Now by definition of the functions $Q_{\|X_0\|_{\mathbb{H}}}$ and $G_{\|X_0\|_{\mathbb{H}}}$, $\{u > 0 : u < H_{\|X_0\|_{\mathbb{H}}}(\mathbb{P}(\|X_0\|_{\mathbb{H}} > t))\} = \{u > 0 : t < Q_{\|X_0\|_{\mathbb{H}}} \circ G_{\|X_0\|_{\mathbb{H}}}(u)\}$, and (3.32) implies that

$$\mathbb{E}(\|X_0\|_{\mathbb{H}}\|\mathbb{E}(X_k|\mathcal{M}_0)\|_{\mathbb{H}}) \leqslant \int_0^{\gamma_k} Q_{\|X_0\|_{\mathbb{H}}} \circ G_{\|X_0\|_{\mathbb{H}}}(u)\,\mathrm{d}u. \tag{3.33}$$

The last point is to prove that (α) implies (β). Since $Q_{\|X_0\|_{\mathbb{H}}} \circ G_{\|X_0\|_{\mathbb{H}}}$ is nonincreasing, we infer from (3.33) that

$$\int_0^{\gamma_k} Q_{\|X_0\|_{\mathbb{H}}} \circ G_{\|X_0\|_{\mathbb{H}}}(u)\,\mathrm{d}u \leqslant 18\int_0^{\gamma_k/18} Q_{\|X_0\|_{\mathbb{H}}} \circ G_{\|X_0\|_{\mathbb{H}}}(u)\,\mathrm{d}u.$$

Since $H_{\|X_0\|_{\mathbb{H}}}$ is absolutely continuous and monotonic, we can make the change-of-variables $u = H_{\|X_0\|_{\mathbb{H}}}(z)$ (see Theorem 7.26 in Rudin (1987) and the example given page 156). Then we get

$$\int_0^{\gamma_k} Q_{\|X_0\|_{\mathbb{H}}} \circ G_{\|X_0\|_{\mathbb{H}}}(u)\,\mathrm{d}u \leqslant 18\int_0^{G_{\|X_0\|_{\mathbb{H}}}(\gamma_k/18)} Q^2_{\|X_0\|_{\mathbb{H}}}(u)\,\mathrm{d}u.$$

Consequently, the result will be proved if we show that $G_{\|X_0\|_{\mathbb{H}}}(\gamma_k/18) \leqslant \alpha_k$. Define the $\mathcal{M}_0$-measurable variable $Y = \mathbb{E}(X_k|\mathcal{M}_0)/\|\mathbb{E}(X_k|\mathcal{M}_0)\|_{\mathbb{H}}$ (interpret 0/0 to be 0). Clearly $\gamma_k = \mathbb{E}(\langle Y, X_k\rangle_{\mathbb{H}})$. Since $\|Y\|_{\mathbb{H}} \leqslant 1$, we have $Q_{\|Y\|_{\mathbb{H}}} \leqslant 1$. We now use an extension of Rio's covariance inequality (1993) to separable Hilbert spaces. This inequality, due to Merlevède et al. (1997), implies that

$$\gamma_k = \mathbb{E}(\langle Y, X_k\rangle_{\mathbb{H}}) \leqslant 18\int_0^{\alpha_k} Q_{\|X_0\|_{\mathbb{H}}}(u)\,\mathrm{d}u.$$

This means exactly that $G_{\|X_0\|_{\mathbb{H}}}(\gamma_k/18) \leqslant \alpha_k$, and the result follows.

3.2.6. *Proof of Corollary 3*

For any positive integer i, let $Y_{k,i} = \langle X_k, e_i\rangle_{\mathbb{H}}$. Since $P_0(Y_{k,i}) = \langle P_0(X_k), e_i\rangle_{\mathbb{H}}$, from (2.5), we infer that for any $i \geqslant 1$

$$\mathbb{E}(Y_{0,i}|\mathcal{M}_{-\infty}) = 0 \text{ a.s.} \quad \text{and} \quad \sum_{k\geqslant 1}\|P_0(Y_{k,i})\|_2 < \infty. \tag{3.34}$$

Proof of s2(a): It suffices to prove that, for any positive integer i,

$$\lim_{N\to\infty}\limsup_{n\to\infty}\frac{1}{n}\left\|\sum_{k=N}^{n}\mathbb{E}(Y_{k,i}|\mathcal{M}_0)\right\|_2^2=0. \tag{3.35}$$

Using the operator P_m and the fact that $\mathbb{E}(Y_{0,i}|\mathcal{M}_{-\infty})=0$ a.s., we have the equalities

$$\begin{aligned}\left\|\sum_{k=N}^{n}\mathbb{E}(Y_{k,i}|\mathcal{M}_0)\right\|_2^2&=\sum_{k=N}^{n}\sum_{\ell=N}^{n}\mathbb{E}(\mathbb{E}(Y_{k,i}|\mathcal{M}_0)\mathbb{E}(Y_{\ell,i}|\mathcal{M}_0))\\&=\sum_{k=N}^{n}\sum_{\ell=N}^{n}\mathbb{E}\left(\sum_{m=0}^{\infty}P_{-m}(Y_{k,i})P_{-m}(Y_{\ell,i})\right).\end{aligned}$$

Using Hölder's inequality and the stationarity of $(X_k)_{k\in\mathbb{Z}}$, we infer that

$$\begin{aligned}\frac{1}{n}\left\|\sum_{k=N}^{n}\mathbb{E}(Y_{k,i}|\mathcal{M}_0)\right\|_2^2&\leqslant\frac{1}{n}\sum_{m=0}^{\infty}\sum_{k=N+m}^{n+m}\sum_{\ell=N+m}^{n+m}\|P_0(Y_{k,i})\|_2\|P_0(Y_{\ell,i})\|_2\\&\leqslant\left(\sum_{k=N}^{\infty}\|P_0(Y_{k,i})\|_2\right)^2,\end{aligned}$$

and (3.35) follows from (3.34).

Proof of s2(b): For any positive integer i, let $S_{n,i}=Y_{1,i}+\cdots+Y_{n,i}$. Clearly,

$$\mathbb{E}(S_{n,i}S_{n,j}|\mathcal{M}_0)=\mathbb{E}((S_{n,i}-\mathbb{E}(S_{n,i}|\mathcal{M}_0))(S_{n,j}-\mathbb{E}(S_{n,j}|\mathcal{M}_0))|\mathcal{M}_0)+\mathbb{E}(S_{n,i}|\mathcal{M}_0)\mathbb{E}(S_{n,j}|\mathcal{M}_0),$$

and we know from (3.35) that $n^{-1}\|\mathbb{E}(S_{n,i}|\mathcal{M}_0)\mathbb{E}(S_{n,j}|\mathcal{M}_0)\|_1$ tends to zero as n tends to infinity. Setting $Z_{k,i}=Y_{k,i}-\mathbb{E}(Y_{k,i}|\mathcal{M}_0)$, we infer that s2(b) is equivalent to: for any positive integers i,j,

$$\lim_{n\to\infty}\left\|\eta_{i,j}-\mathbb{E}\left(\frac{1}{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n}Z_{k,i}Z_{\ell,j}|\mathcal{M}_0\right)\right\|_1=0, \tag{3.36}$$

for some integrable and $\mathcal{M}_0$-measurable random variable $\eta_{i,j}$.

Define the variable $\eta_{i,j}(N)=\mathbb{E}(Y_{0,i}Y_{0,j}|\mathcal{I})+\mathbb{E}(Y_{0,i}S_{N-1,j}|\mathcal{I})+\mathbb{E}(Y_{0,j}S_{N-1,i}|\mathcal{I})$ for any positive integer N. We shall prove that

$$\lim_{N\to\infty}\limsup_{n\to\infty}\left\|\eta_{i,j}(N)-\mathbb{E}\left(\frac{1}{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n}Z_{k,i}Z_{\ell,j}|\mathcal{M}_0\right)\right\|_1=0. \tag{3.37}$$

From (3.37) we easily deduce that both $n^{-1}\mathbb{E}(\sum_{k=1}^{n}\sum_{\ell=1}^{n}Z_{k,i}Z_{\ell,j}|\mathcal{M}_0)$ and $\eta_{i,j}(N)$ are Cauchy sequences in $\mathbb{L}^1$. Consequently $n^{-1}\mathbb{E}(\sum_{k=1}^{n}\sum_{\ell=1}^{n}Z_{k,i}Z_{\ell,j}|\mathcal{M}_0)$ converges in $\mathbb{L}^1$ to a $\mathcal{M}_0$-measurable variable $\eta_{i,j}$ (so that (3.36) holds), and $\eta_{i,j}(N)$ converges in $\mathbb{L}^1$ to $\eta_{i,j}$.

It remains to prove (3.37). Define the two sets

$$G_N=[1,n]^2\cap\{(k,\ell)\in\mathbb{Z}^2:|k-\ell|<N\},\quad\text{and}\quad\bar{G}_N=[1,n]^2-G_N.$$

Write first

$$\left\|\eta_{i,j}(N) - \mathbb{E}\left(\frac{1}{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n} Z_{k,i}Z_{\ell,j}|\mathcal{M}_0\right)\right\|_1 \leqslant \left\|\eta_{i,j}(N) - \mathbb{E}\left(\frac{1}{n}\sum_{G_N} Z_{k,i}Z_{\ell,j}|\mathcal{M}_0\right)\right\|_1 + \frac{1}{n}\left\|\sum_{\bar{G}_N}\mathbb{E}(Z_{k,i}Z_{\ell,j}|\mathcal{M}_0)\right\|_1. \quad (3.38)$$

From Claim 1(a) in Dedecker and Rio (2000), we know that $\eta_{i,j}(N) = \mathbb{E}(\eta_{i,j}(N)|\mathcal{M}_0)$ almost surely. Using this result, we obtain that the first term on right-hand side in (3.38) is less than

$$\left\|\eta_{i,j}(N) - \frac{1}{n}\sum_{G_N} Y_{k,i}Y_{\ell,j}\right\|_1 + \frac{1}{n}\sum_{l=-N+1}^{N-1}\sum_{k=1}^{n}\|\mathbb{E}(Y_{k,i}|\mathcal{M}_0)\mathbb{E}(Y_{k+\ell,j}|\mathcal{M}_0)\|_1. \quad (3.39)$$

Applying the $\mathbb{L}^1$-ergodic theorem, the first term in (3.39) tends to zero as n tends to infinity. Since $\|\mathbb{E}(Y_{k,i}|\mathcal{M}_0)\mathbb{E}(Y_{k+\ell,j}|\mathcal{M}_0)\|_1 \leqslant \|X_0\|_{\mathbb{L}^2_{\mathbb{H}}}\|\mathbb{E}(Y_{k,i}|\mathcal{M}_0)\|_2$, we infer that the second term tends to zero as n tends to infinity provided that

$$\lim_{K\to\infty}\limsup_{n\to\infty}\frac{1}{n}\sum_{k=K}^{n}\|\mathbb{E}(Y_{k,i}|\mathcal{M}_0)\|_2 = 0. \quad (3.40)$$

Using the operators P_m, we have that

$$\frac{1}{n}\sum_{k=K}^{n}\|\mathbb{E}(Y_{k,i}|\mathcal{M}_0)\|_2 \leqslant \frac{1}{n}\sum_{m=0}^{\infty}\sum_{k=K}^{n}\|P_{-m}(Y_{k,i})\|_2 = \frac{1}{n}\sum_{m=0}^{\infty}\sum_{k=K+m}^{n+m}\|P_0(Y_{k,i})\|_2$$

$$\leqslant \sum_{k=K}^{\infty}\|P_0(Y_{k,i})\|_2,$$

and (3.40) follows from (3.34). Consequently, the first term on right-hand side in (3.38) tends to zero as n tends to infinity.

It remains to control the second term on right-hand side in (3.38). Write first

$$\frac{1}{n}\left\|\sum_{\bar{G}_N}\mathbb{E}(Z_{k,i}Z_{\ell,j}|\mathcal{M}_0)\right\|_1 \leqslant \frac{1}{n}\sum_{k=1}^{n}\sum_{\ell=N}^{\infty}\|\mathbb{E}(Z_{k,i}Z_{k+\ell,j}|\mathcal{M}_0)\|_1 + \frac{1}{n}\sum_{\ell=1}^{n}\sum_{k=N}^{\infty}\|\mathbb{E}(Z_{\ell+k,i}Z_{\ell,j}|\mathcal{M}_0)\|_1 \quad (3.41)$$

Using the fact that $Z_{k,i} = \sum_{m=1}^{k} P_m(Y_{k,i})$, we obtain

$$\frac{1}{n}\sum_{k=1}^{n}\sum_{\ell=N}^{\infty}\|\mathbb{E}(Z_{k,i}Z_{k+\ell,j}|\mathcal{M}_0)\|_1 \leqslant \frac{1}{n}\sum_{k=1}^{n}\sum_{\ell=N}^{\infty}\sum_{m=1}^{k}\|P_m(Y_{k,i})P_m(Y_{k+\ell,j})\|_1$$

$$\leqslant \frac{1}{n}\sum_{k=1}^{n}\sum_{m=-\infty}^{k}\|P_m(Y_{k,i})\|_2\left(\sum_{\ell=N}^{\infty}\|P_m(Y_{k+\ell,j})\|_2\right),$$

and by stationarity, we conclude that

$$\frac{1}{n}\sum_{k=1}^{n}\sum_{\ell=N}^{\infty}\|\mathbb{E}(Z_{k,i}Z_{k+\ell,j}|\mathscr{M}_0)\|_1 \leqslant \left(\sum_{k=0}^{\infty}\|P_0(Y_{k,i})\|_2\right)\left(\sum_{\ell=N}^{\infty}\|P_0(Y_{\ell,j})\|_2\right).$$

Of course, the same arguments apply to the second term on right-hand side in (3.41), and we infer from (3.34) that

$$\lim_{N\to\infty}\limsup_{n\to\infty}\frac{1}{n}\left\|\sum_{\bar{G}_N}\mathbb{E}(Z_{k,i}Z_{\ell,j}|\mathscr{M}_0)\right\|_1 = 0.$$

This competes the proof of (3.37), and s2(b) follows.

Proof of s2*(c*): For any positive integer i define $S^*_{n,i} = \max_{1\leqslant k\leqslant n}\{0, S_{k,i}\}$. According to Proposition 6 of Dedecker and Merlevède (2002), for any two sequences of nonnegative numbers $(a_m)_{m\geqslant 0}$ and $(b_m)_{m\geqslant 0}$ such that $K=\sum_{m\geqslant 0} a_m^{-1}$ is finite and $\sum_{m\geqslant 0} b_m = 1$, we have

$$\frac{1}{n}\,\mathbb{E}\left((S^*_{n,i}-M\sqrt{n})_+^2\right) \leqslant 4K\sum_{m=0}^{\infty}a_m\mathbb{E}\left(\frac{1}{n}\sum_{k=1}^{n}P^2_{k-m}(Y_{k,i})\mathbb{1}_{\Gamma(m,n,b_mM\sqrt{n})}\right), \tag{3.42}$$

where $\Gamma(m,n,\lambda)=\{\max_{1\leqslant k\leqslant n}\{0,\sum_{\ell=1}^{k} P_{\ell-m}(Y_{\ell,i})\} > \lambda\}$. Here, we take $b_m = 2^{-m-1}$ and $a_m = (\|P_0(Y_{m,i})\|_2 + (m+1)^{-2})^{-1}$. According to (3.34), $\sum a_m^{-1}$ is finite. Since for all $m\geqslant 0$

$$a_m\mathbb{E}\left(\frac{1}{n}\sum_{k=1}^{n}P^2_{k-m}(Y_{k,i})\mathbb{1}_{\Gamma(m,n,b_mM\sqrt{n})}\right) \leqslant \frac{\|P_0(Y_{m,i})\|_2^2}{\|P_0(Y_{m,i})\|_2+(m+1)^2} \leqslant \|P_0(Y_{m,i})\|_2,$$

we infer from (3.42) and (3.34) that for any $\varepsilon>0$, there exists $N(\varepsilon)$ such that

$$\frac{1}{n}\,\mathbb{E}\left((S^*_{n,i}-M\sqrt{n})_+^2\right) \leqslant \varepsilon + 4K\sum_{m=0}^{N(\varepsilon)}a_m\mathbb{E}\left(\frac{1}{n}\sum_{k=1}^{n}P^2_{k-m}(Y_{k,i})\mathbb{1}_{\Gamma(m,n,b_mM\sqrt{n})}\right). \tag{3.43}$$

Now by Doob's maximal inequality

$$\mathbb{P}(\Gamma(m,n,b_mM\sqrt{n})) \leqslant \frac{4\sum_{k=1}^{n}\|P_{k-m}(Y_{k,i})\|_2^2}{b_m^2M^2n} = \frac{4\|P_0(Y_{m,i})\|_2^2}{b_m^2M^2},$$

and consequently

$$\lim_{M\to\infty}\sup_{n>0}\mathbb{P}(\Gamma(m,n,b_mM\sqrt{n})) = 0. \tag{3.44}$$

Since $n^{-1}\sum_{k=1}^{n} P^2_{k-m}(Y_{k,i})$ converges in $\mathbb{L}^1$ (apply the ergodic theorem), we infer from (3.44) that

$$\lim_{M\to\infty}\limsup_{n\to\infty}\mathbb{E}\left(\frac{1}{n}\sum_{k=1}^{n}P^2_{k-m}(Y_{k,i})\mathbb{1}_{\Gamma(m,n,b_mM\sqrt{n})}\right) = 0. \tag{3.45}$$

Combining (3.43) and (3.45), we conclude that

$$\lim_{M\to\infty}\limsup_{n\to\infty}\frac{1}{n}\,\mathbb{E}((S^*_{n,i}-M\sqrt{n})_+^2) = 0. \tag{3.46}$$

Of course, the same arguments apply to the sequence $(-Y_{k,i})_{k\in\mathbb{Z}}$ so that (3.46) holds for $\max_{1\leqslant k\leqslant n}|S_{k,i}|$ instead of $S^*_{n,i}$. This completes the proof.

Proof of s2*(d*): We start from (3.28), and for each $\ell\geqslant m+1$, we apply Lemma 1.5 in McLeish (1975b). For any sequence of nonnegative numbers $(a_i)_{i\geqslant 0}$ such that $K=\sum_{i\geqslant 0}a_i^{-1}$ is finite, we have

$$\mathbb{E}\left(\max_{1\leqslant i\leqslant n}\frac{\|(I_{\mathbb{H}}-P^m)S_i\|_{\mathbb{H}}^2}{n}\right)\leqslant\frac{4}{n}K\sum_{\ell=m+1}^{\infty}\sum_{i=0}^{\infty}a_i\left(\sum_{k=1}^{n}\mathbb{E}(\langle P_{k-i}(X_k),e_\ell\rangle_{\mathbb{H}}^2)\right).$$

Using first Fubini and next stationarity, we obtain

$$\mathbb{E}\left(\max_{1\leqslant i\leqslant n}\frac{\|(I_{\mathbb{H}}-P^m)S_i\|_{\mathbb{H}}^2}{n}\right)\leqslant\frac{4}{n}K\sum_{i=0}^{\infty}a_i\left(\sum_{k=1}^{n}\mathbb{E}\|(I_{\mathbb{H}}-P^m)P_{k-i}(X_k)\|_{\mathbb{H}}^2\right)$$

$$\leqslant 4K\sum_{i=0}^{\infty}a_i\mathbb{E}\|(I_{\mathbb{H}}-P^m)P_0(X_i)\|_{\mathbb{H}}^2.$$

Considering (2.5), we can choose $a_i=((\mathbb{E}\|P_0(X_i)\|_{\mathbb{H}}^2)^{1/2}+(i+1)^{-2})^{-1}$. Consequently, using the fact that $\mathbb{E}\|(I_{\mathbb{H}}-P^m)P_0(X_i)\|_{\mathbb{H}}^2\leqslant\mathbb{E}\|P_0(X_i)\|_{\mathbb{H}}^2$, we get

$$\mathbb{E}\left(\max_{1\leqslant i\leqslant n}\frac{\|(I_{\mathbb{H}}-P^m)S_i\|_{\mathbb{H}}^2}{n}\right)\leqslant 4K\sum_{i=0}^{\infty}\|(I_{\mathbb{H}}-P^m)P_0(X_i)\|_{\mathbb{L}^2_{\mathbb{H}}}.$$

Now (2.5) together with the dominated convergence theorem imply s2*(d*).

3.2.7. *Proof of Remark 6*

We start with the orthogonal decomposition

$$X_k=\mathbb{E}(X_k|\mathcal{M}_{-\infty})+\sum_{i=0}^{\infty}P_{k-i}(X_k). \tag{3.47}$$

Since (1.2) implies that $\mathbb{E}(X_k|\mathcal{M}_{-\infty})=0$, we infer from (3.47) and the stationarity of $(X_i)_{i\in\mathbb{Z}}$ that

$$\sum_{k>0}L_k\|\mathbb{E}(X_k|\mathcal{M}_0)\|_{\mathbb{L}^2_{\mathbb{H}}}^2=\sum_{k>0}L_k\sum_{i\leqslant 0}\|P_i(X_k)\|_{\mathbb{L}^2_{\mathbb{H}}}^2=\sum_{i>0}\left(\sum_{k=1}^{i}L_k\right)\|P_0(X_i)\|_{\mathbb{L}^2_{\mathbb{H}}}^2.$$

Setting $b_i=L_1+\cdots+L_i$, we infer that (1.2) is equivalent to

$$\mathbb{E}(X_0|\mathcal{M}_{-\infty})=0,\quad \sum_{i\geqslant 1}b_i\|P_0(X_i)\|_{\mathbb{L}^2_{\mathbb{H}}}^2<\infty\quad\text{and}\quad\sum_{i\geqslant 1}\frac{1}{b_i}<\infty. \tag{3.48}$$

Now, Hölder's inequality in ℓ^2 gives

$$\sum_{i\geqslant 1}\|P_0(X_i)\|_{\mathbb{L}^2_{\mathbb{H}}}\leqslant\left(\sum_{i>0}\frac{1}{b_i}\right)^{1/2}\left(\sum_{i\geqslant 1}b_i\|P_0(X_i)\|_{\mathbb{L}^2_{\mathbb{H}}}^2\right)^{1/2}<\infty,$$

which shows that (1.2) implies (2.5).

3.3. The general case

In this section, we prove Theorem 3. For any ℓ in $\mathbb{Z}$ set $X_0^{(\ell)} = \mathbb{E}(X_0|\mathcal{M}_\ell)$ and let $S_n^{(\ell)} = X_0^{(\ell)} \circ T + \cdots + X_0^{(\ell)} \circ T^n$. We start the proof with two preliminary lemmas.

Lemma 2. *Assume that* $\mathbb{E}\|X_0\|_{\mathbb{H}}^p < \infty$. *Under condition* (1.3), *we have*

$$\lim_{\ell\to\infty} \limsup_{n\to\infty} \frac{1}{n}\, \mathbb{E}\|S_n - S_n^{(\ell)}\|_{\mathbb{H}}^2 = 0.$$

Proof. Set $Y_0^{(\ell)} := X_0 - X_0^{(\ell)}$ and $Y_i^{(\ell)} := Y_0^{(\ell)} \circ T^i$. Since $Y_0^{(\ell)}$ is orthogonal to $\mathbb{L}^2(\mathcal{M}_\ell)$, we have for any positive i,

$$\mathbb{E}\langle Y_0^{(\ell)}, Y_{-i}^{(\ell)}\rangle_{\mathbb{H}} = \mathbb{E}\langle Y_0^{(\ell)}, X_{-i} - \mathbb{E}(X_{-i}|\mathcal{M}_{\ell-i})\rangle_{\mathbb{H}} = \mathbb{E}\langle Y_0^{(\ell)}, X_{-i}\rangle_{\mathbb{H}}.$$

Now using first the fact that $Y_0^{(\ell)}$ is orthogonal to $\mathbb{L}^2(\mathcal{M}_\ell)$ and secondly that $X_{-i} - \mathbb{E}(X_{-i}|\mathcal{M}_\ell))_{\mathbb{H}}$ is orthogonal to $\mathbb{L}^2(\mathcal{M}_\ell)$, we successively derive that

$$\mathbb{E}\langle Y_0^{(\ell)}, Y_{-i}^{(\ell)}\rangle_{\mathbb{H}} = \mathbb{E}\langle Y_0^{(\ell)}, X_{-i} - \mathbb{E}(X_{-i}|\mathcal{M}_\ell)\rangle_{\mathbb{H}} = \mathbb{E}\langle X_0, X_{-i} - \mathbb{E}(X_{-i}|\mathcal{M}_\ell)\rangle_{\mathbb{H}}.$$

Hence by stationarity

$$\begin{aligned}\frac{1}{n}\, \mathbb{E}\|S_n - S_n^{(\ell)}\|_{\mathbb{H}}^2 &= \frac{1}{n}\sum_{N=1}^{n}\left(\mathbb{E}\|Y_0^{(\ell)}\|_{\mathbb{H}}^2 + \frac{2}{n}\sum_{N=2}^{n}\sum_{m=1}^{N-1}\mathbb{E}\langle Y_0^{(\ell)}, Y_{m-N}^{(\ell)}\rangle_{\mathbb{H}}\right)\\ &= \frac{1}{n}\sum_{N=1}^{n}\left(\mathbb{E}\|Y_0^{(\ell)}\|_{\mathbb{H}}^2 + \frac{2}{n}\sum_{N=2}^{n}\sum_{i=1}^{N-1}\mathbb{E}\langle Y_0^{(\ell)}, Y_{-i}^{(\ell)}\rangle_{\mathbb{H}}\right)\\ &= \frac{1}{n}\sum_{N=1}^{n}\left(\mathbb{E}\|X_0 - X_0^{(\ell)}\|_{\mathbb{H}}^2 + 2\sum_{i=1}^{N-1}\mathbb{E}\langle X_0, X_{-i} - \mathbb{E}(X_{-i}|\mathcal{M}_\ell)\rangle_{\mathbb{H}}\right).\end{aligned}$$

Therefore Lemma 2 holds via Cesaro's mean convergence theorem provided that

$$\lim_{\ell\to\infty}\limsup_{n\to\infty}\left(\mathbb{E}\|X_0 - X_0^{(\ell)}\|_{\mathbb{H}}^2 + 2\sum_{i=1}^{n}\mathbb{E}\langle X_0, X_{-i} - \mathbb{E}(X_{-i}|\mathcal{M}_\ell)\rangle_{\mathbb{H}}\right) = 0. \tag{3.49}$$

Using first Hölder's inequality and next stationarity, we obtain that

$$\left|\sum_{i=1}^{n}\mathbb{E}\langle X_0, X_{-i} - \mathbb{E}(X_{-i}|\mathcal{M}_\ell)\rangle_{\mathbb{H}}\right| \leqslant \mathbb{E}\|X_0\|_{\mathbb{L}_{\mathbb{H}}^p}\left\|\sum_{m=1+\ell}^{n+\ell}(X_{-m} - \mathbb{E}(X_{-m}|\mathcal{M}_0))\right\|_{\mathbb{L}_{\mathbb{H}}^q}.$$

Finally condition (1.3) implies (3.49) and Lemma 2 follows. □

Lemma 3. *Assume that* $\mathbb{E}\|X_0\|_{\mathbb{H}}^p < \infty$. *Under condition* (1.3), *the sequence* $(X_i^{(\ell)})_i = (X_0^{(\ell)} \circ T^i)_i$ *adapted to the filtration* $(\mathcal{M}_{\ell+i})_{i\in\mathbb{Z}}$ *satisfies condition* (γ) *of Corollary* 2:

$$\|\mathbb{E}(X_0|\mathcal{M}_\ell)\|_{\mathbb{H}}\mathbb{E}(S_n|\mathcal{M}_\ell) \text{ converges in } \mathbb{L}_{\mathbb{H}}^1. \tag{3.50}$$

Proof. Applying Hölder's inequality we have

$$\mathbb{E}(\|\mathbb{E}(X_0|\mathscr{M}_\ell)\|_{\mathbb{H}}\|\mathbb{E}(S_n - S_m|\mathscr{M}_\ell)\|_{\mathbb{H}})$$
$$\leqslant (\mathbb{E}\|\mathbb{E}(X_0|\mathscr{M}_\ell)\|_{\mathbb{H}}^p)^{1/p}(\mathbb{E}\|\mathbb{E}(S_n - S_m|\mathscr{M}_\ell)\|_{\mathbb{H}}^q)^{1/q},$$

and by stationarity

$$\lim_{m\to\infty}\sup_{n>m}\mathbb{E}(\|\mathbb{E}(X_0|\mathscr{M}_\ell)\|_{\mathbb{H}}\|\mathbb{E}(S_n - S_m|\mathscr{M}_\ell)\|_{\mathbb{H}})$$
$$\leqslant \lim_{m\to\infty}\sup_{n>m}\|X_0\|_{\mathbb{L}^p_{\mathbb{H}}}\left\|\sum_{j=m-\ell+1}^{n-\ell}\mathbb{E}(X_j|\mathscr{M}_0)\right\|_{\mathbb{L}^q_{\mathbb{H}}}$$

which equals zero by (1.3) and the fact that $\mathbb{E}\|X_0\|_{\mathbb{H}}^p < \infty$. Lemma 3 is proved. □

Proof of Theorem 3. From Lemma 3 and Corollary 2 we derive that $n^{-1/2}S_n^{(\ell)}$ satisfies s1. In particular the sequence $n^{-1}\|S_n^{(\ell)}\|_{\mathbb{H}}^2$ is uniformly integrable. Via Lemma 2, this implies that $n^{-1}\|S_n\|_{\mathbb{H}}^2$ is also uniformly integrable. Hence we need only prove s1(φ) for any continuous bounded function φ from $\mathbb{H}$ to $\mathbb{R}$.

For any $m \geqslant 1$ and any $v \in \mathbb{R}^m$, set $V_m(x) = \sum_{i=1}^m v_i\langle x, e_i\rangle_{\mathbb{H}}$. According to the proof of Theorem 1, s1(φ) holds for any continuous bounded function φ as soon as: for any $m \geqslant 1$ and any v in $\mathbb{R}^m$

$$\lim_{n\to\infty}\left\|\mathbb{E}(\exp(\mathrm{i}n^{-1/2}V_m(S_n)) - \int \exp(\mathrm{i}V_m(x))P_\Lambda^\varepsilon(\mathrm{d}x)|\mathscr{M}_k)\right\|_1 = 0 \tag{3.51}$$

and

$$\mu_n[Z_n] \text{ is relatively compact in } \mathbb{H}. \tag{3.52}$$

Since for any ℓ in $\mathbb{Z}$ the sequence $n^{-1/2}S_n^{(\ell)}$ satisfies condition (γ) of Corollary 2, there exists a random linear operator $\Lambda^{(\ell)}$ belonging to $\mathscr{S}(\mathbb{H}, \mathscr{M}_\ell)$, such that for any φ in $\mathscr{H}$ and any positive integer k

$$\lim_{n\to\infty}\left\|\mathbb{E}\left(\varphi(n^{-1/2}S_n^{(\ell)}) - \int \varphi(x)P_{\Lambda^{(\ell)}}^\varepsilon(\mathrm{d}x)\middle|\mathscr{M}_k\right)\right\|_1 = 0. \tag{3.53}$$

Moreover $\langle\Lambda^{(\ell)}e_i, e_j\rangle_{\mathbb{H}} = \eta_{i,j}^{(\ell)}$ and $\eta_{i,j}^{(\ell)}$ is the limit in $\mathbb{L}^1$ of the sequence obtained from (2.1) by replacing X_i by $X_i^{(\ell)}$. From (3.53) we obtain that: for any $m \geqslant 1$, any v in $\mathbb{R}^m$, any ℓ in $\mathbb{Z}$ and any positive integer k

$$\lim_{n\to\infty}\left\|\mathbb{E}\left(\exp(\mathrm{i}n^{-1/2}V_m(S_n^{(\ell)})) - \int \exp(\mathrm{i}V_m(x))P_{\Lambda^{(\ell)}}^\varepsilon(\mathrm{d}x)\middle|\mathscr{M}_k\right)\right\|_1 = 0. \tag{3.54}$$

Consequently to show (3.51), it suffices to prove that

$$\lim_{\ell\to\infty}\lim_{n\to\infty}\|\exp(\mathrm{i}n^{-1/2}V_m(S_n)) - \exp(\mathrm{i}n^{-1/2}V_m(S_n^{(\ell)}))\|_1 = 0, \tag{3.55}$$

and that there exists a random linear random operator Λ belonging to $\mathscr{S}(\mathbb{H},\mathscr{A})$ such that

$$\lim_{\ell\to\infty}\left\|\int \exp(\mathrm{i}V_m(x))P^{\varepsilon}_{\Lambda^{(\ell)}}\,\mathrm{d}x-\int \exp(\mathrm{i}V_m(x))P^{\varepsilon}_{\Lambda}\,\mathrm{d}x\right\|_1=0. \tag{3.56}$$

Note first that (3.55) follows straightforwardly from Lemma 2. To prove (3.56), we have to define the random linear operator Λ that we are going to consider. We shall prove that for all i,j in $\mathbb{N}^*$

$(\eta^{(\ell)}_{i,j})_\ell$ converges in $\mathbb{L}^1$ to some random variable $\eta_{i,j}$ and

$$\sum_{\ell=1}^{\infty}\mathbb{E}(\eta_{\ell,\ell})<\infty. \tag{3.57}$$

Using (3.57), it is easy to see that inequality (3.1) holds for the random linear operator L from $\mathrm{span}\{e_i, i>0\}$ to $\mathbb{H}$ defined by $\langle Le_i,e_j\rangle_{\mathbb{H}}=\eta_{i,j}$. We then define the random linear operator Λ as in Section 3.2.1. To prove (3.57), we need the following elementary lemma:

Lemma 4. *Let $(B,\|.\|_B)$ be a Banach space. Assume that the sequences $(u_{n,\ell}),(u_n)$ and (v_ℓ) of elements of B satisfy*

$$\lim_{\ell\to+\infty}\limsup_{n\to+\infty}\|u_{n,\ell}-u_n\|_B=0 \quad \text{and} \quad \lim_{n\to+\infty}u_{n,\ell}=v_\ell.$$

Then the sequence (v_ℓ) converges in B.

Now apply Lemma 4 with $B=\mathbb{L}^1$, $v_\ell=\eta^{(\ell)}_{i,j}$, $u_n=n^{-1}\mathbb{E}(\langle S_n,e_i\rangle_{\mathbb{H}}\langle S_n,e_j\rangle_{\mathbb{H}}|\mathscr{I})$ and $u_{n,\ell}=n^{-1}\mathbb{E}(\langle S^{(\ell)}_n,e_i\rangle_{\mathbb{H}}\langle S^{(\ell)}_n,e_j\rangle_{\mathbb{H}}|\mathscr{I})$. From the decomposition

$$\begin{aligned}\|u_n-u_{n,\ell}\|_B&=\frac{1}{n}\,\mathbb{E}|\mathbb{E}(\langle S_n,e_i\rangle_{\mathbb{H}}\langle S_n,e_j\rangle_{\mathbb{H}}|\mathscr{I})-\mathbb{E}(\langle S^{(\ell)}_n,e_i\rangle_{\mathbb{H}}\langle S^{(\ell)}_n,e_j\rangle_{\mathbb{H}}|\mathscr{I})|\\&=\frac{1}{n}\mathbb{E}|\mathbb{E}(\langle S_n-S^{(\ell)}_n,e_i\rangle_{\mathbb{H}}\langle S_n,e_j\rangle_{\mathbb{H}}|\mathscr{I})\\&\quad+\mathbb{E}(\langle S^{(\ell)}_n,e_i\rangle_{\mathbb{H}}\langle S_n-S^{(\ell)}_n,e_j\rangle_{\mathbb{H}}|\mathscr{I})|,\end{aligned}$$

we easily derive that

$$\|u_n-u_{n,\ell}\|_B\leqslant\sqrt{\frac{1}{n}\,\mathbb{E}||S_n-S^{(\ell)}_n||^2_{\mathbb{H}}}\left(\sqrt{\frac{1}{n}\,\mathbb{E}||S_n||^2_{\mathbb{H}}}+\sqrt{\frac{1}{n}\,\mathbb{E}||S^{(\ell)}_n||^2_{\mathbb{H}}}\right). \tag{3.58}$$

Applying Lemma 4, there exists ℓ_0 such that

$$\text{for } \ell\geqslant\ell_0,\ \limsup_{n\to\infty}\left|\frac{\mathbb{E}\|S_n\|^2_{\mathbb{H}}}{n}-\frac{\mathbb{E}\|S^{(\ell)}_n\|^2_{\mathbb{H}}}{n}\right|\leqslant 1, \tag{3.59}$$

and hence $n^{-1}\mathbb{E}\|S_n\|^2_{\mathbb{H}}$ is bounded. Applying again Lemma 2, inequality (3.58) yields

$$\lim_{\ell\to\infty}\limsup_{n\to\infty}\|u_n-u_{n,\ell}\|_B=0. \tag{3.60}$$

Moreover, Proposition 1(i) combined with Cesaro's mean convergence theorem implies that $u_{n,\ell}$ converges to v_ℓ in $\mathbb{L}^1$. Applying Lemma 4 we obtain the first assertion of (3.57). We now prove the second assertion. Applying Fatou's lemma we obtain

$$\sum_{i=1}^{\infty} \mathbb{E}(\eta_{i,i}) \leqslant \liminf_{\ell\to\infty} \sum_{i=1}^{\infty} \mathbb{E}(\eta_{i,i}^{(\ell)}) = \liminf_{\ell\to\infty} \lim_{n\to\infty} \frac{\mathbb{E}\|S_n^{(\ell)}\|_{\mathbb{H}}^2}{n},$$

which is finite via (3.59).

We now complete the proof of (3.56). Since $P^{\varepsilon}_{\Lambda^{(\ell)}}$ and $P^{\varepsilon}_{\Lambda}$ are two Gaussian measures, we have

$$\left\| \int \exp(\mathrm{i}V_m(x)) P^{\varepsilon}_{\Lambda^{(\ell)}}\,\mathrm{d}x - \int \exp(\mathrm{i}V_m(x)) P^{\varepsilon}_{\Lambda}\,\mathrm{d}x \right\|_1 \leqslant \frac{1}{2} \left\| \sum_{i=1}^{m} \sum_{j=1}^{m} v_i v_j (\eta_{i,j}^{(\ell)} - \eta_{i,j}) \right\|_1 .$$

This inequality combined with (3.57) yields (3.56). Collecting (3.54)–(3.56) we obtain (3.51).

To complete the proof of Theorem 3, it remains to prove (3.52). Following the proof of (3.7), (3.52) will hold as soon as

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{\mathbb{E}\|(I_{\mathbb{H}} - P^m)S_n\|_{\mathbb{H}}^2}{n} = 0 \tag{3.61}$$

and

$$\lim_{m\to\infty} \mathbb{E}\left(\int \|(I_{\mathbb{H}} - P^m)x\|_{\mathbb{H}}^2 P^{\varepsilon}_{\Lambda}(\mathrm{d}x) \right) = 0. \tag{3.62}$$

Since Λ belongs to $\mathscr{S}(\mathbb{H}, \mathscr{A})$, (3.62) follows from the fact that

$$\mathbb{E}\left(\int \|(I_{\mathbb{H}} - P^m)x\|_{\mathbb{H}}^2 P^{\varepsilon}_{\Lambda}(\mathrm{d}x) \right) = \sum_{i=m+1}^{\infty} \mathbb{E}\langle \Lambda e_i, e_i \rangle_{\mathbb{H}}.$$

From Lemma 3 we know that (3.61) holds for $S_n^{(\ell)}$. This combined with Lemma 2 yields (3.61) and the proof of Theorem 3 is complete. □

3.4. Linear processes taking their values in $\mathbb{H}$

3.4.1. Proof of Theorem 4

We first show that the series in (2.7) is convergent in $\mathbb{L}^2_{\mathbb{H}}$. Note that for any sequence of linear bounded operators $(d_k)_{k\in\mathbb{Z}}$ on $\mathbb{H}$, and for any $-\infty < p < q < \infty$, we have

$$\mathbb{E}\left\| \sum_{k=p}^{q} d_k \xi_k \right\|_{\mathbb{H}}^2 = \mathbb{E}\left\| \sum_{k=p}^{q} d_k \sum_{j=-\infty}^{k} P_j(\xi_k) \right\|_{\mathbb{H}}^2 = \mathbb{E}\left\| \sum_{j=-\infty}^{q} P_j \left(\sum_{k=p\vee j}^{q} d_k \xi_k \right) \right\|_{\mathbb{H}}^2 .$$

For any functions f and g in $\mathbb{L}^2_{\mathbb{H}}(\mathbb{P})$ and $i \neq j$ we have $\mathbb{E}\langle P_j(f), P_i(g) \rangle_{\mathbb{H}} = 0$. Consequently,

$$\mathbb{E}\left\| \sum_{k=p}^{q} d_k \xi_k \right\|_{\mathbb{H}}^2 = \sum_{j=-\infty}^{q} \mathbb{E}\left\| \sum_{k=p\vee j}^{q} P_j(d_k \xi_k) \right\|_{\mathbb{H}}^2 \leqslant \sum_{j=-\infty}^{q} \left(\sum_{k=p\vee j}^{q} \|d_k\|_{L(\mathbb{H})} \|P_j(\xi_k)\|_{\mathbb{L}^2_{\mathbb{H}}} \right)^2 .$$

Applying Cauchy Schwarz's inequality, we obtain

$$\mathbb{E}\left\|\sum_{k=p}^{q} d_k\xi_k\right\|_{\mathbb{H}}^2 \leqslant \sum_{j=-\infty}^{q}\left(\sum_{k=p\vee j}^{q}\|d_k\|_{L(\mathbb{H})}^2\|P_j(\xi_k)\|_{\mathbb{L}^2_{\mathbb{H}}}\right)\left(\sum_{k=p\vee j}^{q}\|P_j(\xi_k)\|_{\mathbb{L}^2_{\mathbb{H}}}\right)$$

$$\leqslant \left(\sum_{k=0}^{\infty}\|P_0(\xi_k)\|_{\mathbb{L}^2_{\mathbb{H}}}\right)\left(\sum_{k=p}^{q}\|d_k\|_{L(\mathbb{H})}^2\sum_{j=-\infty}^{k}\|P_j(\xi_k)\|_{\mathbb{L}^2_{\mathbb{H}}}\right).$$

Hence, for any sequence of linear bounded operators $(d_k)_{k\in\mathbb{Z}}$ and $-\infty < p < q < \infty$,

$$\mathbb{E}\|\sum_{k=p}^{q} d_k\xi_k\|_{\mathbb{H}}^2 \leqslant \sum_{k=p}^{q}\|d_k\|_{L(\mathbb{H})}^2\left(\sum_{\ell=0}^{\infty}\|P_0(\xi_\ell)\|_{\mathbb{L}^2_{\mathbb{H}}}\right)^2. \tag{3.63}$$

Consequently, under (2.8) there exists a positive constant K such that

$$\mathbb{E}\left\|\sum_{k=p}^{q} d_k\xi_k\right\|_{\mathbb{H}}^2 \leqslant K\sum_{k=p}^{q}\|d_k\|_{L(\mathbb{H})}^2. \tag{3.64}$$

Inequality (3.64) together with Proposition 1.1 in Merlevède et al. (1997) imply that under (2.8) and (2.10), the series in (2.7) is convergent in $\mathbb{L}^2_{\mathbb{H}}$.

Now to show that if condition (2.8) is replaced by (2.9), the series in (2.7) still converges in $\mathbb{L}^2_{\mathbb{H}}$, it suffices to obtain a bound of type (3.64). Note first that

$$\mathbb{E}\left\|\sum_{j=p}^{q} d_j\xi_j\right\|_{\mathbb{H}}^2 \leqslant \mathbb{E}\|\xi_0\|_{\mathbb{H}}^2\left(\sum_{j=p}^{q}\|d_j\|_{L(\mathbb{H})}^2\right) + 2\sum_{i=p}^{q-1}\sum_{j=i+1}^{q}\mathbb{E}\langle d_i\xi_i, d_j\xi_j\rangle_{\mathbb{H}}$$

$$= \mathbb{E}\|\xi_0\|_{\mathbb{H}}^2\left(\sum_{j=p}^{q}\|d_j\|_{L(\mathbb{H})}^2\right) + 2\sum_{i=p}^{q-1}\sum_{j=i+1}^{q}\mathbb{E}\langle d_i\xi_i, d_j(\mathbb{E}(\xi_j|\mathscr{M}_i^{\xi}))\rangle_{\mathbb{H}}.$$

Since

$$\mathbb{E}\langle d_i\xi_i, d_j(\mathbb{E}(\xi_j|\mathscr{M}_i^{\xi}))\rangle_{\mathbb{H}} \leqslant \|d_i\|_{L(\mathbb{H})}\|d_j\|_{L(\mathbb{H})}\mathbb{E}(\|\xi_0\|_{\mathbb{H}}\|\mathbb{E}(\xi_{j-i}|\mathscr{M}_0^{\xi})\|_{\mathbb{H}})$$

$$\leqslant 1/2(\|d_i\|_{L(\mathbb{H})}^2 + \|d_j\|_{L(\mathbb{H})}^2)\mathbb{E}(\|\xi_0\|_{\mathbb{H}}\|\mathbb{E}(\xi_{j-i}|\mathscr{M}_0^{\xi})\|_{\mathbb{H}}),$$

we infer that

$$\sum_{i=p}^{q-1}\sum_{j=i+1}^{q}\mathbb{E}\langle d_i\xi_i, d_j(\mathbb{E}(\xi_j|\mathscr{M}_i^{\xi}))\rangle_{\mathbb{H}} \leqslant \sum_{i=p}^{q}\|d_i\|_{L(\mathbb{H})}^2\sum_{j=1}^{q}\mathbb{E}\{\|\xi_0\|_{\mathbb{H}}\|\mathbb{E}(\xi_j|\mathscr{M}_0^{\xi})\|_{\mathbb{H}}\}.$$

Therefore

$$\mathbb{E}\left\|\sum_{j=p}^{q} d_j\xi_j\right\|_{\mathbb{H}}^2 \leqslant 2\left(\sum_{j=p}^{q}\|d_j\|_{L(\mathbb{H})}^2\right)\sum_{k=0}^{q}\mathbb{E}\left(\|\xi_0\|_{\mathbb{H}}\|\mathbb{E}(\xi_k|\mathscr{M}_0^{\xi})\|_{\mathbb{H}}\right). \tag{3.65}$$

which proves (3.64).

Now note that under (2.8) (resp. (2.9)), Corollary 3 (resp. 2) ensures that for any φ in $\mathscr{H}$ and any positive integer k,

$$\lim_{n\to\infty}\left\|\mathbb{E}\left(\varphi\left(n^{-1/2}\sum_{k=1}^{n}\xi_k\right)-\int\varphi(x)P^{\varepsilon}_{\Lambda^{\xi}}(\mathrm{d}x)\,\middle|\,\mathscr{M}^{\xi}_{k}\right)\right\|_1=0,$$

so that

$$\lim_{n\to\infty}\left\|\mathbb{E}\left(\varphi\left(n^{-1/2}A\sum_{k=1}^{n}\xi_k\right)-\int\varphi(x)P^{\varepsilon}_{\Lambda^{\xi}_{A}}(\mathrm{d}x)\,\middle|\,\mathscr{M}^{\xi}_{k}\right)\right\|_1=0. \tag{3.66}$$

Now according to the proof of Theorem 1, (2.11) holds as soon as conditions (3.51) and (3.52) are satisfied with Λ^{ξ}_{A} replacing Λ. Following the proof of Theorem 3, we infer that these conditions hold under (3.66) provided that

$$\lim_{n\to\infty}\frac{1}{n}\,\mathbb{E}\left\|\sum_{k=1}^{n}X_k-A\sum_{k=1}^{n}\xi_k\right\|^2_{\mathbb{H}}=0. \tag{3.67}$$

According to Proposition 1 in Merlevède et al. (1997), this holds as soon as a result of type (3.64) holds. This completes the proof of Theorem 4. □

3.4.2. *Proof of Theorem 5*

According to the proof of Theorem 4, the series in (2.6) is convergent in $\mathbb{L}^2_{\mathbb{H}}$ under (2.8) and (2.10). Since $P_0(\xi_m)=0$ as soon as $m\leqslant -1$, we have

$$\|P_0(X_k)\|_{\mathbb{L}^2_{\mathbb{H}}}=\left\|\sum_{j\geqslant 0}a_jP_0(\xi_{k-j})\right\|_{\mathbb{L}^2_{\mathbb{H}}}=\left\|\sum_{j=0}^{k}a_jP_0(\xi_{k-j})\right\|_{\mathbb{L}^2_{\mathbb{H}}},$$

and consequently

$$\|P_0(X_k)\|_{\mathbb{L}^2_{\mathbb{H}}}\leqslant\sum_{j=0}^{k}\|a_jP_0(\xi_{k-j})\|_{\mathbb{L}^2_{\mathbb{H}}}\leqslant\sum_{j=0}^{k}\|a_j\|_{L(\mathbb{H})}\|P_0(\xi_{k-j})\|_{\mathbb{L}^2_{\mathbb{H}}}.$$

Summing in k, we obtain that

$$\sum_{k=0}^{\infty}\|P_0(X_k)\|_{\mathbb{L}^2_{\mathbb{H}}}\leqslant\sum_{j=0}^{\infty}\|a_j\|_{L(\mathbb{H})}\sum_{k=j}^{\infty}\|P_0(\xi_{k-j})\|_{\mathbb{L}^2_{\mathbb{H}}},$$

and we infer that (2.5) is satisfied under (2.8) and (2.10). Now Corollary 3 implies that there exists a random linear operator $\tilde{\Lambda}$ belonging to $\mathscr{S}(\mathbb{H},\mathscr{M}^{\xi}_{0})$ such that for any φ in $\mathscr{H}^*$ and any positive integer k,

$$\lim_{n\to\infty}\left\|\mathbb{E}\left(\varphi(n^{-1/2}W_n)-\int\varphi(x)W_{\tilde{\Lambda}}(\mathrm{d}x)\,\middle|\,\mathscr{M}^{\xi}_{k}\right)\right\|_1=0.$$

Moreover according to Remark 5, for any ℓ,m in $\mathbb{N}^*$, $\langle\tilde{\Lambda}e_\ell,e_m\rangle_{\mathbb{H}}=\tilde{\eta}_{\ell,m}$ where, $\tilde{\eta}_{\ell,m}$ is the limit in $\mathbb{L}^1$ of the sequence defined in (2.1). Applying Theorem 4, we easily infer that $\tilde{\Lambda}=A\circ\Lambda^{\xi}\circ A^*$ almost surely, which ends the proof of (2.12).

Acknowledgements

We would like to thank the referee not only for his valuable advices and suggestions which helped us greatly in the writing of the revised version, but also for having pointed out a gap in the proof of Corollary 3 of a previous version of the paper.

References

Aldous, D.J., Eagleson, G.K., 1978. On mixing and stability of limit theorems. Ann. Probab. 6, 325–331.
Araujo, A., Giné, E., 1980. The Central Limit Theorem for Real and Banach Valued Random Variables. Wiley, New York.
Billingsley, P., 1968. Convergence of Probability Measures. Wiley, New York.
Bingham, M.S., 2000. Approximate martingale central limit theorems on Hilbert space. C.R. Math. Rep. Acad. Sci. Canada 22, 111–117.
Bosq, D., 2000. Linear Processes in Function Spaces. Theory and Applications. Lecture Notes in Statistics, Vol. 149. Springer, Berlin.
Bradley, R.C., 1997. On quantiles and the central limit question for strongly mixing sequences. J. Theoret. Probab. 10, 507–555.
Brockwell, P., Davis, R., 1987. Time Series: Theory and Methods. Springer, Berlin.
Chen, X., White, H., 1998. Central limit and functional central limit theorems for Hilbert-valued dependent heterogeneous arrays with applications. Econometric Theory 14, 260–284.
Davydov, Yu.A., 1973. Mixing conditions for Markov chains. Theory Probab. Appl. 18, 312–328.
Dedecker, J., Doukhan, P., 2003. A new covariance inequality and applications. Stochastic Process. Appl. 106, 63–80.
Dedecker, J., Merlevède, F., 2002. Necessary and sufficient conditions for the conditional central limit theorem. Ann. Probab. 30, 1044–1081.
Dedecker, J., Rio, E., 2000. On the functional central limit theorem for stationary processes. Ann. Inst. H. Poincaré Probab. Statist. 36, 1–34.
Dehling, H., 1983. Limit theorems for sums of weakly dependent Banach space valued random variables. Z. Wahrsch. Verw. Gebiete 63, 393–432.
Doukhan, P., Massart, P., Rio, E., 1994. The functional central limit theorem for strongly mixing processes. Ann. Inst. H. Poincaré Probab. Statist. 30, 63–82.
Eagleson, G.K., 1976. Some simple conditions for limit theorems to be mixing. Teor. Verojatnost. i Primenen. 21, 653–660.
Fréchet, M., 1957. Sur la distance de deux loi de probabilité. C.R. Acad. Sci. Paris 244, 689–692.
Gordin, M.I., 1969. The central limit theorem for stationary processes. Dokl. Akad. Nauk SSSR 188, 739–741.
Gordin, M.I., 1973. Abstracts of Communication, T.1: A-K. International Conference on Probability Theory, Vilnius.
Hall, P., Heyde, C.C., 1980. Martingale Limit Theory and its Applications. Academic Press, New York, London.
Heyde, C.C., 1974. On the central limit theorem for stationary processes. Z. Wahrsch. Verw. Gebiete 30, 315–320.
Hoffman-Jorgensen, J., Pisier, G., 1976. The law of large numbers and the central limit theorem in Banach spaces. Ann. Probab. 4, 587–599.
Ibragimov, I.A., Linnik, Yu.V., 1971. Independent and Stationary Sequences of Random Variables. Wolters-Noordholf, Groningen.
Jain, N.C., 1977. Central limit theorem and related question in Banach spaces. Proc. Symposium in Pure Mathematics, American Mathematical Society, Vol. XXXI, Providence, RI, pp. 55–65.
Jakubowski, A., 1980. On limit theorems for sums of dependent Hilbert space valued random variables. In: Lecture Notes in Statistics, Vol. 2. Springer, New York, pp. 178–187.

Kuelbs, J., 1973. The invariance principle for Banach space valued random variables. J. Multivariate Anal. 3, 161–172.

McLeish, D.L., 1975a. Invariance principles for dependent variables. Z. Wahrsch. Verw. Gebiete 32, 165–178.

McLeish, D.L., 1975b. A maximal inequality and dependent strong laws. Ann. Probab. 3, 829–839.

McLeish, D.L., 1977. On the invariance principle for non stationary mixingales. Ann. Probab. 5, 616–621.

Merlevède, F., 1995. Sur l'inversibilité des processus linéaires à valeurs dans un espace de Hilbert. C.R. Acad. Sci. Paris 321 (Série I), 477–480.

Merlevède, F., 2003. On the central limit theorem and its weak invariance principle for strongly mixing sequences with values in a Hilbert space via martingale approximation. J. Theor. Probab., to appear.

Merlevède, F., Peligrad, M., 2000. The functional central limit theorem for strong mixing sequences of random variables. Ann. Probab. 28 (3), 1336–1352.

Merlevède, F., Peligrad, M., Utev, S., 1997. Sharp conditions for the CLT of linear processes in a Hilbert space. J. Theor. Probab. 10, 681–693.

Rényi, A., 1963. On stable sequences of events. Sankhyā Ser. A 25, 293–302.

Rosenblatt, M., 1956. A central limit theorem and a strong mixing condition. Proc. Nat. Acad. Sci. USA 42, 43–47.

Rio, E., 1993. Covariance inequalities for strongly mixing processes. Ann. Inst. H. Poincaré Probab. Statist. 29, 587–597.

Rudin, W., 1987. Real and Complex Analysis, 3rd Edition. McGraw-Hill Book Co., New York.

van der Waart, A.W., Wellner, J.A., 1996. Weak Convergence and Empirical Processes. Springer, Berlin.

Walk, H., 1977. An invariance principle for the Robbins-Monro process in a Hilbert space. Z. Wahrsch. Verw. Gebiete 39, 135–150.